Diameters, centers, and approximating trees of δ-hyperbolic geodesic spaces and graphs *

[Extended Abstract]

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ABSTRACT

δ-Hyperbolic metric spaces have been defined by M. Gromov in 1987 via a simple 4-point condition: for any four points u, v, w, x, the two larger of the distance sums $d(u,v)+d(w,x), d(u,w)+d(v,x), d(u,x)+d(v,w)$ differ by at most 2δ. They play an important role in geometric group theory, geometry of negatively curved spaces, and have recently become of interest in several domains of computer science.

Given a finite set S of points of a δ-hyperbolic space, we present simple and fast methods for approximating the diameter of S with an additive error 2δ and computing an approximate radius and center of a smallest enclosing ball for S with an additive error 3δ. These algorithms run in linear time for classical hyperbolic spaces and for δ-hyperbolic graphs and networks. Furthermore, we show that for δ-hyperbolic graphs $G = (V, E)$ with uniformly bounded degrees of vertices, the exact center of S can be computed in linear time $O(|E|)$. We also provide a simple construction of distance approximating trees of δ-hyperbolic graphs $G = (V, E)$ on n vertices with an additive error $O(\delta \log_2 n)$. This construction has an additive error comparable with that given by M. Gromov for n-point δ-hyperbolic spaces, but can be implemented in linear time $O(|E|)$ (instead of $O(n^2)$). Finally, we establish that several geometrically defined classes of graphs have bounded hyperbolicity.

Categories and Subject Descriptors: F.2.2 [**Analysis of Algorithms and Problem Complexity**]: Nonnumerical Algorithms and Problems

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1. INTRODUCTION

Given a finite set S of points of a metric space (X, d), the *diameter* $diam(S)$ of S is the maximum distance between any two points of S. A *diametral pair* of S is any pair of points $x, y \in S$ such that $d(x, y) = diam(S)$. For a point $x \in X$, the set $F_S(x)$ of *furthest neighbors* of S consists of all points of S located at the maximum distance from x. The *eccentricity* $ecc_S(x)$ of the point $x \in X$ is the distance from x to any point of $F_S(x)$. The *center* $C(S)$ of S is the set of all points of X having minimum eccentricity; the points of $C(S)$ are called *central points*. The *radius* $rad(S)$ of S is the eccentricity of central points. In other words, $rad(S)$ is the smallest radius of a ball of (X, d) enclosing all points of S (recall that a (closed) *ball* $B(c, r)$ of radius r centered at $c \in X$ consists of all points $x \in X$ at distance at most r to c, i.e., $B(c, r) = \{x \in X : d(c, x) \leq r\}$).

Computing the diameter, the radius, and the center of a point set in geometric or discrete metric spaces are basic algorithmic problems in computational geometry and graph theory having numerous applications in operation research, data clustering, and analysis of complex networks (social networks and the internet). In computational geometry, the diameter and center problems have been investigated for point sets in two-, three-, or high-dimensional vector spaces endowed with usual metrics [17, 22, 38, 42] and for polygonal or polyhedral domains endowed with the geodesic [31, 41] or link metrics [19, 35, 46] (the cited papers represent just a small sample of references). For graphs and networks, the problems were introduced by Hakimi [30], and efficient algorithms for these problems are known for several classes of graphs [6, 13, 15, 18, 21]. Quite surprisingly, most of these algorithms are based on geometric and metric properties of classes of graphs in question.

The investigation of the diameter and center problem for trees and tree-networks goes back to C. Jordan, who established that the center of a tree consists of one or two adjacent vertices and the center of a tree-network is a single point. It is folklore that the diameter $diam(S)$ of a

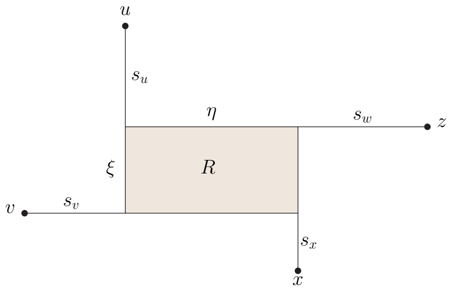

Figure 1: Realization of a 4-point metric in the rectilinear plane.

set S in a tree or a tree-network T can be found in linear time by running the following simple algorithm: *pick an arbitrary point or vertex u of T, run a Breadth-First-Search (BFS) starting from u to find $v \in F_S(u)$, then run a second BFS starting from v to find $w \in F_S(v)$.* Then $d(v,w) = diam(S)$, i.e., v, w is a diametral pair of S. To find the center of S it suffices to take the middle point c of the unique (u,v)-path if T is a tree-network or to take one or two adjacent middle vertices of this path if T is a graphic tree. This shows that $diam(S) = 2rad(S)$ in tree-networks and $diam(S) = 2rad(S)$ or $2rad(S) - 1$ in trees.

In this paper, we establish that this approach can be adapted to provide fast and accurate approximations of the diameter, radius, and center of finite sets S of δ-hyperbolic geodesic spaces and graphs. We show that if $v \in F_S(u)$ and $w \in F_S(v)$, then $d(v,w) \geq diam(S) - 2\delta$ and that $rad(S) \leq d(v,w)/2 + 3\delta$. We also prove that the center $C(S)$ of S is contained in the ball of radius 5δ ($5\delta + 1$ for graphs) centered at the middle point c of any (v,w)-geodesic. This provides a linear time algorithm for computing the center of δ-hyperbolic graphs with uniformly bounded degrees of vertices. We also give a simple linear-time construction of distance approximating trees of δ-hyperbolic graphs $G = (V,E)$ on n vertices with an additive error $O(\delta \log_2 n)$. Finally, we establish that several geometrically defined classes of graphs have bounded hyperbolicity.

1.1 δ-Hyperbolicity

Introduced by Gromov [26], δ-hyperbolicity measures, to some extent, the deviation of a metric from a tree metric. Recall that a metric space (X,d) embeds into a tree network (with positive real edge lengths), that is, d is a *tree metric,* iff for any four points u, v, w, x, the two larger of the distance sums $d(u,v) + d(w,x), d(u,w) + d(v,x), d(u,x) + d(v,w)$ are equal. A metric space (X,d) is called *δ-hyperbolic* if the two largest distance sums differ by at most 2δ. A connected graph $G = (V,E)$ equipped with standard graph metric d_G is *δ-hyperbolic* if the metric space (V, d_G) is δ-hyperbolic. Every 4-point metric d (tree-realizable or not) has a canonical representation in the rectilinear plane. In Fig. 1, the three distance sums are ordered from small to large, thus implying $\xi \leq \eta$. Then η is half the difference of the largest and the smallest sum, while ξ is half the largest minus the medium sum. Hence, a metric space (X,d) is δ-hyperbolic iff ξ does not exceed δ for any four points u, v, w, x of X. 0-Hyperbolic metric spaces are exactly the tree metrics. On the other hand, the Poincaré half space in $\mathbb{R}^k$ with the hyperbolic metric is δ-hyperbolic with $\delta = \log_2 3$. Several classes of geodesic metric spaces are known to be hyperbolic (a metric space (X,d) is called *hyperbolic* if it is δ-hyperbolic for some constant δ and the exact value of δ does not matter).

δ-Hyperbolic metric spaces play an important role in geometric group theory and in geometry of negatively curved spaces [3, 25, 26]. δ-Hyperbolicity captures the basic common features of "negatively curved" spaces like the classical real-hyperbolic space $\mathbb{H}^k$, Riemannian manifolds of strictly negative sectional curvature, and of discrete spaces like trees and the Caley graphs of word-hyperbolic groups. It is remarkable that a strikingly simple concept leads to such a rich general theory [3, 25, 26]. More recently, the concept of δ-hyperbolicity emerged in discrete mathematics, algorithms, and networking. For example, it has been shown empirically in [44] that the internet topology embeds with better accuracy into a hyperbolic space than into a Euclidean space of comparable dimension. A few algorithmic problems in hyperbolic spaces and hyperbolic graphs have been considered in recent papers [16, 23, 24, 34]. Krauthgamer and Lee [34] present a PTAS for the Traveling Salesman Problem when the set of cites lie in $\mathbb{H}^k$. They also show how to preprocess a finite subset of a δ-hyperbolic geodesic space with a uniformly bounded local geometry to efficiently answer nearest-neighbor queries with an additive error $O(\delta)$. Chepoi and Estellon [16] establish a relationship between the minimum number of balls of radius $R + 2\delta$ covering a finite subset S of a δ-hyperbolic geodesic space and the size of the maximum R-packing of S and show how to compute such coverings and packings in polynomial time.

1.2 Related work on diameters and centers

The simple schema for computing the diameters and centers of trees has been used several times in more general contexts. For example, Malandain and Boissonnat [37] called one computation of a furthest neighbor a *FP scan* and use repetitive FP scans to approximate or compute the Euclidean diameter of high-dimensional pointsets ([18] consider a similar approach for fast approximation of diameter of some classes of graphs). Lenhart *et al.* [35] establish that the diameter and radius of a simple polygon P with the link metric satisfies the inequality $diam(P) \geq 2rad(P) - 2$ and use this relationship to find a simple approximation of the link center of P in $O(n \log n)$ time and to compute it in $O(n^2)$ time ($O(n \log n)$-time algorithms for computing the link center and link diameter have been presented in [19, 46]). The papers [11, 12] show that a similar relationship holds for the diameter and radius of a simple rectilinear polygon with the rectilinear link metric (recently this result has been rediscovered by Magazanik and Perles [36] in the more general context of simply connected planar domains). It was also shown in [11, 12] that for rectilinear link distance, as for trees, after two FP scans, the returned distance $d(v,w)$ is a good approximation of the diameter of P and that a cut c passing via a middle edge of any shortest (v,w)-path is close to the center of P. Then using methods of computational geometry it is possible to compute in linear time a central point of P (another linear time algorithm for this problem has been proposed in [39]). As shown in [13], the approach to link centers can be appropriately modified to compute the centers of chordal graphs (graphs in which all induced cycles have length 3) in linear time $O(|E|)$. It was also shown that $diam(G) \geq 2rad(G) - 2$ holds for such graphs. Although $d(v,w)$ gives an approximation of the diameter of a chordal graph G with an additive error ≤ 1, computing the exact diameter of G in subquadratic time seems to be not easier than a similar problem for general graphs [2].

As we will show below, the simple polygons or simple rectilinear polygons with link metric as well as some classes of graphs comprising chordal graphs are 1- or 2-hyperbolic. Thus, the results of our paper present a general framework for the properties and methods developed in [11, 12, 13, 18, 35] and some other papers. Our general results, directly applied to these graphs, are not sharp, which is quite natural, because the setting of link graphs or chordal graphs allow to get more accurate results.

2. GEODESIC δ-HYPERBOLIC SPACES

Let (X, d) be a metric space. A *geodesic segment* joining two points x and y from X is a map ρ from the segment $[a, b]$ of length $|a-b| = d(x, y)$ to X such that $\rho(a) = x, \rho(b) = y$, and $d(\rho(s), \rho(t)) = |s-t|$ for all $s, t \in [a, b]$. A metric space (X, d) is *geodesic* if every pair of points in X can be joined by a geodesic. Every graph $G = (V, E)$ equipped with its standard distance d_G can be transformed into a geodesic (networklike) space (X, d) by replacing every edge $e = (u, v)$ by a segment $[u, v]$ of length 1; the segments may intersect only at common ends. Then (V, d_G) is isometrically embedded in a natural way in (X, d). More generally, any polygonal, simplicial, or cubical complex can be transformed into a geodesic metric space by introducing an intrinsic metric on its geometric realization; for details see [9].

In case of geodesic metric spaces, there exist several equivalent definitions of δ-hyperbolicity involving different but comparable values of δ [3, 9, 25, 26]. A *geodesic triangle* $\Delta(x, y, z)$ with vertices $x, y, z \in X$ is union $[x, y] \cup [x, z] \cup [y, z]$ of three geodesic segments connecting these vertices. Let m_x be the point of the geodesic segment $[y, z]$ located at distance $\alpha_y := (d(y, x) + d(y, z) - d(x, z))/2$ from y. Then m_x is located at distance $\alpha_z := (d(z, y) + d(z, x) - d(y, x))/2$ from z because $\alpha_y + \alpha_z = d(y, z)$. Analogously, define the points $m_y \in [x, z]$ and $m_z \in [x, y]$ both located at distance $\alpha_x := (d(x, y) + d(x, z) - d(y, z))/2$ from x; see Fig. 2 for an illustration. There exists a unique isometry φ which maps $\Delta(x, y, z)$ to a star $\Upsilon(x', y', z')$ consisting of three solid segments $[x', m'], [y', m']$, and $[z', m']$ of lengths α_x, α_y, and α_z, respectively. This isometry maps the vertices x, y, z of $\Delta(x, y, z)$ to the respective leaves x', y', z' of $\Upsilon(x', y', z')$ and the points m_x, m_y, and m_z to the center m of this tripod. Any other point of $\Upsilon(x', y', z')$ is the image of exactly two points of $\Delta(x, y, z)$. A geodesic triangle $\Delta(x, y, z)$ is called *δ-thin* if for all points $u, v \in \Delta(x, y, z)$, $\varphi(u) = \varphi(v)$ implies $d(u, v) \leq \delta$. A geodesic triangle $\Delta(x, y, z)$ is called *δ-slim* if for any point u on the side $[x, y]$ the distance from u to $[x, z] \cup [z, y]$ is at most δ. The notions of geodesic triangles, δ-slim and δ-thin triangles can be also defined in case of graphs. The single difference is that for graphs, the center of the tripod is not necessarily the image of any vertex on the geodesic of $\Delta(x, y, z)$. Nevertheless, if a point of the tripod is the image of a vertex of one side of $\Delta(x, y, z)$, then it is also the image of another vertex located on another side of $\Delta(x, y, z)$. The following result shows that hyperbolicity of a geodesic space is equivalent to having thin or slim geodesic triangles (the same result holds for graphs).

PROPOSITION 1. *[3, 9, 26, 25] Geodesic triangles of geodesic δ-hyperbolic spaces are 4δ-slim and 4δ-thin. Conversely, geodesic spaces with δ-thin triangles are 2δ-hyperbolic and geodesic spaces with δ-slim triangles are 8δ-hyperbolic.*

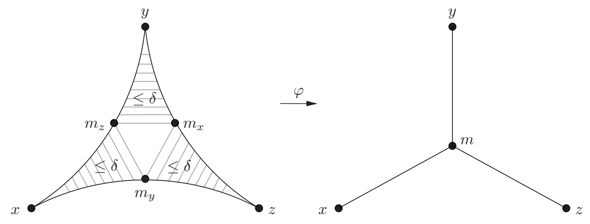

Figure 2: A geodesic triangle $\Delta(x, y, z)$, the points m_x, m_y, m_z, and the tripod $\Upsilon(x', y', z')$

Gromov [26, 25] established that any δ-hyperbolic metric on n points can be approximated in $O(n^2)$ time by a tree-metric with an additive error $O(\delta \log_2 n)$:

THEOREM 1. *For a δ-hyperbolic space (X, d) on n points with a root-point s there exists a weighted tree T and a mapping $\varphi : X \mapsto T$ such that $d_T(\varphi(s), \varphi(x)) = d(s, x)$ for any $x \in X$ and $d(x, y) - 2\delta \log_2 n \leq d_T(\varphi(x), \varphi(y)) \leq d(x, y)$ for any $x, y \in X$.*

3. DIAMETERS, RADII, AND CENTERS

3.1 Some properties of δ-hyperbolic spaces

In this subsection, we establish two new metric properties of δ-hyperbolic spaces which will be used in subsequent subsections on diameters and centers.

LEMMA 1. *Let (X, d) be a δ-hyperbolic space and x, y, v, u be its four arbitrary points. If $d(v, u) \geq \max\{d(y, u), d(x, u)\}$, then $d(x, y) \leq \max\{d(v, x), d(v, y)\} + 2\delta$.*

PROOF. Consider the canonical realization of the 4-point metric space $\{u, v, x, y\}$ in $\mathbb{R}^2$ consisting of a rectangle R and four segments s_u, s_v, s_x, and s_y incident to u, v, x, and y, respectively as shown in Fig. 1. Denote by d_u, d_v, d_x, and d_y the lengths of the segments s_u, s_v, s_x, and s_y, respectively. By definition of δ-hyperbolicity, we have $\xi \leq \delta$.

First suppose that the segments s_u and s_v are incident to opposite corners of the rectangle R (then other two segments are incident to two other opposite corners of R). Then $d(u, v) = d_u + \eta + \xi + d_v$ and $d(x, y) = d_x + \eta + \xi + d_y$. Assume without loss of generality that $d(u, x) = d_u + \eta + d_x$ and $d(u, y) = d_u + \xi + d_y$. We have $d(u, v) \geq \max\{d(u, x), d(u, y)\}$. From previous equalities we conclude that $\xi + d_v \geq d_x$. Hence, we obtain

$$\begin{aligned} d(v, y) &= d_v + \eta + d_y \geq d_x - \xi + \eta + d_y \\ &= d_x + \xi + \eta + d_y - 2\xi \\ &= d(x, y) - 2\xi \geq d(x, y) - 2\delta. \end{aligned}$$

Now suppose that the segments s_u and s_v are incident to corners of R defining a side of length ξ of R. Assume without loss of generality that the segments s_u and s_y are incident to opposite corners of R. Then, $d(u, v) = d_u + \xi + d_v$ and $d(u, y) = d_u + \xi + \eta + d_y$. From these equalities and since $d(u, v) \geq d(u, y)$, we conclude that $d_v \geq \eta + d_y$. Hence,

$$\begin{aligned} d(v, x) &= d_v + \xi + \eta + d_x \geq d_y + \xi + d_x + 2\eta \\ &= d(x, y) + 2\eta \geq d(x, y). \end{aligned}$$

Finally, suppose that the segments s_u and s_v are incident to corners of R defining a side of length η of R. Assume without loss of generality that the segments s_u and s_x are incident to opposite corners of R. Then, $d(u, v) = d_u + \eta + d_v$

and $d(u,x) = d_u + \xi + \eta + d_x$. Since $d(u,v) \geq d(u,x)$, we conclude that $d_v \geq \xi + d_x$. Therefore,

$$\begin{aligned} d(v,y) &= d_v + \xi + \eta + d_y \geq d_x + 2\xi + \eta + d_y \\ &= d(x,y) + 2\xi \geq d(x,y). \end{aligned}$$

Thus, in all three cases, we obtain that $\max\{d(v,x), d(v,y)\} \geq d(x,y) - 2\delta$. □

Next, we establish a Helly-type property for balls in δ-hyperbolic geodesic spaces and graphs. This result was first proved in [16] in case of geodesic spaces with δ-thin geodesic triangles. Since geodesic triangles of a δ-hyperbolic space are only 4δ-thin [3, 25], in order to get a sharper result we present a proof employing the basic definition of δ-hyperbolicity. Let (X,d) be a δ-hyperbolic geodesic space or graph. Let S be a finite subset of points of X and let $r : S \to \mathbb{R}^+$ ($r : S \to \mathbb{N}$ in case of graphs) be a radius function associating with each point $s \in S$ a positive number $r(s)$. We say that a subset S' of S is *r-dominated* by a point $c \in X$ if $d(c,s) \leq r(s)$ for each $s \in S'$. For each point $x \in S$, define the set $S_x := \{y \in S : r(x) + r(y) \geq d(x,y)\}$. We continue with the following important auxiliary result.

LEMMA 2. *For any finite or compact subset S of points of a δ-hyperbolic geodesic space (X,d) and for any radius function $r : S \to \mathbb{R}^+$, there exist two points $x \in S$ and $c \in X$ such that $d(c,y) \leq r(y) + 2\delta$ for any point $y \in S_x$, i.e., the set S_x is $(r+2\delta)$-dominated by c. An analogous result holds for any finite subset S of vertices of a δ-hyperbolic graph G and for any radius function $r : S \to \mathbb{N}$.*

PROOF. If G is a graph, then we consider it as a geodesic space as defined above. Denote the resulting metric by d (notice that $d(u,v) = d_G(u,v)$ if u,v are vertices of G). Let z be an arbitrary point of X (an arbitrary vertex of G) and let x be a point (vertex) of S maximizing the value $M := d(x,z) - r(x)$ (such a point exists because S is compact). If $M \leq \delta$, then z $(r+\delta)$-dominates all points of S and we can set $c := z$. Suppose now that $M > \delta$. Pick a geodesic segment $[x,z]$ between x and z, and let c be the point of $[x,z]$ located at distance $r(x)$ from x. (If G is a graph, since $r(x)$ is an integer, c is a vertex of G.) Consider any $y \in S_x$, i.e., y is a point of S such that $r(x) + r(y) \geq d(x,y)$. We assert that $d(y,c) \leq r(y) + 2\delta$. To show this, pick any two geodesic segments $[x,y]$ and $[y,z]$ between the pairs x,y and y,z. Let $\Delta(x,y,z) := [x,y] \cup [x,z] \cup [y,z]$ be the geodesic triangle formed by the three geodesic segments and let m_x, m_y, and m_z be the three points on these geodesics as defined above. Notice that if G is a graph, then m_x, m_y, and m_z are not necessarily vertices of G. We distinguish between two cases.

First suppose that c belongs to the portion of $[x,z]$ comprised between the points x and m_y, i.e., $r(x) \leq \alpha_x$. Let $\varepsilon = d(x,m_y) - d(x,c)$. Consider the three distance sums defined by the quadruplet x,y,c,z. We have

$$\begin{aligned} d(x,y) + d(c,z) &= \alpha_x + \alpha_y + \varepsilon + \alpha_z, \\ d(x,z) + d(c,y) &= r(x) + \varepsilon + \alpha_z + d(c,y), \\ d(x,c) + d(y,z) &= r(x) + \alpha_y + \alpha_z. \end{aligned}$$

Since $\alpha_x \geq r(x)$, we conclude that $d(x,y)+d(c,z) \geq d(x,c)+d(y,z)$. Notice also that $d(x,z) + d(c,y) \geq d(x,c) + d(y,z)$ because $\alpha_y + \alpha_z = d(y,z)$ and $d(y,z) \leq d(z,c) + d(c,y) = \alpha_z + \varepsilon + d(c,y)$ by triangle inequality. If $d(x,y) + d(c,z) > d(x,z) + d(c,y)$, then $\alpha_x + \alpha_y > r(x) + d(c,y)$. Hence $d(c,y) < d(x,y) - r(x) \leq r(y)$, establishing the required property. Now, let $d(x,z) + d(c,y) \geq d(x,y) + d(c,z)$. By δ-hyperbolicity, we obtain $d(x,z) + d(c,y) - d(x,y) - d(c,z) \leq 2\delta$, yielding $r(x) + d(c,y) - d(x,y) \leq 2\delta$. Since $d(x,y) \leq r(x)+r(y)$, we obtain $d(c,y) \leq d(x,y)+2\delta-r(x) \leq r(x) + r(y) + 2\delta - r(x) = r(y) + 2\delta$, establishing that y is $(r+2\delta)$-dominated by c.

Now suppose that c belongs to the portion of $[x,z]$ comprised between z and m_y, i.e., $\alpha_x < r(x)$. Let again $\varepsilon = d(c,m_y)$. The choice of the vertices x and z yields $d(y,z) - r(y) \leq d(x,z) - r(x)$. Since $d(x,z) = \alpha_x + \alpha_z$ and $d(y,z) = \alpha_y + \alpha_z$, we conclude that $\alpha_y - r(y) \leq \alpha_x - r(x)$. Thus $\varepsilon = d(c,m_y) = r(x) - \alpha_x \leq r(y) - \alpha_y$. The three distance sums for the quadruplet x,y,z,c have the form:

$$\begin{aligned} d(x,y) + d(c,z) &= \alpha_x + \alpha_y - \varepsilon + \alpha_z, \\ d(x,z) + d(c,y) &= \alpha_x + \alpha_z + d(c,y), \\ d(x,c) + d(y,z) &= \alpha_x + \varepsilon + \alpha_y + \alpha_z. \end{aligned}$$

Obviously, we have $d(x,y)+d(c,z) \leq d(x,c)+d(y,z)$. If the second distance sum takes the smallest value, then $d(x,z) + d(c,y) \leq d(x,y) + d(c,z) \leq r(x) + r(y) + d(x,z) - r(x)$, yielding $d(c,y) \leq r(y)$ and we are done. Now, let the first sum be the smallest one. First, if $d(x,z)+d(c,y) > d(x,c)+d(y,z)$, then δ-hyperbolicity and $\varepsilon \leq r(y) - \alpha_y$ imply that

$$d(c,y) \leq \varepsilon + \alpha_y + 2\delta \leq r(y) - \alpha_y + \alpha_y + 2\delta = r(y) + 2\delta.$$

On the other hand, if $d(x,z) + d(c,y) \leq d(x,c) + d(y,z)$, then $\varepsilon \leq r(y) - \alpha_y$ implies that

$$d(c,y) \leq \varepsilon + \alpha_y \leq r(y) - \alpha_y + \alpha_y = r(y).$$

In both cases we conclude that $d(c,y) \leq r(y) + 2\delta$. □

The following result can be viewed as an analog of the classical Helly property for balls.

PROPOSITION 2. *Let S be a finite or a compact subset of a δ-hyperbolic geodesic space (X,d) or a δ-hyperbolic graph $G = (V,E)$. Let $r : S \to \mathbb{R}^+$ (or $r : S \to \mathbb{N}$ for graphs) be a radius function such that the balls of the family $\mathcal{F} = \{B(x,r(x)) : x \in S\}$ pairwise intersect. Then the balls $\{B(x,r(x)+2\delta) : x \in S\}$ have a nonempty common intersection.*

PROOF. Since $d(x,y) \leq r(x)+r(y)$, for each pair $x,y \in S$, the equality $S_x = S$ holds for the center x of any ball. By Lemma 2, the set S is $(r+2\delta)$-dominated by a single point (vertex) c. Obviously c belongs to all balls $B(x,r(x)+2\delta)$, $x \in S$, establishing the result. □

Note that a similar result with 1 instead of 2δ has been established in [35] for balls of simple polygons endowed with the link distance.

3.2 Diameters and radii

In this subsection, we use the properties established in previous subsection to get fast approximations for diameters and radii of finite subsets S of δ-hyperbolic spaces (X,d).

We start with the analysis in δ-hyperbolic spaces (X,d) of the simple heuristic for computing a diametral pair in trees using two FP scans. Recall, it consists in picking any point u of the space X, finding $v \in F_S(u)$, then finding $w \in F_S(v)$, and returning the pair $\{v,w\}$. This algorithm can be implemented in $O(|S|)$ time if computing the distance between two points in (X,d) can be done in constant time. In particular, this is the case when (X,d) is a model of the hyperbolic plane. On the other hand, in graphs $G = (V, E)$ (and, more generally, networklike spaces) one FP scan can be done in $O(|E|)$ time by BFS or Dijkstra's algorithms. Thus, in this case the pair $\{u, v\}$ can be computed in linear $O(|E|)$ time. It remains to establish how well $d(v, w)$ approximates the diameter $diam(S)$ of S.

Let x, y be a diametral pair of S. Since $v \in F_S(u)$, we conclude that $d(u, v) \geq \max\{d(v, x), d(v, y)\}$. Applying Lemma 1 to the quadruplet $\{u, v, x, y\}$, we deduce that $\max\{d(v, x), d(v, y)\} \geq d(x, y) - 2\delta$. Now, since $d(v, w) = ecc_S(v) \geq \max\{d(v, x), d(v, y)\}$ and $d(x, y) = diam(S)$, we obtain the inequality $d(v, w) \geq diam(S) - 2\delta$. Concluding, we obtain the following result which holds for all δ-hyperbolic spaces (X, d) :

Proposition 3. *For a finite subset S of a δ-hyperbolic space (X, d) and any point $u \in X$, if $v \in F_S(u)$ and $w \in F_S(v)$, then $d(v, w) \geq diam(S) - 2\delta$. The pair $\{v, w\}$ can be computed using $O(|S|)$ distance calculations. If $S \subset \mathbb{H}^2$, then $\{v, w\}$ can be computed in $O(|S|)$ time. For a δ-hyperbolic graph $G = (V, E)$ the pair $\{v, w\}$ can be computed in $O(|E|)$ time.*

From the triangle inequality it immediately follows that the inequality $diam(S) \leq 2rad(S)$ holds for all metric spaces. Now, we will establish a stronger relationship between diameters and radii of δ-hyperbolic geodesic spaces and graphs.

Proposition 4. *For any finite subset S of a geodesic δ-hyperbolic space or δ-hyperbolic graph, we have $diam(S) \geq 2rad(S) - 4\delta$ and $diam(S) \geq 2rad(S) - 4\delta - 1$, respectively.*

Proof. First consider the case of geodesic spaces (X, d). For each point $s \in S$ consider the ball $B(s, diam(S)/2)$ of radius $diam(S)/2$ centered at s. These balls pairwise intersect because for any pair of points $s, s' \in S$ we have $d(s, s') \leq diam(S) = diam(S)/2 + diam(S)/2$. By Proposition 2, there exists a point $c \in X$ belonging to all balls of radius $diam(S)/2 + 2\delta$ centered at the points of S. This implies that $rad(S) \leq ecc(c) \leq diam(S)/2 + 2\delta$, yielding $diam(S) \geq 2rad(S) - 4\delta$. Now, in case of graphs $G = (V, E)$, in order to ensure that the balls centered at vertices of S pairwise intersect in V we must take balls of radius $diam(S)/2$ if $diam(S)$ is even and of radius $\lfloor diam(S)/2 \rfloor + 1$ if $diam(S)$ is odd. Then the same reasoning yields the inequality $diam(S) \geq 2rad(S) - 4\delta - 1$. □

Combining the two previous results, we obtain that, if $v \in F_S(u)$ and $w \in F_S(v)$, then $d(v, w) \geq 2rad(S) - 6\delta$ for geodesic δ-hyperbolic spaces and $d(v, w) \geq 2rad(S) - 6\delta - 1$ for δ-hyperbolic graphs. This provides a fast approximation of the radius of S :

Corollary 1. *For any finite subset S of a δ-hyperbolic geodesic space or graph, we have $rad(S) \leq d(v, w)/2 + 3\delta$ and $rad(S) \leq \lfloor d(v, w) + 1)/2 \rfloor + 3\delta$, respectively.*

3.3 Centers

Here we show that the center $C(S)$ of S is contained in the ball $B(c, 5\delta)$ centered at the middle point c of any geodesic between the points v, w defined in previous subsection. In other words, $B(c, 5\delta)$ can be viewed as a *corset* [1] for the center problem in δ-hyperbolic geodesic spaces (the ball $B(c, 5\delta + 1)$ for graphs). First we establish an upper bound on the diameter of the center $C(S)$.

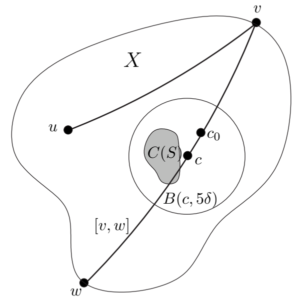

Figure 3: To the algorithm

Proposition 5. *$diam(C(S)) \leq 4\delta$ for δ-hyperbolic geodesic spaces and $diam(C(S)) \leq 4\delta + 1$ for δ-hyperbolic graphs.*

Proof. Consider the case of geodesic spaces (the proof for graphs is analogous up to ceiling). Pick $x, y \in C(S)$ and let z be a middle point of any geodesic between x and y. Pick any furthest neighbor s of z. Then $d(z, s) = ecc(z) \geq rad(S)$. On the other hand, $d(x, s) \leq rad(S)$ and $d(y, s) \leq rad(S)$ because $x, y \in C(S)$. Hence $d(z, s) \geq \max\{d(x, s), d(y, s)\}$. By Lemma 1 applied to the quadruplet $\{x, y, z, s\}$, we conclude that $d(x, y) \leq \max\{d(x, z), d(z, y)\} + 2\delta = d(x, y)/2 + 2\delta$, showing that indeed $d(x, y) \leq 4\delta$. □

In the next "coreset" result, the point c is determined in the following way: pick any point u of the space (vertex of the graph), find any furthest neighbor v of u, find any furthest neighbor w of v, and take as c the middle point (a middle vertex in case of graphs) of any geodesic segment $[v, w]$ between v and w (for an illustration, see Fig. 3). Let also c_0 be the point (vertex) of $[v, w]$ located at distance $rad(S)$ from w. Since $rad(S) \geq d(w, c) = d(v, w)/2 \geq rad(S) - 3\delta$, we conclude that for geodesic spaces, $d(c, c_0) \leq 3\delta$ and that c_0 is located on the geodesic $[c, v]$ between c and v (for graphs, $d(c, c_0) \leq 3\delta + 1$).

Proposition 6. *The inequalities $ecc(c) \leq rad(S) + 5\delta$ and $ecc(c_0) \leq rad(S) + 2\delta$ hold for all δ-hyperbolic geodesic spaces and graphs. Moreover $C(S) \subseteq B(c, 5\delta)$ for δ-hyperbolic geodesic spaces and $C(G) \subseteq B(c, 5\delta + 1)$ for δ-hyperbolic graphs.*

Proof. Since $d(c, c_0) \leq 3\delta$, it suffices to show that $ecc(c_0) \leq rad(S) + 2\delta$. For this, we will apply the proof of Lemma 2 assuming that $r(s) = rad(S)$ for each point $s \in S$ and that v plays the role of the point z. Since w is a furthest neighbor of v, the point $w \in S$ can be selected as the point maximizing the difference $M = d(v, s) - r(s)$ over all $s \in S$. Now, for any point $s \in S$ we have $d(w, s) \leq diam(S) \leq 2rad(S) = r(w) + r(s)$, yielding $S_w = S$. Since c_0 is located at distance $rad(S) = r(w)$ from w, by Lemma 2 we deduce that $d(c_0, s) \leq r(s) + 2\delta = rad(S) + 2\delta$ for each point $s \in S$. Thus $ecc(c_0) \leq rad(S) + 2\delta$.

Now we will show that $C(S) \subseteq B(c, 5\delta)$ for geodesic spaces (the proof of similar inclusion for graphs is analogous). Pick any central point $c' \in C(S)$. Then $\max\{d(c', v), d(c', w)\} \leq rad(S)$. Consider the three distance sums $d(v, w) + d(c, c')$, $d(v, c) + d(c', w)$, and $d(c, w) + d(c', v)$ and suppose, without loss of generality, that $d(v, c) + d(c', w) \geq d(c, w) + d(c', v)$, i.e., $d(c', w) \geq d(c', v)$. Notice that the difference between the second and the first sums is $d(c', w) - d(c, w) - d(c, c')$. Since $d(c', w) \leq rad(S)$ and $d(c, w) \geq rad(S) - 3\delta$, the second sum is larger than the first one only if $d(c, c') \leq 3\delta$.

So, suppose that the first sum is larger than the second sum. Then $0 \leq d(c,w) + d(c,c') - d(c',w) \leq 2\delta$. If $d(c,w) \geq d(c',w)$, then necessarily $d(c,c') \leq 2\delta$. On the other hand, if $d(c,w) \leq d(c',w)$, then $d(c',w) - d(c,w) \leq 3\delta$ because $d(c,w) \geq rad(S) - 3\delta$ and $d(c',w) \leq rad(S)$. In this case we deduce that $d(c,c') \leq 5\delta$. This shows that in all cases $d(c,c') \leq 5\delta$ holds, establishing the desired inclusion. □

Notice that since $rad(S)$ is unknown, we cannot find c_0 algorithmically. In case of graphs with bounded hyperbolicity, we can compute in $O(|E|)$ time the eccentricities of all 3δ vertices of the geodesic segment $[c,w] \cap B(c,3\delta)$ and return the vertex c_0' with smallest eccentricity, instead of c_0. Clearly, $ecc(c_0') \leq ecc(c_0) \leq rad(S) + 2\delta$. For δ-hyperbolic graphs or networks $G = (V,E)$ with uniformly bounded degrees, the ball $B(c, 5\delta + 1)$ contains a constant number of vertices (here, we assume that δ is bounded by a constant, too) and therefore the center $C(S)$ can be found in $O(|E|)$ time by computing the eccentricities of all vertices of $B(c, 5\delta + 1)$ and choosing among them the vertices with smallest eccentricity. Notice that the requirement that the degrees are uniformly bounded is closely related to locally doubling condition on δ-hyperbolic spaces occurring in the results of Krautghamer and Lee [34].

COROLLARY 2. *For a finite subset $S \subseteq V$ of a δ-hyperbolic graph $G = (V,E)$ with maximum degree $\Delta(G)$ and δ bounded by a constant, a vertex c with $ecc(c) \leq rad(S)+2\delta$ can be computed in $O(|E|)$ time and the center $C(S)$ can be computed in $O(|\Delta(G)|^{5\delta+1}|E|)$ time. If the degrees of vertices of G are uniformly bounded, then $C(S)$ can be computed in linear $O(|E|)$ time.*

In classical real-hyperbolic spaces $\mathbb{H}^k$ or other models of hyperbolic geometry (which are all isometric to $\mathbb{H}^k$), the distances and the geodesics between points can be computed analytically. For example, in the Poincaré disk model, the points of the geometry are the points of an open k-dimensional Euclidean unit disk, and the lines are segments of circles contained in the disk orthogonal to the boundary of the disk, or else diameters of the disk. If u and v are two points of this geometry, then the distance is computed by the formula

$$d(u,v) = arccosh(1 + \frac{2||u-v||^2}{(1-||u||^2)(1-||v||^2)}),$$

where $||\cdot||$ is the usual Euclidean norm. Thus in all classical models, the points $v, w \in S$ and the middle point $c \in [v,w]$ with $ecc(c) \leq rad(S)+5\delta$ can be found using $O(|S|)$ distance computations. Let c' be the point of the geodesic $[v,w]$ located between c and v at distance 3δ from c. The distance function on $\mathbb{H}^k$ is convex, therefore the point $c_0' \in [c,c']$ with $ecc(c_0') \leq rad(S) + 2\delta$ can be found by running a binary search on the geodesic segment $[c,c'] = [c,v] \cap B(c,3\delta)$. For each tested point we compute its eccentricity and return the point with smallest eccentricity. This way, we can find a point $c_0'' \in [c,c']$ with $ecc(c_0'') \leq rad(S) + (2+\epsilon)\delta$ using $\log_2 \frac{3\delta}{\epsilon}$ eccentricity computations.

For general δ-hyperbolic geodesic spaces (for example, some polyhedral complexes), our algorithm needs a subroutine for computing distances and geodesics. Notice also that if $[v,w]$ is available, then the point c_0'' with $ecc(c_0') \leq rad(S) + (2+\epsilon)\delta$ can be found by subdividing the geodesic segment $[c,c'] = [c,v] \cap B(c,3\delta)$ into segments of length 2ϵ and computing the eccentricities of the subdivision points.

COROLLARY 3. *For a finite subset S of a δ-hyperbolic geodesic space (X,d), a point c with $ecc(c) \leq rad(S)+5\delta$ can be found using $O(|S|)$ distance computations and one computation of a geodesic segment. For given $\epsilon > 0$, a point c_0'' with $ecc(c_0'') \leq rad(S) + (2+\epsilon)\delta$ can be found by computing the eccentricity of $O(\frac{1}{\epsilon})$ points ($O(\log_2 \frac{1}{\epsilon})$ points if $S \subset \mathbb{H}^k$).*

4. APPROXIMATING TREES

In this section, we present a simple method which constructs for any δ-hyperbolic graph $G = (V,E)$ with n vertices a distance $O(\delta \log n)$-approximating tree in optimal time $O(|E|)$. A tree $T = (V,F)$ is called a *distance κ-approximating tree* of a graph $G = (V,E)$ if $|d_G(x,y) - d_T(x,y)| \leq \kappa$ for each pair of vertices $x,y \in V$. Our result and the definition of a distance approximating tree are comparable with Theorem 1. The approximation of distances used in Theorem 1 is stronger because the mapping φ is non-expansive. On the other hand, distance approximating trees have the same set of vertices as G while the trees occurring in the theorem of Gromov may have Steiner points (in fact our construction can be easily modified to be non-expansive by accepting edges of length 1/2 and Steiner points). The error incurred by our result is slightly weaker (but of the same order), however the construction of our approximating tree T is simpler and can be done in linear $O(|E|)$ time while the construction in Theorem 1 needs $O(|V|^2)$ time.

We start with a property of δ-hyperbolic graphs formulated and proven in several texts on Gromov hyperbolic spaces (in particular, in [9]) for all δ-hyperbolic spaces. This result is used in the proof of the fundamental property of δ-hyperbolic spaces established in [26] that geodesics in such spaces diverge at exponential rate; for a proof, see also [3, 9]. For a simple path ρ of a graph G, let $l(\rho)$ denote its length.

PROPOSITION 7. *Let $G = (V,E)$ be a graph with δ-thin geodesic triangles and let ρ be a simple path connecting two vertices p,q of G. If $[p,q]$ is a geodesic segment between p and q, then for every vertex $x \in [p,q]$, the distance from x to a closest vertex y of ρ is at most $1 + \delta \log_2 l(\rho)$.*

PROOF. To explain why $\delta \log_2 l(\rho)$ occurs in this result, we sketch its (nice) proof, at the same time bringing it some computer science flavor (a detailed proof is given on p. 401 of [9]). Take the cycle constituted by the geodesic $[p,q]$ and the path ρ and "triangulate" it in the following way. If ρ consists of a single edge, then return this edge. Otherwise, pick the middle vertex r of the path ρ and include in the triangulation the geodesic triangle Δ having $[p,q]$ and two geodesic segments $[p,r]$ and $[r,q]$ as sides. Then recursively apply this algorithm twice to the geodesic segments $[p,r]$ and $[r,q]$ and the subpaths ρ' and ρ'' of ρ comprised between p and r, and r and q, respectively. The resulting triangulation $\mathcal{T}$ can be viewed as a binary tree rooted at Δ whose nodes are the triangles of $\mathcal{T}$ and two triangles are adjacent iff they share a common geodesic segment. Since the length of the current simple path is divided by 2 at each iteration, the number of levels h of this binary tree satisfies the inequality $l(\rho)/2^{h+1} < 1 \leq l(\rho)/2^h$.

For a vertex $x \in [p,q]$, the distance from x to one of the sides $[p,r]$ or $[r,q]$ of the geodesic triangle Δ is at most δ, because Δ is δ-thin. Suppose that $d_G(x,x') \leq \delta$ for a vertex $x' \in [p,r]$. Let Δ' be the geodesic triangle sharing the side $[p,r]$ with Δ. Repeating recursively the same operation to x' and Δ', we will construct a path from the initial vertex x to

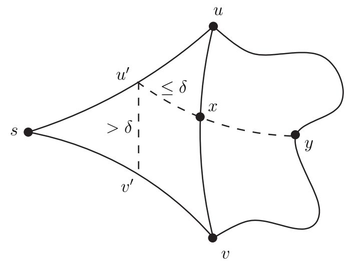

Figure 4: To the proof of Proposition 8

a vertex y of ρ consisting of at most h geodesic segments of length at most δ each. Hence $d_G(x,y) \leq 1 + \delta \log_2 l(\rho)$. □

Let $G = (V, E)$ be a connected graph with a distinguished vertex s. A *layering* of G with respect to s is the partition of V into the *spheres* $L^i = \{u \in V : d(s,u) = i\}$, $i = 0, 1, 2, \ldots$ A *layering partition* of G is a partition of each L^i into clusters $L^i_1, \ldots, L^i_{p_i}$ such that two vertices $u, v \in L^i$ belong to the same cluster L^i_j if and only if they can be connected by a path outside the ball $B_{i-1}(s)$ of radius $i-1$ centered at s (this partition has been introduced in [8, 14] and also used in [20]). We continue by showing that if G is a graph with n vertices and with δ-thin geodesic triangles, then the diameters of clusters of a layering partition of G are bounded by a function of δ and $\log_2 n$. Set $\Lambda_n := 4 + 3\delta + 2\delta \log_2 n$.

Proposition 8. *Let L^i_j be a cluster of a layering partition of a graph G with δ-thin geodesic triangles and n vertices, and let $u, v \in L^i_j$. Then $d_G(u,v) \leq \Lambda_n$.*

Proof. Suppose, by way of contradiction, that u, v belong to a common cluster L^i_j but $d_G(u,v) > \Lambda_n$. Let ρ be a simple path connecting the vertices u, v outside the ball $B_{i-1}(s)$. Let $[u,v]$ be a geodesic segment connecting the vertices u and v. Set $r := 2 + \delta + \delta \log_2 n$. On the sphere L^{i-r} pick two vertices u', v' of G such that u' lies on a geodesic segment $[s,u]$ between the root s and the vertex u and v' lies on a geodesic segment $[s,v]$ between s and v; see Fig. 4. Since $d_G(u,v) > 2\delta \log_2 n + 3\delta + 4$, we conclude that $d_G(u',v') > \delta$. Since the geodesic triangle formed by the geodesic segments $[s,u], [s,v], [u,v]$ is δ-thin, $d_G(s,u') = d_G(s,v')$, and $d_G(u',v') > \delta$, we conclude that $d(u',x) \leq \delta$ for some vertex x of G lying on the geodesic segment $[u,v]$. By Proposition 7, the path ρ contains a vertex y such that $d_G(x,y) \leq \delta \log_2 l(\rho) + 1 \leq \delta \log_2 n + 1$. Thus $d_G(s,y) \leq d_G(s,u') + d_G(u',x) + d_G(x,y) \leq i - r + \delta + \delta \log_2 n + 1$. On the other hand, since y belongs to the path ρ, we must have $d_G(s,y) \geq i$. Thus $i \leq i - r + \delta + \delta \log_2 n + 1$, hence $2 + \delta + \delta \log_2 n = r \leq 1 + \delta + \delta \log_2 n$, a contradiction. □

Let Γ be a graph whose vertex set is the set of all clusters L^i_j in a layering partition of G and two vertices L^i_j and $L^{i'}_{j'}$ are adjacent in Γ if and only if there exist $u \in L^i_j$ and $v \in L^{i'}_{j'}$ such that u and v are adjacent in G. It is shown in [14] that Γ is a tree, called the *layering tree* of G, and that Γ is computable in linear time in the size of G. To construct the tree $T = (V, F)$, for each cluster $C := L^i_j$ we select a vertex v_C of L^{i-1} which is adjacent in G with at least one vertex of C and make v_C adjacent in T to all vertices of C. Since Γ is a tree, T is a tree as well.

Proposition 9. *$T = (V, F)$ is a Λ_n-approximating tree for a graph $G = (V, E)$ with δ-thin geodesic triangles and n vertices. In particular, $T = (V, F)$ is a $4\Lambda_n$-approximating tree for a δ-hyperbolic graph.*

Proof. It can be easily shown that the tree T preserves the distances to the root s, i.e., $d_T(x,s) = d_G(x,s)$ for any $x \in V$. From Proposition 8, if x, y belong to a common cluster, then $d_T(x,y) = 2$ and $d_G(x,y) \leq \Lambda_n$. Now, suppose that x and y belong to different clusters of Γ, say $x \in C' := L^{i'}_{j'}$ and $y \in C'' := L^{i''}_{j''}$. Let $C := L^i_j$ be the cluster which is the nearest common ancestor of C' and C'' in the tree Γ. By definition of clusters, any path of G connecting the vertices x and y will traverse the clusters on the unique path $P(C', C'')$ of the tree Γ connecting C' and C''. In particular, any shortest (x,y)-path will intersect the cluster C. Since $d_G(x,z) \geq i' - i$ and $d_G(z,y) \geq i'' - i$ for any vertex $z \in C$, we conclude that $d_G(x,y) \geq i' + i'' - 2i$. On the other hand, any (x,y)-path of G sharing a single vertex with each cluster (except C) of the path $P(C', C'')$ and intersecting the cluster C in a shortest path has length at most $i' + i'' - 2i + \Lambda_n$, thus $i' + i'' - 2i \leq d_G(x,y) \leq i' + i'' - 2i + \Lambda_n$. Now, notice that $d_T(x,y) = i' + i'' - 2i + 2$ or $d_T(x,y) = i' + i'' - 2i$ if the two clusters of $P(C', C'')$ incident to C have the same neighbor in T. In both cases, we conclude that $|d_G(x,y) - d_T(x,y)| \leq \Lambda_n$. Now, since geodesic triangles of a δ-hyperbolic graph G are 4δ-thin, the second assertion is immediate. □

By using edges of length $\frac{1}{2}$ and Steiner points, the tree T can be easily transformed into a tree $T_{\frac{1}{2}}$ which has the same approximating performances and satisfies the non-expansive property. For this, for each cluster $C := L^i_j$ we introduce a Steiner point w_C, and add an edge of length $\frac{1}{2}$ between any vertex of C and w_C and an edge of length $\frac{1}{2}$ between w_C and the vertex v_C defined above.

5. GRAPHS WITH BOUNDED HYPERBOLICITY

In this section, we establish that two classes of geometric graphs are 1-hyperbolic and some other classes of graphs have bounded hyperbolicity. Notice that in case of δ-hyperbolic graphs, δ has the form $k/2$ for some natural number k. 0-hyperbolic graphs are exactly the graphs in which all blocks induce complete subgraphs (they distance-function is a tree-distance). A full characterization of 1/2-hyperbolic graphs has been given in [4] (see also [33] for a partial characterization): these are the graphs in which all balls are convex and which do not contain six isometric subgraphs. Chordal graphs and distance-hereditary graphs (graphs in which all induced paths are geodesic paths) are 1-hyperbolic [5, 33].

5.1 Sufficient conditions for hyperbolicity

We continue with several graph-theoretical notions. All graphs $G = (V, E)$ in this section are connected, undirected, but not necessarily finite. The *interval* $I(u,v)$ between two vertices u and v of G consists of all vertices (metrically) *between* u and v: $I(u,v) := \{x \in V : d(u,x) + d(x,v) = d(u,v)\}$. The interval $I(u,v)$ is *δ-thin* if $d(x,y) \leq \delta$ for all vertices $x, y \in I(u,v)$ such that $d(u,x) = d(u,y)$. An induced subgraph of G is called *convex* if it includes the interval of G between any of its vertices. An induced subgraph H of G is *isometric* if the distance between any pair of vertices in H is the same as that in G. The ball $B(C,r)$ centered at set C is the union of all balls $B(c,r)$ with centers c from C.

Three vertices v_1, v_2, v_3 of a graph G form a *metric triangle* $v_1v_2v_3$ if the intervals $I(v_1,v_2), I(v_2,v_3)$ and $I(v_3,v_1)$ pairwise intersect only in the common end vertices. If $d(v_1,v_2) = d(v_2,v_3) = d(v_3,v_1) = k$, then this metric triangle is called *equilateral* of *size* k. A metric triangle $v_1v_2w_2$ of G is a *quasi–median* of the triplet x,y,z if the following metric equalities are satisfied:

$$d(x,y) = d(x,v_1) + d(v_1,v_2) + d(v_2,y),$$
$$d(y,z) = d(y,v_2) + d(v_2,v_3) + d(v_3,z),$$
$$d(z,x) = d(z,v_3) + d(v_3,v_1) + d(v_1,x).$$

Every triplet x,y,z of a graph has at least one quasi-median: first select any vertex v_1 from $I(x,y) \cap I(x,z)$ at maximal distance to x, then select a vertex v_2 from $I(y,v_1) \cap I(y,z)$ at maximal distance to y, and finally select any vertex v_3 from $I(z,v_1) \cap I(z,v_2)$ at maximal distance to z. *Median graphs* are the graphs in which all metric triangles have size 0 and any triplet of vertices x,y,z admits a unique quasi-median. Median graphs are closely related to cubical complexes of global non-positive curvature [5]. *Bridged graphs* are the graphs in which all balls $B(S,r)$ centered at convex sets S are also convex. It has been shown in [27, 45] that bridged graphs are exactly the graphs in which all isometric cycles have length 3. For a detailed account on median structures and bridged graphs, see the survey [5].

An important class of median graphs is that of plane graphs in which all inner faces have length 4 and all inner vertices have degrees ≥ 4. This class of plane median graphs comprises the graph $H(4,5)$ of the regular $\{4,5\}$ hyperbolic tessellation (this is a tiling of the plane by regular hyperbolic squares, five squares meeting at each vertex) and any subgraph of $H(4,5)$ induced by the vertices lying on or inside the region bounded by a simple cycle of $H(4,5)$ (which we call $(4,5)$-*polygons*). Among examples of bridged graphs are the chordal graphs and the plane triangulations in which all inner vertices have degrees ≥ 6. If we impose that all inner vertices of such a triangulation have degree 7, then among such graphs we find the graph $H(3,7)$ of the regular hyperbolic tessellation $\{3,7\}$ and $(3,7)$-*polygons.* Notice also that each metric triangle of size k of a bridged graph G defines an isometric subgraph of G which is isomorphic to the tessellation of the equilateral Euclidean triangle of size k into equilateral triangles of size 1 (we call it a k-*deltoid*).

To establish hyperbolicity of a graph G (or of a geodesic metric space) sometime it is easier to show that geodesic triangles of G are δ-slim or δ-thin for some δ. We continue by showing that all graphs G with thin intervals and metric triangles having bounded sides are hyperbolic. Papasoglu [40] already proved that graphs with μ-thin intervals are $f(\mu)$-hyperbolic for an exponential function f.

PROPOSITION 10. *If the intervals of a graph $G = (V,E)$ are μ-thin and the metric triangles of G have sides of length at most ν, then the geodesic triangles of G are $(2\mu + \nu/2)$-slim and G is $(16\mu + 4\nu)$-hyperbolic.*

PROOF. Consider a geodesic triangle $\Delta(x,y,z)$ of G and pick an arbitrary vertex u on the side between x and y. Let $v_1v_2v_3$ be a quasi-median of the triplet x,y,z as defined above. Now, let v be a vertex on a geodesic path P between x,y passing via v_1 and v_2 such that $d(x,v) = d(x,u)$. Since $u,v \in I(x,y)$, we have $d(u,v) \leq \mu$. First suppose that v belongs to the subpath of P between x and v_1 or between v_2 and y, say the first. Then $v \in I(x,v_1) \subset I(x,z)$. Let w be the vertex on the side $[x,z]$ of $\Delta(x,y,z)$ such that $d(x,w) = d(x,v)$. Since $v,w \in I(x,z)$ and $I(x,z)$ is μ-thin, we have $d(v,w) \leq \mu$. Hence $d(u,w) \leq 2\mu$, establishing that the distance from u to two other sides of $\Delta(x,y,z)$ is bounded by $2\mu + \nu/2$. Now, let $v \in I(v_1,v_2)$. Since $d(v_1,v_2) \leq \nu$, the distance from v to v_1 or v_2, say to v_1, is at most $\nu/2$. Applying the same reasoning to $v_1 \in I(x,z)$ as we did in previous case with v, we conclude that $d(u,w) \leq d(u,v) + d(v,v_1) + d(v_1,w) \leq 2\mu + \nu/2$, establishing the first assertion. The second assertion follows from Proposition 1. □

The following two results show that in case of bridged and median graphs both conditions can be reformulated in terms of forbidden isometric subgraphs.

COROLLARY 4. *If the deltoids of a bridged graph G have size at most ν, then the geodesic triangles of G are $(5\nu/2)$-slim and G is 20ν-hyperbolic.*

PROOF. We will show that any interval $I(u,v)$ of G is ν-thin. Pick any two vertices $x,y \in I(u,v)$ such that $d(u,x) = d(u,y) = k'$ and $d(v,x) = d(x,y) = k''$, with $k' + k'' = d(u,v)$. We assert that any quasi-median of the triplet u,x,y has the form $u'xy$. Suppose not and let $u'x'y'$ be a quasi-median of u,x,y such that $x' \neq x$. Since $x' \in I(u,x)$ and $x' \neq x$, we infer that $d(u,x') < k'$. Thus $d(x',v) > k''$ because $x' \in I(u,x) \subset I(u,v)$. On the other hand, $x' \in I(x,y)$ and x,y belong to the ball $B(v,k'')$ of radius k'' centered at v. This contradicts the convexity of $B(v,k'')$. Thus any quasi-median of u,x,y has the form $u'xy$, yielding $d(x,y) \leq \nu$ because $u'xy$ defines a deltoid of size at most ν. □

To establish δ-hyperbolicity of a graph G for a small value of δ one has to show that each quadruplet u,v,w,x of G is δ-hyperbolic, i.e., the two largest of distance sums $d(u,v) + d(w,x), d(u,w)+d(v,x), d(u,x)+d(v,w)$ differ by at most 2δ. We will assume that $d(u,v) + d(w,x) \leq d(u,w) + d(v,x) \leq d(u,x)+d(v,w)$ and say that the ordered quadruplet $(uvxw)$ is a *quadrangle* Q with sides $d(u,v), d(v,x), d(x,w), d(w,u)$. In all subsequent proofs, we will proceed by induction on the total distance sum $d(u,x)+d(v,w)+d(u,v)+d(u,w)+d(v,x)+d(w,x)$. As noticed in [4], the induction hypothesis guarantees that $I(u,v) \cap I(u,w) = \{u\}, I(v,u) \cap I(v,x) = \{v\}, I(w,u) \cap I(w,x) = \{w\}, I(x,v) \cap I(x,w) = \{x\}$. In this case, we say that the quadruplet u,v,w,x satisfies the *condition (A).*

COROLLARY 5. *Any median graph $G = (V,E)$ not containing the rectilinear grid $\delta \times \delta$ as an isometric subgraph is δ-hyperbolic. In particular, $H(4,5)$ and any $(4,5)$-polygon is 2-hyperbolic.*

PROOF. If the quadruplet u,v,w,x satisfies the condition (A), then $d(u,v) = d(w,x) = k$ and $d(u,w) = d(v,x) = m$ for some $k \leq m$. Hence, the two larger distance sums differs by $2k$. It can be shown by induction on k and m that G contains an isometric rectilinear grid $k \times m$ having the vertices u,v,w,x as corners. Thus, $k \leq \delta$. □

5.2 Graphs of 7-systolic simplicial complexes

A *simplicial complex* $\mathcal{K}$ is a collection of sets (called *simplices*) such that $\sigma \in \mathcal{K}$ and $\sigma' \subseteq \sigma$ implies $\sigma' \in \mathcal{K}$. Denote by $V(\mathcal{K})$ and $E(\mathcal{K})$ the sets of all 0-dimensional and 1-dimensional simplices of $\mathcal{K}$. Then $G(\mathcal{K}) = (V(\mathcal{K}), E(\mathcal{K}))$ is called the *graph* of $\mathcal{K}$. Conversely, for a graph G one can derive a simplicial complex $\mathcal{K}(G)$ by taking all complete subgraphs as simplices. $\mathcal{K}$ is a *flag complex* if any set of vertices is included in a face whenever each pair of its vertices does. A flag complex can be recovered from its graph $G(\mathcal{K})$: the complete subgraphs of $G(\mathcal{K})$ are exactly the simplices of $\mathcal{K}$. $\mathcal{K}$ is called *simply connected* if it is connected and if every continuous mapping of the 1-dimensional sphere $\mathbb{S}^1$ into the geometric realization $|\mathcal{K}|$ of $\mathcal{K}$ can be extended to a continuous mapping of the disk $\mathbb{D}^2$ with boundary $\mathbb{S}^1$ into $|\mathcal{K}|$. The *systole* of $\mathcal{K}$ is the minimum number of edges in a cycle of $G(\mathcal{K})$ which is a full subcomplex of $\mathcal{K}$. The *residue* of a simplex σ of $\mathcal{K}$ is the union of all simplices containing σ.

A *k-systolic complex* is a simply connected simplicial flag complex $\mathcal{K}$ in which the systole of the residue of each simplex of $\mathcal{K}$ is at least k. If $k = 6$, such a complex is called *systolic*. Systolic complexes have been recently introduced by Januszkiewicz and Swiatkowski [32] and Haglund [28]. Papers [32, 28, 29] investigate these complexes in relationship with CAT(0) geometry and geometric group theory. Many results established in [32, 28, 29] concern the metric and convexity properties of the underlying graphs of systolic complexes. This is not a surprise because, as it was previously shown in Theorem 8.1 of [10], the simplicial complexes derived from bridged graphs G (called bridged complexes) are exactly the systolic complexes, namely these are simply connected flag complexes in which the neighborhood of any vertex does not contain induced 4- and 5-cycles. It was shown in [32] that the graphs of 7-systolic complexes are 11-hyperbolic, which, as they mention, "is by no means optimal". We continue by showing that these graphs are in fact 1-hyperbolic.

PROPOSITION 11. *Graphs of 7-systolic complexes are 1-hyperbolic.*

PROOF. Let G be a graph of a 7-systolic complex. Then G is bridged and does not contain 6-wheels, i.e., an induced 6-cycle plus an extra vertex adjacent to all vertices of this cycle. Since any metric triangle xyz of size k of a bridged graph defines an isometric k-deltoid, we must have $k \leq 2$, otherwise G would contain a 6-wheel. Now, notice that the intervals $I(u, v)$ of G are 1-thin (this is also Lemma 3.4 of [28]). Indeed, if $x, y \in I(u, v)$ with $d(u, x) = d(u, y)$, then, as noticed in the proof of Proposition 4, any quasi-median of u, x, y has the form $u'xy$ and any quasi-median of v, x, y has the form $v'xy$. Since the metric triangles have size at most 2, we conclude that $d(x, y) \leq 2$. Now, if $d(x, y) = 2$, then we will get two deltoids of size 2 with corners u', x, y and v', v, y. Together they will define a 2×2 triangulated lozenge containing a 6-wheel. This shows that $d(x, y) = 1$.

To establish 1-hyperbolicity of G, pick a quadruplet u, v, w, x satisfying the condition (A). Additionally, suppose that among all such quadruplets, let u, v, w, x also minimizes the perimeter $d(u, v)+d(w, x)+d(u, w)+d(v, x)$ of the quadrangle $Q = (uvxw)$. Notice that if one of the sides of Q has length 1, say $d(u, v) = 1$, then $d(u, x) \leq d(v, x) + 1$ and $d(v, w) \leq d(u, w) + 1$, whence the quadruplet u, v, w, x is 1-hyperbolic. So, suppose that all sides of Q have length ≥ 2. Since $I(u, v) \cap I(u, w) = \{u\}$ by condition (A), any quasi-median of u, v, w has the form $uv'w'$. The analogous conclusion holds for all triplets of u, v, w, x.

Next, we assert that for each vertex, say v, at least one of two incident sides of Q has length ≤ 3. Suppose not, and let $d(v, u) \geq 4, d(v, x) \geq 4$. Let $uv'w'$ and $xv''w''$ be any two quasi-medians of the triplets u, v, w and x, v, w. Since these quasi-medians have size ≤ 2, we deduce that $d(v, v') \geq 2$ and $d(v, v'') \geq 2$. Pick vertices $a \in I(v, v')$ and $b \in I(v, v'')$ at distance 2 from v. Since $v', v'' \in I(v, w)$, we conclude that $a, b \in I(v, w)$. Since $I(v, w)$ is 1-thin, $d(a, b) \leq 1$. Analogously, $d(a', b') \leq 1$ for common neighbors a' of a, v and b' of v, b. If $a = b$ or $a' = b'$, we obtain a contradiction with condition (A). Thus $d(a, b) = d(a', b') = 1$, and the vertices a, a', b, b' define a 4-cycle. Since bridged graphs do not contain induced 4-cycles, either a and b' are adjacent or b and a' are adjacent. In both cases, we obtain a contradiction with $I(v, u) \cap I(v, x) = \{v\}$. This contradiction proves our assertion. The same contradiction holds if $v' = u$ and $v'' = x$, or, more generally if $d(v, v') \geq 2$ and $d(v, v'') \geq 2$. Therefore we cannot have $u, x \in I(v, w)$ or $v, w \in I(u, x)$.

Previous assertion implies that at least the smallest distance sum is ≤ 6. On the other hand, since we cannot have $u, x \in I(v, w)$, if say $u \notin I(v, w)$, then any quasi-median of u, v, w has the form $uv'w'$ with v' and w' different from u. Hence we can find two adjacent neighbors $a \in I(u, v')$ and $b \in I(u, w')$ of u. Since the perimeters of the quadrangles obtained by replacing u either with a or with b are both strictly smaller than the perimeter of Q (and the total distance sum of new quadruplets is not larger than that of u, v, w, x), the quadruplets a, v, w, x and b, v, w, x are 1-hyperbolic. Comparing the evolution of the three distance sums of these quadruplets with those of u, v, w, x, we note that in the first case, the smallest distance sum decreases by 1 and the second distance sum remains the same, while in the second case the smallest distance sum remains the same and the second distance sum decreases by 1. This means that either u, v, w, x is 1-hyperbolic as well or that the two smallest distance sums of u, v, w, x must be equal.

Hence, let $4 \leq r := d(u, v) + d(w, x) = d(u, w) + d(v, x) \leq 6$. If $r = 4$, then the sides of the quadrangle Q all have length 2. Since u and x cannot both belong to $I(v, w)$, we infer $d(v, w) \leq 3$. Analogously, $d(u, x) \leq 3$, yielding $d(u, x)+d(v, w) \leq 6 = r+2$. If $r = 6$, then the sides of Q all have length 3. We assert that $d(u, x) \leq 4$ and $d(v, w) \leq 4$. Suppose by way of contradiction that $d(v, w) \geq 5$. Then any quasi-median $uv'w'$ of u, v, w and any quasi-median $xv''w''$ of xvw must have size ≤ 1 each. But this implies that $d(v, v') \geq 2$ and $d(v, v'') \geq 2$, which is impossible. Finally, let $r = 5$ and suppose that $d(u, w) = d(w, x) = 2$, $d(u, v) = d(v, x) = 3$. First notice that $d(v, w) \leq 4$, otherwise, if $d(v, w) \geq 5$, we will get the same contradiction as in previous case. Hence, if $d(u, x) \leq 3$, then we are done. Now suppose that $d(u, x) = 4$. Then $w \in I(u, x)$ and any quasi-median $vu'x'$ of vux has size 2. Let w' be a common neighbor of u', x'. Since $w' \in I(u', x')$ and $u', x' \in B(v, 2)$, the convexity of the ball $B(v, 2)$ implies that $d(v, w') \leq 2$. Since $w', w \in I(u, x)$ and $d(u, w) = d(u, w') = 2$, the fact that $I(u, x)$ is 1-thin implies that w and w' are adjacent. Hence, $d(v, w) \leq d(v, w') + d(w', w) = 2 + 1 = 3$. Thus, $d(u, x) + d(v, w) \leq 4 + 3 = 7 = r + 2$ holds also in this case. This establishes the required 4-point condition, proving that G is 1-hyperbolic. □

5.3 Link graphs of simple polygons

The *link distance* [35, 46] $d(x,y)$ between two points x,y of a simple polygon P of $\mathbb{R}^2$ is the minimum number of segments in a polygonal path connecting x and y in P. A polygon P endowed with link distance can be viewed as a graph $LG(P)=(P,E)$ (called the *link graph* of P) having the points of P as vertices and two vertices x,y are adjacent iff the segment between x and y belongs to P. It is shown in [35] that the balls of $LG(P)$ are convex, which indicates a close relationship of link graphs with bridged graphs and 1/2-hyperbolic graphs. The following result confirms these ties.

PROPOSITION 12. *For any simple polygon P, its link graph $LG(P)$ is 2-hyperbolic.*

PROOF. Consider an arbitrary but fixed triangulation $T(P)$ of polygon P, and let T be the dual graph of that triangulation, which is known to be a tree. Let $u,v,w,x\in P$ be any quadruplet of $LG(P)$. Denote by U,V,W,X the triangles of the triangulation $T(P)$ (nodes of T) containing the points u,v,w,x, respectively. Root the tree T at node W and let M be the lowest common ancestor in T of U,V, and X. By the choice of M, the two paths of T connecting one of the nodes U,V,X, say X, to two other nodes U,V pass via M. Denote by β_x the distance in $LG(P)$ between x and a closest to x point of the triangle M. The distances β_u, β_v, and β_w are defined analogously. Clearly, for any pair of points a,b from the set $\{u,v,w,x\}$ except possibly the pair u,v, any link-path between a and b has to pass via triangle M. This shows that $d(a,b)\geq\beta_a+\beta_b-1$. Therefore, the two largest sums among $d(x,u)+d(v,w)$, $d(x,v)+d(u,w)$, and $d(x,w)+d(u,v)$ are larger than or equal to $\beta_u+\beta_v+\beta_w+\beta_x-2=:\beta-2$. On the other hand, since the link-distance between any two points of a triangle is 1, we conclude that $d(a,b)\leq\beta_a+\beta_b+1$ for any pair of vertices a,b from $\{u,v,w,x\}$. This shows that each of the three distance sums is at most $\beta+2$. From the established upper and lower bounds for the two largest sums, we conclude that they differ by at most 4. Since this is true for all quadruplets of vertices, the graph $LG(P)$ is 2-hyperbolic. □

The *rectilinear link graph* $RLG(P)$ of a simple rectilinear polygon P has the points of P as vertices and $x,y\in P$ are adjacent iff they lie on a common vertical or horizontal line and the segment between x and y belongs to P (for rectilinear link metric, see [7]). Analogously to previous result one can show that the graph $RLG(P)$ is 2-hyperbolic. In the full version of this paper, we will show that in fact the graphs $LG(P)$ and $RLG(P)$ are 1-hyperbolic (the proof of this is more involved).

5.4 Tree-length λ graphs

We now recall the definition of tree-decomposition introduced by Robertson and Seymour in their work on graph minors [43]. A *tree-decomposition* of a graph G is a tree T whose vertices, called *bags*, are subsets of $V(G)$ such that:

1. $\bigcup_{X\in V(T)}X=V(G)$;
2. for all $uv\in E(G)$, there exists $X\in V(T)$ such that $u,v\in X$;
3. for all $X,Y,Z\in V(T)$, if Y is on the path from X to Z in T then $X\cap Z\subseteq Y$.

The *length* of tree-decomposition T of a graph G is $\max_{X\in V(T)}\max_{u,v\in X}d_G(u,v)$, and the *tree-length of* G [20] is the minimum, over all tree-decompositions T of G, of the length of T. The following two results establish a relationship between the δ-hyperbolic graphs and the graphs with bounded tree length.

PROPOSITION 13. *If a graph $G=(V,E)$ has a tree decomposition of length λ, then G is λ-hyperbolic.*

PROOF. The proof of this result is quite analogous to the proof of Proposition 12. Consider a decomposition tree T of G with bags of diameter $\leq\lambda$. Let $x,y,z,w\in V$ be any quadruplet of G. Denote by X,Y,Z,W some bags of T containing the vertices x,y,z,w, respectively. Root the decomposition tree T at W and let M be the lowest common ancestor in T of X,Y, and Z. By the choice of M, the two paths connecting one of the bags X,Y,Z, say X, to two other bags Y,Z, pass via M. Denote by β_x the distance in G between x and a closest to x vertex of the bag M. The distances β_y, β_z, and β_w are defined analogously. By definition of the decomposition tree T, the bag M is a separator in G for any pair of vertices a,b from the set $\{x,y,z,w\}$ except possibly the pair y,z. This shows that $d_G(a,b)\geq\beta_a+\beta_b$. Therefore, the two largest sums among $d_G(x,y)+d_G(z,w)$, $d_G(x,z)+d_G(y,w)$, and $d_G(x,w)+d_G(y,z)$ are larger than or equal to $\beta_x+\beta_y+\beta_z+\beta_w=:\beta$. On the other hand, since the bag M has diameter at most λ, we conclude that $d_G(a,b)\leq\beta_a+\beta_b+\lambda$ for any pair of vertices a,b from $\{x,y,z,t\}$. This shows that each of the three sums $d_G(x,y)+d_G(z,w)$, $d_G(x,z)+d_G(y,w)$, $d_G(x,w)+d_G(y,z)$ is at most $\beta+2\lambda$. From the established upper and lower bounds for the two largest sums, we conclude that they differ by at most 2λ. Since this is true for all quadruplets of vertices, the graph G is λ-hyperbolic. □

PROPOSITION 14. *A δ-hyperbolic graph $G=(V,E)$ with n vertices has a tree decomposition of length at most $4\Lambda_n+1$.*

PROOF. Following [20], we define a tree-decomposition T derived from the layering-tree Γ. Let T be a copy of Γ. For each node L^i_j of Γ except the root, denote by $L^{i-1}_{j'}$ the father of L^i_j in Γ and replace in T the node L^i_j by $L^i_j\cup X^i_j$ where X^i_j are vertices of $L^{i-1}_{j'}$ that are adjacent in G to vertices of L^i_j. We claim that T is a tree decomposition of G. First, note that Γ is a partition of V, thus each vertex v of G is contained in exactly one node of Γ. By construction of T, v belongs to the corresponding node of T and to some of its children. Therefore, the set of nodes of T that contain a given vertex v is a subtree of T. This establishes the third condition in the definition of a tree-decomposition. Now consider an edge uv of G. If $d_G(u,s)=d_G(v,s)$, then u and v belong to the same node of Γ and thus to at least one node of T. Otherwise, if $d_G(u,s)\neq d_G(v,s)$, let L^i_j and $L^{i'}_{j'}$ be the two nodes of Γ that contain u and v. By construction, these two nodes are adjacent in Γ and, therefore, there exists a node of T that contains both u and v. This establishes that T is a tree-decomposition of G. It remains to show that the diameters of bags in the tree-decomposition T are at most $4\Lambda_n+1$. Pick two vertices u and v that belong to a bag $L^i_j\cup X^i_j$ of this decomposition. By Proposition 8, if $u,v\in L^i_j$ or $u,v\in X^i_j\subseteq L^{i-1}_{j'}$, then $d_G(u,v)\leq 4\Lambda_n$. Now, suppose that $u\in L^i_j$ and $v\in X^i_j$. By construction, v is adjacent to some vertex of L^i_j yielding that $d_G(u,v)\leq 4\Lambda_n+1$ in this case. □

6. REFERENCES

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