Separation axiom S_3 for geodesic convexity in graphs

Victor Chepoi, LIS, Marseille, France

Symposium on Metric Graph Theory honoring Manoj Changat
May 15, 2024

Based on my personal convexity journey (from 1981 to 2024) and sprinkled with photos of one group convexity trip in Kerala in 2006.

V. Chepoi, Separation axiom S3 for geodesic convexity in graphs, arXiv:2405.07512v1, 2024.

Convexity spaces

Definition

- A *convexity space* (or a *closure space*) is a pair $(X, \mathfrak{C})$ where X is a set and $\mathfrak{C}$ is a family of subsets of X such that $\varnothing, X \in \mathfrak{C}$ and $\mathfrak{C}$ is closed by taking intersections.

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- A convexity space $(X, \mathfrak{C})$ has *arity* n if $A \in \mathfrak{C}$ if and only if $\mathfrak{c}(A') \subset A$ for any $A' \subset A$ with $|A'| \leq n$.

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  S_3 is equivalent to the condition that all semispaces are halfspaces.
- *Join-Hull Commutativity (JHC):* for any point x and any convex set C, $\mathfrak{c}(x \cup C) = \bigcup_{y \in C} \mathfrak{c}(x, c)$.

First stop, at hotel Manoj (with Martyn Mulder)

History of separation

Separations theorems in linear spaces (with numerous applications in geometry, functional analysis, optimization, machine learning):

- Farkas's lemma (1902);
- Minkowski's separation theorems (1911);
- Hahn-Banach theorem (1927, 1929);
- Kakutani (1937) and Tukey (1942) S_4-separation theorem (Stone (1937) for distributive lattices);
- Definition of semispaces: Hammer (1955), Motzkin (in $\mathbb{R}^3$, 1951), Köthe (1960);
- Characterization of semispaces: Hammer (1955), Klee (1956), and Köthe (1960).

History of separation

Separations theorems and abstract convexity spaces:

- Axiomatic approach to convexity: Levi (1951);
- Ellis (1952): characterization of JHC S_4-spaces via the Pasch axiom;
- Axiomatic approach to convexity: Calder (1971, Ekhoff (1968), Hammer (1955,1965), Kay and Womble (1971), Jamison (1974), van de Vel (1983), Soltan (1984), Prenowitz and Jantosiak (1979),. . . ;
- Soltan (1976): characterization of finite-dimensional normed S_4-spaces;
- Jamison (1974): characterization of S_4 spaces by the separation of polytopes;
- van de Vel (1984): characterization of S_4 spaces by screening;
- Chepoi (1986): characterization of S_4 by convexity of shadows, characterization of S_4 n-ary spaces by separation of n-polytopes.

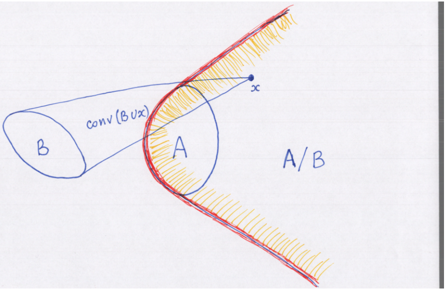
Shadows and S_4

Definition (Shadow)

Given two sets A, B of a convexity space $(X, \mathfrak{C})$ the *shadow* of A with respect to B is the set

$$A/B = \{x \in X : \mathfrak{c}(B \cup \{x\}) \cap A \neq \varnothing\}.$$

Defined by Chepoi (1986) and called *penumbra* (*twilight*?).



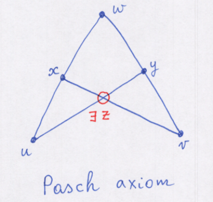
Shadows and S_4

Theorem (Chepoi, 1986)

(1) *A convexity space* $(X,\mathfrak{C})$ *is* S_4 *iff* A/B *and* B/A *are convex for any* $A,B\in\mathfrak{C}$.

(2) *If* $(X,\mathfrak{C})$ *has arity* n, *then* $(X,\mathfrak{C})$ *is* S_4 *iff for any* n*-polytope* A *and* $(n-1)$*-polytope* B, *the shadow* A/B *is convex and iff any two disjoint* n*-polytopes* A *and* B *can be separated by halfspaces.*

(3) *If* $(X,\mathfrak{C})$ *has arity 2, then* $(X,\mathfrak{C})$ *is* S_4 *iff Pasch axiom holds:* $\forall u,v,w\in X,\ x\in\mathfrak{c}(w,u), y\in\mathfrak{c}(w,v),\ \exists z\in\mathfrak{c}(u,y)\cap\mathfrak{c}(v,x)$.



What about S_3?

Proposition (Chepoi, 1986)

A convexity space $(X,\mathfrak{C})$ satisfies S_3 iff for any polytope P and any point $x_0 \notin P$, the shadow x_0/P is convex.

Remark

While the characterizations of S_4 for n-ary convexities is efficient, the characterization of S_3 is not efficient. No efficient characterizations of S_3 are known in arity n or even arity 2. The following questions are open:

- **Q.1:** Characterize (if possible) S_3-convexity spaces of arity n (arity 2 or geodesic convexity in graphs) via a condition (a) on specific subsets or (b) on subsets with a fixed number of points.
- **Q.2:** What is the complexity of deciding if a convexity space is S_3?
- **Q.3:** Characterize the semispaces in convexity spaces of arity n (2).

Second stop: drinking some (partially hidden) beer with Manoj (and not only him)

Properties of S_3-graphs

Let $G = (V, E)$ be a connected, simple, non necessarily finite graph endowed with the standard graph-metric.

Definition

- *Metric interval:* $[u, v] = \{z \in V : d(u, z) + d(z, v) = d(u, v)\}$.
- *Geodesic convexity:* $\forall u, v \in A$, $[u, v] \subseteq A$.
- *S_3-graph:* the geodesic convexity of G satisfies S_3.

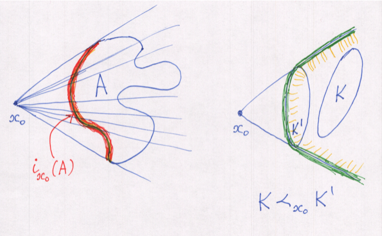
(1) If G is an S_3-graph, then the intervals $[u, v]$ and the shadows x_0/A with A convex are convex sets.

(2) If S is a semispace of G, then there exists x_0 adjacent to S, such that S is a semispace at x_0.

Imprints and proximal sets

Definition

- *Imprint:* $\imath_{x_0}(A) = \{z \in A : [x,z] \cap A = \{z\}\}$.
- *Proximal set:* A set $K \subseteq V$ of G is x_0-*proximal* if
  - (P1) $\imath_{x_0}(K) = K$ and $x_0 \sim K$;
  - (P2) $\mathfrak{c}(K)$ of K does not contain the vertex x_0.
- *Maximal proximal sets:* Maximal elements of the partial order: for x_0-proximal sets K, K', define $K \preceq_{x_0} K'$ if and only if $K \subseteq K'/x_0$.
- $\mathrm{Max}(\Upsilon^*_{x_0})$ is the set of all maximal x_0-proximal sets.



With some abstraction, an illustration of imprint $\imath_{x_0}(A)$ in $\mathbb{R}^2$, where x_0 is at bottom-middle and A is the barque. The imprint $\imath_{x_0}(A)$ is x_0-proximal.

Suspense about shadows: will they be used again?

S_3-graphs and their semispaces

Theorem (C., 2024)

Let $G = (V, E)$ be an S_3-graph and x_0 be an arbitrary vertex of G. If S is a semispace at x_0 adjacent to S, then $\iota_{x_0}(S) \in \mathrm{Max}(\Upsilon^*_{x_0})$ and $S = \iota_{x_0}(S)/x_0$. Conversely, if $K \in \mathrm{Max}(\Upsilon^*_{x_0})$, then K/x_0 is a semispace at x_0 adjacent to x_0. Consequently, there exists a bijection between the semispaces at x_0 adjacent to x_0 and the sets of $\mathrm{Max}(\Upsilon^*_{x_0})$.

Theorem (C., 2024)

For a graph $G = (V, E)$, the following conditions are equivalent:

(i) *G is an S_3-graph;*

(ii) *for any $x_0 \in V$ and $K \in \mathrm{Max}(\Upsilon^*_{x_0})$, the shadows K/x_0 and x_0/K are convex and disjoint;*

(iii) *for any $x_0 \in V$ and $K \in \mathrm{Max}(\Upsilon^*_{x_0})$, x_0 and $\mathfrak{c}(K)$ can be separated by halfspaces.*

S_3-graphs satisfying (TC): semispaces

Definition

- *Triangle condition (TC):* for any $u, v, w \in V$ with $1 = d(v, w) < d(u, v) = d(u, w)$ there exists a common neighbor x of v and w such that $d(u, x) = d(u, v) - 1$.
- *Pointed maximal clique:* a pair (x_0, K), where K is a clique, $x_0 \notin K$, and $K \cup \{x_0\}$ is a maximal by inclusion clique of G.

Theorem (C., 2024)

If G is an S_3-graph satisfying (TC), then S is a semispace at x_0 adjacent to x_0 if and only if there exists a pointed maximal clique (x_0, K) such that $S = K/x_0$.

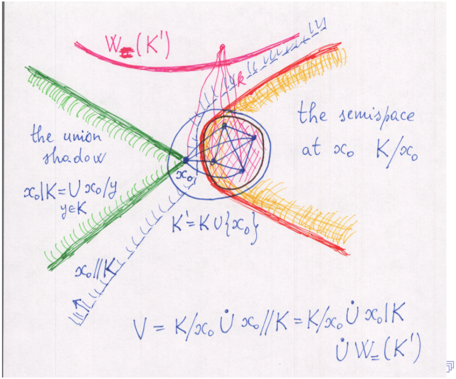
Corollary (C. 2024)

The semispaces of a finite S_3-graph G satisfying (TC) can be enumerated in output polynomial time.

S_3-graphs satisfying (TC): extended shadows

Definition (Extended shadow)

For a clique $K' = K \cup \{x_0\}$, the vertex set V of G is the *disjoint union* of the sets $K/x_0, W_=(K') := \{v \in V : d(v,y) = d(v,z) \text{ for all } y,z \in K'\}$, and $x_0|K := \bigcup_{y \in K} x_0/y$. Let $x_0//K = x_0|K \cup W_=(K')$ and call it the *extended shadow* of K with respect to x_0.



S_3-graphs satisfying (TC): characterization

Theorem (C., 2024)

For a graph $G = (V, E)$ satisfying (TC) the following conditions are equivalent:

(i) *G is an S_3-graph;*

(ii) *for any pointed maximal clique (x_0, K), x_0 and K can be separated by complementary halfspaces;*

(iii) *for any pointed maximal clique (x_0, K), the shadow K/x_0 and the extended shadow $x_0//K$ are convex.*

Third stop: in the search for a non-existing tiger

Meshed graphs

Definition (Meshed graph (Bandelt, Mulder, Soltan, 1994))

A graph $G = (V, E)$ is called *meshed* if for any vertex u its distance function d satisfies the following *Weak Quadrangle Condition* (QC$^-$):

- for any $u, v, w \in V$ with $d(v, w) = 2$, there exists a common neighbor x of v and w such that $2d(u, x) \leq d(u, v) + d(u, w)$.

Meshed graphs comprise large and important classes of graphs:

- Weakly modular graphs are meshed (median, modular, bridged, Helly, dual polar, sweakly modular);
- Basis graphs of matroids and of even Δ-matroids are meshed;
- Meshed graphs satisfy triangle condition (TC).

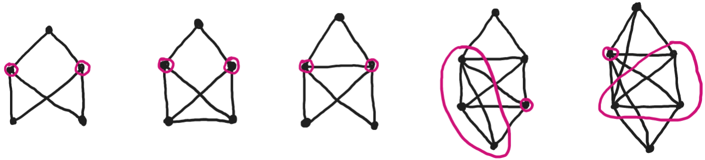
Meshed S_3-graphs

Theorem (C., 2024)

A meshed graph $G = (V, E)$ is S_3 if and only if it does not contain the following five graphs as induced subgraphs.

Corollary

Meshed S_3-graphs can be recognized in polynomial time.



Main steps of the proof

(1) (Local convexity implies convexity) A connected induced subgraph H of a meshed graph G is convex if and only if H is locally-convex.

(2) If G is a meshed graph not containing the 5 forbidden graphs, then:

- Intervals of G are convex;
- G satisfies the Positionning condition (PC);
- Shadows x/y are convex;
- for each maximal pointed clique (x_0, K) the shadow K/x_0 is convex;
- for each maximal pointed clique (x_0, K) the extended shadow $x_0//K$ is convex.

Examples of S_3-graphs

The following graphs are S_3:

- partial cubes,
- partial Hamming graphs,
- partial Johnson graphs satisfying (TC),
- planar (3,6)-,(4,4)-, and (6,3)-graphs,
- the Petersen graph and the 1-skeleta of Platonic solids.

Additionally, the following graphs are meshed S_3-graphs:

- hyperoctahedra, the complete graphs, the icosahedron, and the graph Γ from the paper,
- basis graphs of matroids,
- median, quasi-median, and weakly median graphs,
- the 2-dimensional ℓ_∞-grid $\mathbb{Z}^2_\infty$ and any its subgraph contained in the region of $\mathbb{R}^2$ bounded by a simple closed path of the grid.

Fourth stop: back to luxury nature

Halfspace separation problem

Definition (Seiffart, Horváth, and Wrobel, 2023)

Given a pair (A, B) of sets of a convexity space $(X, \mathfrak{C})$, decide if A and B are separable by complementary halfspaces H', H'' and find H', H'' if they exist.

Halfspace enumeration method

- Enumerate the complementary halfspaces of $(X, \mathfrak{C})$.
- Given (A, B), test if $\mathfrak{c}(A) \cap \mathfrak{c}(B) = \varnothing$. If "yes", then test all pairs of (H', H'') of complementary halfspaces and find one that separate $\mathfrak{c}(A)$ and $\mathfrak{c}(B)$. Return "not" if such a pair does not exist.

Is polynomial when $(X, \mathfrak{C})$ has a polynomial number of halfspaces.

Classes of graphs with a few halfspaces

Known results

Glantz and Meyerhenke (2017) proved that bipartite graphs have at most $O(|E|)$ halfspaces and planar graphs have at most $O(n^5)$ halfspaces and can enumerate them in polynomial time.

Theorem (C., 2024)

If $(X,\mathfrak{C})$ is a convexity space on n points and Radon number r, then $(X,\mathfrak{C})$ has at most $O(n^r)$ halfspaces. If $\mathfrak{C}$ is a convexity with connected sets on a graph $G=(V,E)$ not containing K_{k+1} as a minor, then $\mathfrak{C}$ has at most $O(n^{2k})$ halfspaces. If G is planar, then $\mathfrak{C}$ has at most $O(n^5)$ halfspaces.

Corollary

A Helly graph G has at most $O(n^{\omega(G)})$ halfspaces and any chordal graph G has at most $O(n^{\omega(G)+1})$ halfspaces ($\omega(G)$ is the clique number of G).

Classes of graphs with a few halfspaces

Proposition (C., 2024)

The halfspace enumeration and the halfspace separation problems can be solved for geodesic convexity in a graph G with n vertices in the following classes of graphs:

(1) $O(\mathrm{poly}(n))$ *time if* G *is bipartite;*

(2) *(Glantz and Meyerhenke, 2017)* $O(\mathrm{poly}(n))$ *if* G *is planar;*

(3) $O(\mathrm{poly}(n)n^{\omega(G)})$ *if* G *is chordal or Helly;*

(4) $O(\mathrm{poly}(n)n^{2\eta(G)})$ *if* G *is meshed and admits a hereditary dismantling order.*

Shadow-closed and osculating pairs

Definition

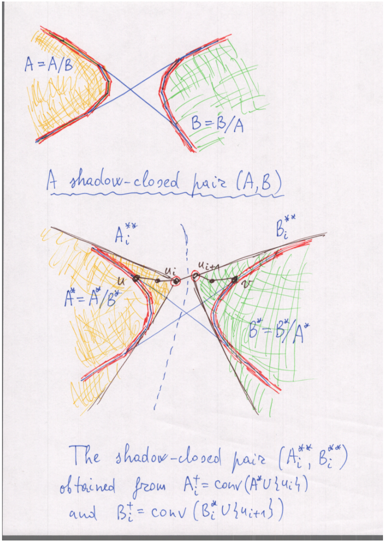
Shadow-closed pair: $A = A/B = \mathfrak{c}(A/B)$ and $B = B/A = \mathfrak{c}(B/A)$.

Osculating pair: $A \cap B = \varnothing$ and $d(A, B) = 1$.

Remarks:

(1) The pair (A, B) is separable iff the pair $(A/B, B/A)$ of shadows is separable iff the pair $(\mathfrak{c}(A/B), \mathfrak{c}(B/A))$ is separable. Thus passing from a pair (A, B) to the shadow-closure (A^*, B^*) does not change separability.
(2) A shadow-closed pair (A^*, B^*) is separable iff any shortest path P between any pair of closest vertices $u \in A^*, v \in B^*$ contains an edge $u_i u_{i+1}$ such that $\mathfrak{c}(A^* \cup \{u_i\})$ and $\mathfrak{c}(B^* \cup \{u_{i+1}\})$ are separable.

Illustrations of the method



The three-step method

(1) Compute the shadow-closed pair (A^*, B^*) for (A, B). Return "not" if $A^* \cap B^* \neq \varnothing$.

(2) For (A^*, B^*), pick a shortest path (A, B)-path P and for each edge $u_i u_{i+1}$ of P, set $A_i^+ = \mathfrak{c}(A^* \cup \{u_i\})$ and $B_i^+ = \mathfrak{c}(B^* \cup \{u_{i+1}\})$ and for (A_i^+, B_i^+) compute the shadow-closed pair (A_i^{**}, B_i^{**}) using Step 1. If $A_i^{**} \cap B_i^{**} \neq \varnothing$ for all edges of P, return "not".

(3) For each shadow-closed osculating pair (A_i^{**}, B_i^{**}), solve the separation problem using a case-oriented algorithm. If "yes" is returned for at least one pair, then return "yes", otherwise, return "not".

The three steps method works for:

- gated convexity in graphs.
- monophonic convexity in graphs.
- some classes of graphs with geodesic convexity.

For monophonic convexity, different solutions were obtained by Elaroussi, Nourine, and Vilmin (arXiv:2404.17564v1, 26 Apr 2024) and Bressan, Esposito, and Thiessen (arXiv:2405.00853v1, 1 May 2024).

Last stop...

Merci!


HAPPY
Retirement
Manoj !

Questions?

Discussions about convexity in the middle of the nature