GEOMETRY OF LOPSIDED SETS

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Abstract. Lopsided sets introduced by Jim Lawrence in 1983 can be regarded as finite partial hypercubes for which the intersections with any fibers (alias faces) yield isometric subgraphs. In a previous article we characterized lopsided sets in various combinatorial ways. A particularly attractive feature of lopsided sets, which we study here, is their geometric realization as cubihedra by which they can be characterized. Several other characterizations can be established which are inspired by the geometry of l_1-spaces. This testifies to the naturalness of the concept of lopsided sets in the context of cubical complexes.

1. Introduction

This paper is the follow-up of [3], in which we presented a list of combinatorial, recursive, and graph-theoretical characterizations of lopsided sets first introduced and investigated by Lawrence [8]. In the present paper, we provide several geometric characterizations of lopsided sets, each emphasizing one or another feature of lopsidedness and its relationships with some fundamental concepts from the geometry of l_1-spaces. We will repeatedly refer to the first part [3] for definitions, properties and characterizations of lopsided sets.

Throughout this paper, X denotes a finite set with $n := \#X$ elements, $\{\pm 1\}^X$ is the set of all ("sign") maps from X into the two-element set $\{\pm 1\}$, and $\mathcal{S}$ denotes a subset of $\{\pm 1\}^X$. The set-theoretic complement of $\mathcal{S}$ is written as $\mathcal{S}^*$:

$$\mathcal{S}^* := \{\pm 1\}^X - \mathcal{S}.$$

The *Hamming distance* $D(s', s'')$ between two sign maps $s', s'' \in \{\pm 1\}^X$ equals the cardinality of the *difference set*

$$\Delta(s', s'') = \{x \in X | s'(x) \neq s''(x)\}.$$

We may view the set $\{\pm 1\}^X$ as the vertex set of the graphic hypercube $G(\{\pm 1\}^X)$ representing the 1-skeleton of the (solid) hypercube $[-1, +1]^X$ where two vertices $s', s'' \in \{\pm 1\}^X$ are adjacent if and only if $D(s', s'') = 1$. Note however that the solid edges of the 1-skeleton have length 2 and hence twice the length of their discrete counterparts in the graph $G(\{\pm 1\}^X)$. Further, we can associate the induced subgraph $G(\mathcal{S}) = (\mathcal{S}, E(\mathcal{S}))$ of the graphic hypercube $G(\{\pm 1\}^X)$ to any subset $\mathcal{S}$ of $\{\pm 1\}^X$. The set $\mathcal{S}$ is called *connected* if $G(\mathcal{S})$ is connected, and it is called *isometric* if every pair of vertices s', s'' of $\mathcal{S}$ can be connected in $G(\mathcal{S})$ by a path of length $D(s', s'')$.

Given any subset Y of X, one can always associate two subsets $\mathcal{S}_Y$ and $\mathcal{S}^Y$ of $\{\pm 1\}^{X-Y}$ with an arbitrary set $\mathcal{S} \subseteq \{\pm 1\}^X$ of sign maps:

$$\mathcal{S}_Y := \{t \in \{\pm 1\}^{X-Y} |\text{ some extension } s \in \{\pm 1\}^X \text{ of } t \text{ belongs to } \mathcal{S}\},$$
$$\mathcal{S}^Y := \{t \in \{\pm 1\}^{X-Y} |\text{ every extension } s \in \{\pm 1\}^X \text{ of } t \text{ belongs to } \mathcal{S}\}.$$

$\mathcal{S}_Y = \{s|_{X-Y}|\ s \in \mathcal{S}\}$ encodes the projection of $\mathcal{S}$ onto the graphic subhypercube $\{\pm 1\}^{X-Y}$. In contrast to $\mathcal{S}_Y$, the smaller set $\mathcal{S}^Y$ requires the existence of a full fiber isomorphic to $\{\pm 1\}^Y$ within $\mathcal{S}$ rather than just one point from $\mathcal{S}$. The two operators defined by (3) and (4) suggest two ways to derive a simplicial complex from $\mathcal{S}$:

$$\overline{\mathcal{X}}(\mathcal{S}) := \{Y \subseteq X|\ \mathcal{S}_{X-Y} = \{\pm 1\}^Y\},$$
$$\underline{\mathcal{X}}(\mathcal{S}) := \{Y \subseteq X|\ \mathcal{S}^Y \neq \varnothing\}.$$

To give an example, let $X = \{1, 2, 3\}$ and consider the subset $\mathcal{S}$ of $\{\pm 1\}^X$ that consists of all sign maps except the two constant ones. Then $\mathcal{S}$ encodes an isometric 6-cycle in $G(\{\pm 1\}^X)$. For every singleton Y the projection of $\mathcal{S}$ onto $\{\pm 1\}^{X-Y}$ is surjective, but $\mathcal{S}$ does not include a full fiber isomorphic to this 4-cycle. Therefore $\overline{\mathcal{X}}(\mathcal{S})$ comprises all proper subsets of $\{1, 2, 3\}$, whereas $\underline{\mathcal{X}}(\mathcal{S})$ consists of the empty set and the three singletons. In [3] it is shown that

$$\#\underline{\mathcal{X}}(\mathcal{S}) \leq \#\mathcal{S} \leq \#\overline{\mathcal{X}}(\mathcal{S})$$

holds in general. We called a set $\mathcal{S}$ *ample* if the equality $\#\mathcal{S} = \#\overline{\mathcal{X}}(\mathcal{S})$ holds. Ampleness turned out to be preserved when passing to the complementary set $\mathcal{S}^*$ and to the sets $\mathcal{S}^Y, \mathcal{S}_Y$, and to imply connectedness (and, even more, isometricity) of the subgraph $G(\mathcal{S})$ induced by $\mathcal{S}$ in the graphic hypercube $G(\{\pm 1\}^X)$. It followed that $\mathcal{S}_Y$ and $\mathcal{S}^Y$ had to be connected (isometric) subgraphs of $G(\{\pm 1\}^{X-Y})$ for every ample subset $\mathcal{S}$ of $\{\pm 1\}^X$. Conversely, connectivity (or isometricity) of $\mathcal{S}^Y$ for all $Y \subseteq X$ turned out to imply ampleness, suggesting to call such subsets *superconnected* or *superisometric*. Further investigation finally resulted in recognizing that our ample sets coincided exactly with Lawrence's lopsided sets and that an amazingly rich and multi–facetted theory regarding such subsets of $\{\pm 1\}^X$ could be developed. Here is a list of the most remarkable properties of lopsided sets established in [3], each of which could be used to define them:

superisometry: $\mathcal{S}^Y$ is isometric for all $Y \subseteq X$,

superconnectivity: $\mathcal{S}^Y$ is connected for all $Y \subseteq X$,

iso-recursivity: $\mathcal{S}$ is isometric, and both $\mathcal{S}_e$ and $\mathcal{S}^e$ are lopsided for some $e \in X$,

con-recursivity: $\mathcal{S}$ is connected, and $\mathcal{S}^e$ is lopsided for every $e \in X$,

commutativity: $(\mathcal{S}^Y)_Z = (\mathcal{S}_Z)^Y$ holds for any disjoint $Y, Z \subseteq X$,

ampleness: $\#\mathcal{S} = \#\overline{\mathcal{X}}(\mathcal{S})$,

sparseness: $\#\mathcal{S} = \#\underline{\mathcal{X}}(\mathcal{S})$, *hereditary Euler characteristic 1:* for every face F of $[-1,+1]^X$ intersecting $\mathcal{S}$,

$$\sum_{i\geq 0}(-1)^i f_i(\mathcal{S}\cap F) = 1$$

holds where $f_i(\mathcal{S}\cap F)$ counts the number of graphic i-cubes contained in the intersection of $\mathcal{S}$ with F.

Wiedemann [13] in his PhD thesis of 1986 rediscovered lopsided sets under the name "simple sets" in terms of the last property. Note that our Corollary 2 of [3], which is covered by his results, was not correctly formulated in that the equation can only hold for the faces F that intersect $\mathcal{S}$.

Every subset $\mathcal{S}$ of $\{\pm 1\}^X$ gives rise to a (not necessarily connected) *cubical complex* comprising all faces of the hypercube $[-1,+1]^X$ all of whose vertices belong to $\mathcal{S}$; cf. [12]. This (compact) cubical polyhedron (a *cubihedron* for short) will be denoted by $|\mathcal{S}|$ and called the *geometric realization* of $\mathcal{S}$. The vertices of $|\mathcal{S}|$ are exactly the elements in $\mathcal{S}$. Actually, $|\mathcal{S}|$ is the largest subcomplex of $[-1,+1]^X$ with this property. The 1-skeleton $|\mathcal{S}|_1$ of this cubihedron is the scale 2 "geometric" graph realizing the graph $G(\mathcal{S})$ defined above; recall that the distance $d_1(s,t)$ in $|\mathcal{S}|_1$ between any two vertices s and t of $G(\mathcal{S})$ equals $2D(s,t)$. If $\mathcal{S}$ is connected, then $|\mathcal{S}|$ is connected as well and therefore can be endowed with an intrinsic l_1-metric $d_{|\mathcal{S}|}$. The resulting metric space $(|\mathcal{S}|, d_{|\mathcal{S}|})$ is geodesic but not necessarily a metric subspace of $(\mathbb{R}^X, ||\cdot||_1)$. For example, if $\mathcal{S}$ comprises the six vertices of an isometric 6-cycle in the 3-cube, then $|\mathcal{S}|$ is a solid 6-cycle of $\mathbb{R}^3$. The l_1-distance between the midpoints of two opposite sides of this cycle is 2, while the intrinsic l_1-distance between the same points is 3. In this paper, we will establish that path-l_1-isometry of the associated cubical complex in $\mathbb{R}^X$ is yet another characteristic feature of lopsidedness, thus demonstrating that lopsided sets constitute a fundamental domain for l_1-geometry:

path-l_1-isometry: $|\mathcal{S}|$ endowed with the intrinsic path-metric is a metric subspace of $(\mathbb{R}^X, ||\cdot\ ||_1)$.

The primary motivation of Lawrence in his paper [8] was to investigate and generalize those subsets

$$\mathcal{S}(K) := \{s\in\{\pm 1\}^X \mid \{t\in K \mid t(x)s(x)\geq 0 \text{ for all } x\in X\}\neq\varnothing\}$$

of $\{\pm 1\}^X$ which represent the closed orthants of $\mathbb{R}^X$ intersecting a convex subset K of $\mathbb{R}^X$. He (as well as Wiedemann [13]) showed that such sets are lopsided. However, not every lopsided set encodes the orthant intersection pattern for a convex set in Euclidean space; see [8]. It comes close, though. As we will show below, in order to have a full geometric representation, one has to resort to a weaker concept:

$\mathcal{S}$ encodes the orthant intersection pattern for some geodesic metric subspace K of $(\mathbb{R}^n, ||\cdot||_1)$, that is, a sign vector x belongs to $\mathcal{S}$ exactly when the orthant determined by x also includes a point from K.

Lopsided sets $\mathcal{S}$ can also be characterized via their "cocircuits", i.e. the barycentric maps of the facets (maximal faces) of the associated cubihedron $|\mathcal{S}|$, which are particular mappings from X to $\{\pm 1, 0\}^X$. Every subset $\mathcal{R}$ of $\{\pm 1, 0\}^X$ can be extended to the cubihedron $[\mathcal{R}]$ consisting of all cubes whose barycenters belong to $\mathcal{R}$. We will then show that the cubihedron $[\mathcal{R}]$ is path-l_1-isometric if and only if $\mathcal{R}$ satisfies the signed-circuit axiom from the theory of oriented matroids [4]:

signed-circuit axiom: for all $r_1, r_2 \in \mathcal{R}$ and $e \in X$ with $r_1(e) \cdot r_2(e) = -1$ there exists some $r_0 \in \mathcal{R}$ such that $r_0(e) = 0$ and $r_0(x) \in \{0, r_1(x), r_2(x)\}$ for all $x \in X$.

Then we establish that the cocircuits of lopsided sets are exactly the sign maps that satisfy the signed-circuit axiom and consist of pairwise non-dominated maps (maximality). The characterization of cocircuits $\mathcal{R} \subseteq \{\pm 1, 0\}^X$ of oriented matroids also involves maximality, the signed-circuit axiom, but, additionally, requires the symmetry ($r \in \mathcal{R}$ implies that $-r \in \mathcal{R}$) [4]. Nevertheless, the lack of symmetry implies that lopsided sets can differ substantially from oriented matroids regarding their combinatorial and geometric structure.

The paper is organized in the following way. In Section 2 we discuss the intrinsic path metrics associated with an arbitrary metric space $(\mathcal{M}, d)$. Sections 3 and 4 present characterizations of lopsided cubihedra via metric conditions and via projections. Then the geometric realization of lopsided sets in terms of intersection pattern with orthants is established in Section 5. The final Section 6 investigates the concepts of circuits and cocircuits and their relation to the geometric structure of lopsided cubihedra.

2. Intrinsic path metrics

In this section we recall some basic notions about intrinsic path metrics and the length of paths, which are relevant for the intrinsic metrics of cubihedra. In an arbitrary metric space $(\mathcal{M}, d)$ one can trivially define the *length $\ell(P)$ of a finite x, y-path P* of points $x =: t_0, t_1, \ldots, t_k := y$ as the sum

$$\ell(P) := \Sigma_{i=1}^k d(t_{i-1}, t_i).$$

The *modulus* of this path P is the maximum of all single step lengths $d(t_{i-1}, t_i)$ for $i = 1, ..., k$. Then the infimum

$$\inf\{\ell(P) |\ P \text{ is a finite } x, y\text{-path in } \mathcal{M} \text{ with modulus}(P) < \epsilon\}$$

of the lengths of all finite paths between two points x and y in $\mathcal{M}$ having modulus smaller than ϵ exists because the length of every finite path between x and y is bounded below by $d(x, y)$ by virtue of the triangle inequality. Then the supremum

$$\delta_{\mathcal{M},d}(x, y) := \sup\{\inf\{\ell(P) |\ P \text{ is a finite } x, y\text{-path in } \mathcal{M} \text{ with modulus}(P) < \epsilon\} |\ \epsilon \in \mathbb{R}^+\}$$

will be called the *(intrinsic) finite-path distance* between x and y in the metric space $(\mathcal{M}, d)$, which may formally take the value ∞ of the extended real line when the supremum does not exist. If, however, all values are real numbers then we could speak of the *(intrinsic) finite-path metric* of $(\mathcal{M}, d)$, since evidently $(\mathcal{M}, d)$ satisfies the triangle inequality.

If there exists a (general) x, y-*path* in $(\mathcal{M}, d)$, that is, a continuous map $\gamma : [0,1] \to \mathcal{M}$ with $\gamma(0) = x$ and $\gamma(1) = y$, then its length [10, p.11]

$$L(\gamma) := \sup \Big\{ \sum_{i=1}^{k} d(\gamma(t_{i-1}), \gamma(t_i)) \mid 0 \le t_0 \le t_1 \le \ldots \le t_k \le 1 \text{ where } k \ge 1 \Big\}$$

is an upper bound for the lengths $\ell(P)$ of all finite x, y-paths P contained in γ. If an x, y-path γ exists with $L(\gamma) < \infty$, then x and y are said to be *connected by a rectifiable path* in $(\mathcal{M}, d)$ [10, p.35]. Taking the infimum over all general x, y-paths, we obtain

$$\delta^{\mathcal{M},d}(x, y) := \inf\{L(\gamma) | \ \gamma \text{ is an } x, y\text{-path in } \mathcal{M}\},$$

referred to as the *path* (or *length*) *distance* in $\mathcal{M}$ relative to d (in the book [10] the length distance is denoted by d_ℓ). If each pair of points in $\mathcal{M}$ is connected by a rectifiable path, then by [10, Proposition 2.1.5] $\delta^{\mathcal{M},d}$ is a metric on $\mathcal{M}$, referred to the *(intrinsic) path metric* of $\mathcal{M}$ associated to d. We thus have the inequalities

$$d(x, y) \le \delta_{\mathcal{M},d}(x, y) \le \delta^{\mathcal{M},d}(x, y),$$

with the understanding that either δ value could equal ∞. In the extreme case, $\delta_{\mathcal{M},d}$ could be a metric whereas $\delta^{\mathcal{M},d}$ is the constant ∞ map on pairs of distinct points. Consider, for example, the rational unit interval $M = \mathbb{Q}[0, 1]$ with its natural metric d: since paths of any modulus exist, the intrinsic path metric coincides with the natural metric, but geodesics between distinct points do not exist, whence the intrinsic length distance is infinite. Therefore the advantage of using $\delta_{\mathcal{M},d}$ rather than $\delta^{\mathcal{M},d}$ is that we do not need to impose existence of rectifiable paths. In the important case that $(\mathcal{M}, d)$ is a *length* (or *path-metric) space* [10, p.35], that is, d coincides with $\delta^{\mathcal{M},d}$, both concepts of intrinsic metrics can be equated with the original metric d. The property $\delta_{\mathcal{M},d} = d$ carries over to dense subspaces of $(\mathcal{M}, d)$. In an arbitrary length space $(\mathcal{M}, d)$ general x, y-paths of length $d(x, y)$, that is, x, y-geodesics, need not exist. When however they exist for all pairs of points then the space is called *geodesic*. A geodesic space $(\mathcal{M}, d)$ is called a *real tree* if every geodesic constitutes the unique path between its end points. Note that a compact real tree $(\mathcal{M}, d)$ may have infinitely many *branching points*, i.e. points p for which $\mathcal{M} - \{p\}$ has at least three connected components.

Every finite graph $G = (V, E)$ has a trivial geometric realization as a 1-dimensional cell complex obtained by replacing each edge $\{x, y\}$ by a solid link, that is, a copy $[u_x, u_y]$ of the unit interval $[0, 1]$ of the real line such that two copies intersect in an endpoint exactly when the corresponding edges are incident. The resulting geometric graph (alias network) $\mathcal{M}$ inherits its metric δ from the standard-graph metric d of G and the usual distance of $[0, 1]$. Informally speaking, the distance $\delta(p, q)$ between a point p from a link $[u_v, u_w]$ and a point q of a link $[u_x, u_y]$ in the geometric graph $\mathcal{M}$ is the smallest of the four sums $|p-u_v|+d(v, x)+|q-u_x|$, $|p - u_v| + d(v, y) + |q - u_y|$, $|p - u_w| + d(w, x) + |q - u_x|$, and $|p - u_w| + d(w, y) + |q - u_y|$. More formally, one can apply the general construction of the length metric as described in [10, Example 2.1.3(iv)] or [5, Section I.1.9]. In this way, one obtains a geodesic space $(\mathcal{M}, \delta)$. Similarly, in ad hoc manner, one could deal with the geometric realization of any finite cubical complex, but for lopsided sets we can obtain the length space property for the associated geometric realizations in a canonical way; see below.

A metric space $(\mathcal{M}, d)$ is *Menger-convex* if for any two distinct points $x, y \in \mathcal{M}$ there exists some point z *between* x and y, that is, z belongs to the *segment*

$$[x, y]_{\mathcal{M}} = \{z \in \mathcal{M} |\ d(x, z) + d(z, y) = d(x, y)\}$$

such that, in addition, z is different from x and y. Menger [9] himself has shown that Menger-convexity in a complete metric space $(\mathcal{M}, d)$ entails that $(\mathcal{M}, d)$ is geodesic (see also [1] or [11, §18.5]). Completeness is an essential prerequisite here. Indeed, the intersection $(\mathcal{M}, d)$ of the Cantor set with the irrational (open) unit interval is Menger-convex but $\delta_{\mathcal{M},d}(x, y) = \infty$ holds for all pairs x, y of distinct points. In the absence of Menger-convexity the requirement $\delta_{\mathcal{M},d} < \infty$ together with some local compactness condition can at least guarantee that $(\mathcal{M}, \delta_{\mathcal{M},d})$ is a geodesic space. We say that a metric space is *boundedly compact* if every closed bounded subset is compact. Note that in the context of Riemannian geometry and length spaces one usually calls these spaces “proper”.

Lemma 1. *If a boundedly compact metric space $(\mathcal{M}, d)$ admits a finite-path metric $\delta_{\mathcal{M},d} < \infty$, then $(\mathcal{M}, \delta_{\mathcal{M},d})$ is a geodesic space, whence $\delta_{\mathcal{M},d} = \delta^{\mathcal{M},d}$ holds.*

Proof. For any distinct points x, y we can approximate $\alpha := \delta_{\mathcal{M},d}(x, y)$ by finite x, y-paths of moduli $1/n$ $(n \to \infty)$. Specifically, for every $n \in \mathbb{N}$ there exists a finite x, y-path P_n of length smaller than $\alpha(1 + 1/(2n))$ and modulus smaller than $\alpha/(2n)$. On each path P_n pick the first point y_n for which the initial x, y_n-subpath P'_n exceeds length $\alpha/2$, whence the length of the final y_n, y-subpath of P_n is then smaller than $\alpha(1 + 1/(2n))$. Since the sequence (y_n) is contained in the closed 2α-ball centered at x, it includes a subsequence (z_n) converging to some point z such that, say, $d(z_n, z) < \alpha/(2n)$. Inserting this limit point z into each P_n directly after y_n yields a path consisting of an initial x, z-subpath plus a final z, y-subpath both of modulus smaller than $\alpha/(2n)$ and with lengths between $\alpha/2 - 1/(2n)$ and $\alpha/2 + 1/n$. Therefore

$$\delta_{\mathcal{M},d}(x, z) + \delta_{\mathcal{M},d}(z, y) \leq \alpha/2 + \alpha/2 = \alpha = \delta_{\mathcal{M},d}(x, y),$$

whence by virtue of the triangle inequality z is the midpoint of the segment between x and y relative to the finite path metric $\delta_{\mathcal{M},d}$. We can now iterate this midpoint construction by applying the procedure first to the pairs x, z and z, y, and so on. Then we eventually obtain a dense subset Z of the segment $[x, y]$ relative to the metric $\delta_{\mathcal{M},d}$, admitting an isometry ζ from $(Z, \delta_{\mathcal{M},d}|_{Z\times Z})$ to the set of α-multiples of the dyadic fractions of 1 endowed with the natural metric.

Every non-dyadic number τ from the unit interval is the limit of some sequence (τ_n) of dyadic fractions of 1, for which the α-multiples each have a pre-image u_n under ζ in Z. Then (u_n) is a Cauchy sequence relative to $\delta_{\mathcal{M},d}$ and hence to d, which converges to some point u due to completeness of $(\mathcal{M}, d)$. For any two points v and w of Z with $\zeta(v) < \alpha\tau$ and $\zeta(w) > \alpha\tau$ almost all u_n are between v and w relative to $\delta_{\mathcal{M},d}$. Therefore one can approximate $\delta_{\mathcal{M},d}(v, w)$ by pairs of concatenated finite v, u_n-paths and u_n, w-paths of moduli converging to 0 and total lengths converging to $\delta_{\mathcal{M},d}(v,w)$ $(n \to \infty)$. Substituting u_n by u in all paths yields a sequence of concatenated paths approximating both $\delta_{\mathcal{M},d}(v,u)+\delta_{\mathcal{M},d}(u,w)$ and $\delta_{\mathcal{M},d}(v,w)$. This shows that u is between v and w relative to $\delta_{\mathcal{M},d}$. In summary, we have thus established an isometry from the closure $\overline{Z}$ of Z in $(\mathcal{M}, \delta_{\mathcal{M},d})$ to the interval $[0,\alpha]$, where $\overline{Z}$ is a x,y-geodesic in $(\mathcal{M}, \delta_{\mathcal{M},d})$. □

Let A and B be two sets of maps from some disjoint nonempty sets Y and Z, respectively, to a set Λ. Then we write $A \times B$ for the set of all maps $r : Y \cup Z \to \Lambda$ for which $r|_Y$ belongs to A and $r|_Z$ belongs to B. For singletons A or B, set brackets are omitted. Given a finite set X and a nonempty subset Y of X, the set $\Lambda^Y \times r_0$ for any $r_0 \in \Lambda^{X-Y}$ is called a *fiber* of the Cartesian power Λ^X, namely the Y-*fiber* of Λ^X *at* $q_0 \times r_0$ for any $q_0 \in \Lambda^Y$. If Λ is endowed with a (natural) metric, then we will denote by d the product metric d^X on the the product space Λ^X, where X is a finite set. When Λ is connected, a subspace $\mathcal{R}$ of Λ^X is called *fiber-connected* if the intersection of $\mathcal{R}$ with each fiber of Λ^X is connected (or empty). Similarly, when Λ is a geodesic space, $\mathcal{R} \subseteq \Lambda^X$ is said to be *fiber-geodesic* if $\mathcal{R}$ intersects each fiber of Λ^X in a geodesic subspace (or the empty set). The X-fiber of Λ^X is understood to be the entire space Λ^X.

Corollary 1. *Let Λ be a boundedly compact geodesic space. For a closed subset $\mathcal{R}$ of some finite power Λ^X of Λ (endowed with the product metric $d = d^X$), the following statements are equivalent:*

- (i) *the intrinsic finite-path metric $\delta_{\mathcal{R},d}$ of $\mathcal{R}$ coincides with the restriction $d|_{\mathcal{R}}$ of the product distance $d = d^X$ of Λ^X;*
- (ii) *$\mathcal{R}$ is a geodesic subspace of Λ^X;*
- (iii) *$\mathcal{R}$ is fiber-geodesic.*

Proof. Clearly, (ii)⇒(i)&(iii) and (iii)⇒(ii). Since Λ is boundedly compact and X is finite, Λ^X is also boundedly compact. This carries over to $(\mathcal{R}, d|_{\mathcal{R}})$ because $\mathcal{R}$ is a closed subset of Λ^X. If (i) holds, then $(\mathcal{R}, \delta_{\mathcal{R},d|_{\mathcal{R}}})$ is a geodesic space by Lemma 1, because $\delta_{\mathcal{R},d|_{\mathcal{R}}} = d|_{\mathcal{R}} < \infty$. Therefore $\mathcal{R}$ is a geodesic subspace of Λ^X, thus establishing the implication (i)⇒(ii). □

Corollary 1 can be applied, for instance, to the closed subsets $\mathcal{R}$ of the Euclidean space $\mathcal{R}^X$: thus, the restriction of the Euclidean (l_2-)metric to $\mathcal{R}$ constitutes the intrinsic (finite-)path metric of $\mathcal{R}$ relative to the Euclidean metric exactly when $\mathcal{R}$ is closed under taking line segments, that is, $\mathcal{R}$ is convex. In the l_1 case condition (iii) of Corollary 1 can be weakened further depending on the space Λ. Namely, in order to replace fiber-geodesity by fiber-connectedness, the factors need to be real trees.

Lemma 2. *Let Λ be a boundedly compact real tree (equipped with the natural metric) in which every segment includes only finitely many branching points. Then a closed subset $\mathcal{R}$ of some finite power Λ^X of Λ is a geodesic subspace of Λ^X if and only if $\mathcal{R}$ if fiber-connected.*

Proof. By Corollary 1, any closed geodesic subspace of Λ^X is fiber-geodesic and hence fiber-connected. Conversely assume that $\mathcal{R}$ is a fiber-connected closed subspace of Λ^X. Since

(Λ^X, d) is complete and $\mathcal{R}$ is closed, $(\mathcal{R}, d|_{\mathcal{R}})$ is also a complete metric space. Thus to prove that $\mathcal{R}$ is a geodesic subspace of Λ^X, it suffices to establish that $\mathcal{R}$ is Menger-convex with respect to d. We proceed by induction on $\#X$. Since any proper fiber $\mathcal{R}_0$ of $\mathcal{R}$ is a closed fiber-connected subset of $\Lambda^Y \times r_0$ for a proper subset Y of X and a map $r_0 \in \Lambda^{X-Y}$, by induction assumption we can assume that $\mathcal{R}_0$ is a geodesic subspace of $\Lambda^Y \times \{r_0\}$ and Λ^X. Suppose by way of contradiction that $\mathcal{R}$ is not Menger-convex. In view of the induction hypothesis, we may then suppose that there exist $r_1, r_2 \in \mathcal{R}$ with

$$[r_1, r_2]_{\mathcal{R}} = \mathcal{R} \cap [r_1, r_2]_{\Lambda^X} = \{r_1, r_2\}$$

such that $r_1(x) \neq r_2(x)$ holds for all $x \in X$.

For any $x \in X$, let Λ^x be the xth factor (a copy of Λ) of the product Λ^X. Then $\Lambda^{X-\{x\}} \times r_1(x)$ is the $(X-\{x\})$-fiber of Λ^X that contains the point r_1 of $\mathcal{R}$. By $\Lambda^x_+ = \Lambda^x_+(r_1, r_2)$ denote the set of all points λ of the real tree Λ^x for which $r_1(x)$ is not between λ and $r_2(x)$, that is,

$$\Lambda^x_+ := \{\lambda \in \Lambda^x | \ [\lambda, r_1(x)]_{\Lambda^X} \cap [r_1(x), r_2(x)]_{\Lambda^X} \neq \{r_1(x)\}\}.$$

Then Λ^x_+ is an open subset of the real tree Λ^x and its closure $\overline{\Lambda^x_+} = \Lambda^x_+ \cup \{r_1(x)\}$ is a boundedly compact subtree of Λ^x. Trivially, the closure of $\Lambda^X_+ := \prod_{x \in X} \Lambda^x_+$ equals

$$\overline{\Lambda^X_+} = \prod_{x \in X} (\Lambda^x_+ \cup \{r_1(x)\}).$$

(Note that Λ^X_+ resp. $\overline{\Lambda^X_+}$ equal the open resp. closed first orthant of $\mathbb{R}^X$ in the case that $\Lambda = \mathbb{R}$, $r_1 = \mathbf{0}$ and $r_2 > \mathbf{0}$.) We claim that

$$\mathcal{R} \cap \overline{\Lambda^X_+} = (\mathcal{R} \cap \Lambda^X_+) \cup \{r_1(x)\}.$$

Suppose the contrary, that is, suppose that there exists a point

$$r \in (\mathcal{R} - \{r_1\}) \cap \prod_{y \in X} (\Lambda^y_+ \cup \{r_1(y)\} \text{ with } r(x) = r_1(x)$$

for some $x \in X$. Then both r and r_1 belong to the fiber $\Lambda^{X-\{x\}} \times r_1(x)$ and hence are connected by a geodesic γ in $\mathcal{R}$, according to the induction hypothesis. On the other hand, as Λ is a real tree, there exists a point s of Λ^X (the median point of r_1, r, s) such that

$$[r_1, s]_{\Lambda^X} = [r_1, r_2]_{\Lambda^X} \cap [r_1, r]_{\Lambda^X}.$$

The complement $Y = \{y \in X | \ r_1(y) \neq r(y)\}$ of the equalizer of r_1 and r does not contain the element x. Then the geodesic γ between r_1 and r is included in the Y-fiber at r_1 (and r). By the initial hypothesis, $r(y) \in \Lambda^y_+$ and hence $r_1(y) \neq s(y)$ for each $y \in Y$. Let ϵ be the minimum of the distances of $r_1(y)$ and $s(y)$ for $y \in Y$ in the copies of the real tree Λ. Then the intersection of γ with the closed ball of radius ϵ centered at r_1 is included in the box

$$[r_1|_Y, s|_Y]_{\Lambda^Y} \times r_1|_{X-Y} \subseteq [r_1, r_2]_{\Lambda^X},$$

whence

$$\{r_1\} \subsetneqq \gamma \cap [r_1, r_2]_{\Lambda^X} \subseteq [r_1, r_2]_{\mathcal{R}},$$

contrary to the assumption that $[r_1, r_2]_{\mathcal{R}} = \{r_1, r_2\}$.

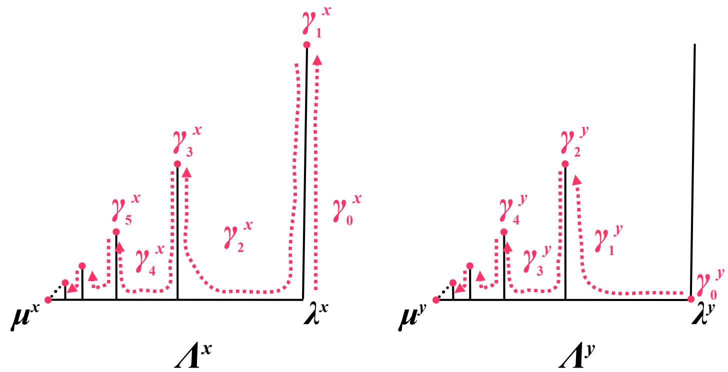

FIGURE 1. Projection of a geodesic γ in $\Lambda^x \times \Lambda^y$ to the factors

The intersection $\mathcal{R}_+ := \mathcal{R} \cap \Lambda_+^X$ is an open set in the (topological) subspace $\mathcal{R}$ of Λ^X that includes r_2 but not r_1. We wish to show that $\mathcal{R}_+$ is closed as well. By what has just been shown,

$$\overline{\mathcal{R}_+} \subseteq \mathcal{R}_+ \cup \{r_1\} \subseteq \mathcal{R}.$$

So suppose that $r_1 \in \overline{\mathcal{R}_+}$. Then there exists a sequence (s_n) in $\mathcal{R}_+$ converging to r_1. The sequence (t_n) of median points t_n of r_1, r_2, and s_n ($n \in \mathbb{N}$) also converges to r_1. Since (s_n) is fully included in Λ_+^X, so is (t_n). It follows from $[r_1, r_2]_{\mathcal{R}} = \{r_1, r_2\}$ that either $t_n(x) = r_2(x)$ or $t_n(x)$ is a branching point of Λ^x for each coordinate x and index n. This, however, conflicts with the initial requirement on Λ that all segments of Λ contain only finitely many branching points. We conclude that $\overline{\mathcal{R}_+} = \mathcal{R}_+$ as asserted. Therefore $\mathcal{R}_+$ is both open and closed in $\mathcal{R}$, which finally contradicts connectedness of $\mathcal{R}$, and the proof is complete. $\square$

A standard example shows that the finiteness condition on branching points in segments cannot be dropped in Lemma 2. Consider the compact tree Λ shown in Figure 2 in two copies Λ^x and Λ^y. It connects three vertices of a right-angled triangle with legs of length 1 where the sequence of leaves is located on the hypotenuse and converges to the intersection point μ of the hypothenuse and one leg. The branching points of Λ lie on that leg and converge also to μ. The total length of Λ equals 3. A path $\gamma = \gamma_0 \cup \gamma_1 \cup \gamma_2 \cup \ldots \cup \{(\mu^x, \mu^y)\}$ between two points (λ^x, λ^y) and (μ^x, μ^y) in $\Lambda^x \times \Lambda^y$ can be constructed by alternating x-fibers $\gamma_0, \gamma_2, \ldots$ and y-fibers $\gamma_1, \gamma_3, \ldots$, with (μ^x, μ^y) as limiting point; see Figure 1 for the two projections of γ on Λ^x and Λ^y. Then γ intersects the segment between (λ^x, λ^y) and (μ^x, μ^y) only in its two end points. Thus, the closed subset γ of $\Lambda^x \times \Lambda^y$ is fiber-connected but not Menger-convex and hence does not constitute a geodesic subspace.

3. METRIC CHARACTERIZATIONS

For a connected subset $\mathcal{S}$ of the finite graphic hypercube $G(\{\pm 1\}^X)$, its geometric realization $|\mathcal{S}|$ within the hypercube $[-1, +1]^X$ (endowed with the l_1-metric d) admits an intrinsic path metric

$$d_{|\mathcal{S}|} := \delta_{|\mathcal{S}|, d|_{|\mathcal{S}|}} = \delta^{|\mathcal{S}|, d|_{|\mathcal{S}|}}.$$

Indeed, any pair of points in $|\mathcal{S}|$ can be connected by a rectifiable path in $|\mathcal{S}|$ relative to d, whence $d_{|\mathcal{S}|}$ exists by virtue of Lemma 1. Although $(\mathcal{S}, d_{|\mathcal{S}}|)$ is a geodesic space in its own right, it is not necessarily a metric subspace of $([-1,+1]^X, d)$, even when $\mathcal{S}$ is isometric in $G(\{\pm 1\}^X)$. We say that a subset $\mathcal{R}$ of $[-1,+1]^X$ is *path-l_1-isometric* if the restriction of the l_1-metric d on $\mathcal{R}$ constitutes the intrinsic path metric of $\mathcal{R}$. The theorem proved in this section describes under which circumstances this holds in the case of $\mathcal{R} = |\mathcal{S}|$.

A *face* $\mathcal{F}$ of $[-1,+1]^X$ is a Y-fiber of the form

$$\mathcal{F} = [-1,+1]^Y \times s_0 \text{ for some } s_0 \in \{\pm 1\}^{X-Y}$$

with $Y \subseteq X$. By convention, the entire hypercube is its X-fiber and its vertices are the faces that are $\varnothing$-fibers. We say that two faces of $[-1,+1]^X$ are *parallel* if they are both Y-fibers with respect to the same subset Y of X. Parallel faces thus arise by intersecting the hypercube with parallel (affine) Y*-planes*

$$\{p \in \mathbb{R} |\ p|_{X-Y} = c\} \text{ for } c \in [-1,+1]^{X-Y}.$$

The faces of $[-1,+1]^X$ are known to be gated in the following sense. A subset $\mathcal{A}$ of any metric space $(\mathcal{M}, d)$ is called *gated* [7] if for every point $x \in \mathcal{M}$ there exists a (necessarily unique) point $x' \in \mathcal{A}$, the *gate* of x in $\mathcal{A}$, for which

$$d(x,y) = d(x,x') + d(x',y) \text{ for all } y \in \mathcal{A}.$$

Given $\mathcal{S} \subseteq \{\pm 1\}^X$, the faces of the geometric realization of $\mathcal{S}$ are those faces $\mathcal{F}$ of the hypercube for which $\mathcal{F} \cap \{\pm 1\}^X \subseteq \mathcal{S}$. For a point r of $|\mathcal{S}|$ we denote by $[r]$ the smallest face of $[-1,+1]^X$ containing r. Note that this face (necessarily included in $|\mathcal{S}|$ by definition) is determined by the coordinates $r(z)$ $(z \in X)$ for which $-1 < r(z) < +1$:

$$\begin{aligned}
[r] &= [-1,+1]^{X(r)} \times r|_{X-X(r)}, \text{ where} \\
X(r) &:= \{z \in X|\ -1 < r(z) < +1\}, \text{ and} \\
r|_{X-X(r)} &\in \{\pm 1\}^{X-X(r)}.
\end{aligned}$$

The *barycenter* of the face $[r]$ is given by the map from X to $\{\pm 1, 0\}$ that is the zero map on $X(r)$ and coincides with $r|_{X-X(r)}$ elsewhere. Then $\mathrm{Baryc}(|\mathcal{S}|)$ collects the barycenters of all faces of $|\mathcal{S}|$:

$$\begin{aligned}
\mathrm{Baryc}(|\mathcal{S}|) : &= \{t \in \{\pm 1, 0\}^X |\ \exists r \in |\mathcal{S}| \text{ with } r(x) \cdot t(x) > 0\ \forall x \in X \text{ with } r(x) \neq 0\} \\
&= |\mathcal{S}| \cap \{\pm 1, 0\}^X.
\end{aligned}$$

We now introduce the notation $\mathcal{R}^Y$ and $\mathcal{R}_Y$ for (connected) subsets $\mathcal{R}$ of $[-1,+1]^X$ in analogy to $\mathcal{S}^Y$ and $\mathcal{S}_Y$ for subsets $\mathcal{S}$ of $\{\pm 1\}^X$. Namely,

$$\mathcal{R}_Y := \{r|_{X-Y} |\ r \in \mathcal{R}\} \subseteq [-1,+1]^{X-Y}$$

is isomorphic to the orthogonal projection of $\mathcal{R}$ onto the $(X-Y)$-plane, whereas

$$\mathcal{R}^Y := \{r|_{X-Y} |\ [-1,+1]^Y \times r|_{X-Y} \in \mathcal{R}\} \subseteq [-1,+1]^{X-Y}$$

encodes the location of the Y-fibers in $\mathcal{R}$ that are also Y-fibers (faces) of $[-1,+1]^X$.

Theorem 1. *For a subset $\mathcal{S}$ of $\{\pm 1\}^X$, the following statements are equivalent:*

(i) *$\mathcal{S}$ is lopsided;*

(ii) $|\mathcal{S}|$ *is a path-l_1-isometric subset of* $[-1,+1]^X$;
(iii) $\mathcal{S}$ *is isometric in* $G(\{\pm 1\}^X)$ *and every face of* $|\mathcal{S}|$ *is gated in* $(|\mathcal{S}|, d_{|\mathcal{S}|})$;
(iv) *the restrictions of the intrinsic metric* $d_{|\mathcal{S}|}$ *and the* l_1*-metric* d *on Baryc*$(|\mathcal{S}|)$ *coincide;*
(v) *the barycenters* u *and* v *of any two parallel faces of* $|\mathcal{S}|$ *have the same distance with respect to the intrinsic metric and* l_1*-metric:*

$$d_{|\mathcal{S}|}(u,v) = d(u,v).$$

Proof. The two implications (ii)⇒(iv)⇒(v) are trivial. The implication (ii)⇒ (iii) is also obvious: For a face $\mathcal{F}$ of $|\mathcal{S}|$ and a point $r \in |\mathcal{S}|$, take the gate r' of r in $\mathcal{F}$ relative to the l_1-metric d of the hypercube $[-1,+1]^X$. Since $r' \in \mathcal{F} \subseteq |\mathcal{S}|$ and $|\mathcal{S}|$ is path-l_1-isometric, we have $d_{|\mathcal{S}|}(r,r') = d(r,r')$, whence r' also serves as the corresponding gate within the cubihedron $|\mathcal{S}|$.

To establish (i)⇒(ii), we only need to show that $|\mathcal{S}|$ is fiber-connected, by virtue of Corollary 1 and Lemma 2. A fiber $\mathcal{F}$ of $|\mathcal{S}|$ is the intersection of $|\mathcal{S}|$ with some Y-plane, say,

$$\mathcal{F} = |\mathcal{S}| \cap ([-1,+1]^Y \times r|_{X-Y}) \text{ for some } r \in |\mathcal{S}| \text{ and } Y \subseteq X.$$

Note that the smallest face of $|\mathcal{S}|_Y$ containing $r|_{X-Y}$ has the form

$$[r|_{X-Y}] = [-1,+1]^Z \times r|_{X-Y-Z} \text{ where}$$
$$Z = X(r) - Y = \{z \in X - Y | \; -1 < r(z) < +1\}.$$

Thus, the smallest face of $|\mathcal{S}|$ containing any point of $\mathcal{F}$ has $[-1,+1]^Z$ as its factor:

$$[-1,+1]^Z \times \mathcal{F}_Z \subseteq |\mathcal{S}|.$$

This can also be expressed by saying that $\mathcal{F}_Z = \{q|_{X-Z} | \; q \in \mathcal{F}\}$ is the Y-fiber of $|\mathcal{S}^Z|$ containing $r|_{X-Z}$. Since the positions at which the points of $\mathcal{F}_Z$ have their coordinates properly between -1 and +1 all belong to Y, we infer that $\mathcal{F}_Z$ equals the geometric realization of the Y-fiber of $\mathcal{S}^Z$ at $r|_{X-Z}$. Since $\mathcal{S}^Z$ and its fibers are lopsided and hence connected by [3, Theorem 3], we conclude that $\mathcal{F}_Z$ is connected and so is $\mathcal{F}$.

Next we show that (iii)⇒(v) holds. If u and v are the barycenters of two parallel faces, then these faces are $[u]$ and $[v]$ having the say dimension k, say. Then they lie on parallel Y-planes for some subset $Y \subseteq X$ with $|Y| = k$. Let q be any vertex of $[u]$ (necessarily belonging to $\mathcal{S}$) and r be the corresponding vertex from $[v]$ (and $\mathcal{S}$), thus satisfying $q|_Y = r|_Y$. Since $\mathcal{S}$ is isometric and u is the barycenter of $[u]$, we have

$$\begin{aligned} d_{|\mathcal{S}|}(u,r) &\leq d_{|\mathcal{S}|}(u,q) + d_{|\mathcal{S}|}(q,r) \\ &= d(u,q) + d(q,r) = d(u,r), \end{aligned}$$

whence equality holds. Therefore the gate of u in $[v]$ must have distance $k/2$ to all vertices of $[v]$. The unique point in $[v]$ with this property is the barycenter v. Consequently, v is the gate of u in $[v]$ relative to the intrinsic metric $d_{|\mathcal{S}|}$. In particular, $d_{|\mathcal{S}|}(u,v) = d(q,r) = d(u,v)$, as required.

Finally, we establish the implication (v)⇒(i). Connect the barycenters u and v of two parallel Y-faces $[u]$ and $[v]$ by a geodesic γ in $|\mathcal{S}|$. Then every point r of γ has the same projection on $[-1,+1]^Y$ as u and v. Therefore $[r]$ has a Y-cube as a factor, and consequently, the geodesic γ projects onto a geodesic γ^Y of $|\mathcal{S}|^Y$ which connects the vertices $u|_{X-Y}$ and $v|_{X-Y}$ of $\mathcal{S}^Y$. To show that $u|_{X-Y}$ and $v|_{X-Y}$ are at distance $d(u,v)$ in $\mathcal{S}^Y$, we use induction on the (l_1-)distance between $u|_{X-Y}$ and $v|_{X-Y}$. Let r be the point of γ at distance 2 from u. Then $[r]$ is some $(Y \cup Z)$-cube within $\mathcal{S}$ with $\varnothing \neq Z \subseteq X - Y$, which necessarily includes $[u]$ as a face. Then every point in $[r|_{X-Y}]$ belongs to $|\mathcal{S}|^Y$. In particular, the neighbor s' of $r|_{X-Y}$ with $r|_{X-Y-\{e\}} = s'|_{X-Y-\{e\}}$ and $s'(e) = -r(e)$ for some $e \in Z$ also belongs to $|\mathcal{S}|^Y$ and is between $u|_{X-Y}$ and $v|_{X-Y}$. By virtue of the induction hypothesis s' and $v|_{X-Y}$ are at distance $d(s', v|_{X-Y}) = d(u,v) - 2$. Therefore $\mathcal{S}^Y$ is isometric and consequently $\mathcal{S}$ is lopsided. $\square$

4. Projection and dimension

We will now show that for a subset $\mathcal{S}$ of $\{\pm 1\}^X$ the two operators of (orthogonal) projection and geometric realization commute exactly when $\mathcal{S}$ is lopsided. To this end we will make use of the calculus involving sets of the form $(\mathcal{S}^Z)_Y$, as developed in [3]. First observe that, essentially by definition, we have

$$|\mathcal{S}| = \bigcup_{Z \subseteq X} \bigcup_{s \in \mathcal{S}^Z} [-1,+1]^Z \times s. \tag{1}$$

Therefore the (topological) dimension of the cubihedron $|\mathcal{S}|$ can be expressed as

$$\dim |\mathcal{S}| = \max\{\#Z \mid \mathcal{S}^Z \neq \varnothing\} = \max\{\#Z \mid Z \in \underline{\mathcal{X}}(\mathcal{S})\},$$

using the terminology of [3]; see also the Introduction. Applying the above equation to $\mathcal{S}_Y$ instead of $\mathcal{S}$ yields

$$|\mathcal{S}_Y| = \bigcup_{Z \subseteq X-Y} \bigcup_{t \in (\mathcal{S}_Y)^Z} [-1,+1]^Z \times t. \tag{2}$$

For the projection from $[-1,+1]^X$ to $[-1,+1]^{X-Y}$ applied to $\mathcal{R} = |\mathcal{S}|$ we compute

$$\begin{aligned} |\mathcal{S}|_Y &= \left(\bigcup_{Z \subseteq X} \bigcup_{s \in \mathcal{S}^Z} [-1,+1]^Z \times s \right)_Y \\ &= \bigcup_{Z \subseteq X} \bigcup_{s \in \mathcal{S}^Z} [-1,+1]^{Z-Y} \times s|_{(X-Z)-Y} \\ &= \bigcup_{Z \subseteq X} \bigcup_{t \in (\mathcal{S}^Z)_{Y-Z}} [-1,+1]^{Z-Y} \times t \\ &\subseteq \bigcup_{Z \subseteq X} \bigcup_{t \in (\mathcal{S}^{Z-Y})_Y} [-1,+1]^{Z-Y} \times t \end{aligned}$$

because for every subset Z of X we have the inclusion

$$(\mathcal{S}^Z)_{Y-Z} = ((\mathcal{S}^{Z-Y})^{Z \cap Y})_{Y-Z} \subseteq ((\mathcal{S}^{Z-Y})_{Z \cap Y})_{Y-Z} = (\mathcal{S}^{Z-Y})_Y.$$

Note that here the first and last equations use formulas from (13) of [3], whereas the inclusion is derived from formula (12) of [3]. If Z is chosen to be disjoint from Y, then the preceding chain of expressions just collapses to $(\mathcal{S}^Z)_Y$ and $[-1,+1]^Z$ equals $[-1,+1]^{Z-Y}$. This shows that actually equality holds above:

$$|\mathcal{S}|_Y = \bigcup_{Z \subseteq X-Y} \bigcup_{t \in (\mathcal{S}^Z)_Y} [-1,+1]^Z \times t. \tag{3}$$

Since $(\mathcal{S}^Z)_Y \subseteq (\mathcal{S}_Y)^Z$ by (13) of [3], we infer from (2) and (3) the inclusion

$$|\mathcal{S}|_Y \subseteq |\mathcal{S}_Y|.$$

Using (3), the dimension of the projected complex can be expressed as

$$\begin{aligned} \dim |\mathcal{S}|_Y &= \max\{\#Z|\ (\mathcal{S}^Z)_Y \neq \varnothing \text{ for some } Z \subseteq X-Y\} \\ &= \max\{\#Z|\ Z \subseteq X-Y \text{ with } \mathcal{S}^Z \neq \varnothing\}. \end{aligned} \tag{4}$$

Now, the prerequisites for proving the announced result are all in place.

Theorem 2. *For a subset $\mathcal{S} \subseteq \{\pm 1\}^X$, the following statements are equivalent:*

(i) *$\mathcal{S}$ is lopsided;*
(ii) *$|\mathcal{S}|_Y = |\mathcal{S}_Y|$ holds for all $Y \subseteq X$;*
(iii) *$\dim(|\mathcal{S}|_Y) = \dim(|\mathcal{S}_Y|)$ holds for all $Y \subseteq X$.*

Proof. If $\mathcal{S}$ is lopsided, then $(\mathcal{S}^Z)_Y = (\mathcal{S}_Y)^Z$ according to [3, Theorem 2] and consequently $|\mathcal{S}|_Y$ and $|\mathcal{S}_Y|$ are equal by (2) and (3). The latter equality trivially implies equality of the corresponding dimensions.

To complete the proof assume that

$$\dim |\mathcal{S}|_{X-Y} = \dim |\mathcal{S}_{X-Y}| \text{ for all } Y \in \overline{\mathcal{X}}(\mathcal{S}),$$

that is, $Y \subseteq X$ with $\mathcal{S}_{X-Y} = \{\pm 1\}^Y$. Then

$$\begin{aligned} \#Y &= \dim[-1,+1]^Y = \dim |\mathcal{S}_{X-Y}| \\ &= \dim |\mathcal{S}|_{X-Y} \\ &= \max\{\#Z|\ Z \subseteq Y \text{ with } \mathcal{S}^Z \neq \varnothing\}, \end{aligned}$$

whence $\mathcal{S}^Y \neq \varnothing$, that is, $Y \in \underline{\mathcal{X}}(\mathcal{S})$. Therefore $\mathcal{S}$ is lopsided by [3, Theorem 2]. □

5. Orthant intersection pattern

Every closed orthant of $\mathbb{R}^X$ includes a unique sign map s from $\{\pm 1\}^X$ by which it can be identified as $\mathcal{O} = \mathcal{O}(s)$:

$$\mathcal{O}(s) := \{r \in \mathbb{R}^X|\ r(x) \cdot s(x) \geq \text{ for all } x \in X\}.$$

Conversely, given $r \in \mathbb{R}^X$, the set

$$\mathrm{Sign}(r) := \{s \in \{\pm 1\}^X|\ r(x) \cdot s(x) \geq 0 \text{ for all } x \in X\},$$

which constitutes a fiber of $\{\pm 1\}^X$, indicates the orthants to which r belongs. Then given any subset $\mathcal{R}$ of $\mathbb{R}^X$, the following set $\mathrm{Sign}(\mathcal{R})$ encodes the intersection pattern of $\mathcal{R}$ with the closed orthants determined by the sign maps of $\{\pm 1\}^X$:

$$\mathrm{Sign}(\mathcal{R}) := \bigcup_{r \in \mathcal{R}} \mathrm{Sign}(r) = \{s \in \{\pm 1\}^X \mid \mathcal{R} \cap \mathcal{O}(s) \neq \varnothing\}.$$

Thus $s \in \mathrm{Sign}(\mathcal{R})$ exactly when there exists some $r \in \mathcal{R}$ with $r(x) \cdot s(x) \geq 0$ for all $x \in X$.

This construction was performed for convex sets by Lawrence [8]. He showed that $\mathrm{Sign}(\mathcal{R})$ is lopsided whenever $\mathcal{R}$ is a convex set in the Euclidean space $\mathbb{R}^X$, but not every lopsided set $\mathcal{S}$ can be realized in this way.

The clue for a realization within a wider class of subsets of $\mathbb{R}^X$ with some weaker convexity properties comes from the rather obvious realization

$$\mathrm{Sign}(|\mathcal{S}|) = \mathcal{S} \text{ if } \mathcal{S} \subseteq \{\pm 1\}^X \text{ is lopsided.} \tag{5}$$

Indeed, the inclusion $\mathcal{S} \subseteq \mathrm{Sign}(|\mathcal{S}|)$ is trivial. Now, if $t \in \mathcal{S}^* = \{\pm 1\}^X - \mathcal{S}$, then the corresponding closed orthant $\mathcal{O}(t)$ does not intersect any fiber $\mathcal{F} \subseteq \mathcal{S}$ of $\{\pm 1\}^X$ and hence is disjoint from the face $|\mathcal{F}|$ of $|\mathcal{S}|$, whence $t \notin |\mathcal{S}|$.

We will now show that for a closed subset $\mathcal{R}$ of $\mathbb{R}^X$ path-l_1-isometricity suffices to ensure that $\mathrm{Sign}(\mathcal{R})$ is lopsided. To this end we may consider only compact subsets of $[-1,+1]^X$. Since $\mathcal{S} = \mathrm{Sign}(\mathcal{R})$ is finite, there exists a finite subset $\mathcal{R}_0$ of $\mathcal{R}$ with $\mathcal{S} = \mathrm{Sign}(\mathcal{R}_0)$. We may scale $\mathcal{R}$ with some $\lambda > 0$ such that $\lambda\mathcal{R}_0 \subseteq [-1,+1]^X$. Then

$$\widetilde{\mathcal{R}} := [-1,+1]^X \cap \lambda\mathcal{R}$$

is compact and path-l_1-isometric in $[-1,+1]^X$ with $\mathcal{S} = \mathrm{Sign}(\widetilde{\mathcal{R}})$.

Theorem 3. *A subset $\mathcal{S}$ of $\{\pm 1\}^X$ is lopsided if and only if there exists a closed path-l_1-isometric subset $\mathcal{R}$ of $\mathbb{R}^X$ (or, equivalently, a compact path-l_1-isometric subset $\mathcal{R}$ of $[-1,+1]^X$) with $\mathcal{S} = \mathrm{Sign}(\mathcal{R})$.*

Proof. It remains to show that $\mathcal{S} = \mathrm{Sign}(\mathcal{R})$ is lopsided whenever $\mathcal{R}$ is a closed path-l_1-isometric subset of $[-1,+1]^X$. We proceed by induction on $\#X$. For $e \in X$ with $\mathcal{S}^e \neq \emptyset$ we claim that $\mathcal{S}^e = \mathrm{Sign}(\mathcal{R}')$, where

$$\mathcal{R}' = \{r|_{X-\{e\}} \mid r \in \mathcal{R} \text{ with } r(e) = 0\}.$$

Clearly, $\{\pm 1\} \times \mathrm{Sign}(\mathcal{R}') \subseteq \mathcal{S}$. Conversely, if $s' \in \mathcal{S}^e$, then both extensions $s_1, s_2 \in \{\pm 1\}^X$ of s' (with $s_1(e) = -1$ and $s_2(e) = +1$) belong to $\mathcal{S}$. Hence there exist $r_1, r_2 \in \mathcal{R}$ with $r_1(e) = -1, r_2(e) = +1$ and $r_1(x) \cdot s'(x) \geq 0$, $r_2(x) \cdot s'(x) \geq 0$ for all $x \in X - \{e\}$. Then any geodesic γ connecting r_1 and r_2 in $\mathcal{R}$ must contain a point r with $r(e) = 0$, so that necessarily $r(x) \cdot s'(x) \geq 0$ for all $x \in X$ holds as well. This establishes $\mathcal{S}^e = \mathrm{Sign}(\mathcal{R}')$. Obviously, the intersection of $\mathcal{R}$ with the e-hyperplane through 0 is path-l_1-isometric in $[-1,+1]^{X-\{e\}}$. Therefore $\mathcal{S}^e$ is lopsided by the induction hypothesis.

Finally, to prove that $\mathcal{S}$ is isometric, assume that for some subset Y with $\#Y > 1$ we have $s_1, s_2 \in \mathcal{S}$ with $s_1|_{X-Y} = s_2|_{X-Y}$ and $\{s_1(y), s_2(y)\} = \{\pm 1\}$ for all $y \in Y$. For notational convenience, assume that $s_1(y) = -1$ and $s_2(y) = +1$ for all $y \in Y$. By definition of $\mathcal{S}$ there exist $r_1, r_2 \in \mathcal{R}$ with $r_i(x) \cdot s_i(x) \geq 0$ for $i = 1, 2$ and all $x \in X - Y$ but $r_1(y) \leq 0 \leq r_2(y)$ for all $y \in Y$. Any geodesic γ connecting r_1 and r_2 in $\mathcal{R}$ contains a point r with $r(z) = 0$ for some $z \in Y$ such that all interior points on the subgeodesic of γ connecting r_1 and r have negative coordinates at Y. Necessarily $r(x) \cdot s_i(x) \geq 0$ for $i = 1, 2$ and all $x \in X - Y$. Therefore the neighbor s of s_1 on a shortest path between s_1 and s_2 in the graphic hypercube $G(\{\pm 1\}^X)$ satisfying $s(z) = +1$ and $s|_{X-\{z\}} = s_1|_{X-\{z\}}$ belongs to $\mathrm{Sign}(r) \subseteq \mathcal{S}$. A trivial induction on $\#Y$ thus shows that $\mathcal{S}$ is isometric. Then by [3, Theorem 4] we conclude that $\mathcal{S}$ is lopsided. $\square$

6. Circuits and cocircuits

In this section we will characterize the lopsided sets $\mathcal{S}$ via the barycenter maps of the maximal faces of their cubihedra $|\mathcal{S}|$. It turns out that only a single axiom is required, which is analogous to the "weak elimination" axiom for (signed) circuits in oriented matroids (see [4, Definition 3.2.1]). We say that a subset $\mathcal{T} \subseteq \{\pm, 0\}^X$ satisfies the *signed-circuit axiom* if the following condition holds:

(SCA) for all $t_1, t_2 \in \mathcal{T}$, and $e \in X$ with $t_1(e) \cdot t_2(e) = -1$ there exists some $t_0 \in \mathcal{T}$ such that $t_0(e) = 0$ and $t_0(x) \in \{0, t_1(x), t_2(x)\}$ for all $x \in X$.

This property shows up naturally with the intersection patterns of path-l_1-isometric subsets of $\mathbb{R}^X$ with open orthants and coordinate (hyper-)planes:

Lemma 3. *In $\mathcal{R}$ is a path-l_1-isometric subset of $[-1, +1]^X$, then*

$$\mathrm{Sign}_0(\mathcal{R}) := \{t \in \{\pm 1, 0\}^X \mid \exists r \in \mathcal{R} \text{ with } r(x) \cdot t(x) > 0 \ \forall x \in X \text{ with } r(x) \neq 0\}$$

satisfies the sign-circuit axiom (SCA).

Proof. Assume that $t_1(e) = -1$ and $t_2(e) = +1$ hold for some $t_1, t_2 \in \mathrm{Sign}_0(\mathcal{R})$ and $e \in X$. Then there exist $r_1, r_2 \in \mathcal{R}$ such that $r_1(e) \leq 0 \leq r_2(e)$. Since $\mathcal{R}$ is path-l_1-isometric in $[-1, +1]^X$, there must exist some $r_0 \in \mathcal{R}$ on a geodesic connecting r_1 and r_2 such that $r_0(e) = 0$ and $r(x)$ is between $r_1(x)$ and $r_2(x)$ for all $x \in X - \{e\}$. Define $t_0 \in \{\pm 1, 0\}^X$ by

$$t_0(x) := \begin{cases} -1 & \text{if } r_0(x) < 0, \\ +1 & \text{if } r_0(x) > 0, \\ \mathrm{med}(t_1(x), t_2(x), 0) & \text{if } r_0(x) = 0. \end{cases}$$

Then $t_0(x)$ takes the same sign as one of $t_1(x), t_2(x)$ if $r_0(x) \neq 0$. If $r_0(x) = 0$, then $t_0(x)$ is between $t_1(x)$ and $t_2(x)$; in particular, for $x = e$ the median choice guarantees $t_0(e) = 0$. Therefore t_0 qualifies as a member of $\mathrm{Sign}_0(\mathcal{R})$ as recognized by $r_0 \in \mathcal{R}$ and it satisfies the requirement in (SCA) relative to the given maps t_1 and t_2. $\square$

In order to express inclusion of faces of $[-1, +1]^X$ in terms of the corresponding barycenter maps we use the standard ordering $\prec$ of signs $-1, +1, 0$ for which -1 and $+1$ are incomparable, $0 \prec -1$ and $0 \prec +1$; see Fig. 2(a). The product ordering $\prec^X$ on $\{\pm 1, 0\}^X$ will also be denoted

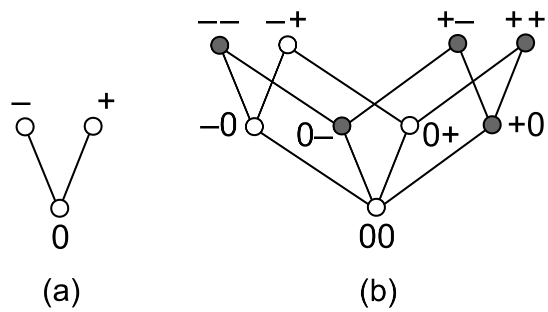

FIGURE 2. (a) Ordering of signs. (b) Product ordering on $\{\pm 1, 0\}^2$; the shaded nodes correspond to the barycenter maps of a lopsided set

by $\prec$. The undirected Hasse diagram of $(\{\pm 1, 0\}^X, \prec)$ is a grid graph (viz., the Cartesian X-power of a path with two edges) and will be denoted by $G(\{\pm 1, 0\}^X)$; see Fig 2(b) for $\#X = 2$. Thus, $t_1 \prec t_2$ for two maps $t_1, t_2 \in \{\pm 1, 0\}^X$ holds if and only if $t_1(x) \in \{0, t_2(x)\}$ for all $x \in X$, or equivalently, if for the associated faces the inclusion $[t_2] \subseteq [t_1]$ holds.

For a given subset $\mathcal{T} \subseteq \{\pm 1, 0\}^X$ one defines the *upper set* $\uparrow\mathcal{T}$ relative to the ordering $\prec$ by

$$\uparrow\mathcal{T} = \{t' \in \{\pm 1, 0\}^X \mid t \prec t' \text{ for some } t \in \mathcal{T}\}.$$

Note that the upper set $\uparrow\mathcal{T}$ satisfies (SCA) exactly when the stronger version

(SCA$\uparrow$) $\quad$ for all $t_1, t_2 \in \uparrow\mathcal{T}$, and $e \in X$ with $t_1(e) \cdot t_2(e) = -1$ there exists some $t_0 \in [t_1, t_2] \cap \uparrow\mathcal{T}$ with $t_0(e) = 0$

holds, where $[t_1, t_2]$ is a segment (box) of $[-1, +1]^X$. Indeed (SCA) yields some $t \in \uparrow\mathcal{T}$ with $t(x) \in \{0, t_1(x), t_2(x)\}$ for all $x \in X - \{e\}$ and $t(e) = 0$. Then the map t_0 defined by

$$t_0(x) := \begin{cases} t_1(x) & \text{if } t(x) = 0 \text{ and } x \neq e, \\ t(x) & \text{otherwise} \end{cases}$$

dominates t and hence belongs to $\uparrow\mathcal{T} \cap [t_1, t_2]$.

To $\mathcal{T} \subseteq \{\pm 1, 0\}^X$ one associates the set of sign maps

$$\mathrm{Sign}(\mathcal{T}) = \{s \in \{\pm 1\}^X \mid \exists t \in \mathcal{T} \text{ with } s(x) \cdot t(x) \geq 0 \ \forall x \in X\} = \uparrow\mathcal{T} \cap \{\pm 1\}^X = \mathrm{Sign}(\uparrow\mathcal{T})$$

and then the corresponding geometric realization $|\mathrm{Sign}(\mathcal{T})|$. This realization can be compared to the union $[\mathcal{T}]$ of the smallest faces $[t]$ of $[-1, +1]^X$ containing t for $t \in \mathcal{T}$:

$$[\mathcal{T}] := \bigcup_{t \in \mathcal{T}} [t] \ \subseteq \ |\mathrm{Sign}(\mathcal{T})|.$$

One can retrieve $\uparrow\mathcal{T}$ from $[\mathcal{T}]$ as $\uparrow\mathcal{T} = \mathrm{Sign}_0([\mathcal{T}]) = \mathrm{Baryc}([\mathcal{T}])$.

Theorem 4. *The following statements are equivalent for a set $\mathcal{T} \subseteq \{\pm 1, 0\}^X$:*

(i) $[\mathcal{T}]$ *is a path-l_1-isometric subset of* $[-1, +1]^X$*;*
(ii) $\mathcal{T}$ *satisfies (SCA);*
(iii) $\uparrow\mathcal{T}$ *satisfies (SCA);*

(iv) $\uparrow\mathcal{T}$ *is an isometric subset of the grid graph* $G(\{\pm 1, 0\}^X)$*;*
(v) $\uparrow\mathcal{T} \cap \{\pm 1\}^X$ *is lopsided such that* $[\mathcal{T}] = |\uparrow\mathcal{T} \cap \{\pm 1\}^X|$.

Proof. If $[\mathcal{T}]$ is path-l_1-isometric, then $\uparrow\mathcal{T} = \mathrm{Sign}_0(\mathcal{T})$ satisfies (SCA) by Lemma 3. This establishes (i)$\Rightarrow$(iii).

Now, we will show that (ii)$\Longleftrightarrow$(iii). Trivially, $\mathcal{T}$ and $\uparrow\mathcal{T}$ have the same set of minimal elements. Assume that $\mathcal{T}$ satisfies (SCA) and let $t'_1, t'_2 \in\uparrow\mathcal{T}$ with $t_1 \prec t'_1$ and $t_2 \prec t'_2$ for some $t_1, t_2 \in \mathcal{T}$. If $t'_1(e) \cdot t'_2(e) = -1$ for some $e \in X$, then t_0 can be chosen to be one of t_1, t_2 in case that $0 \in \{t_1(e), t_2(e)\}$. Otherwise, one may choose t_0 in $\mathcal{T}$ with $t_0(e) = 0$ and $t_0(x) \in \{0, t_1(x), t_2(x)\}$ for all $x \in X$. Conversely, assume that $\uparrow\mathcal{T}$ satisfies (SCA). Since the minimal choices relative to $\prec$ for establishing (SCA) in $\uparrow\mathcal{T}$ belong to $\mathcal{T}$, it follows that (SCA) holds for $\mathcal{T}$ as well.

To prove that (iii)$\Rightarrow$(iv) holds, suppose that there were some distinct $t_1, t_2 \in\uparrow\mathcal{T}$ which are not adjacent in the graph $G(\{\pm 1, 0\}^X)$ such that the box $[t_1, t_2]$ in $[-1, +1]^X$ intersects $\uparrow\mathcal{T}$ only in t_1 and t_2. If for some $e \in X$ we have $t_1(e) \cdot t_2(e) = -1$, then there exists some $t_0 \in [t_1, t_2] \cap \uparrow\mathcal{T}$ with $t_0(e) = 0$ by (SCA$\uparrow$). Then $t_0 \notin \{t_1, t_2\}$, however, conflicts with the initial hypothesis. Therefore $t_1(x) \cdot t_2(x) \geq 0$ for all $x \in X$, that is, all coordinates of t_1 and t_2 have comparable signs. Consequently, the join t of t_1 and t_2 exists in the ordered set $(\{\pm 1, 0\}^X, \prec)$ and is given by $t(x) = \min\{t_1(x), t_2(x)\}$ for all $x \in X$. Then $t \in [t_1, t_2] \cap \uparrow\mathcal{T} = \{t_1, t_2\}$, say $t_2 \prec t = t_1$. Since t_1 and t_2 are not adjacent in the graph $G(\{\pm 1, 0\}^X)$, there exist at least two distinct coordinates e and f at which they differ, whence $t_2(e) = t_2(f) = 0$ and $t_1(e), t_1(f) \in \{\pm 1\}$. Then the map t defined by

$$t(x) := \begin{cases} 0 & \text{if } x = e, \\ t_1(x) & \text{if } x \neq e \end{cases}$$

is different from t_1 and t_2 but belongs to $[-1, +1]^X \cap \uparrow\mathcal{T}$, yielding a final contradiction.

For the proof of (iv)$\Rightarrow$(v) we first claim that the smallest isometric subset $\mathcal{U}$ of the grid graph $G(\{\pm 1, 0\}^X)$ that includes some Y-fiber of $\{\pm 1\}^X$ at some vertex $s \in \{\pm 1\}^X$ is the Y-fiber of $\{\pm 1, 0\}^X$ at s. In fact, as $\{\pm 1\}^Y \times s|_{X-Y} \subseteq \mathcal{U}$ and in each coordinate 0 is needed to connect -1 to $+1$, we may assume by induction on $\#Y$ that for some $e \in Y$,

$$\{\pm 1, 0\}^{Y - \{e\}} \times \{\pm 1\}^e \times s|_{X-Y} \subseteq \mathcal{U},$$

whence $\{\pm 1, 0\}^{Y-\{e\}} \times \{0\}^e \times s|_{X-Y}$ constitutes the set of unique common neighbors in the grid graph for the pairs t_1, t_2 with $t_1|_{Y-\{e\}} = t_2|_{Y-\{e\}}$, $\{t_1(e), t_2(e)\} = \{\pm 1\}$, and $t_1|_{X-Y} = s|_{X-Y} = t_2|_{X-Y}$. Hence $\{\pm 1, 0\}^Y \times s|_{X-Y} \subseteq \mathcal{U}$, as asserted.

Since $\uparrow\mathcal{T}$ is isometric in $G(\{\pm 1, 0\}^X)$, we can apply the preceding observation to infer that for $\mathcal{S} =\uparrow\mathcal{T} \cap \{\pm 1\}^X$ the upper set $\uparrow\mathcal{T}$ encompasses $\mathrm{Baryc}(|\mathcal{S}|)$. Since the reverse inclusion is trivial because $\uparrow\mathcal{T}$ is an upper set, we have thus established $\uparrow\mathcal{T} = \mathrm{Baryc}(|\mathcal{S}|)$, whence it follows that

$$[\mathcal{T}] = [\uparrow\mathcal{T}] = [\mathrm{Baryc}(|\mathcal{S}|)] = |\mathcal{S}|,$$

as required. Isometry of $\uparrow\mathcal{T}$ also entails that $\mathcal{S}^Y$ is isometric for every $Y \subseteq X$. Indeed, for each pair $s_1, s_2 \in \mathcal{S}^Y$, the barycenter maps t_i corresponding to $\{\pm 1\}^Y \times s_i$ $(i = 1, 2)$

belong to $\uparrow\mathcal{T}$ by what has just been observed. Then any isometric path connecting t_1 and t_2 in $\uparrow\mathcal{T}$ projects to an isometric path connecting s_1 and s_2 in $G(\{\pm 1, 0\}^{X-Y})$ because $t_1|_Y = t_2|_Y$ is the zero map on Y and $t_i|_{X-Y} = s_i$ for $i = 1, 2$. To show that s_1 and s_2 are actually connected by a shortest path in $G(\{\pm 1\}^{X-Y})$ (which is scale 2 embedded in $G(\{\pm 1, 0\}^{X-Y})$), we use a trivial induction on the (l_1-)distance between s_1 and s_2. Let the neighbor w of s_1 on the projected path equal 0 at the coordinate $e \in X - Y$. Then the sign map s'_1 with $s'_1|_{X-Y-\{e\}} = s_1|_{X-Y-\{e\}} = w|_{X-Y-\{e\}}$ and $s'_1(e) = -s_1(e) = s_2(e)$ belongs to $\mathcal{S}^Y$ because $w \prec s'_1$. Since s'_1 is a neighbor of s_1 between s_1 and s_2 in $G(\{\pm 1\}^{X-Y})$, we conclude that $\mathcal{S}^Y$ is isometric. This finally shows that $\mathcal{S}$ is lopsided.

Finally, if (v) holds, then the geometric realization $|\mathcal{S}|$ of the lopsided set $\mathcal{S} =\uparrow\mathcal{T} \cap \{\pm 1\}^X$ is a path-l_1-isometric subset of $[-1, +1]^X$ according to Theorem 1 and coincides with $[\mathcal{T}]$ by (v). This establishes (i), and the proof is complete. □

Conditions (ii) to (v) of the preceding theorem are purely combinatorial and their equivalence can be proven without employing the metric/topological features of the entire geometric realization. In fact, our combinatorial proof established (ii)$\Longleftrightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(v). The implication left, (v)$\Rightarrow$(iii) can be shown directly, without the (de)tour through cubihedra: Assuming that $\mathcal{T} \subseteq \{\pm 1, 0\}^X$ satisfies (v), the "top" set $\mathcal{S} =\uparrow\mathcal{T} \cap \{\pm 1\}^X$ of sign maps is lopsided, and the second part of (v) guarantees that all barycentric maps of the cubihedron $|\mathcal{S}|$ belong to $\uparrow\mathcal{T}$. Let t_1 and t_2 be two members of $\uparrow\mathcal{T}$ with $t_1(e) = -1$ and $t_2(e) = +1$ for some $e \in X$. The zero coordinates of t_i determine the set $Y_i \subsetneqq X$, so that t_i encodes some Y_i-cube of $G(\{\pm 1\}^X)$ for $i = 1, 2$. The Y_1-cube and Y_2-cube admit mutually nearest vertices (gates) s_1 and s_2 within $G(\{\pm 1\}^X)$. Then $t_1 \prec s_1$ and $t_2 \prec s_2$. By the choice of e and the gate property for s_1 and s_2, we have

$$e \in \Delta(s_1, s_2) = \{x \in X |\ s_1(x) \neq s_2(x)\} \subseteq X - (Y_1 \cup Y_2).$$

Since $\mathcal{S}$ is lopsided, $\mathcal{S}^{Y_1 \cap Y_2}$ is isometric, whence there exists a shortest path P in $\mathcal{S}$ connecting s_1 and s_2, which projects onto a shortest path between $s_1|_{Y_1 \cap Y_2}$ and $s_2|_{Y_1 \cap Y_2}$ in $\mathcal{S}^{Y_1 \cap Y_2}$. Necessarily, P passes through two adjacent vertices s'_1 and s'_2 with $s'_1(e) = -1$ and $s'_2(e) = +1$. Thus, there exist $Y_1 \cap Y_2$-cubes at s'_1 and s'_2, which are fibers of a $(Y_1 \cap Y_2) \cup \{e\}$-cube containing s'_1 and s'_2. Let t_0 denote the barycenter map of this latter cube. Then

$$t_0(x) := \begin{cases} 0 & \text{if } x \in (Y_1 \cap Y_2) \cup \{e\}, \\ s_2(x) = t_2(x) & \text{if } x \in \Delta(s_1, s'_1), \\ s_1(x) = t_1(x) & \text{if } x \in \Delta(s'_1, s_2), \\ s_1(x) = s_2(x) & \text{otherwise.} \end{cases}$$

Note that for $y \in Y_i - Y_j$ one has $s_i(y) = t_0(y) = t_j(y)$, where $\{i, j\} = \{1, 2\}$. Therefore t_0 satisfies the requirements in (SCA).

The preceding theorem demonstrates that the set of minimal elements in $\mathrm{Baryc}(|\mathcal{S}|)$ relative to the order $\prec$ on $\{\pm, 0\}$ deserves naming:

$$\mathrm{Cocirc}(\mathcal{S}) := \{t \in \mathrm{Baryc}(|\mathcal{S}|) |\ t' \prec t \text{ implies } t' = t \text{ for all } t' \in \ \mathrm{Baryc}(|\mathcal{S}|)\}.$$

The members of $\mathrm{Cocirc}(\mathcal{S})$ are referred to as the *cocircuits* of $\mathcal{S}$, since their characteristic feature (SCA) in the case of lopsided sets $\mathcal{S}$ bears resemblance with one of the axioms for circuits in an oriented matroid. The cocircuits of $\mathcal{S}$ correspond to the facets of the cubihedron $|\mathcal{S}|$. Then

$$\begin{aligned}\mathrm{Baryc}(|\mathcal{S}|) = \uparrow\mathrm{Cocirc}(\mathcal{S}) &= \{t \in \{\pm 1, 0\}^X \mid \ \uparrow\{t\} \cap \{\pm 1\}^X \subseteq \mathcal{S}\} \\ &= \{t \in \{\pm 1, 0\}^X \mid t|_{X - X(t)} \in \mathcal{S}^{X(t)}\} \\ &= \{t \in \{\pm 1, 0\}^X \mid t|_{X - X(t)} \notin (\mathcal{S}^*)_{X(t)}\},\end{aligned}$$

where $\mathcal{S}^* = \{\pm 1\}^X - \mathcal{S}$ and $X(t) = \{x \in X \mid \ -1 < t(x) < +1\} = t^{-1}(\{0\})$ for $t \in \{\pm 1, 0\}$ were defined previously.

The set $\mathrm{Circ}(\mathcal{S})$ of *circuits* of $\mathcal{S}$ is defined to be the set $\mathrm{Cocirc}(\mathcal{S}^*)$ of cocircuits of $\mathcal{S}^*$. Clearly, $t \in \{\pm 1, 0\}^X$ is contained in $\mathrm{Cocirc}(\mathcal{S}^*)$ if and only if $t|_{X - X(t)} \notin \mathcal{S}_{X(t)}$ holds, that is, if and only if, for every $s \in \mathcal{S}$, there exists some $x \in X$ with $t(x) \cdot s(x) = -1$ or – equivalently – if and only if $s \in \{\pm 1\}^X$ and $t \prec s$ implies $s \notin \mathcal{S}$. So, $\mathrm{Circ}\,(\mathcal{S})$ consists of the minimal elements in $\{\pm 1, 0\}^X$ with that property. It is also easy to see that $\mathcal{S}$ coincides with the set of all sign maps $s \in \{\pm 1\}^X$ with $t \prec s$ for some $t \in \mathrm{Cocirc}(\mathcal{S})$ as well as with the set of all sign maps $s \in \{\pm 1\}^X$ with $t \not\prec s$ for all $t \in \mathrm{Circ}(\mathcal{S})$. We then obtain our final result essentially as a corollary to the preceding theorem:

Theorem 5. *The following statements are equivalent for a set $\mathcal{S} \subseteq \{\pm 1\}^X$:*

(i) $\mathcal{S}$ *is lopsided;*
(ii) $\mathrm{Baryc}(|\mathcal{S}|)$ *satisfies (SCA);*
(iii) $\mathrm{Cocirc}(\mathcal{S})$ *satisfies (SCA);*
(iv) $\mathrm{Baryc}(|\mathcal{S}^*|)$ *satisfies (SCA);*
(v) $\mathrm{Circ}(\mathcal{S})$ *satisfies (SCA);*

Proof. Given a set $\mathcal{S} \subseteq \{\pm 1\}^X$, the associated subset $\mathcal{T} := \mathrm{Baryc}(|\mathcal{S}|)$ is upward closed, i.e., $\uparrow\mathcal{T} = \mathcal{T}$, and yields $\mathcal{S}$ back as $\mathcal{T} \cap \{\pm 1\}^X$. In particular, $[\mathcal{T}] = |\mathcal{S}|$ holds by definition of the two cubihedra. Therefore, if $\mathcal{S}$ is lopsided, then condition (v) of Theorem 4 is satisfied. This establishes (i)$\Rightarrow$(ii) (or (iii), respectively). Trivially, $\uparrow\mathrm{Cocirc}(\mathcal{S}) = \mathcal{T}$, whence (ii)$\Longleftrightarrow$(iii) immediately follows from the equivalence of (iii) and (ii) in Theorem 4. If $\mathcal{T}$ satisfies (SCA), then $\mathcal{S}$ is lopsided by the implication from (ii) to (v) in Theorem 4. Summarizing, we have shown that the first three statements (i),(ii),(iii) are equivalent. Since $\mathrm{Circ}(\mathcal{S}) = \mathrm{Cocirc}(\mathcal{S}^*)$ and $\mathcal{S}$ is lopsided exactly when its complement $\mathcal{S}^*$ is (cf. [3, Theorem 2]), statements (iv) and (v) are also equivalent to (i). $\square$

Remark. It follows that $r \in \mathrm{Circ}(\mathcal{S})$ for some lopsided subset $\mathcal{S}$ of $\{\pm 1\}^X$ implies $\mathcal{S}|_Y = \{\pm 1\}^Y$ for every proper subset Y of $X - X(r)$ and hence

$$\mathcal{X}(\mathcal{S}_{X(r)}) = \mathcal{P}(X - X(r)) - \{X - X(r)\}.$$

In other words, for every circuit $r \in \mathrm{Circ}(\mathcal{S})$, the support $X - X(r)$ is a "circuit" of $\mathcal{X}(\mathcal{S})$, that is, a minimal subset of X not contained in $\mathcal{X}(\mathcal{S})$, while $r|_{X - X(r)}$ is the unique element in $\{\pm 1\}^{X-X(r)}$ not contained in $\mathcal{S}_{X(r)}$. In particular, we have $\#\mathrm{Circ}(\mathcal{S}) = \#\mathrm{Circ}(\mathcal{X}(\mathcal{S}))$ with

$$\mathrm{Circ}(\mathcal{X}(\mathcal{S})) := \{Y \in \mathcal{P}(X) - \mathcal{X}(\mathcal{S}) |\ Z \in \mathcal{X}(\mathcal{S}) \text{ for all } Z \subsetneqq Y\}$$

for every lopsided subset $\mathcal{S}$ of $\{\pm 1\}^X$.

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