# Packing and covering $\delta$-hyperbolic spaces by balls*

Victor Chepoi and Bertrand Estellon

Laboratoire d'Informatique Fondamentale de Marseille,
Faculté des Sciences de Luminy, Universitée de la
Méditerranée, F-13288 Marseille Cedex 9, France,
{chepoi,estellon}@lif.univ-mrs.fr

**Abstract.** We consider the problem of covering and packing subsets of $\delta$-hyperbolic metric spaces and graphs by balls. These spaces, defined via a combinatorial Gromov condition, have recently become of interest in several domains of computer science. Specifically, given a subset $S$ of a $\delta$-hyperbolic graph $G$ and a positive number $R$, let $\gamma(S,R)$ be the minimum number of balls of radius $R$ covering $S$. It is known that computing $\gamma(S,R)$ or approximating this number within a constant factor is hard even for 2-hyperbolic graphs. In this paper, using a primal-dual approach, we show how to construct in polynomial time a covering of $S$ with at most $\gamma(S,R)$ balls of (slightly larger) radius $R+\delta$. This result is established in the general framework of $\delta$-hyperbolic geodesic metric spaces and is extended to some other set families derived from balls. This covering algorithm is used to design better than in general case approximation algorithms for the augmentation problem of $\delta$-hyperbolic graphs with diameter constraints and slackness $\delta$ and for the $k$-center problem in $\delta$-hyperbolic graphs.

## 1 Introduction

The *set cover problem* is a classical question in computer science [39] and combinatorics [9]. In this problem, given a collection $\mathcal{S}$ of subsets of a domain $U$ with $n$ elements, the task is to find a subcollection of $\mathcal{S}$ of minimum size $\gamma(\mathcal{S})$ whose union is $U$. It was one of Karp's 21 NP-complete problems. More recently, it has been shown that, under the assumption $P \neq NP$, set cover cannot be approximated in polynomial time to within a factor of $c \cdot \ln n$, where $c$ is a small constant; see [3] and the references cited therein. On the other hand, set cover can be approximated in polynomial time to within a factor of $\ln n + 1$ using several algorithms [39], in particular, using the greedy algorithm. The *set packing problem* asks to find a maximum number $\pi(\mathcal{S})$ of pairwise disjoint subsets of $\mathcal{S}$. Another problem closely related to set cover is the hitting set problem. A subset $T$ is called a *hitting set* of $\mathcal{S}$ if $T \cap S \neq \emptyset$ for any $S \in \mathcal{S}$. The *minimum hitting set problem* asks to find a hitting set of $\mathcal{S}$ of smallest cardinality $\tau(\mathcal{S})$.

* This research was partly supported by the ANR grant BLAN06-1-138894 (projet OPTICOMB)

Numerous algorithmic and optimization problems can be formulated as set cover or set packing problems for structured set families. For example, many papers consider cover and packing problems with set families like intervals and unions of intervals of a line, subtrees of a tree, or cliques, cuts, paths, and balls of a graph. For example, in case of covering with balls, one can expect that the specific metric properties of graphs in question yield better algorithmic results in comparison with the general set cover. Although the set cover problem can be viewed as a particular instance of covering with unit balls of rather special graphs, for several graphs classes polynomial time algorithms have been designed. These algorithms resides on the treelike structure of those graphs and on the equality between ball covering and packing numbers of such graphs.

In this note, we consider the problem of covering and packing by balls and union of balls of hyperbolic metric spaces and graphs. The *ball* $B(x, R)$ of center $x$ and radius $R \geq 0$ consists of all points of a metric space $(X, d)$ at distance at most $R$ from $x$. In our paper, we will consider covering and packing problems of the following type: given a finite subset $S$ of points of $X$, a radius $R$, and a slack parameter $\delta$, find a good covering of $S$ with balls of radius at most $R + \delta$. We show that if the metric space $(X, d)$ is $\delta$-hyperbolic, then in polynomial time we can construct a covering of $S$ with balls of radius $R + \delta$ and a set of the same size of pairwise disjoint balls of radius $R$ centered at points of $S$. This type of results is obtained for arbitrary subfamilies of balls and for set-families consisting of unions of $\kappa$ balls of $(X, d)$. We apply these results to design better approximation algorithms for the $k$-center problem and the augmentation problem with diameter constraints in $\delta$-hyperbolic graphs.

### 1.1 Geodesic and $\delta$-hyperbolic metric spaces

Let $(X, d)$ be a metric space. A *geodesic segment* joining two points $x$ and $y$ from $X$ is a map $\rho$ from the segment $[a, b]$ of length $|a - b| = d(x, y)$ to $X$ such that $\rho(a) = x, \rho(b) = y$, and $d(\rho(s), \rho(t)) = |s - t|$ for all $s, t \in [a, b]$. A metric space $(X, d)$ is *geodesic* if every pair of points in $X$ can be joined by a geodesic. We will denote by $[x, y]$ any geodesic segment connecting the points $x$ and $y$. Every graph $G = (V, E)$ equipped with its standard distance $d_G$ can be transformed into a (network-like) geodesic space $(X, d)$ by replacing every edge $e = (u, v)$ by a segment $[u, v]$ of length 1. These segments may intersect only at their commons ends. Then $(V, d_G)$ is isometrically embedded in a natural way in $(X, d)$.

Introduced by Gromov [29], $\delta$-hyperbolicity measures, to some extent, the deviation of a metric from a tree metric. Recall that a metric space $(X, d)$ embeds into a tree network (with positive real edge lengths), that is, $d$ is a *tree metric,* if and only if for any four points $u, v, w, x$ the two larger ones of the distance sums $d(u, v) + d(w, x), d(u, w) + d(v, x), d(u, x) + d(v, w)$ are equal. Now, a metric space $(X, d)$ is called $\delta$-*hyperbolic* if the two larger distance sums differ by at most $\delta$. A connected graph $G = (V, E)$ equipped with standard graph metric $d_G$ is $\delta$-hyperbolic if $(V, d_G)$ is a $\delta$-hyperbolic metric space.

In case of geodesic metric spaces, there exist several equivalent definitions of $\delta$-hyperbolic metric spaces involving different but comparable values of $\delta$ [5, 28, 29]. In this paper, we will use the definition employing $\delta$-thin geodesic triangles. A *geodesic triangle* $\Delta(x, y, z)$ with vertices $x, y, z \in X$ is a union $[x, y] \cup [x, z] \cup [y, z]$ of three geodesic segments connecting these vertices. Let $m_x$ be the point of the geodesic segment $[y, z]$ located at distance $\alpha_y := (d(y, x) + d(y, z) - d(x, z))/2$ from $y$. Then $m_x$ is located at distance $\alpha_z := (d(z, y) + d(z, x) - d(y, x))/2$ from $z$ because $\alpha_y + \alpha_z = d(y, z)$. Analogously, define the points $m_y \in [x, z]$ and $m_z \in [x, y]$ both located at distance $\alpha_x := (d(x, y) + d(x, z) - d(y, z))/2$ from $x$; see Fig. 1 for a construction. There exists a unique isometry $\varphi$ which maps the geodesic triangle $\Delta(x, y, z)$ to a star $\Upsilon(x', y', z')$ consisting of three solid segments $[x', m']$, $[y', m']$, and $[z', m']$ of lengths $\alpha_x, \alpha_y$, and $\alpha_z$, respectively. This isometry maps the vertices $x, y, z$ of $\Delta(x, y, z)$ to the respective leaves $x', y', z'$ of $\Upsilon(x', y', z')$ and the points $m_x, m_y$, and $m_z$ to the center $m$ of this tripod. Any other point of $\Upsilon(x', y', z')$ is the image of exactly two points of $\Delta(x, y, z)$. A geodesic triangle $\Delta(x, y, z)$ is called *$\delta$-thin* [5] if for all points $u, v \in \Delta(x, y, z)$, $\varphi(u) = \varphi(v)$ implies $d(u, v) \leq \delta$. A geodesic metric space $(X, d)$ is called *$\delta$-hyperbolic* if all geodesic triangles of $X$ are $\delta$-thin. Note that our $\delta$-hyperbolic metric spaces will be $2\delta$-hyperbolic if we will use the first definition of $\delta$-hyperbolicity; see the proof of Proposition 2.1 of [5].

Throughout this paper, we will suppose that all metric spaces are either geodesic or graphic with $\delta$-thin geodesic triangles. Additionally, in case of geodesic spaces $(X, d)$, we will assume the following computational assumption: *there exists an oracle which, given two points $x, y \in X$, it returns a geodesic segment* $[x, y]$. In case of graph-distance $d_G$ or of geodesic spaces derived from graphs, the role of this oracle is played by any shortest path algorithm.

### 1.2 $r$-Domination and $r$-packing

Now, we will formulate the *$r$-domination* and *$r$-packing* problems, which correspond to covering and packing by balls. Let $S$ be a subset of not necessarily distinct points of a metric space $(X, d)$ and let $r : S \to \mathbb{R}_+$ be a map associating to each point $s \in S$ a positive number $r(s)$. We say that a subset $C$ of $X$ *$r$-dominates* $S$ if for each point $s \in S$ there exists a point $c \in C$ such

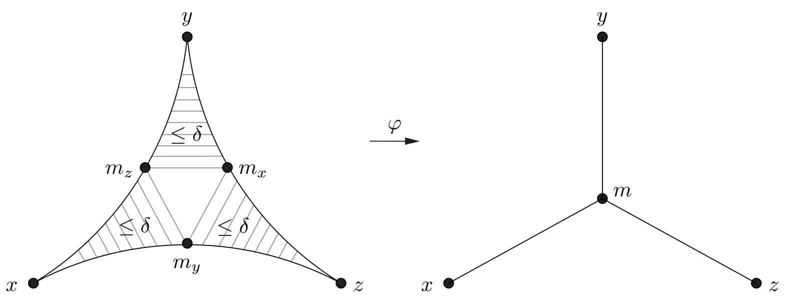


**Fig. 1.** A geodesic triangle $\Delta(x, y, z)$, the points $m_x, m_y, m_z$, and the tripod $\Upsilon(x', y', z')$

that $d(s,c) \leq r(s)$. In other words, $C$ is a hitting set for the family of balls $\mathcal{B}_{S,r} = \{B(s,r(s)) : s \in S\}$. A subset $P$ of $S$ is called an *$r$-packing* of $S$, if for each pair $x, x'$ of points of $P$ we have $r(x)+r(x') < d(x,x')$ (in other words, the family $B(x,r(x)), x \in P$, consists of pairwise disjoint balls). The *$r$-domination problem* is to find an $r$-dominating set with minimum size $\gamma(S,r)$ and the *$r$-packing problem* is to find an $r$-packing set with maximum size $\pi(S,r)$. Then $\gamma(S,r) = \tau(\mathcal{B}_{S,r})$ and $\pi(S,r)$ are called the *$r$-domination* and the *$r$-packing numbers* of $(X,d)$ (these numbers are well-defined when $S$ is finite). If $S$ is a subset of vertices of a graph $G = (V,E)$ and the $r$-dominating set $C$ is also contained in $V$, then we denote the respective $r$-domination and $r$-packing numbers by $\gamma_G(S,r)$ and $\pi_G(S,r)$. If $r(s) \equiv R$ for all $s \in S$, then we obtain the problem of covering $S$ with a minimum number of balls of radius $R$; in the particular case $r(s) \equiv 1$ and $S \subseteq V$, we obtain the well-known domination problem of a graph.

The $r$-domination problem is closely related with the $k$-center clustering problem [10, 30, 31, 39]. In the *$k$-center problem,* given a set $S$ of $n$ points in a metric space $(X,d)$, the goal is to find the smallest $R^*$ and the position of $k$ centers, such that any point of $S$ is at distance of at most $R^*$ from one of those centers (in other words, $R^*$ is the least radius such that $S$ can be covered with at most $k$ balls of radius $R^*$).

A *$\kappa$-ball* ${}^{\kappa}B$ of a metric space $(X,d)$ is the union of $\kappa$ balls $B(x_1,r_1),\ldots, B(x_\kappa,r_\kappa)$, i.e., ${}^{\kappa}B = \bigcup_{j=1}^{\kappa} {}^{\kappa}B(x_j,r_j)$. It extends the notions of $d$-intervals of [1, 7]; these are unions of $d$ closed intervals of $\mathbb{R}$. Indeed, each interval $[a,b]$ can be viewed as a closed ball of $\mathbb{R}$ of radius $(b-a)/2$ centered at the point $(a+b)/2$. As in the case of $r$-domination, any finite family ${}^{\kappa}\mathcal{B}_{S,\mathbf{r}}$ of $\kappa$-balls can be defined via the set $S$ of centers of all balls and a multi-valued map $\mathbf{r} : S \to \mathbb{R}_+$ which associates to each point $s \in S$ the list of radii of the balls from $\bigcup {}^{\kappa}\mathcal{B}_{S,\mathbf{r}}$ centered at $s$. Thus two $\kappa$-balls may have two balls centered at the same point. The *$(\kappa,\mathbf{r})$-domination problem* consists in finding a hitting set $C$ for a family ${}^{\kappa}\mathcal{B}_{S,\mathbf{r}}$ of $\kappa$-balls of minimum cardinality $\gamma(S,\mathbf{r})$. Analogously, the *$(\kappa,\mathbf{r})$-packing problem* is to find a maximum number $\pi(S,\mathbf{r})$ of pairwise disjoint $\kappa$-balls of ${}^{\kappa}\mathcal{B}_{S,\mathbf{r}}$.

### 1.3 Augmentation under diameter constraints

In Section 4, we apply our results on covering $\delta$-hyperbolic graphs with balls to the following *augmentation problem under diameter constraints* (problem ADC): Given a graph $G = (V,E)$ with $n$ vertices and a positive integer $D$, add a minimum number $\mathrm{OPT}(D)$ of new edges $E'$ such that the augmented graph $G' = (V, E \cup E')$ has diameter at most $D$. ADC can be viewed as a network improvement problem where $G$ is the initial communication network and a minimum number of additional communication links must be added so that the upgraded network $G'$ ensures a low communication delay.

### 1.4 Our results

Using the notation established in previous subsections, the main algorithmic results of our paper can be formulated in the following way: for geodesic $\delta$-

hyperbolic spaces and $\delta$-hyperbolic graphs, $\gamma(S, r+\delta) \leq \pi(S, r)$ and $\gamma(S, \mathbf{r}+2\delta) \leq 2\kappa^2\pi(S, \mathbf{r})$ hold. Moreover, it is possible to construct in polynomial time an $(r+\delta)$-dominating set $C$ and an $r$-packing $P$ such that $|C| = |P|$, and a $(\kappa, \mathbf{r}+2\delta)$-dominating set $C$ and a $(\kappa, \mathbf{r})$-packing $P$ such that $|C| \leq 2\kappa^2|P|$. Using these results, we show that one can augment in polynomial time a $\delta$-hyperbolic graph $G = (V, E)$ to a graph of diameter $2R + 2\delta$ using at most $2\mathrm{OPT}(2R)$ new edges. These results also show that for $\delta$-hyperbolic graphs, the well-known 2-approximation algorithm [31] for the $k$-center problem returns a solution of radius at most $\mathrm{OPT} + \delta$. Notice also that the problem of approximating $\gamma(S, r)$ within a constant is hard already for 2-hyperbolic graphs and $r(s) \equiv 1$ for all $s \in S$, because the split graphs, which encode the general set cover problem (the elements $x$ of the domain $U$ form a clique and the sets $S$ of $\mathcal{S}$ form a stable set so that the vertices $x$ and $S$ are adjacent if and only if $x \in S$), are chordal, and therefore, 2-hyperbolic.

### 1.5 Related work

We briefly review the known results related with the subject of our paper. The inequality $\gamma(S, r) \geq \pi(S, r)$ holds in any metric space $(X, d)$, because two points of an $r$-packing cannot be $r$-dominated by the same point. On the other hand, the equality $\gamma_G(S, r) = \pi_G(S, r)$ holds for trees [13, 14], strongly chordal graphs [15, 25], dually chordal graphs [11], and it is at the heart of linear-time algorithms for $r$-covering and $r$-packing problems for these graphs. The paper [21] proposes an exact fixed-parameter algorithm for the ($NP$-hard) problem of covering planar graphs with a minimum number of balls of radius $R$. Finally, [17] shows that every planar graph of diameter $2R$ can be covered with a fixed number of balls of radius $R$. Covering and packing problems for special families of subtrees of a tree have been considered in [8]. Alon [1, 2] established that if ${}^{\kappa}\mathcal{I}$ is a family of $\kappa$-intervals of the line (or a family consisting of unions of $\kappa$ subtrees of a tree), then $\tau({}^{\kappa}\mathcal{I}) \leq 2\kappa^2\pi({}^{\kappa}\mathcal{I})$. In case of $\kappa$-intervals, Bar-Yehuda et al. [7] presented a factor $2\kappa$ algorithm for approximating $\pi({}^{\kappa}\mathcal{I})$. Their algorithm is based on rounding a fractional solution of the linear relaxation of the problem and construction of a respective packing using the local ratio technique.

The $k$-center problem is a well-studied $k$-clustering and facility location problem [10, 30, 39]. The general problem is NP-hard to approximate with a factor smaller than 2 (see Theorem 5.7 of [39]). The analogous problem in Euclidean spaces is NP-hard to approximate with a factor smaller than 1.822 [26]. Hochbaum and Shmoys [31] present a (best possible) factor 2 approximation algorithm for the general $k$-center problem.

The augmentation problem of graphs with diameter constraints has been introduced in [20]. It is already non-trivial when the input graph is a path [4, 24]. Approximation algorithms for this augmentation problem has been designed in [18, 19, 22, 32]. In particular, [18, 19] propose factor 2 approximation algorithms for the augmentation problem of trees and dually chordal graphs with even and odd diameters $2R$ and $2R + 1$ based on particular coverings of trees with balls of radius $R - 1$ and $R$.

$\delta$-Hyperbolic metric spaces play an important role in geometric group theory and in geometry of negatively curved spaces [5, 28, 29]. $\delta$-Hyperbolicity captures the basic common features of "negatively curved" spaces like the hyperbolic space $\mathbb{H}^k$, Riemannian manifolds of strictly negative sectional curvature, and of discrete spaces like trees and the Caley graphs of word-hyperbolic groups. It is remarkable that a strikingly simple concept leads to such a rich general theory [5, 28, 29]. More recently, the concept of $\delta$-hyperbolicity emerged in discrete mathematics, algorithms, and networking. For example, it has been shown empirically in [38] that the internet topology embeds with better accuracy into a hyperbolic space than into a Euclidean space of comparable dimension. A few algorithmic problems in hyperbolic spaces and hyperbolic graphs have been considered in recent papers [23, 27, 33, 35]. 0-Hyperbolic metric spaces are exactly the tree metrics. On the other hand, the Poincaré half space in $\mathbb{R}^k$ with the hyperbolic metric is $\delta$-hyperbolic with $\delta = \log 3$. A full characterization of 1-hyperbolic graphs has been given in [6]; see also [34] for a partial characterization. Chordal graphs (graphs in which all induced cycles have length 3) are 2-hyperbolic [34]. For chordal graphs as well as dually chordal and strongly chordal graphs one can construct trees approximating the graph-distances within a constant 2 or 3 [12], from which follows that those graphs have low $\delta$-hyperbolicity (this result has been extended in [16] to all graphs in which the largest induced cycle is bounded by some constant $\delta$; this result implies that those graphs are $\delta$-hyperbolic). In general, the distance of a $\delta$-hyperbolic space on $n$ points can be approximated within a factor of $2\delta \log n$ by a tree metric [29, 28] and this approximation is sharp.

## 2 $r$-Domination and $r$-packing

Let $(X, d)$ be a geodesic $\delta$-hyperbolic space. Given an instance $(S, r)$ of the $r$-domination and $r$-packing problems, denote by $r + \delta$ the function defined by setting $(r+\delta)(x) := r(x) + \delta$ for all $x \in S$. For each point $x \in S$, define the set $S_x := \{y \in S : r(x) + r(y) \geq d(x, y)\}$ of all points which cannot belong to the same $r$-packing set as $x$. Next auxiliary result shows that in any compact subset $S$ of $X$ one can always find a point $x$ such that $x$ and all points of the set $S_x$ can be $(r+\delta)$-dominated by a common point $c \in X$.

**Lemma 1.** *For any compact subset $S$ of $X$, there exist two points $x \in S$ and $c \in X$ such that $d(c, y) \leq r(y) + \delta$ for any point $y \in S_x$, i.e., $S_x$ is $(r+\delta)$-dominated by $c$.*

*Proof.* Let $x, z$ be a pair of points of $S$ maximizing the value $M := d(x, z) - r(x)$ (such a pair exists because $S$ is compact). If $M \leq \delta$, then the point $z$ $(r+\delta)$-dominates all points of $S$ and we can set $c := z$. Suppose now that $M > \delta$. Pick a geodesic segment $[x, z]$ between $x$ and $z$, and let $c$ be the point of $[x, z]$ located at distance $r(x)$ from $x$. Consider any point $y \in S$ such that $r(x) + r(y) \geq d(x, y)$. We assert that $d(y, c) \leq r(y) + \delta$. For this pick any two geodesic segments $[x, y]$ and $[y, z]$ between the pairs $x, y$ and $y, z$. Let $\Delta(x, y, z) := [x, y] \cup [x, z] \cup [y, z]$ be the geodesic triangle formed by the three geodesic segments and let $m_x, m_y$, and $m_z$ be the three points on these geodesics as defined above. We distinguish two cases. First suppose that $c$ belongs to the portion of $[x, z]$ comprised between the points $x$ and $m_y$. In this case, since $d(x, m_y) = d(x, y) - \alpha_y$, we obtain

$$d(c, m_y) = d(x, y) - \alpha_y - r(x) \leq r(x) + r(y) - \alpha_y - r(x) = r(y) - \alpha_y.$$

Since $\Delta(x, y, z)$ is $\delta$-thin, the triangle condition yields

$$d(c, y) \leq d(c, m_y) + d(m_y, m_x) + d(m_x, y) \leq d(c, m_y) + \delta + \alpha_y \leq r(y) + \delta.$$

On the other hand, if $c$ belongs to the portion of $[x, z]$ comprised between $z$ and $m_y$, then the choice of the points $x$ and $z$ yields $d(y, z) - r(y) \leq d(x, z) - r(x)$. Since $d(x, z) = \alpha_x + \alpha_z$ and $d(y, z) = \alpha_y + \alpha_z$, we conclude that $\alpha_y - r(y) \leq \alpha_x - r(x)$. Thus $d(c, m_y) = r(x) - \alpha_x \leq r(y) - \alpha_y$. As a result, we deduce that $d(c, y) \leq d(c, m_y) + \delta + \alpha_y \leq r(y) + \delta$. □

The following result can be viewed as the variant for $\delta$-hyperbolic spaces of the classical Jung theorem asserting that each subset $S$ of the Euclidean space $\mathbb{E}^m$ with finite diameter $D$ is contained in a ball of radius at most $\sqrt{\frac{m}{2(m+1)}}D$.

**Corollary 1.** *If the diameter of a compact geodesic $\delta$-hyperbolic metric space $(X, d)$ is $D := 2R$, then $X$ can be covered by single ball of radius $R + \delta$, i.e., the radius of $X$ is at most $R + \delta$.*

*Proof.* Let $S := X$ and $r(x) \equiv R$. Since $d(x, y) \leq 2R = r(x) + r(y)$ for any pair $x, y \in X$, we conclude that $S_x = X$ for any point $x \in X$. Since $X$ is compact, by Lemma 1, there exist a point $x \in S = X$ and a point $c \in X$ such that $X = S_x \subseteq B(c, r(x) + \delta) = B(c, R + \delta)$. □

The following result, generalizing Corollary 1, can be viewed as the analogy of the classical Helly property for balls.

**Corollary 2.** *If $B(x_i, r_i), i \in I$, is a collection of pairwise intersecting balls of a geodesic $\delta$-hyperbolic metric space $(X, d)$ with a compact set $S := \{x_i : i \in I\}$ of centers, then the balls $B(x_i, r_i + \delta), i \in I$, have a nonempty intersection.*

*Proof.* Set $r(x_i) := r_i$. Then, as in previous result, since $d(x_i, x_j) \leq r_i + r_j = r(x_i) + r(x_j)$, the equality $S_x = S$ holds for any point $x_i$ of $S$. By Lemma 1, $S$ is $(r + \delta)$-dominated by a single point $c$. Obviously this point belongs to all balls $B(x_i, r_i + \delta)$, establishing the result. □

Now, we present the main result of this paper. It generalizes the equality $\gamma(S, r) = \pi(S, r)$ for trees to all $\delta$-hyperbolic spaces in the following way:

**Theorem 1.** *Let $S$ be a finite subset of a geodesic $\delta$-hyperbolic metric space $(X, d)$. Then $\gamma(S, r + \delta) \leq \pi(S, r)$. Moreover, a set $C$ $(r + \delta)$-dominating the set $S$ and an $r$-packing $P$ of $S$ such that $|C| = |P|$ can be constructed using a polynomial in $|S|$ number of calls of the oracle for computing a geodesic segment in $(X, d)$.*

*Proof.* The proof of this result is algorithmic: we construct the $r$-packing $P$ and the $(r+\delta)$-dominating set $C$ step by step taking care that the following properties hold: (i) each time when a new point is inserted in $C$, then a new point is also inserted in $P$, and (ii) at the end, the set $P$ is an $r$-packing and $C$ is an $(r+\delta)$-dominating set for $S$.

The algorithm starts with $S' := S$, $C := \emptyset$, and $P := \emptyset$. While the set $S'$ is nonempty, the algorithm applies Lemma 1 to the current set $S'$ in order to obtain a point $x \in S'$ and a point $c \in X$ which $(r+\delta)$-dominates the set $S'_x$. The algorithm adds the point $x$ to $P$ and the point $c$ to $C$, and then it updates the set $S'$ by removing from $S'$ all points which are $(r+\delta)$-dominated by $c$, and so on. The algorithm terminates in at most $|S|$ rounds. Notice also that $|P| = |C|$, because when the point $x$ is inserted in $P$, then at the same step $x$ is removed from $S'$ because $x$ is $(r+\delta)$-dominated by the point $c$ which is included at that step in $C$.

We assert that at the end, $P$ is an $r$-packing of $S$ and $C$ is an $(r+\delta)$-dominating set for $S$. Indeed, $C$ $(r+\delta)$-dominates $S$ because $S'$ is empty when the algorithm halts and that each point $s \in S$ is $(r+\delta)$-dominated by the point which is inserted in $C$ at the iteration when $s$ is removed from $S'$. To show that $P$ is an $r$-packing it suffices to show that after each iteration the updated set $P$ is an $r$-packing. So, suppose that at the current iteration the point $y$ has been inserted in the set $P$, which before this insertion was an $r$-packing. We must show that $P \cup \{y\}$ is an $r$-packing as well. Suppose by way of contradiction that $d(x,y) \leq r(x) + r(y)$ for some point $x \in P$. Consider the iteration at which the point $x$ was inserted in $P$ and suppose that at this iteration the point $c$ was inserted in $C$. Since $y \in S'_x$ and all points of $S'_x$ are $(r+\delta)$-dominated by $c$, the algorithm will remove at this iteration $y$ from $S'$, thus $y$ cannot be inserted in $P$ at a later stage, contrary to our assumption. This ensures that $P$ is an $r$-packing during all execution of the algorithm. □

Consider now the case of $r$-domination and $r$-packing for graphs $G = (V, E)$ such that the underlying geodesic metric space $(X, d)$ is $\delta$-hyperbolic. More precisely, let $S$ be a subset of vertices of $G$, let $r$ be a map from $S$ to $\mathbb{N}_+$, and we are searching for a subset of vertices $C \subseteq V$ which $(r+\delta)$-dominates $S$. Now, if we will run in $(X, d)$ the algorithm described in Theorem 1 with $S$ and $r$ as an input, then the $r$-dominating set $C$ returned by this algorithm must be a subset of $V$. For this, it suffices to notice that each vertex $c \in C$ is defined according to the choice of Lemma 1. The point $c$ in this lemma is located at distance $r(x)$ from $x$ on a geodesic segment $[x, z]$. Since $r(x)$ and $d(x, z)$ are integers, we conclude that $c$ is a vertex of $G$. In case of graphs, we can specify the oracle computing geodesic segments: it suffices to use any shortest-path algorithm in $G$. Finally, notice that if $r(x) := R$ for all $x \in S$, then an $(r+\delta)$-dominating set $C$ corresponds to the set of centers of balls of radius $R + \delta$ covering the set $S$. Summarizing, we obtain the following observation.

**Corollary 3.** *Let $S$ be a subset of vertices of a finite $\delta$-hyperbolic graph $G = (V, E)$. Then $\gamma_G(S, r+\delta) \leq \pi_G(S, r)$. Moreover a set $C \subseteq V$ $(r+\delta)$-dominating* *the set $S$ and an $r$-packing $P$ of $S$ such that $|C| = |P|$ can be constructed in polynomial time.*

## 3 $(\boldsymbol{\kappa}, \mathbf{r})$-Domination and $(\boldsymbol{\kappa}, \mathbf{r})$-packing

Let ${}^\kappa\mathcal{B}_{S,\mathbf{r}}$ be a finite family of $\kappa$-balls of a $\delta$-hyperbolic geodesic metric space. For $\varepsilon > 0$, denote by ${}^\kappa\mathcal{B}_{S,\mathbf{r}+\varepsilon}$ the family of balls obtained by "inflating" each ball $B(s, r(s))$ of $\bigcup {}^\kappa\mathcal{B}_{S,\mathbf{r}}$ until its radius becomes $r(s)+\varepsilon$, i.e., obtained by replacing $\mathbf{r}$ by the function $\mathbf{r}+\varepsilon$. We call the $\kappa$-balls of ${}^\kappa\mathcal{B}_{S,\mathbf{r}+\varepsilon}$ *$\varepsilon$-inflated* $\kappa$-balls. This section is devoted to the proof of the following result:

**Theorem 2.** *Let ${}^\kappa\mathcal{B}_{S,\mathbf{r}} = \{{}^\kappa B_1, \ldots, {}^\kappa B_m\}$ be a family of $\kappa$-balls of a $\delta$-hyperbolic geodesic metric space. Then $\gamma(S, \mathbf{r}+2\delta) \leq 2\kappa^2\pi(S, \mathbf{r})$. Moreover a hitting set $C$ for ${}^\kappa\mathcal{B}_{S,\mathbf{r}+2\delta}$ and a packing $P$ of ${}^\kappa\mathcal{B}_{S,\mathbf{r}}$ such that $|C| \leq 2\kappa^2|P|$ can be constructed with a polynomial in $|S|$ number of calls of the oracle for computing geodesic segments in $(X, d)$.*

For each $\kappa$-ball ${}^\kappa B_i$, denote by $S_i$ the set of centers of the balls constituting ${}^\kappa B_i$. For each $s \in S_i$, let $r_i(s)$ be the radius of the ball of ${}^\kappa B_i$ centered at $s$. Clearly, $S = \bigcup_{i=1}^m S_i$. For a point $v \in X$, let $N[v] := \{i : d(v, s) \leq r_i(s)$ for some $s \in S_i\}$ be the set of indices of all $\kappa$-balls ${}^\kappa B_i$ covering $v$. For any $i = 1, \ldots, m$, let $N[i]$ be the set of indices of all $\kappa$-balls which cannot be included in a common $(\kappa, \mathbf{r})$-packing with ${}^\kappa B_i$, i.e., $N[i] = \bigcup\{N[v] : v \in {}^\kappa B_i\}$. Clearly, if $j \in N[i]$, then $i \in N[j]$. Notice also that $i \in N[i]$.

Now we can formulate a pair of dual linear programs whose optimal solutions $\pi_f(S, \mathbf{r})$ and $\gamma_f(S, \mathbf{r})$ are an optimal fractional packing and an optimal fractional covering for ${}^\kappa\mathcal{B}_{S,\mathbf{r}}$, respectively. For this, we introduce a variable $x_i$ for each $\kappa$-ball ${}^\kappa B_i$ and a dual variable $y_v$ for each point $v \in X$.

$$\begin{cases} \max & \sum_{i=1}^m x_i \\ \text{s.t.} & \sum_{i \in N[v]} x_i \leq 1 \quad \forall\, v \in X \\ & x_i \qquad\quad \geq 0 \quad \forall\, i = 1, \ldots, m \end{cases} \qquad\qquad \mathit{\Pi}(S, \mathbf{r})$$

$$\begin{cases} \min & \int_{v \in X} y_v \\ \text{s.t.} & \int_{v \in {}^\kappa B_i} y_v \geq 1 \quad \forall\, i = 1, \ldots, m \\ & y_v \qquad\quad \geq 0 \quad \forall\, v \in X. \end{cases} \qquad\qquad \mathit{\Gamma}(S, \mathbf{r})$$

Notice that the first linear program contains as many constraints as points in the space $X$, while the second linear program assumes that we can integrate over the balls of $X$. In fact, one can easily rewrite $\mathit{\Pi}(S, \mathbf{r})$ using only a finite number of constraints: since there exists only a finite number of patterns of intersections of balls in $\bigcup {}^\kappa\mathcal{B}_{S,\mathbf{r}}$, we can pick a point $v$ in each type of intersection and write the constraints $\sum_{i \in N[v]} x_i \leq 1$ only for such $v$. Denote the resulting finite set by $V^*$. We can also rewrite $\mathit{\Gamma}(S, \mathbf{r})$ by replacing the integration by a sum over all points $v \in V^*$ belonging to ${}^\kappa B_i$. The resulting linear programs have respectively $m$ variables, $|V^*|$ constraints and, vice-versa, $|V^*|$ variables and $m$ constraints.

Now, we will construct in polynomial time a set $V$ of size $\kappa \cdot m$ and formulate the linear programs on $V$ instead of $X$ or $V^*$. Then we relate the admissible and optimal solutions of resulting linear programs with those of $\Pi(S,\mathbf{r})$ and $\Gamma(S,\mathbf{r})$.

Let $\mathcal{B}$ denote the set of all balls participating in $\kappa$-balls of the family ${}^{\kappa}\mathcal{B}_{S,\mathbf{r}}$. Denote the radius function of these balls by $r$ and by $S = \bigcup_i^m S_i$ the multi-set of centers of the balls from $\mathcal{B}$. The set $V$ is constructed iteratively, starting with $V := \emptyset$, $S' = S$, and $\mathcal{B}' := \mathcal{B}$. At each iteration, given the current set of balls $\mathcal{B}'$ and the set $S'$ of their centers, we apply Lemma 1 to find a point $s \in S'$, a ball $B \in \mathcal{B}'$ centered at $s$, and a point $c_s \in X$ such that the set $S'_s$ is $(r+\delta)$-dominated by $c_s$. Then the point $c_s$ is inserted in $V$ and the ball $B$ is removed from $\mathcal{B}'$. The algorithm halts when $\mathcal{B}'$ becomes empty. Clearly, the returned set $V$ has cardinality $\kappa \cdot m$. Denote by $\pi'_f(S,\mathbf{r})$ and $\gamma'_f(S,\mathbf{r})$ the optimal solutions of the following linear programs:

$$\begin{cases} \max \ \sum_{i=1}^m x_i & \\ \text{s.t.} \ \ \sum_{i \in N[v]} x_i \le 1 & \forall\, v \in V \\ \qquad x_i \ge 0 & \forall\, i = 1,\dots,m \end{cases} \qquad \Pi'(S,\mathbf{r})$$

$$\begin{cases} \min \ \sum_{v \in C} y_v & \\ \text{s.t.} \ \ \sum_{v \in {}^{\kappa}B_i} y_v \ge 1 & \forall\, i = 1,\dots,m \\ \qquad y_v \ge 0 & \forall\, v \in V. \end{cases} \qquad \Gamma'(S,\mathbf{r})$$

**Lemma 2.** *Any admissible solution $\{x_i : i = 1,\dots,m\}$ of $\Pi'(S,\mathbf{r}+\delta)$ is also an admissible solution of $\Pi(S,\mathbf{r})$. Moreover, $\gamma'_f(S,\mathbf{r}+\delta) \le \gamma_f(S,\mathbf{r})$.*

*Proof.* Notice that it suffices to check the inequality $\sum_{i \in N[v]} x_i \le 1$ only for points $v \in X$ for which the set $N[v]$ is nonempty. Then $v$ belongs to at least one ball from the set $\mathcal{B}$. Among such balls, let $B$ be the first ball considered by the algorithm constructing the set $V$. Let $s$ be the center of $B$ and $c_s$ be the point included in $V$ when the ball $B$ is removed from $\mathcal{B}'$. Notice that the set $S(v)$ of centers of all balls of $\mathcal{B}$ containing $v$ belongs to $S'_s$. The definition of $c_s$ yields that $S(v)$, as a part of $S'_s$, is $(r+\delta)$-dominated by the point $c_s$ of $V$. Writing down the constraint of $\Pi'(S,\mathbf{r}+\delta)$ defined by the point $c_s$, we conclude that the sum of $x_i$'s over all $\delta$-inflated $\kappa$-balls containing $c_s$ is at most 1. Since the $\delta$-inflations of all $\kappa$-balls containing $v$ all contain $c_s$, we conclude that $\sum_{i \in N[v]} x_i \le 1$ holds in $\Pi(S,\mathbf{r})$. □

**Lemma 3.** *If $\mathbf{x} = \{x_i : i = 1,\dots,m\}$ is an admissible solution of $\Pi'(S,\mathbf{r}+\delta)$, then there exists a $\kappa$-ball ${}^{\kappa}B_i$ such that $\sum_{j \in N[i]} x_j \le 2\kappa$.*

*Proof.* The proof of this result is inspired by the averaging argument used in the proof of Lemma 4.1 of [7]. Define a graph $\mathbf{N}$ with $1,\dots,m$ as the set of vertices and in which $ij$ is an edge if and only if $j \in N[i]$ (and consequently $i \in N[j]$). For each edge $ij$ of $\mathbf{N}$, set $z(i,j) = x_i \cdot x_j$. Since $i \in N[i]$, define $z(i,i) = x_i^2$. In the sum $\sum_{i=1}^m \sum_{j \in N[i]} z(i,j)$ every $z(i,j)$ is counted twice. On the other hand, an upper bound on this sum can be obtained in the following way. For a point $s \in S$, let $N^\delta[c_s]$ be the set of indices of all $\delta$-inflated $\kappa$-balls which contain the point $c_s$. Now, for each $\kappa$-ball ${}^{\kappa}B_i$ consider its set of centers $S_i$, and for each $s \in S_i$, add up $z(i,j)$ for all $j \in N^{\delta}[c_s]$, and then multiply the total sum by 2. This way we computed the sum $2\sum_{i=1}^{m}\sum_{s\in S_i}\sum_{j\in N^{\delta}[c_s]} z(i,j)$. We assert that this suffices. Indeed, pick any $z(i,j)$ for an edge $ij$ of the graph $\mathbf{N}$. Thus the $\kappa$-balls ${}^{\kappa}B_i$ and ${}^{\kappa}B_j$ contain two intersecting balls $B$ and $B'$, say $B$ is centered at $s \in S_i$. Suppose without loss of generality that the algorithm for constructing the set $V$ considers $B$ before $B'$. Then necessarily $j \in N^{\delta}[c_s]$, because $c_s$ hits the $\delta$-inflation of the ball $B'$. Hence the term $z(i,j)$ will appear at least once in the triple sum, establishing the required inequality

$$\sum_{i=1}^{m}\sum_{j\in N[i]} z(i,j) \le 2\sum_{i=1}^{m}\sum_{s\in S_i}\sum_{j\in N^{\delta}[c_s]} z(i,j).$$

Taking into account that $z(i,j) = x_i{\cdot}x_j = z(j,i)$, this inequality can be rewritten in the following way:

$$\sum_{i=1}^{m} x_i \sum_{j\in N[i]} x_j \le 2\sum_{i=1}^{m} x_i \sum_{s\in S_i}\sum_{j\in N^{\delta}[c_s]} x_j.$$

Now, since $c_s$ hits all $\delta$-inflated $\kappa$-balls from $N^{\delta}[c_s]$ and $\mathbf{x}$ is an admissible solution of $\Pi'(S,\mathbf{r}+\delta)$, we conclude that $\sum_{j\in N^{\delta}[c_s]} x_j \le 1$. Thus $\sum_{s\in S_i}\sum_{j\in N^{\delta}[c_s]} x_j \le |S_i|$. Since $|S_i| \le \kappa$, we deduce that $\sum_{i=1}^{m} x_i \sum_{j\in N[i]} x_j \le 2\kappa\sum_{i=1}^{m} x_i$. Hence, there exists ${}^{\kappa}B_i$ such that $x_i\sum_{j\in N[i]} x_j \le 2\kappa x_i$, yielding $\sum_{j\in N[i]} x_j \le 2\kappa$.□

**Lemma 4.** *It is possible to construct in polynomial time an integer admissible solution* $\mathbf{x}^*$ *of the linear program* $\Pi(S,\mathbf{r})$ *of size at least* $\pi'_f(S,\mathbf{r}+\delta)/(2\kappa)$.

*Proof.* Let $\mathbf{x} = \{x_1,\ldots,x_m\}$ be an optimal (fractional) solution of the linear program $\Pi'(S,\mathbf{r}+\delta)$ (it can be found in polynomial time). We will iteratively use Lemma 3 to $\mathbf{x}$ to derive an integer solution $\mathbf{x}^* = \{x_1^*,\ldots,x_m^*\}$ for the linear program $\Pi(S,\mathbf{r})$. The algorithm starts with the set ${}^{\kappa}\mathcal{B}' := {}^{\kappa}\mathcal{B}_{S,\mathbf{r}}$ of $m$ $\kappa$-balls. By Lemma 3 there exists a $\kappa$-ball ${}^{\kappa}B_i \in {}^{\kappa}\mathcal{B}'$ such that $\sum_{j\in N[i]} x_j \le 2\kappa$. We set $x_i^* := 1$ and $x_j^* := 0$ for all $j \in N[i]\setminus\{i\}$, then we remove all $\kappa$-balls ${}^{\kappa}B_j$ with $j \in N[i]$ from ${}^{\kappa}\mathcal{B}'$. The algorithm continues with the current set ${}^{\kappa}\mathcal{B}'$ of $\kappa$-balls until it becomes empty. Notice that at all iterations of the algorithm the restriction of $\mathbf{x}$ on the $\kappa$-balls of ${}^{\kappa}\mathcal{B}'$ remains an admissible solution of the linear program $\Pi'(S',\mathbf{r}+\delta)$ defined by ${}^{\kappa}\mathcal{B}'$. This justifies the use of Lemma 3 at all iterations of the algorithm.

To show that $\mathbf{x}^*$ is an admissible solution of $\Pi(S,\mathbf{r})$, suppose by way of contradiction that there exist two intersecting $\kappa$-balls ${}^{\kappa}B_i$ and ${}^{\kappa}B_j$ with $x_i^* = 1 = x_j^*$. Suppose that the algorithm selects ${}^{\kappa}B_i$ before ${}^{\kappa}B_j$. Consider the iteration when $x_i^*$ becomes 1. Since $j \in N[i]$, at this iteration $x_j^*$ becomes 0 and ${}^{\kappa}B_j$ is removed from ${}^{\kappa}\mathcal{B}'$. Thus $x_j^*$ cannot become 1 at a later stage. This shows that the $\kappa$-balls ${}^{\kappa}B_i$ with $x_i^* = 1$ indeed constitute a packing for ${}^{\kappa}\mathcal{B}_{S,\mathbf{r}}$.

It remains to compare the costs of the solutions $\mathbf{x}$ and $\mathbf{x}^*$. For this, notice that according to the algorithm, for each $\kappa$-ball ${}^{\kappa}B_i$ with $x_i^* = 1$ we can define a subset $N'[i]$ of $N[i]$ such that $i \in N'[i]$, $x^*_j = 0$ for all $j \in N'[i] \setminus \{i\}$, and $\sum_{j \in N'[i] \cup \{i\}} x_j \leq 2\kappa$. Hence, the $\kappa$-balls of ${}^{\kappa}\mathcal{B}_{S,\mathbf{r}}$ can be partitioned into groups, such that each group contains a single $\kappa$-ball selected in the integer solution and the total cost of the fractional solutions of the balls from each group is at most $2\kappa$. This shows that $\sum_{i=1}^{m} x^*_i \geq (\sum_{i=1}^{m} x_i)/(2\kappa)$. □

**Lemma 5.** *It is possible to construct in polynomial time an integer solution* $\mathbf{y}^*$ *of the linear program* $\Gamma(S, \mathbf{r} + \delta)$ *of size at most* $\kappa\gamma'_f(S, \mathbf{r})$.

*Proof.* Let $\mathbf{y} = \{y_v : v \in V\}$ be an optimal (fractional) solution of the linear program $\Gamma'(S, \mathbf{r})$. Since $\sum_{v \in {}^{\kappa}B_i} y_v \geq 1$ for all $i = 1, \ldots, m$, each $\kappa$-ball ${}^{\kappa}B_i$ contains a ball $B_i$ such that $\kappa \sum_{v \in B_i} y_v \geq 1$. Let $s_i$ be the center of the ball $B_i$ and let $r(s_i)$ be its radius. Set $S = \{s_1, \ldots, s_m\}$. Notice that $\mathbf{y}' = \{y'_v : v \in V\}$ defined by setting $y'_v = \kappa \cdot y_v$ if $v \in \bigcup_{i=1}^{m} B_i$ and $y'_v = 0$ otherwise, is a fractional covering for the family of balls $B_1, \ldots, B_m$. Thus the cost of $\mathbf{y}'$ is at least $\gamma_f(S, r) = \pi_f(S, r)$. Notice also that the cost of $\mathbf{y}'$ is at most $\kappa$ times the cost of $\mathbf{y}$. By Theorem 1, we can construct in polynomial time a set $C$ of size at most $\pi(S, r)$ which $(r + \delta)$-dominates the set $S$. Let $\mathbf{y}^* = \{y_v : v \in V\}$ be defined by setting $y^*_v = 1$ if $v \in C$ and $y^*_v = 0$ otherwise. Since $\pi(S, r) \leq \pi_f(S, r)$, putting all things together, we obtain:

$$\sum_{v \in V} y^*_v = |C| \leq \pi(S, r) \leq \pi_f(S, r) = \gamma_f(S, r) \leq \sum_{v \in V} y'_v \leq \kappa \sum_{v \in V} y_v = \kappa\gamma'_f(S, \mathbf{r}). \square$$

Now, we are ready to complete the proof of Theorem 2. According to Lemma 4 we can construct in polynomial time an integer solution $\mathbf{x}^*$ for $\Pi(S, \mathbf{r})$ of size at least $\pi'_f(S, \mathbf{r} + \delta)/(2\kappa)$. Let $P = \{{}^{\kappa}B_i : x^*_i = 1\}$. On the other hand, applying Lemma 5 for the radius function $\mathbf{r} + \delta$ instead of $\mathbf{r}$, we can construct in polynomial time an integer solution $\mathbf{y}^*$ of the linear program $\Gamma(S, \mathbf{r} + 2\delta)$ of size at most $\kappa\gamma'_f(S, \mathbf{r} + \delta)$. Let $C = \{v \in V : y^*_v = 1\}$. Since, by duality, $\gamma'_f(S, \mathbf{r} + \delta) = \pi'_f(S, \mathbf{r} + \delta)$, we deduce that $|C| \leq 2\kappa^2|P|$, as required.

## 4 Augmentation under diameter constraints

Denote by OPT$(D)$ the minimum number of edges necessary to decrease the diameter of the input $\delta$-hyperbolic graph $G = (V, E)$ until $D$. First suppose that the resulting diameter $D$ is even, say $D = 2R$. We recall the relationship between the augmentation problem ADC and the $r$-domination problem established in [19] for trees. Let $E^*$ be an optimal augmentation, i.e. $|E^*| = \text{OPT}(D)$ and the graph $G^* = (V, E \cup E^*)$ has diameter $D$. Denote by $C$ the set of end-vertices of the edges of $E^*$ and let $V'$ the set of all vertices which are located at distance less than or equal to $R - 1$ from a vertex of $C$ (in other words, $V'$ is the union of all balls or radius $R - 1$ centered at vertices of $C$). Since $E^*$ is a solution for the problem ADC, it can be easily shown (see [19] for missing details) that the diameter in $G$ of the set $Q := V \setminus V'$ is at most $2R$. Since $G$ is $\delta$-hyperbolic, from Corollary 1 and the discussion preceding Corollary 3 we infer that $Q$ can be covered by a single ball $B(c^*, R+\delta)$ of radius $R+\delta$. Let $Q'$ be the set of vertices of $G$ located outside $B(c, R+\delta)$. Since $Q' \subseteq V' = \bigcup\{B(x, R-1) : x \in C\}$ and each edge of $E^*$ has both ends in $C$, we conclude that $\gamma_G(Q', R-1) \leq \gamma_G(V', R-1) \leq |C| \leq 2|E^*| = 2\mathrm{OPT}(D)$.

Now, we turn this analysis of an optimal solution (which we do not know how to construct) into a polynomial time algorithm which instead will find a set $E'$ of new $2\mathrm{OPT}(D)$ edges so that the resulting graph $G' = (V, E \cup E')$ will have diameter at most $D + 2\delta$ (instead of $D$, as required). As for trees [19], the algorithm will try every vertex $c'$ of $G$ as a center of a ball of radius $R+\delta$ and it covers the set $V \setminus B(c', R+\delta)$ with at most $\pi_G(V \setminus B(c', R+\delta), R-1)$ balls of radius $R-1+\delta$. This is done using the procedure described in Theorem 1. Among $|V|$ such coverings, the algorithm selects the one with a minimum number of balls. Let $c'$ be the center of the ball of radius $R+\delta$ providing this covering $C'$. Then the algorithm returns as the set $E'$ of new edges all pairs of the form $c'c$, where $c$ is a center of a ball of radius $R-1+\delta$ from $C'$. Notice that the graph obtained from $G$ after adding the new edges has diameter at most $2R+2\delta$. Finally notice that since the algorithm tested the vertex $c^*$ described above as the center of the ball of radius $R+\delta$, by Theorem 1 we conclude that $|C'| \leq \pi_G(Q', R-1)$, showing that $|C'| \leq 2\mathrm{OPT}(2R)$. We obtain the following result:

**Proposition 1.** *Given a $\delta$-hyperbolic graph $G = (V, E)$ and $R \geq 1$, one can construct in polynomial time an admissible solution for the problem ADC with $D = 2R + 2\delta$ which contains at most $2\mathrm{OPT}(2R)$ edges.*

## 5 *k*-Center problem

Let $G = (V, E)$ be a $\delta$-hyperbolic graph and $S$ be a set of $n$ input vertices of the $k$-center problem. Then, as we noticed already, the $k$-center problem consists in finding the smallest radius $R^*$ such that the set $S$ can be covered with at most $k$ balls of radius $R^*$. The value of $R^*$ belongs to the list $\Delta$ of size $O(|V| \cdot |S|)$ consisting of all possible distinct values of distances from the vertices of $G$ to the set $S$. As in some other minmax problems [30, 31, 39], the approximation algorithm tests the entries of $\Delta$, using a parameter $R$, which is the "guess" of the optimal radius. For current $R \in \Delta$, instead of running the algorithm of Hochbaum and Shmoys [31], we use the algorithm described in Theorem 1 and Corollary 3 with $r(x) = R$ for all $x \in S$. This algorithm either finds a covering of $S$ with at most $k$ balls of radius $R+\delta$ or it returns an $r$-packing $P$ of size greater than $k$. In the second case, we conclude that $\gamma_G(S, r) \geq \pi_G(S, r) > k$, therefore the tested value $R$ is too small, yielding $R < R^*$. Now, if $R$ is the least value for which the algorithm does not return the negative answer, then $R \leq R^*$, and we obtain a solution for the $k$-center problem of radius $R + \delta \leq R^* + \delta$.

**Proposition 2.** *Given a $\delta$-hyperbolic graph $G = (V, E)$, one can construct in polynomial time an admissible solution for the $k$-center problem having radius at most* $\mathrm{OPT} + \delta$.

## References


1. N. Alon, Piercing $d$-intervals, *Discrete and Computational Geometry,* **19** (1998), 333-334.
2. N. Alon, Covering a hypergraph of subgraphs, *Discrete Mathematics,* **257** (2002), 249–254.
3. N. Alon, D. Moshkovitz, and M. Safra, Algorithmic construction of sets for $k$-restrictions, *ACM Transactions on Algorithms,* to appear.
4. N. Alon, A. Gyárfás, and M. Ruszinkó, Decreasing the diameter of bounded degree graphs, *J. Graph Theory,* **35** (2000), 161–172.
5. J.M. Alonso, T. Brady, D. Cooper, V. Ferlini, M. Lustig, M. Mihalik, M. Shapiro, and H. Short, *Notes on word hyperbolic groups,* In Group Theory from a Geometrical Viewpoint, ICTP Trieste 1990 (E. Ghys, A. Haefliger, and A. Verjovsky, eds.), World Scientific, 1991, pp. 3–63.
6. H.-J. Bandelt and V. Chepoi, 1-Hyperbolic graphs, *SIAM J. Discrete Mathematics* **16**(2003) 323–334.
7. R. Bar-Yehuda, M. M. Halldorsson, J. Naor, H. Shachnai, and I. Shapira, Scheduling split intervals, *SIAM Journal on Computing,* **36** (2006), 1–15 and *SODA'* 2002.
8. I. Bárány, J. Edmonds, and L. A. Wolsey, Packing and covering a tree by subtrees, *Combinatorica,* **6** (1986), 221–233.
9. C. Berge, *Hypergraphs,* North-Holland, 1989.
10. M. Bern and D. Eppstein, Approximation algorithms for geometric problems, In *Approximation Algorithms for NP-Hard Problems* (D. S. Hochbaum, ed.), PWS, Boston, MA, 1997, pp. 296–345.
11. A. Brandstädt, V. Chepoi, and F. Dragan, Clique r-domination and clique r-packing problems on dually chordal graphs, *SIAM J. Discrete Mathematics,* **10**(1997), 109–127.
12. A. Brandstädt, V. Chepoi, and F. Dragan, Distance approximating trees for chordal and dually chordal graphs, *J. Algorithms,* **30** (1999), 166–184 and *ESA*'97.
13. R. Chandrasekaran and A. Doughety, Location on tree networks: $p$-center and $q$-dispersion problems, *Mathematics Operations Research,* **6** (1981), 50–57.
14. G.J. Chang, Labeling algorithms for domination problems in sun-free chordal graphs, *Discrete Applied Mathematics,* **22** (1988/1989), 21–34.
15. G. J. Chang and G. L. Nemhauser, The $k$-domination and $k$-stability problems on sun-free chordal graphs, *SIAM J. Alg. Disc. Meth.,* **5** (1986), 332–345.
16. V. Chepoi and F. Dragan, A note on distance approximating trees in graphs, *European J. Combinatorics,* **21** (2000), 761–766.
17. V. Chepoi, B. Estellon, and Y. Vaxès, On covering planar graphs with a fixed number of balls, *Discrete and Computational Geometry* **37** (2007), 237–244.
18. V. Chepoi, B. Estellon, K. Nouioua, and Y. Vaxès, Mixed covering of trees and the augmentation problem with odd diameter constraints, *Algorithmica,* **45** (2006), 209–226.
19. V. Chepoi and Y. Vaxès, Augmenting trees to meet biconnectivity and diameter constraints, *Algorithmica,* **33** (2002), 243–262.
20. F. R. K. Chung and M. R. Garey, Diameter bounds for altered graphs, *J. Graph Theory,* **8** (1984), 511–534.
21. E.D. Demaine, F. V. Fomin, M. Hajiaghayi, and D. M. Thilikos, Fixed-parameter algorithms for $(k, r)$-center in planar graphs and map graphs, *ACM Transactions on Algorithms,* **1** (2005), 33–47 and *ICALP'* 2003.

22. Y. Dodis and S. Khanna, Designing networks with bounded pairwise distance, In *Proc. 31th Annual ACM Symposium on the Theory of Computing STOC'* 1999, 750–759.
23. D. Eppstein, Squarepants in a tree: sum of subtree clustering and hyperbolic pants decomposition, In *Proc. 18th ACM-SIAM Symposium on Discrete Algorithms SODA'* 2007.
24. P. Erdös, A. Gyárfás, and M. Ruszinkó, How to decrease the diameter of triangle–free graphs, *Combinatorica,* **18** (1998), 493–501.
25. M. Farber, Domination, independent domination and duality in strongly chordal graphs, *Discrete Applied Mathematics,* **7** (1984), 115–130.
26. T. Feder and D.H. Greene, Optimal algorithms for approximate clustering, In *Proc. 20th ACM Symp. Theory of Computing, STOC'* 1988, pp. 434–444.
27. C. Gavoille and O. Ly, Distance labeling in hyperbolic graphs, In *ISAAC'* 2005, pp. 1071–1079.
28. E. Ghys and P. de la Harpe eds., *Les groupes hyperboliques d'après M. Gromov,* Progress in Mathematics, Vol. 83, Birkhäuser, 1990.
29. M. Gromov, Hyperbolic Groups, In *Essays in group theory* S.M. Gersten ed., MSRI Series 8 (1987), pp. 75–263.
30. D.S. Hochbaum, Various notions of approximations: good, better, best, and more, In *Approximation Algorithms for NP-Hard Problems* (D. S. Hochbaum, ed.), PWS, Boston, MA, 1997, pp. 346–398.
31. D.S. Hochbaum and D.B. Shmoys, A unified approach to approximation algorithms for bottleneck problems, *Journal of the ACM,* **33** (1986), 533–550.
32. T. Ishii, S. Yamamoto, and H. Nagamochi, Augmenting forests to meet odd diameter requirements, *Discrete Optimization,* **3** (2006), 154–164.
33. R. Kleinberg, Geographic routing using hyperbolic space, In *Proc. 26th Annual IEEE International Conference on Computer Communications, INFOCOM'* 2007.
34. J. Koolen and V. Moulton, Hyperbolic bridged graphs, *European J. Combinatorics,* **23** (2002), 683–699.
35. R. Krauthgamer and J.R. Lee, Algorithms on negatively curved spaces, In *Proc. 47th Annual IEEE Symposium on Foundations of Computer Science, FOCS'* 2006.
36. Ch.-L. Li, S.Th. McCormick, and D. Simchi–Levi, On the minimum-cardinality-bounded-diameter and the bounded-cardinality-minimum-diameter edge addition problems, *Operations Research Letters,* **11** (1992), 303–308.
37. A.A. Schoone, H.L. Bodlaender, and J. van Leeuwen, Diameter increase caused by edge deletion, *J. Graph Theory,* **11** (1987), 409–427.
38. Y. Shavitt and T. Tankel, On internet embedding in hyperbolic spaces for overlay construction and distance estimation, *IEEE/ACM Transactions on Networking* (to appear) and *INFOCOM'* 2004.
39. V. Vazirani, *Approximation Algorithms*, Springer, 2001.