# CENTERS OF TRIANGULATED GRAPHS

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Let $G = (X, U)$ be an ordinary graph with arbitrarily (not necessarily finitely) many vertexes, any two of which are joined by some finite chain. We endow G with a standard metric $d(x, y)$ equal to the number of edges in the chain of shortest length joining vertexes x, y. The eccentricity $e(z)$ of a vertex z is defined as $\{d(z, v): v \in X\}$. The radius $r(G)$ is the least eccentricity of the vertexes, and the diameter $d(G)$ is the largest eccentricity. The center $C(G)$ of G is the subgraph generated by the set of vertexes with minimal eccentricity.

It is well known [1, 2] that any graph G, even if it is not connected, is the center of some graph G', i.e., $G = C(G')$. At the same time, if one confines attention to special classes of graphs, their centers may have rather specific features. Thus, a well-known result of Jordan [3] states that the center of any finite tree consists of one vertex or two adjacent vertexes; that is to say, the only two possibilities are the graphs $K_1$ and $K_2$ (where $K_n$ denotes the complete subgraph on n vertexes). The centers of maximal outerplanar graphs and 2-trees were described in [4, 5]. In this paper we characterize the centers of triangulated graphs. In this connection, we note that metric properties, including in particular properties of the centers, of triangulated graphs have been studied by various authors [6-9]. Our Theorems 1 and 2 were proved in [8], but they are established here in a more general form and the proofs are simpler.

Recall [10] that a graph G is said to be triangulated if, in any simple cycle Γ of length more than 3, there are two vertexes not adjacent in Γ but joined by an edge in G.

Recall moreover that a clique of a graph G is any maximal complete subgraph (with respect to inclusion). The density of a graph G is the cardinality of the largest clique in G (if such a clique exists).

Throughout this paper, the term triangulated graph will mean a triangulated graph of finite diameter without infinite complete subgraphs.

Let CT denote the family of graphs which are centers of triangulated graphs.

We shall need a number of additional concepts and definitions.

Given sets $M \subset X$ in a graph G and any number $r \geq 0$, we put

$$\Sigma_r(M) = \{z \in X:\ d(z, M) \leqslant r\},$$
$$U_r(M) = \{z \in X:\ d(z, M) = r\},$$

where

$$d(z, M) = \min \{d(z, v):\ v \in M\}.$$

Let

$$N_x(M) = \{z \in M:\ d(x, z) = d(x, M)\}$$

denote the metric projection of the vertex x onto M.

A set $B \subset X$ in a graph G is said to be dominant (and the subgraph that it generates is called a dominant subgraph) if, for any vertex $x \in X \setminus B$, there is a vertex in B adjacent to x (i.e., if $\Sigma_1(B) = X$).

A simple chain $(x = x_0, x_1, \ldots, x_{n-1}, x_n = y)$ in a graph G with its ends at vertexes x and y is said to be nonchordal if vertexes $x_i$, $x_j$ are adjacent in G only when $|i - j| = 1$. For example, any shortest chain is nonchordal (the converse is not always true). A set $A \subset X$ is said to be m-convex [6] if it contains all the nonchordal chains with ends at vertexes

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of A. A metric segment with ends at vertexes x, y of G is defined as the set

$$\langle x, y\rangle = \{z \in X: d(x, z) + d(z, y) = d(x, y)\}.$$

LEMMA 1 [6]. Any ball $\Sigma_r(x)$ in a triangulated graph G is an m-convex set. In particular, the center

$$C(G) = \bigcap \{\Sigma_{r(G)}(x): x \in X\}$$

of G is m-convex.

The following two properties of m-convex sets follow innediately from the definition.

LEMMA 2. The metric projection $N_x(M)$ of any vertex x on an m-convex set M in a graph G generates a complete subgraph.

LEMMA 3. For any m-convex set M and vertexes $x \in X$, $y \in M$ in G,

$$\langle x, y\rangle \cap N_x(M) \neq \varnothing.$$

The next lemma follows from the definition of a triangulated graph (see also [9]).

LEMMA 4. If all vertexes of a complete subgraph $K = \{x_1, \ldots, x_n\}$ are equidistant from a vertex x in a triangulated graph G, then there exists a vertex $\overline{x} \in \bigcap_{i=1}^n \langle x, x_i\rangle$ adjacent to all vertexes of K.

LEMMA 5. If vertexes x, y, u, v of a triangulated graph G are such that $x \in \langle u, y\rangle$, $y \in \langle x, v\rangle$, and $d(x, y) = 1$, then

$$d(u, v) \geqslant d(u, x) + d(y, v). \qquad (1)$$

Equality holds in (1) only if there exists a vertex $w \in \langle u, y\rangle \cap \langle x, v\rangle$ adjacent to x and y.

Proof. Let $u_0$, $v_0$ denote the vertexes closest to x and y in the set $\langle u, v\rangle \cap \langle u, x\rangle$, $\langle u, v\rangle \cap \langle v, y\rangle$. Assume that $d(u, v) \leq d(u, x) + d(y, v)$. Then $u_0 \neq x$, $v_0 \neq y$. Let $\Gamma$ be the simple cycle formed by the edge (x, y) and certain shortest chains $\ell_1$, $\ell_2$, $\ell_3$ between the respective pairs $\{x, u_0\}$, $\{y, v_0\}$, $\{u_0, v_0\}$. In $\Gamma$, the edge (x, y) belongs to some cycle of length 3, say (x, y, w). Since $x \in \langle u_0, y\rangle$, $y \in \langle v_0, x\rangle$, it follows that $w \in \ell_3$. Then $d(u_0, w) = d(u_0, x)$, $d(v_0, w) = d(v_0, y)$. It follows from the choice of $u_0$, $v_0$ and from Lemma 4 that $(u_0, w)$, $(u_0, x)$, $(v_0, w)$, $(v_0, y)$ are edges of G. Thus,

$$d(u, v) = d(u, w) + d(w, y) = d(u, x) + d(y, v),$$

and the vertex $w \in \langle u, y\rangle \cap \langle v, x\rangle$ is adjacent to x and y.

LEMMA 6. Let G be a triangulated graph and r a natural number such that $d(G) < 2r$. Then any complete subgraph K of G contains a vertex x such that $\Sigma_r(x) = \Sigma_r(K)$.

Proof. We shall prove that any vertexes $y, z \in K$ satisfy one of the inclusion relations

$$\Sigma_r(y) \subseteq \Sigma_r(z), \quad \Sigma_r(z) \subseteq \Sigma_r(y).$$

Assume the contrary; then the vertexes $u \in \Sigma_r(y) \setminus \Sigma_r(z)$, $v \in \Sigma_r(z) \setminus \Sigma_r(y)$ satisfy the relations $y \in \langle u, z\rangle$, $z \in \langle y, v\rangle$, $d(u, y) = d(z, v) = r$. By Lemma 5,

$$d(u, v) \geqslant d(y, u) + d(z, v) = 2r,$$

contradicting the assumption $d(G) < 2r$.

Thus, the vertexes $x_1, \ldots, x_n$ of the set K may be assumed ordered in such a way that

$$\Sigma_r(x_1) \subseteq \Sigma_r(x_2) \subseteq \ldots \subseteq \Sigma_r(x_n).$$

Consequently,

$$\Sigma_r(K) = \bigcup_{i=1}^n \Sigma_r(x_i) = \Sigma_r(x_n).$$

It follows from Lemma 1 that C(G) is a connected isometric subgraph of G. Consequently, we can define the numbers r(C(G)), d(C(G)) as the radius and diameter of the graph C(G).

THEOREM 1. If G is a triangulated graph, then $d(C(G)) \leq 3$, $r(C(G)) \leq 2$.

*Proof*. We first show that $d(C(G)) \leq 3$. Assuming the contrary, let $x, y \in C(G)$, $d(x, y) = 4$ and $(x = v_1, v_2, v_3, v_4, v_5 = y)$ be some shortest chain between x and y. As $C(G)$ is m-convex, $v_3 \in C(G)$. Hence there exists a vertex v such that $d(v, v_3) = r(G)$. It follows from the m-convexity of the ball $\Sigma_{r-1}(G)$, where $r = r(G)$, that $d(v_i, v) = r$, $i = 1, \ldots, 5$. By Lemma 4, there exist vertexes $u_i \in \langle v_i, v\rangle \cap \langle v_{i+1}, v\rangle$, $i = 1, \ldots, 4$, adjacent to $v_i, v_{i+1}$. Since $u_1, u_2 \in \langle v_2, v\rangle$, $u_2, u_3 \in \langle v_3, v\rangle$, $u_3, u_4 \in \langle v_4, v\rangle$, the vertexes in these pairs are either adjacent or coincide. By Lemma 4, there exist vertexes $w_i \in \langle u_i, v\rangle \cap \langle u_{i+1}, v\rangle$, $i = 1, 2, 3$ adjacent to $u_i, u_{i+1}$. The only coincidences possible in the cycle $\Gamma = (v_2, u_1, w_1, w_2, w_3, u_4, v_4, v_3, v_2)$ are $w_1 = w_2$, $w_2 = w_3$; all other vertexes must be pairwise distinct. Since $d(w_i, v) = r - 2$, $i = 1, 2, 3$, it follows that $\Gamma$ can contain only diagonals $(v_3, u_1)$, $(v_3, u_4)$, $(v_2, u_4)$, $(v_4, u_1)$, $(u_1, u_4)$. But the existence of any such edge contradicts the fact that $d(v_1, v_5) = 4$. Hence $d(C(G)) \leq 3$.

Let K be a clique in $C(G))$. Since $d(C(G)) < 4$, Lemma 6 implies the existence of a vertex $x \in K$ such that $\Sigma_2(K) = \Sigma_2(x)$ in $C(G)$. If $r(C(G)) > 2$, then $d(x, \bar{x}) = 3$ for some vertex x in $C(G)$. Then $d(\bar{x}, K) = 3$, and for some vertex $y \in K$ we have $d(y, \bar{x}) = 3$. It follows from Lemma 4 that there exists a vertex $x_0 \in \cap \{\langle \bar{x}, y\rangle: y \in K\}$ adjacent to all vertexes of K. As the set $C(G)$ is m-convex, $x_0 \in \langle \bar{x}, x\rangle \subset C(G)$, contradicting the fact that K is a clique in $C(G)$.

THEOREM 2. If G is a triangulated graph, then $d(G) \geq 2r(G) - r(C(G))$. In particular $d(G) \geq 2r(G) - 2$.

*Proof*. Let x be an arbitrary vertex of the center $C(G)$. Choose a vertex $\bar{x}$ such that $d(x, \bar{x}) = r(G)$. Let $x_0$ denote any vertex of $N_{\bar{x}}(C(G)) \cap \langle x, \bar{x}\rangle$ (this set is nonempty by Lemma 3). Then $\langle x_0, \bar{x}\rangle \cap C(G) = \{x_0\}$. If $x_0 = \bar{x}$, then $r(G) = r(C(G))$ and there is nothing more to prove. Assume, therefore, that $x_0 \neq \bar{x}$. Put $K = \{z \in \langle x_0, \bar{x}\rangle: (x_0, z) \in U\}$. The set K generates a complete subgraph, and so it is finite. Without loss of generality, we may assume that $d(G) < 2r(G)$. Then K contains a vertex y for which $\Sigma_{r(G)}(y) = \Sigma_{r(G)}(K)$ (Lemma 6). Since $y \notin C(G)$, it follows that $d(y, \bar{y}) = r(G) + 1$ for some $\bar{y}$ in G. For this vertex $d(\bar{y}, K) = r(G) + 1$. By Lemma 5,

$$d(\bar{y}, \bar{x}) \geqslant d(\bar{y}, x_0) + d(y, \bar{x}) \geqslant r(G) + r(G - r(C(G)) - 1.$$

We claim that this inequality is always strict. Indeed, otherwise $d(\bar{y}, \bar{x}) = d(\bar{y}, \bar{x}_0) + d(\bar{x}, y)$, so that there would exist a vertex $w \in \langle y, \bar{y}\rangle \cap \langle x_0, \bar{x}\rangle$ adjacent to $x_0$ and y (Lemma 5). Then $w \in K$ and $d(\bar{y}, w) = d(x_0, \bar{y}) = r(G)$, contradicting the equality $d(\bar{y}, K) = r(G) + 1$. Thus $d(\bar{y}, \bar{x}) > 2r(G) - r(C(G)) - 1$, i.e., $d(G) \geq 2r(G) - r(C(G))$. This completes the proof.

Note that a weaker relationship between the radius and diameter of a triangulated graph was obtained in [7].

LEMMA 7. If G is a triangulated graph of radius $r = r(G)$, the center $C(G)$ is not a complete subgraph and $U_{r-1}(C(G)) = \emptyset$, then there exist vertexes $x_1, x_2 \in U_{r-1}(C(G))$, such that

$$N_{x_1}(C(G)) \cap N_{x_2}(C(G)) = \varnothing.$$

*Proof*. Let $x_1 \in U_{r-1}(C(G))$. By Lemma 5, any vertex in $C(G)) \setminus N_{x_1}(C(G))$ is adjacent to some vertex in $N_{x_1}(C(G))$. Consider a vertex

$$\bar{x} \in \cap \{\langle x, x_1\rangle: x \in N_{x_1}(C(G))\},$$

adjacent to all vertexes of $N_{x_1}(C(G))$. Since $\bar{x} \notin C(G)$, it follows that $d(\bar{x}, x_2) = r + 1$ for some vertex $x_2$ in G. Then for any $x \in N_{x_1}(C(G))$ we have $d(x, x_2) = r$. As $C(G)$ is not complete, $d(x_2, C(G)) < r$, i.e., $N_{x_1}(C(G)) \cap N_{x_2}(C(G)) = \emptyset$. Any vertex in $N_{x_2}(C(G))$ is adjacent to some vertex in $N_{x_1}(C(G))$, and so $d(x_2, C(G)) = r - 1$.

LEMMA 8. If $G \in CT$, then any clique in $C(G)$ is a dominant set in G.

*Proof*. Let $R = \{x_1, \ldots, x_n\}$ be some clique in $C(G)$, and assume that $d(x, R) = 2$ for some x in G. Then $r(G) = 2$ and $N_x(R) = R$ (Theorem 1). Put

$$K = \{z \in \cap_{i=1}^{n} \langle x, x_i\rangle: (x, z) \in U\}.$$

By Lemma 4, this set is not empty, hence it generates a complete subgraph. In K, choose a vertex z for which $\Sigma_2(K) = \Sigma_2(z)$ (this is possible by Lemma 6 and Theorem 1). Since R is a clique in $C(G)$, $z \notin C(G)$, i.e., $d(z, v) = 3$ for some vertex v. Thus $z \in \langle x, x_i\rangle$, $x_i \in \langle z, v\rangle$, i.e., $d(x, v) = 3$ (Lemma 5). Let $(x, u_1, u_2, v)$ be some shortest chain between x, v. Let w denote some vertex in $\cap_{i=1}^{n}\langle v, x_i\rangle$ adjacent to $x_1, \ldots, x_n$, v. Consider the cycle $\Gamma_i = (x, z, x_i, w, v, u_2, u_1, x)$. It is easy to see that $\Gamma_i$ contains diagonals $(x, u_1)$, $(z, u_1)$. Now, in the cycle $\Gamma_i' = (z, x_i, w, v, u_2, u, z)$, the edge $(z, u_1)$ belongs to some cycle of length 3. Since $d(z, v) = 3$, the third vertex of this triangle must be $x_i$. Thus $u_1$ is adjacent to all vertexes in R and to x, i.e., $u_1 \in K$. But $d(u_1, v) = 2$, contradicting the fact that $v \notin \Sigma_2(z) = \Sigma_2(K)$. Consequently, R is a dominant set in G.

LEMMA 9. If G is a triangulated graph with $r(C(G)) = r(G)$, then $G \in CT$.

Proof. We shall show that $G = C(G')$ for some triangulated graph $G' = (X', U')$. This is certainly true if G consists of a single vertex. We may therefore assume that $r(G) > 0$. Attach to every vertex x in C(G) a chain $(x, x', x'')$ of length 2, where x' is adjacent only to x and x'', and x'' is an end vertex. This gives a triangulated graph G'. The m-convexity of C(G) and the equality $r(G) = r(C(G))$ imply that $\max\{d(x, z): x \in C(G)\} = r(G)$ for any z in G and $d(G) \le r(G) + 2$. Consequently, the eccentricity in G' of any vertex of G is equal to $r(G) + 2$. It is readily seen that the eccentricities of x', x'' are respectively $r(G) + 3$, $r(G) + 4$. Thus, $x', x'' \notin C(G')$, i.e., $C(G') = G$.

LEMMA 10. If G is a triangulated graph with $r(C(G)) < r(G)$, then $G \in CT$ if and only if G contains nonintersecting dominant complete subgraphs $R_1$, $R_2$.

Proof. Let $R_1$, $R_2$ be nonintersecting dominant sets generating complete subgraphs. Then $\{d(x, R_1), d(x, R_2)\} = 1$ for any x in G. Define a triangulated graph $G' = (X', U')$ as follows: $X' = X \cup \{u_1, u_2, v_1, v_2\}$, where $v_1$ is adjacent to $u_2$ and all vertexes of $R_2$. The vertexes $u_1$ and $u_2$ will be end vertexes of G'. Then in G' it is true that $e(v_1) = e(v_2) = 4$, $e(u_1) = e(u_2) = 5$ and $e(x) = 3$ for any vertex x in G. Thus G is the center of G'.

Conversely, let $r(C(G)) < r(G)$ and $G = C(G')$ for some triangulated graph $G' = (X', U')$. Put $r' = r(G')$. It follows from Theorems 1, 2 that $r(C(G)) \le 1$, i.e., there exists a vertex $x_0 \in C(G)$ adjacent to all vertexes of C(G). Let $d(x_0, r) = r'$; we shall prove that $d(y, X) = r' - 1$, where X is the set of vertexes of G. Since G is not complete and $r(G) \le 2$, it follows from Lemmas 2 and 3 that $r' - 2 \le d(y, X) < r'$. If $d(y, X) = r' - 2$, then by Lemma 3, $X \subseteq \Sigma_2(N_y(X))$. By Lemma 6, $N_y(X)$ contains a vertex x for which $X \subseteq \Sigma_2(x)$. Since $d(x_0, y) = r'$, we have $d(x_0, N_y(X)) = 2$, i.e., $d(x_0, x) \ge 2$ and $r(G) = 2$. This implies that $x \in C(G)$, contrary to the assumption that $x_0$ is adjacent to all vertexes of C(G).

Thus, $d(y, X) = r' - 1$, i.e., $U_{r'-1}(X) \neq \emptyset$. By Lemma 7, there exist vertexes $x_1, x_2 \in U_{r'-1}(X)$ such that $N_{x_1}(X) \cap N_{x_2}(X) = \emptyset$. It follows from Lemma 3 and the relationships $C(G') = G$, $d(x_1, X) = d(x_2, X) = r' - 1$ that the sets $N_{x_1}(X)$, $N_{x_2}(X)$ are dominant in G.

THEOREM 3. A triangulated graph G is a member of the family CT if and only if it contains a finite system $S = \{R_1, \ldots, R_n\}$ of dominant complete subgraphs such that $\cap_{i=1}^{n} R_i = \emptyset$ and $R_j \not\subseteq \cup_{i=1, i\neq j}^{n} R_i$ for any $R_j$. If $r(G) = r(C(G))$, then $G \in CT$ (i.e., such a system always exists). If $r(C(G)) < r(G)$, then $G \in CT$ only if G contains two nonintersecting dominant complete subgraphs $R_1$, $R_2$. If G is of finite density $\varphi$, the system S may be assumed to contain at most $\varphi$ complete subgraphs.

Proof. In view of Lemmas 9 and 10, it will suffice to show that if $r(G) = r(C(G))$ then G contains a system S of complete subgraphs satisfying the conditions of the theorem. If $r(G) = 1$, then C(G) contains at least two distinct vertexes $x_1$, $x_2$. Then $S = \{x_1, x_2\}$. Suppose now that $r(G) = 2$. Then for any x in C(G) there exists a clique R(X) in C(G) that does not contain x; we need only define R(x) as any clique in C(G) containing a vertex $\overline{x} \in C(G)$ with $d(x, \overline{x}) = 2$.

For an arbitrary fixed vertex $x_0 \in C(G)$, let $R(x_0) = \{x_1, \ldots, x_k\}$. By Lemma 8, the cliques $R(x_0), R(x_1), \ldots, R(x_k)$ are dominant in G. Moreover, they satisfy the condition $\cap_{i=0}^{k} R(x_i) = \emptyset$. Now select from the family $R(x_0), R(x_1), \ldots, R(x_k)$ a system $S = \{R_1, \ldots, R_n\}$, where $R_1 = R(x_{i_1}), \ldots, R_n = R(x_{i_n})$, consisting of the minimal number of elements such that $\cap_{i=1}^{n-1} R_i \neq \emptyset$. We claim that no subgraph S is covered by the others. Assuming the contrary, suppose that $R_n \subset \cup_{i=1}^{n-1} R_i$. By the choice of S, $\cap_{i=1}^{n-1} R_i \neq \emptyset$. If $z \in \cap_{i=1}^{n-1} R_i$, then $z \notin R_n$. Since z is adjacent to all vertexes of $R_n$, this contradicts the fact that $R_n$ is a clique in C(G). If G is of finite density $\varphi$, then the system S just constructed is such that $n \le \varphi$. This follows from the fact that the Helly number for m-convexity in G is $\varphi$ [11].

Remark 1. For any integer $\varphi \geq 2$, there exists a graph $G \in CT$ of density $\varphi$ in which the smallest system S satisfying the conditions of Theorem 3 is of cardinality $\varphi$.

Remark 2. This result can be used as a general scheme for characterizing the centers of different classes of triangulated graphs. For example, using Theorems 1 and 3 one easily enumerates the centers of trees, 2-trees, and maximal outerplanar graphs.

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