Ample sets in Cartesian products

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The goal of the talk is to give the definition and an overview of characterizations and properties of ample sets in Cartesian products.

Ample sets in hypercubes

Let $S \subseteq \{0,1\}^E$ (or $S \subseteq \{-1,+1\}^E$).

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- $Y \subseteq E$ (or $\{0,1\}^Y$) is *shattered* by S if $S_Y = \{0,1\}^Y$.
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- $Y \subseteq E$ (or $\{0,1\}^Y$) is *shattered* by S if $S_Y = \{0,1\}^Y$.
- $\{0,1\}^Y$ is *strongly shattered* by S if S contains a copy of $\{0,1\}^Y$.
- $S \subseteq \{0,1\}^E$ is called *ample* if whenever a cube $\{0,1\}^Y$ is shattered by S, the cube $\{0,1\}^Y$ is strongly shattered by S.

The sandwich lemma

For a set $S \subseteq \{0,1\}^E$, let

- $\overline{X}(S) = \{Y \subseteq E : \{0,1\}^Y$ is shattered by $S\}$;
- $\underline{X}(S) = \{Y \subseteq E : \{0,1\}^Y$ is strongly shattered by $S\}$.

$\overline{X}(S)$ and $\underline{X}(S)$ are simplicial complexes and $\underline{X}(S) \subseteq \overline{X}(S)$. The largest dimension of a simplex of $\overline{X}(S)$ is the *Vapnik-Chervonenkis dimension* of S.

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Sandwich lemma (Dress, 1995) For any set $S \subseteq \{0,1\}^E$,

$$|\underline{X}(S)| \leq |S| \leq |\overline{X}(S)|.$$

The sandwich lemma implies the well-known Sauer-Shelah-Perles lemma $|S| \leq \binom{m}{\leq d}$, where $m = |E|$ and $d = \text{VC-dim}(S)$.

Initially, Dress called a set S ample if $|S| = |\overline{X}(S)|$.

Characterization of ample sets

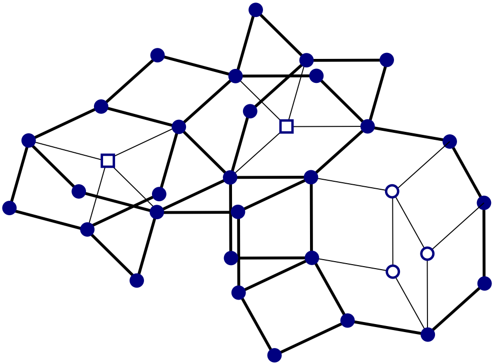
Theorem (Bandelt, C. Dress, Koolen, 2006)

For a set $S \subseteq \{0,1\}^E$, the following conditions are equivalent:

1. S *is ample;*
2. $|\underline{X}(S)| = |S|$;
3. $\underline{X}(S) = \overline{X}(S)$ (*shattering→strong shattering principle*);
4. $S^* = \{0,1\}^E \setminus S$ *is ample;*
5. *for any* $Y \subseteq E$, *either* $\{0,1\}^Y$ *is strongly shattered by* S *or* $\{0,1\}^{E \setminus Y}$ *is strongly shattered by* S^* (*lopsidedness of J. Lawrence, 1983);*
6. S *is commutative;*
7. S *is superisometric/superconnected.*

Brief history of amplenness: *Lopsided sets* first discovered by Lawrence (1983), rediscovered by Wiedemann in 1986 as *simple sets* and by Bollobas and Ratcliffe in 1995 as *extremal sets*.

A gallery of ample sets



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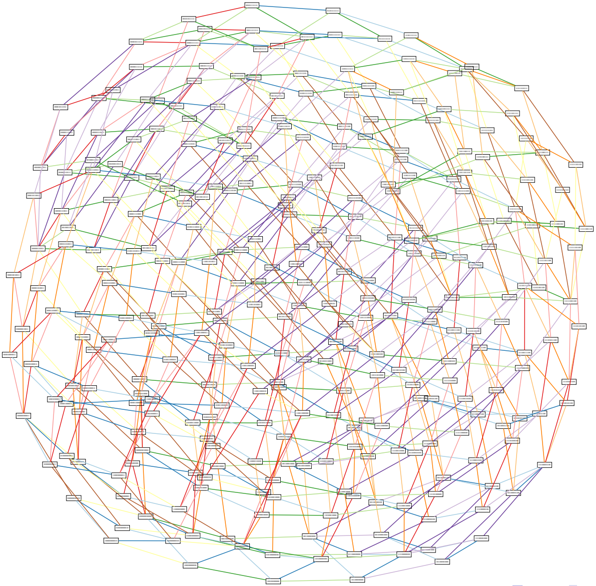

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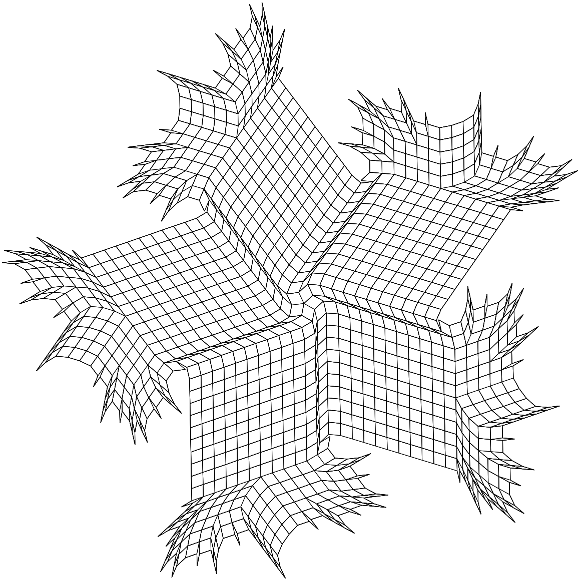
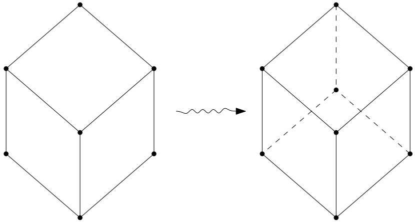


Examples of ample sets

1. (Lawrence) intersection patterns of convex sets with the orthants of $\mathbb{R}^E$;
2. (Bandelt, C., Dress, Koolen) median structures, i.e., 1-skeleta of CAT(0) cube complexes;



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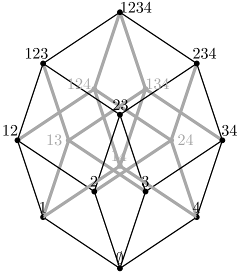
1. (Bandelt, C., Dress, Koolen) convex geometries of Edelman and Jamison (alias, antimatroids of Korte and Lovasz) and their bouquets (conditional antimatroids);



2. (floklore) maximum set systems of VC-dimension d (having exactly $\binom{m}{\leq d}$ sets;
3. (Maat) sets of strategies in (certain) mean payoff games in graphs;
4. (Bandelt, C. Dress, Koolen, unpublished) rhombic tilings, in particular Penrose tilings.

From hypercubes to Hamming graphs

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The setting:

- $S \subseteq U_1 \times \ldots \times U_m$, where $U_1, \ldots, U_m$ are finite sets;
- $U_1 = \ldots = U_m = Y, X = \{1, \ldots, m\}$, and $S \subseteq Y^X$ is the setting of multiclass learning;

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- subproduct $V = V_{i_1} \times \ldots \times V_{i_k}$, where $\emptyset \neq V_{i_j} \subseteq U_{i_j}$, $j = 1, \ldots, k$;
- support $\mathrm{supp}(V) = \{i_1, \ldots, i_k\}$;

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- support $\mathrm{supp}(V) = \{i_1, \ldots, i_k\}$;
- full-dimensional subproduct (box) $\mathrm{supp}(V) = m$, i.e., $V = V_1 \times \ldots \times V_m$;

The setting, bis

- Hamming graph $H(U) = (U, E)$: $u, v \in U$ are adjacent in $H(U)$ iff u and v differ in a single coordinate. $H(U)$ is a product of cliques.
- Hamming distance $d(u, v)$ is the number of coordinates in which $u, v \in U$ differ;

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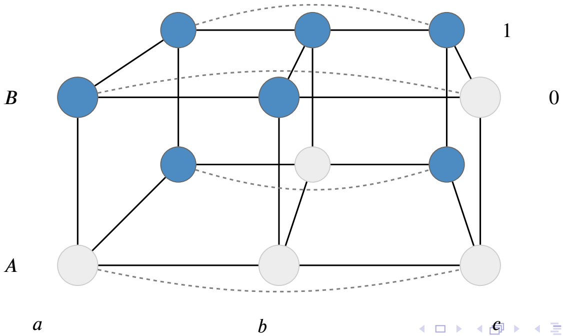
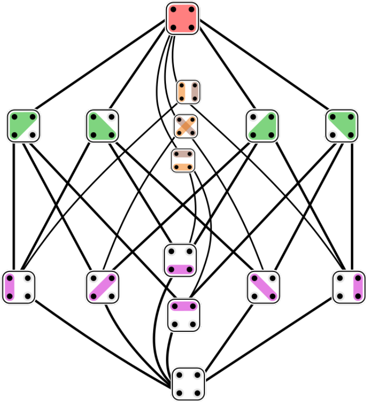
- Hamming distance $d(u, v)$ is the number of coordinates in which $u, v \in U$ differ;
- For $S \subseteq U$, $H(S)$ is the subgraph of $H(U)$ induced by S;
- S is called isometric (connected) if $H(S)$ is an isometric subgraph of $H(U)$ (connected subgraph of $H(U)$).



Lattice of partitions

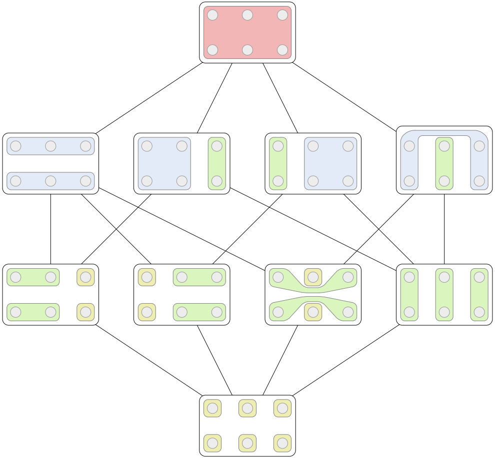
$\mathrm{Part}(Z) = \{\alpha = \{P_1, \dots, P_k\} : Z = P_1 \sqcup P_2 \sqcup \dots \sqcup P_k\}$ with $\alpha \leq \beta$ if α is a refinement of β; meet $\alpha \wedge \beta$ and join $\alpha \vee \beta$.



$\mathrm{Part}(\{1, 2, 3, 4\})$ (source Wikipedia, author Tilman Piesk)

Lattice of generalized partitions

GPart(U)= $\mathrm{Part}(U_1) \times \ldots \times \mathrm{Part}(U_m)$ with $\Lambda \leq \Upsilon$ (where $\Lambda = (\alpha_1, \ldots, \alpha_m), \Upsilon = (\beta_1, \ldots, \beta_m)$) iff $\alpha_i \leq_i \beta_i, i = 1, \ldots, m$.



Box-partitions

Definition (Box-partitions)

A box is a full-dimensional subproduct $B = B_1 \times \ldots \times B_m$ of U. A box-partition of U is a partition $\mathcal{B}$ of U into boxes. $\mathrm{BPart}(U)$ denotes the set of all box-partitions of U.

To each $\Lambda = (\alpha_1, \ldots, \alpha_m) \in \mathrm{GPart}(U)$ corresponds a box-partition $\mathcal{B}(\Lambda, U)$: if $\alpha_i = \{P^i_1, \ldots, P^i_{\ell_i}\}$, then $\mathcal{B}(\Lambda, U)$ consists of $P^1_{j_1} \times \ldots \times P^m_{j_m}$.

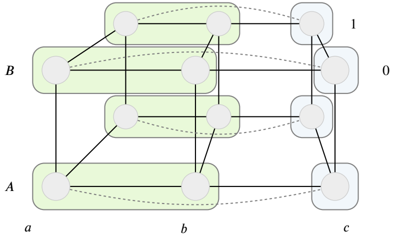


Figure: The box partition of the generalized partition $\Lambda = (\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_1 = \{\{a, b\}, \{c\}\}, \alpha_2 = \{\{A\}, \{B\}\}$, and $\alpha_3 = \{\{0\}, \{1\}\}$.

Minor-subproducts

Definition (Minor-subproducts)

A product $M = M(\Lambda, U) = M_1 \times \ldots \times M_m$ is called a minor-subproduct defined by $\Lambda \in \mathrm{GPart}(U)$ if $M_i = \{w^i_1, \ldots, w^i_{\ell_i}\}$, where w^i_j is a new element to which the block P^i_j of α_i is contracted. $\mathrm{MProd}(U)$ denotes the set of all minor-subproducts $M = M(\Lambda, U)$ with $\Lambda \in \mathrm{GPart}(U)$.

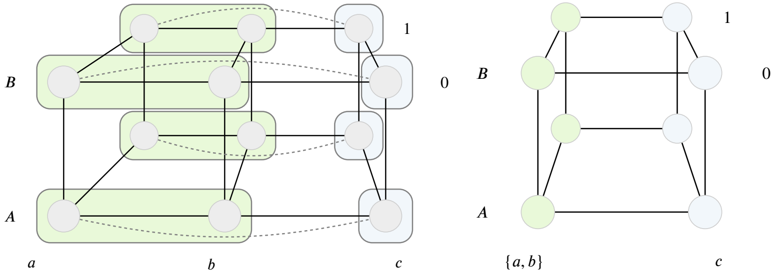


Figure: A box partition (left) and its related minor-subproduct (right).

Shattering and strong-shattering

Definition (Shattering)

A minor-subproduct $M = M(\Lambda, U) \in \mathrm{MProd}(U)$ is shattered by $S \subseteq U$ if S intersects each box of the box-partition $\mathcal{B}(\Lambda, U)$.

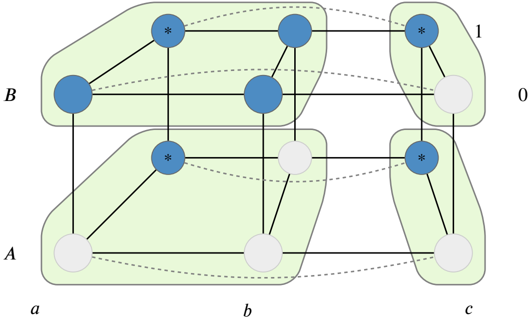


Figure: The set S shatters the minor-subproduct $M = M(\Lambda, U)$, where $\Lambda = (\{\{a, b\}, \{c\}\}, \{A\}, \{B\}, \{\{0, 1\}\})$.

Shattering and strong-shattering

Definition (Copies of minor-subproducts and strong shattering)

Let $M = M(\Lambda, U) = M_1 \times \ldots \times M_m \in \mathrm{MProd}(U)$ with $\Lambda = (\alpha_1, \ldots, \alpha_m)$ and $\alpha_i = \{P_1^i, \ldots, P_{\ell_i}^i\}$. For each $i = 1, \ldots, m$ and each $w_j^i \in M_i$, pick $u_j^i \in P_j^i, j = 1, \ldots, \ell_i$. Set $W_i = \{u_{j_1}^i, \ldots, u_{\ell_i}^i\}$ and consider the subproduct $W = W_1 \times \ldots \times W_m$ of U. Then W is a copy of M. A minor-subproduct $M \in \mathrm{MProd}(U)$ is strongly-shattered by $S \subseteq U$ if S contains a copy of M.

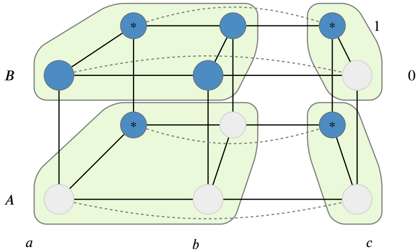


Figure: S strongly shatters M: S contains the copy $\{a, c\} \times \{A, B\} \times \{1\}$ of M.

Ampleness

Definition (Ample sets of Cartesian products)

Let $S \subseteq U = U_1 \times \ldots \times U_m$ and $\mathcal{M} \subseteq \mathrm{MProd}(U)$. Then S is called:

- $\mathcal{M}$-*ample* if each $M \in \mathcal{M}$ shattered by S is strongly shattered by S;
- *ample* if S is $\mathrm{MProd}(U)$-ample.

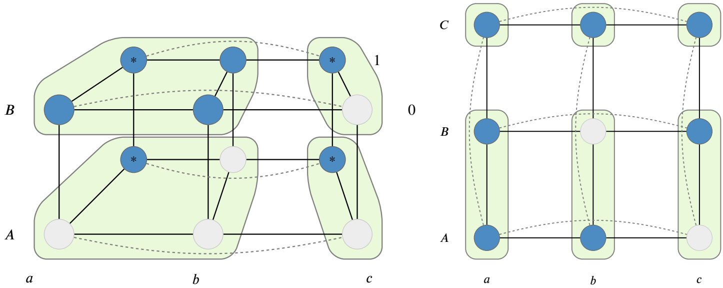


Figure: An ample set (left) and a non-ample set (right). The minor-subproduct of the box-partition from the right picture does not have a copy in S.

Ampleness versus $\mathrm{M_{ext}Prod}(U)$-ampleness

An extended minor-subproduct of U is a minor-subproduct $M = M(\Lambda, U)$ with $\Lambda = (\alpha_1, \ldots, \alpha_m)$ s.t. each partition α_i contains at most one non-trivial block. $\mathrm{M_{ext}Prod}(U)$ denotes the set of all extended minor-subproducts.

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Theorem 1: For a set $S \subseteq U = U_1 \times \ldots \times U_m$, the following conditions are equivalent:

(1) S is ample;

(2) S is $\mathrm{M_{ext}Prod}(U)$-ample;

(3) $S \cap V$ is ample in V for each full-dimensional subproduct V of U.

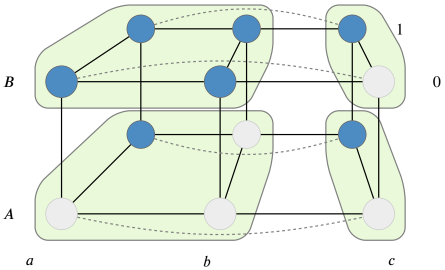
Ample sets are isometric.

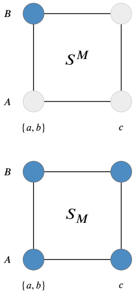
Projection and strong projection

Definition

Let $S \subseteq U = U_1 \times \ldots \times U_m$ and $M = M(\Lambda, U) \in \mathrm{MProd}(U)$.

- box of $t \in M$: box $B(t)$ of the box-partition $\mathcal{B}(\Lambda, U)$ contracted to t;
- projection $S_M = \{t \in M : B(t) \cap S \neq \varnothing\}$;
- strong projection $S^M = \{t \in M : B(t) \subseteq S\}$.





First main characterization

Definition (Projections and strong-projections, bis)

Let $M, M' \in \mathrm{MProd}(U)$ and $S \subseteq M'$. Then
$S_M = \{t \in M \vee M' : B_{M'}(t) \cap S \neq \varnothing\}$ is the projection of S on M and
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Theorem 2: For a set $S \subseteq U$, the following conditions are equivalent:

(1) S is ample;

(2) S^M is isometric for all $M \in \mathrm{MProd}(U)$ (superisometricity);

(3) any pair of parallel boxes of S is connected in S by a geodesic gallery (box-superisometricity);

(4) S^M is connected for all $M \in \mathrm{MProd}(U)$ (superconnectivity);

(5) $(S^M)_{M'} = (S_{M'})^M$ for all $M, M' \in \mathrm{MProd}(U)$ with disjoint supports (commutativity);

(6) $S^* = U \setminus S$ is ample (ampleness of complements).

Second main characterization (economical)

Definition (Elementary minor-subproducts and intervals)

Elementary minor-subproducts are the atoms of $\mathrm{GPart}(U)$. They are the generalized partitions $e = (\alpha_1, \dots, \alpha_m)$, where all α_i are trivial partitions, except one α_j, which consists of singletons and one block $\{a, b\}$ of size 2. For $u, v \in U$, the interval is $[u, v] = \{z \in U : d(u, v) = d(u, z) + d(z, u)\}$.

Remark: We identify e with the block $\{a, b\}$, and denote the projection and the strong-projection by S_e and S^e: S_e is obtained by contracting all (a, b)-edges of $H(S)$ and S^e is an analog of hyperplane.

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Theorem 3: For a set $S \subseteq U$, the following conditions are equivalent:

(1) S is ample;

(2) S is isometric and S^e and S_e are ample for some $e \in \mathrm{M_{el}Prod}(U)$;

(3) S is connected and S^e is ample for every $e \in \mathrm{M_{el}Prod}(U)$;

(4) $S \cap [u, v]$ is ample in $[u, v]$ for all $u, v \in S$.

Consequences

Corollary: For a set $S \subseteq U$, the following hold:

(1) The property of being ample is preserved by isometric embeddings;

(2) Recognizing if $S \subseteq U$ is ample can be done in time polynomial in the size of S.

(3) ampleness can be characterized by hyperplanes of S, which correspond to middles of (a, b)-edges in S.

Box and prism complexes of ample sets

Definition (Box and prism complexes)

(1) box complex of S: $\mathrm{Box}(U)$ denote the set of boxes of U included in S;

(2) prism complex of S: $||\mathrm{Box}(S)||$ is the geometric realization of $\mathrm{Box}(S)$ and is obtained by replacing each box of $\mathrm{Box}(S)$ by a prism (Cartesian product of simplices);

(3) f-vector of S: $f(S) = (f_0(S), f_1(S), \ldots)$, where $f_i(S)$ denotes the number of prisms of $||\mathrm{Box}(S)||$ of dimension i.

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Theorem 4: If $S \subseteq U$ is ample, then:

(1) the f-vector and ampleness are preserved by push downs;

(2) a complete set of push downs on S is an ample set S_* whose prism complex is a bouquet of prisms;

(3) the Euler characteristics $\chi(S) = f_0(S) - f_1(S) + f_2(S) - \ldots$ of S is 1;

(4) the prism complex $||\mathrm{Box}(S)||$ is contractible.

Sectors, cosectors, boundaries, etc

Let $S \subseteq U$, U_i be a factor of U, and $a \in U_i$.

(1) sector $S(a)$: all $u = (u_1, \dots, u_m) \in S$ such that $u_i = a$;

(2) boundary $\partial S(a)$: all $u \in S(a)$ adjacent to $v \in S$ with $v_i \neq a$.

(3) cosector: $S^c(a) = S \setminus S(a)$.

(4) neighborhood $N(S(a))$: all $v \in S^c(a)$ having a neighbor in $S(a)$;

(5) extended sector $S^+(a)$= $S(a) \cup N(S(a))$;

(6) carrier of S: the union $\bigcup_{a \in U_i} \partial S(a)$ of all boundaries of sectors.

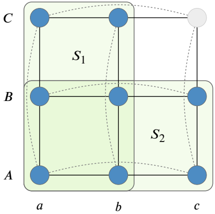


Figure: The sector $S(C)$ for $U_2 = \{A, B, C\}$ has size 2, its neighborhood has size 4. The cosector $S^c(C)$ is S_2 and the extended sector $S^+(C)$ is S_1.

The decomposition theorem

Theorem 5: If $S \subseteq U$ is ample, then for any U_i, all sectors, cosectors, boundaries, neighborhoods, extended sectors, and the carrier are ample.

Each ample set S is obtained from its maximal boxes by amalgams, i.e., by gluing two ample sets S_1, S_2 along a proper ample subset S_0 of both, which is also a $(S_1 \setminus S_0, S_2 \setminus S_0)$-separator.

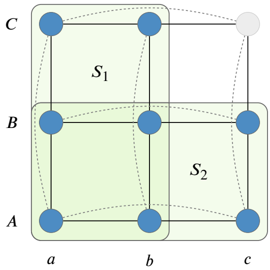


Figure: The set $S \subseteq \{a, b, c\} \times \{A, B, C\}$ is an amalgam of S_1 and S_2, where $S_1 = S^+(C)$ (the extended sector of C) and $S_2 = S^c(C)$ (the cosector of C).

Examples of ample sets

Examples of binary ample sets:

- intersection patterns of convex sets with the orthants of $\mathbb{R}^E$;
- median structures, i.e., 1-skeleta of CAT(0) cube complexes;
- convex geometries (antimatroids) and their bouquets (conditional antimatroids);
- maximum set systems of VC-dimension d;
- sets of strategies in mean payoff games in graphs (special case);
- rhombic tilings, in particular Penrose tilings.

New examples:

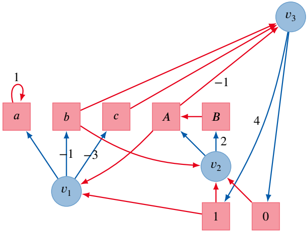
- sets of strategies in mean payoff games in graphs;
- prism-like polyhedra (vertex sets of polyhedra whose face lattice is a sublattice of the face lattice of a prism);
- quasi-median graphs (retracts of Hamming graphs).

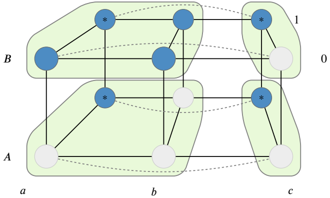
Sets of strategies in mean payoff games

Two players Maximizer and Minimizer on a weighted directed graph $G = (V_{\max} \cup V_{\min}, E, w)$, where $w : E \to \mathbb{Z}$ and $V_{\max} = \{v_1, v_2, \dots, v_m\}$.

- A pebble is placed on v_0.
- At step i, the player at v_i chooses an outgoing arc to send the pebble.
- The outcome is the long term average of weights of encountered arcs.
- Maximizer uses a positional strategy.

Let $N^+(v_i)$ be the successors of $v_i \in V_{\max}$. Any positional strategy is a tuple $\sigma \in N^+(v_1) \times \ldots \times N^+(v_m)$. Let Σ be the set of strategies σ that guarantee a nonnegative outcome for every starting vertex.





Merci!