On The Cost Of Simulating A Parallel Boolean Automata Networks By A Sequential One

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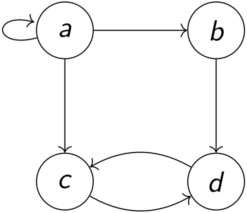
Boolean automata networks (BAN)

BAN $\simeq$ Dynamic systems with n Boolean variable

- Conventional Representative Models for Complex Systems :
  - Neural networks [McCulloch & Pitts 1943]
  - Gene networks [Kauffman 1969, Thomas 1973]
  - Social Networks [Taramasco & Demongeot 2011]
  - Epidemic Diffusion Networks [Demongeot 2013]
  - etc.
- Calculation models :
  - Boolean cellular automata with Bounded space
  - Memoryless computation, Network coding [Gadouleau 2011,2012]

Boolean automata networks (BAN)



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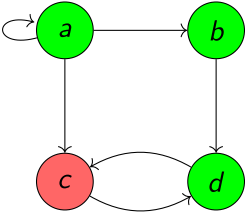

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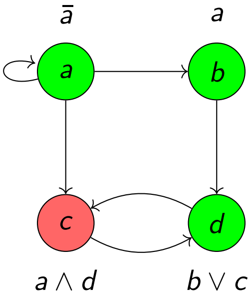

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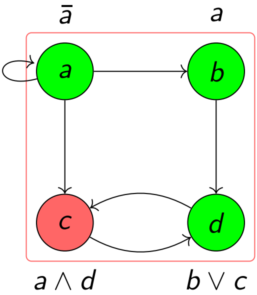

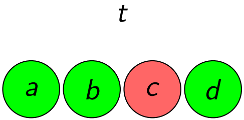

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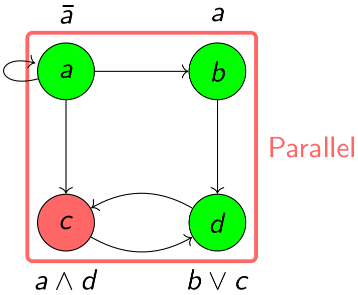

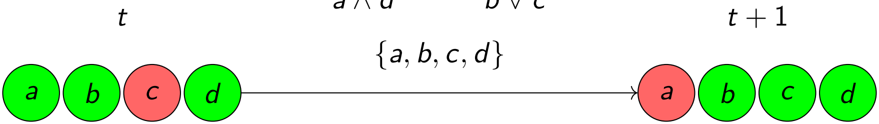

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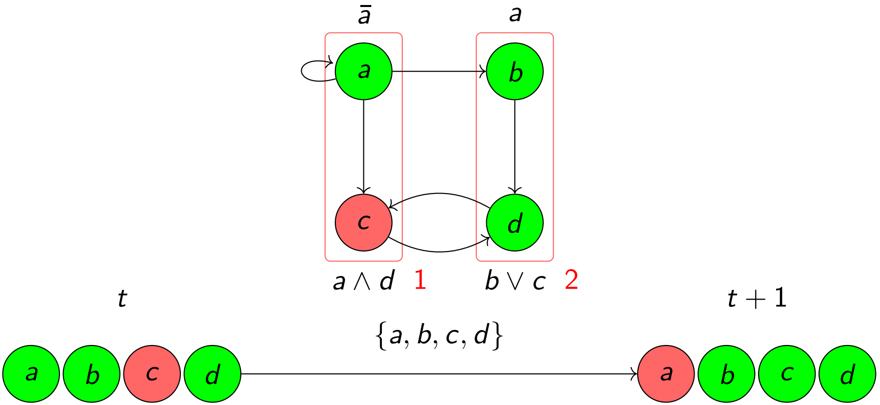

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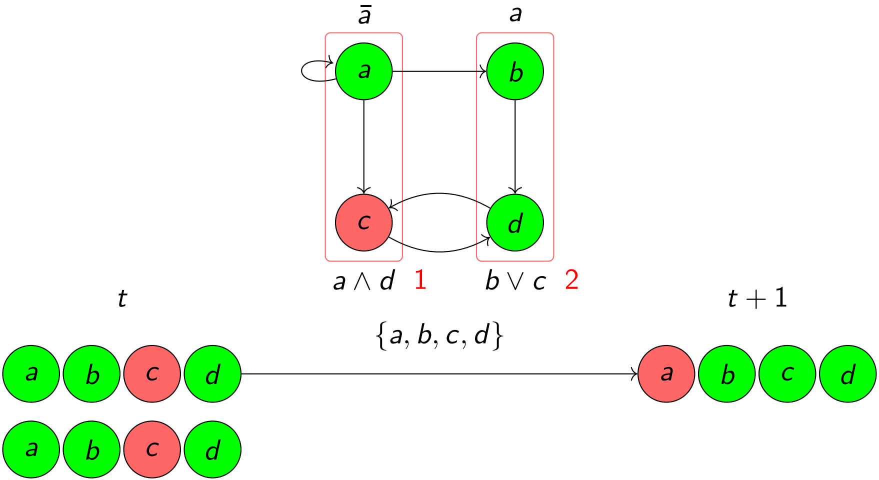

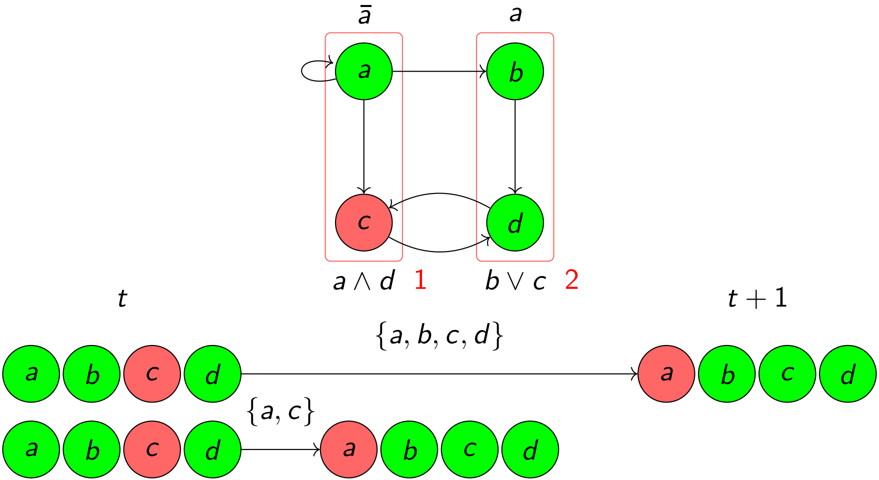
$\bar{a}$
a
a
b
c
d
$a \wedge d$ 1
$b \vee c$ 2
t
$t+1$
$\{a, b, c, d\}$
$\{a, c\}$
a b c d
a b c d
a b c d
a b c d

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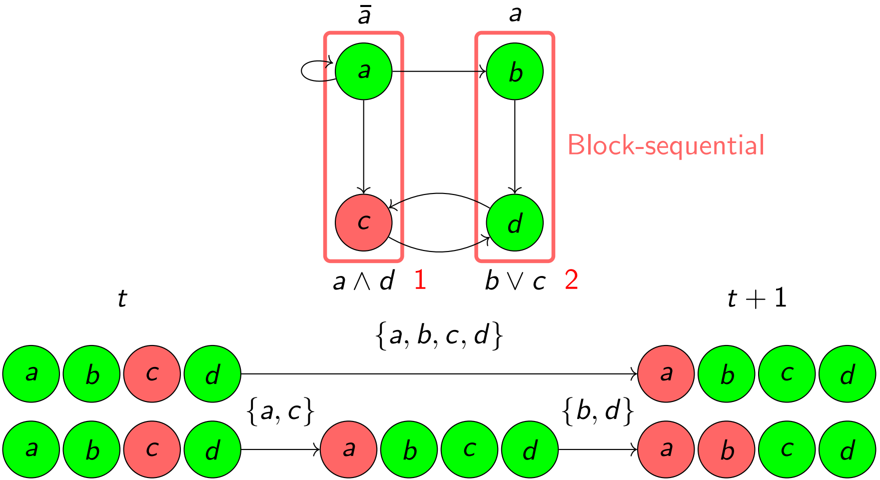
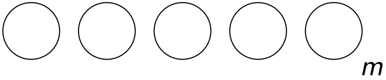


Intrinsic simulation

(N', S')



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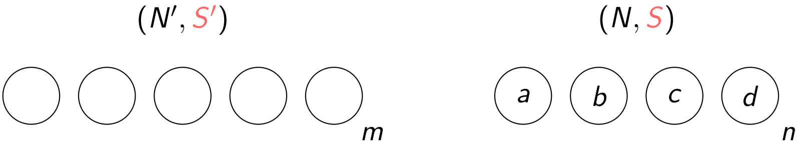

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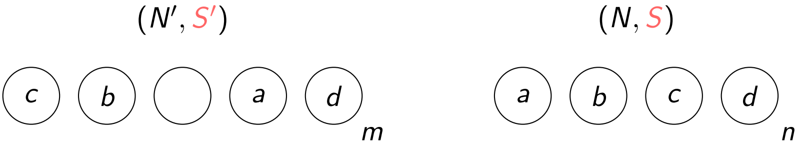

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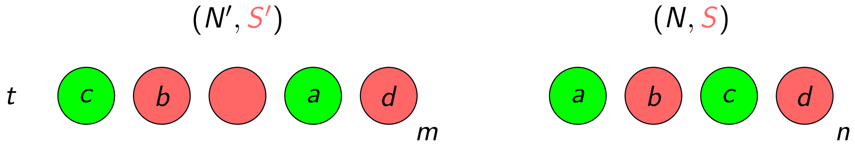

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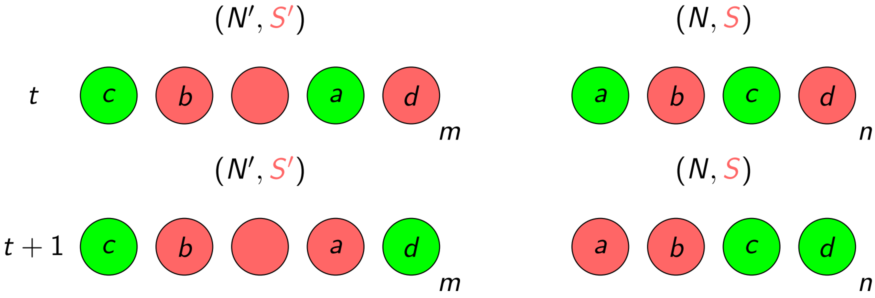

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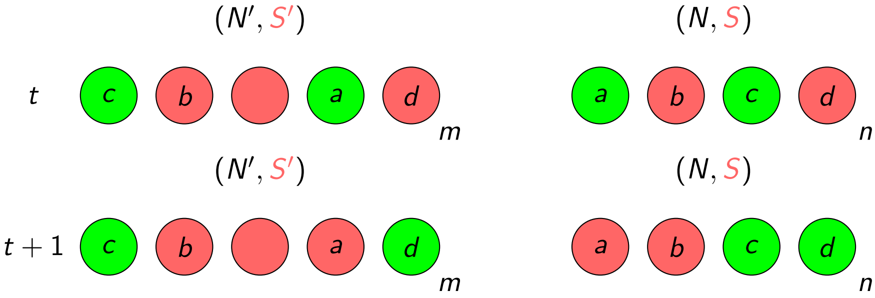
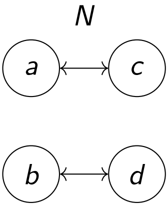


$$(N', S') \triangleright (N, S)$$

The Cost of simulation κ



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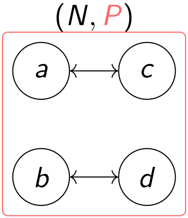

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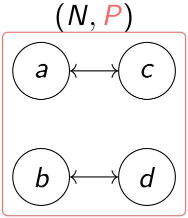

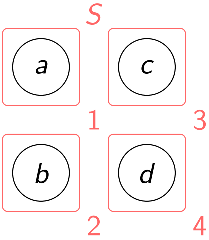

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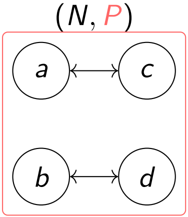
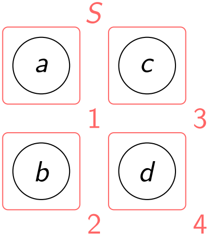
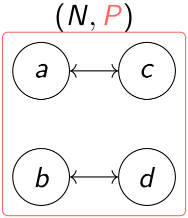
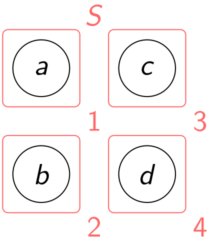
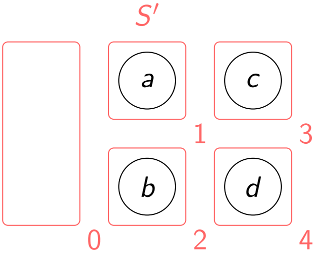




$\kappa(N, S)$ is the minimum additional size of (N', S') such that :







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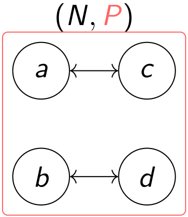

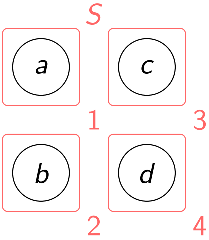

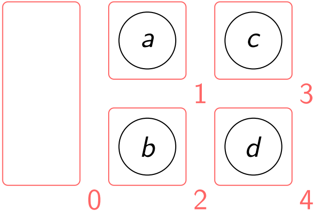

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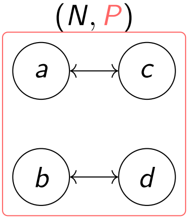

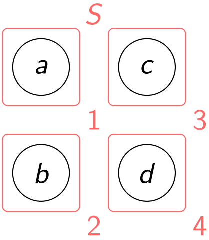

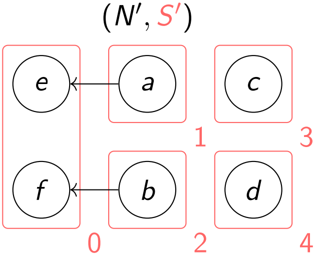

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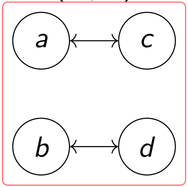
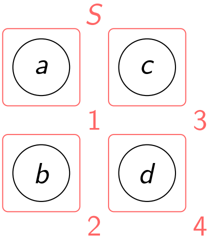
(N, P)





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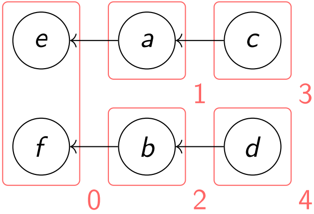

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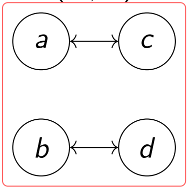

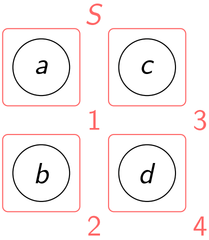

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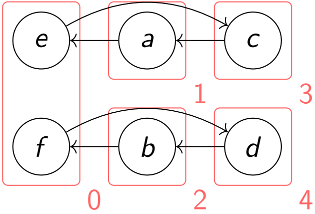

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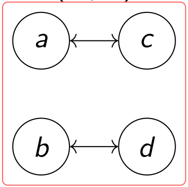

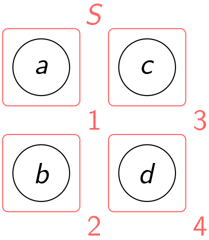

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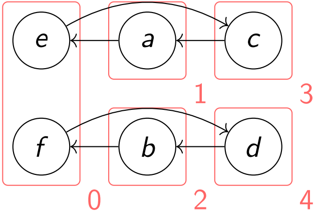

$\kappa(N, S) \leq 2$

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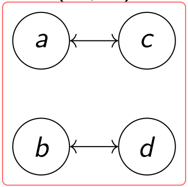
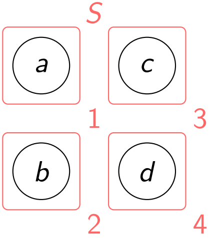
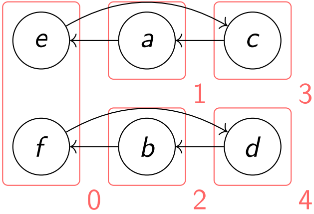
The Cost of simulation κ

(N, P)





(N', S')



$\kappa(N, S) = 2$

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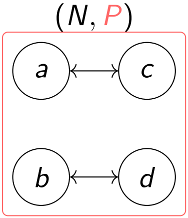

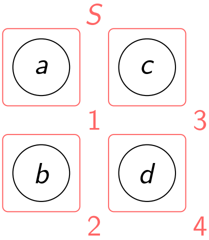

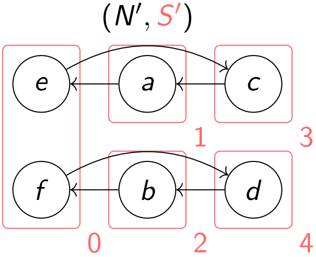

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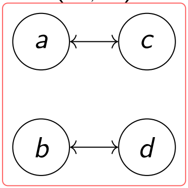
κ_n is the max $\kappa(N, S)$ with $|N| = n$

(N, P)



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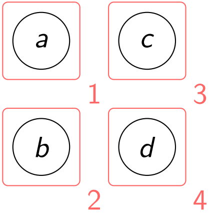
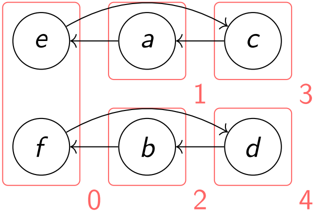


(N', S')



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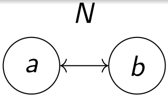
$\kappa(N, S) = 2$ and thus $\kappa_4 \geq 2$

Confusion graph $G_{N,S}$



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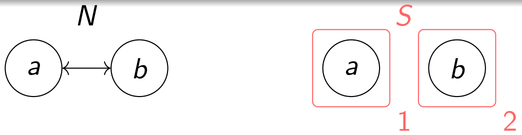

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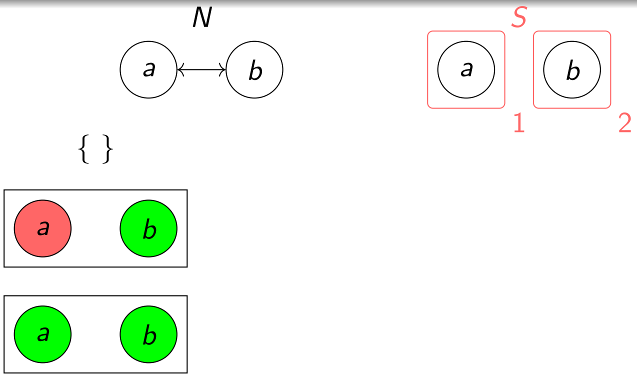

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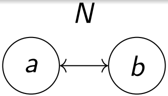

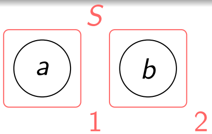

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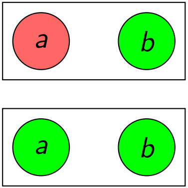

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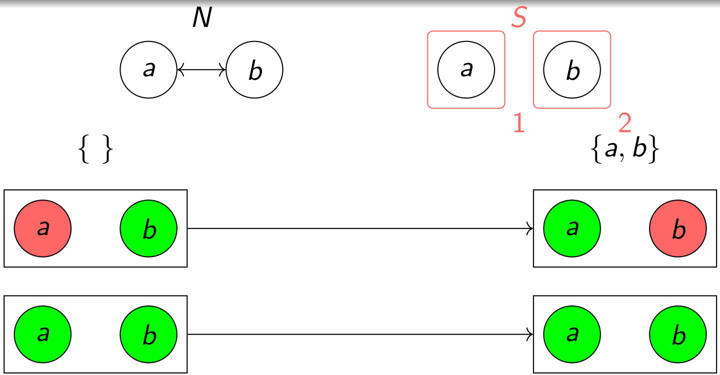

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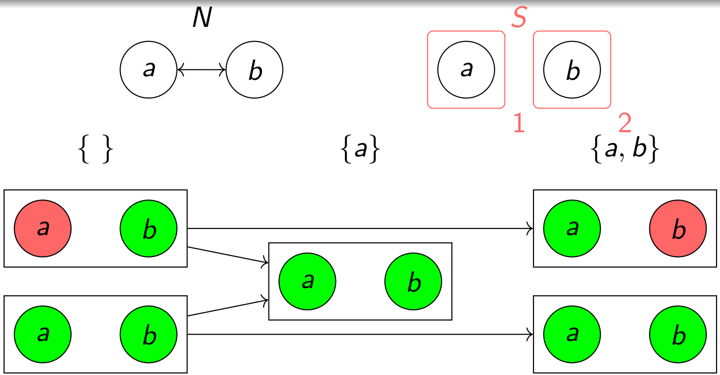

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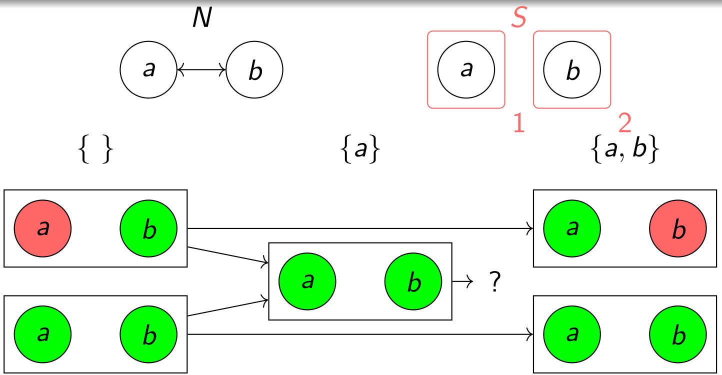

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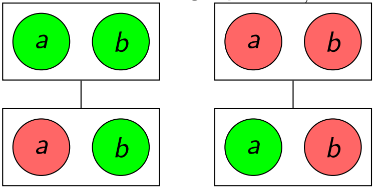

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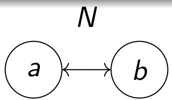

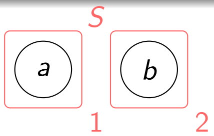

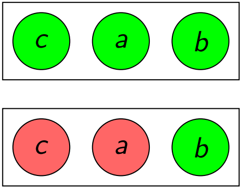

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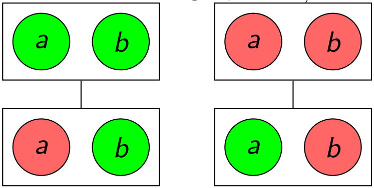
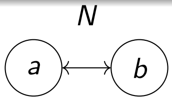
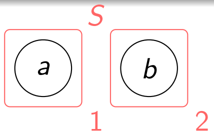
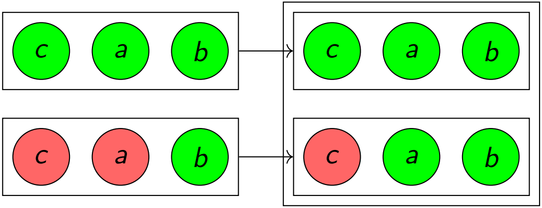


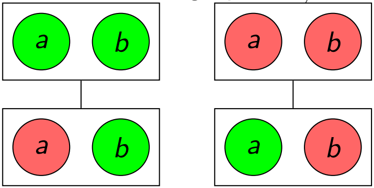
Confusion graph $G_{N,S}$



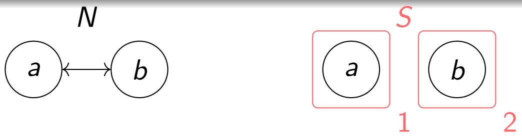


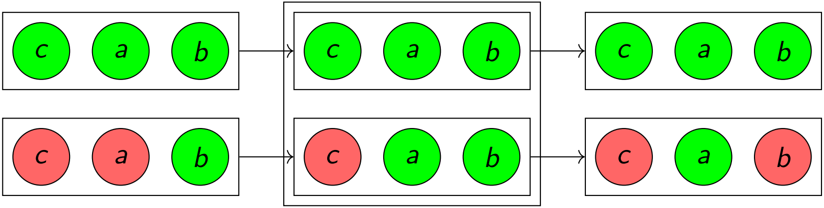


Confusion graph $G_{N,S}$:

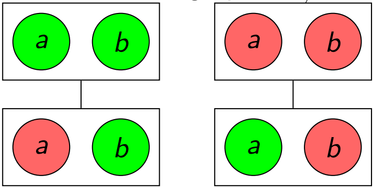


Confusion graph $G_{N,S}$

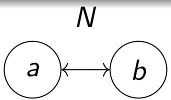


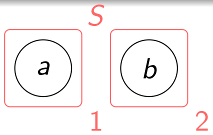


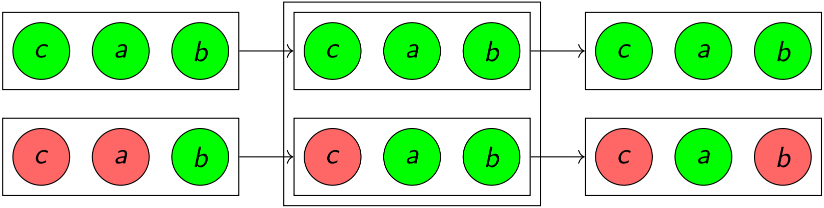
Confusion graph $G_{N,S}$:



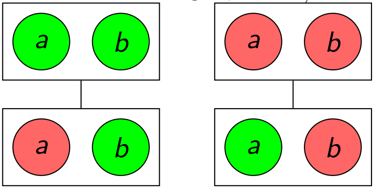
Confusion graph $G_{N,S}$



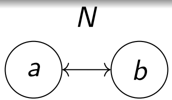


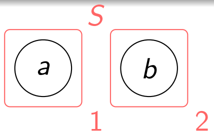


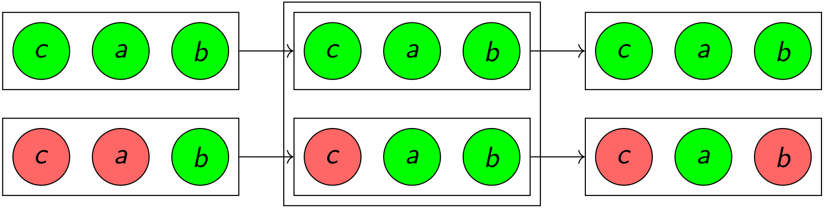
Confusion graph $G_{N,S}$:



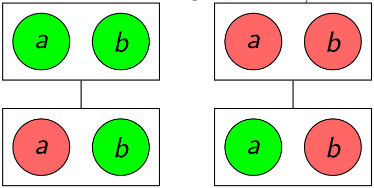
Lemma. $\kappa_{N,S} \geq \lceil \log_2(\chi(G_{N,S})) \rceil$





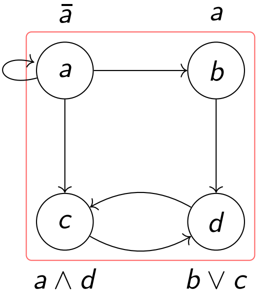


Confusion graph $G_{N,S}$:

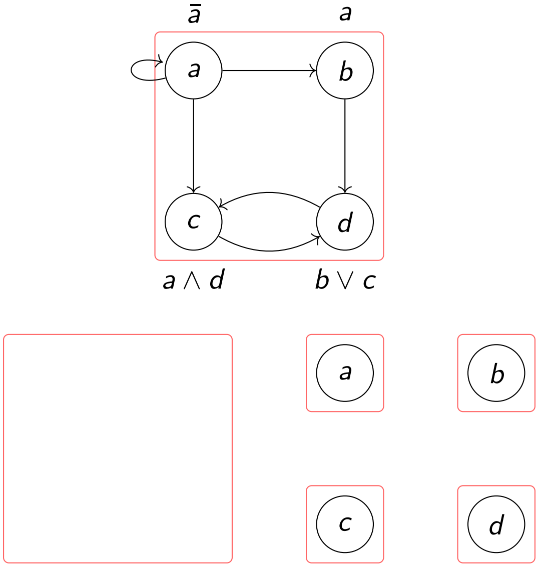


Theorem. $\kappa_{N,S} = \lceil \log_2(\chi(G_{N,S})) \rceil$

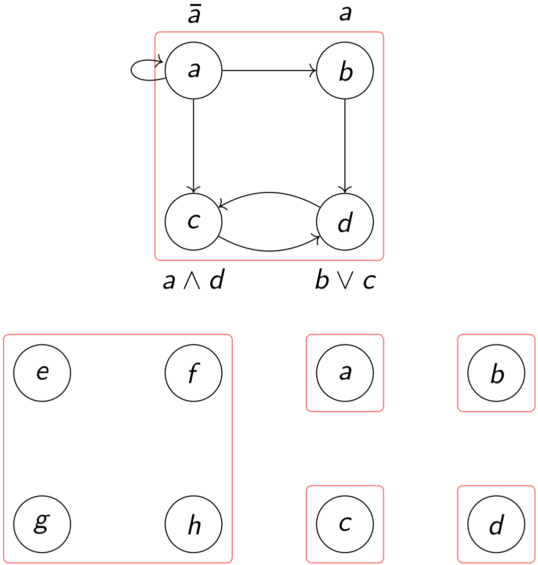
An easy upper bound for κ_n : n



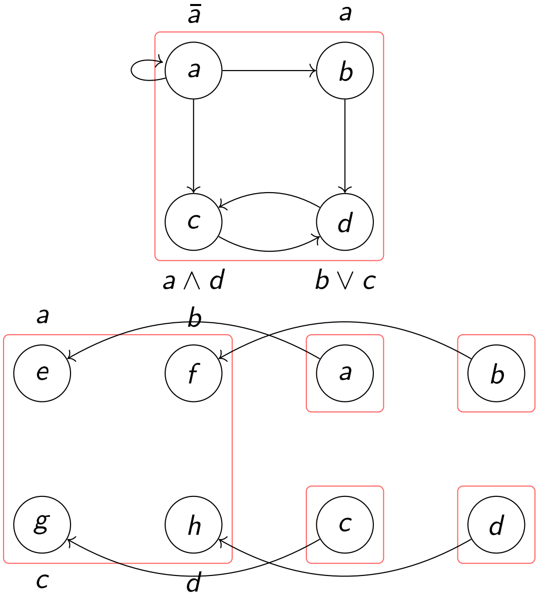
An easy upper bound for κ_n : n



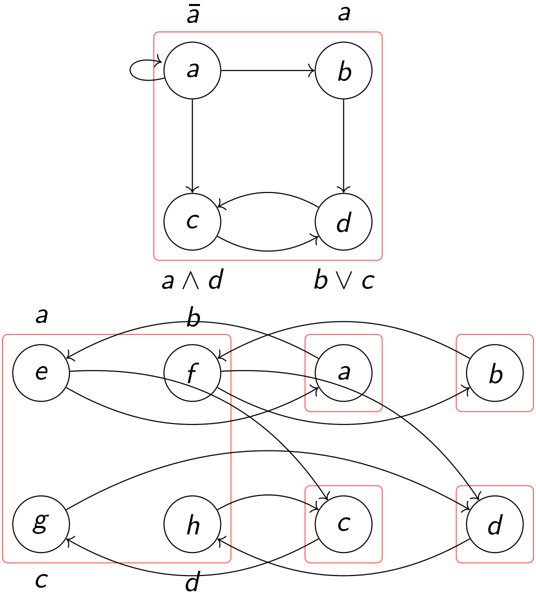
An easy upper bound for κ_n : n



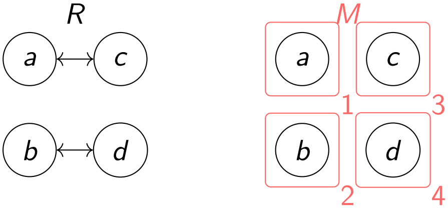
An easy upper bound for κ_n : n



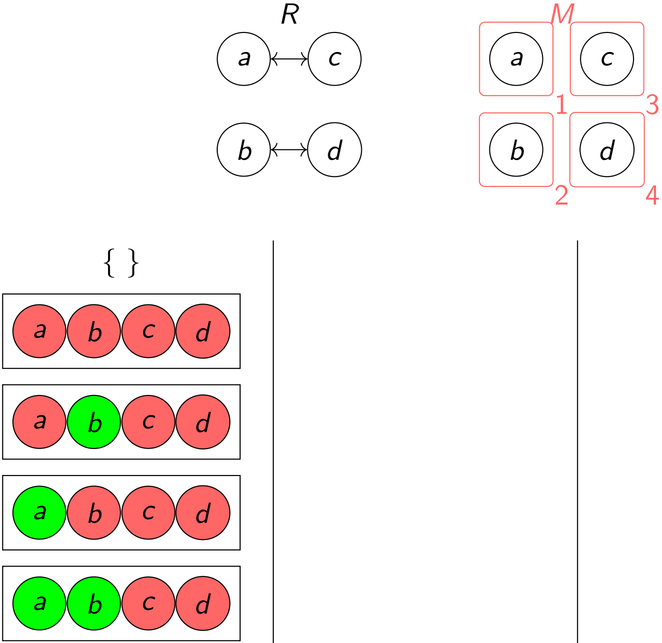
An easy upper bound for κ_n : n



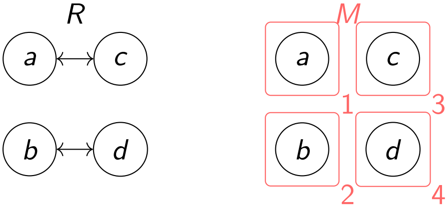
κ_n : Lower bound and conjecture



κ_n : Lower bound and conjecture



κ_n : Lower bound and conjecture



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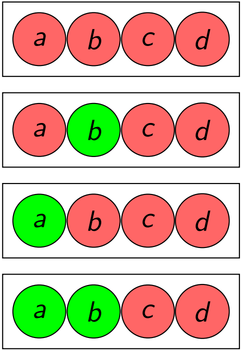
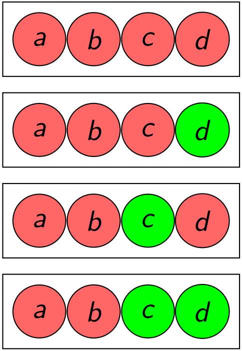
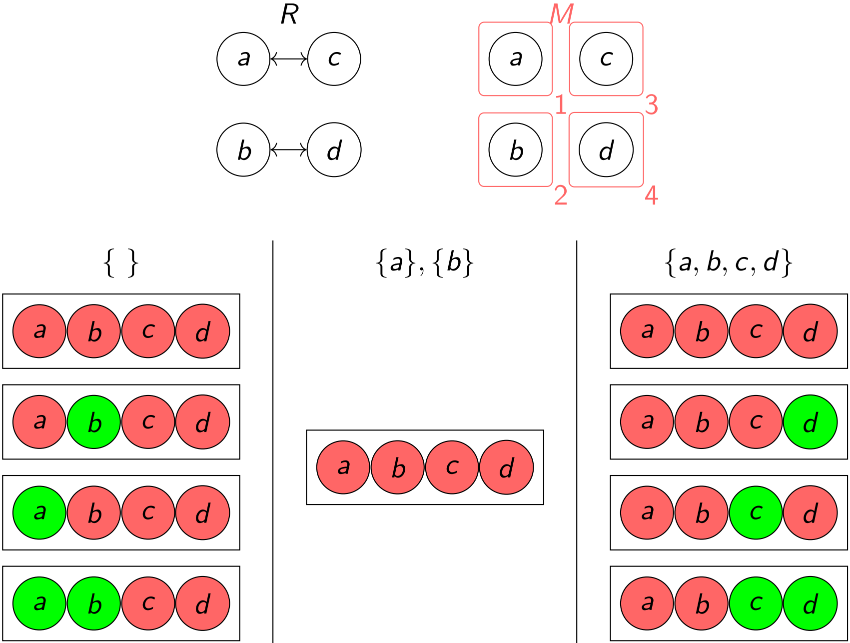


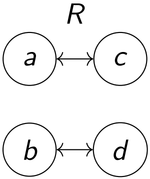
$\{a, b, c, d\}$

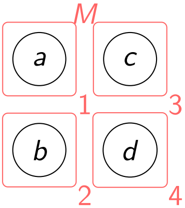


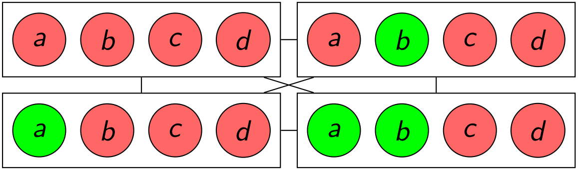
κ_n : Lower bound and conjecture



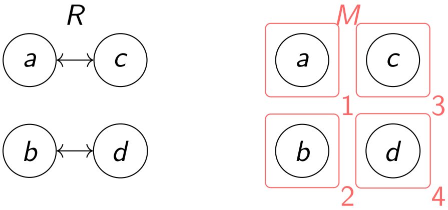
κ_n : Lower bound and conjecture

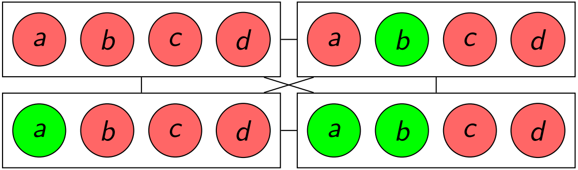






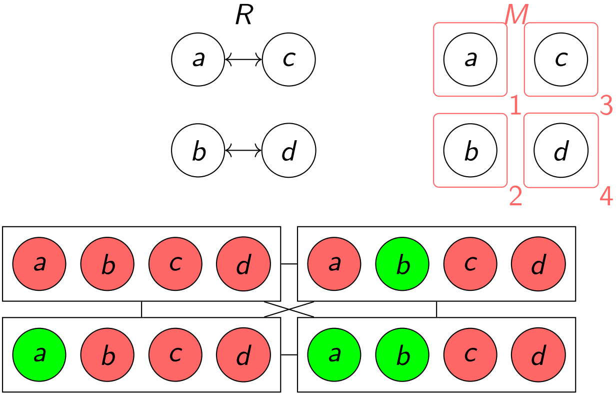
κ_n : Lower bound and conjecture





$\omega(G_{R,M}) \geq 2^2$

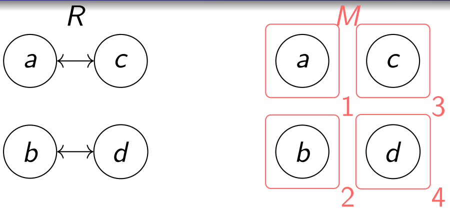
κ_n : Lower bound and conjecture

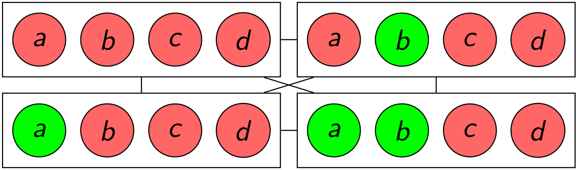


$\omega(G_{R,M}) \geq 2^2$

$\chi(G_{R,M}) \geq 2^2$

κ_n : Lower bound and conjecture



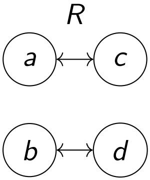


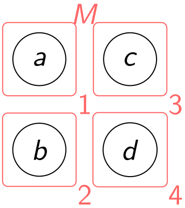
$\omega(G_{R,M}) \geq 2^2$

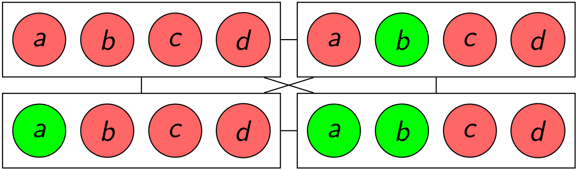
$\chi(G_{R,M}) \geq 2^2$

$\kappa(R, M) \geq 2$

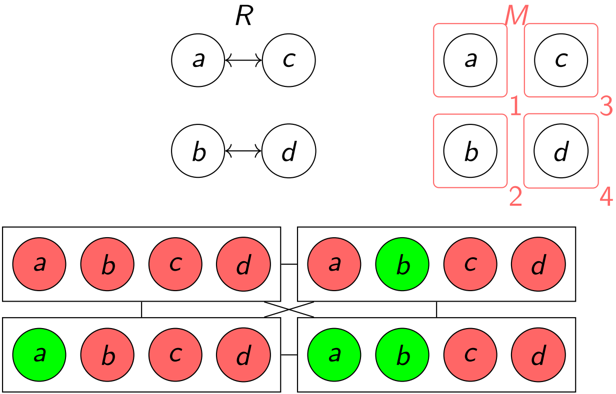
κ_n : Lower bound and conjecture







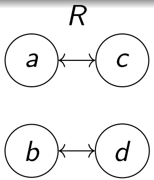
Theorem. $\lfloor n/2 \rfloor \leq \kappa_n \leq n$

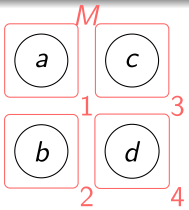


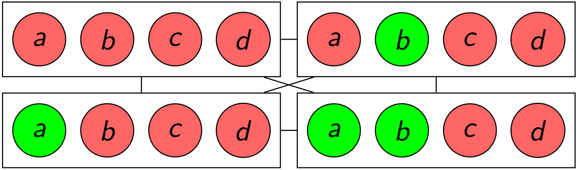
Theorem. $\lfloor n/2 \rfloor \leq \kappa_n \leq n$

Conjecture. $\kappa_n = \lfloor n/2 \rfloor$

κ_n : Lower bound and conjecture





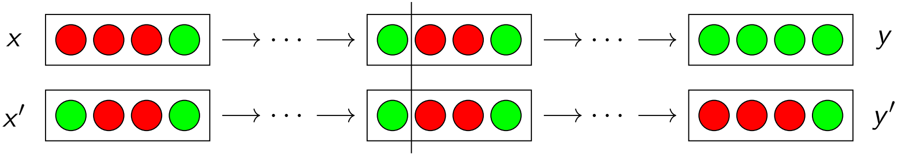


Theorem. $\lfloor n/2 \rfloor \leq \kappa_n \leq n$

Conjecture. $\kappa_n = \lfloor n/2 \rfloor$

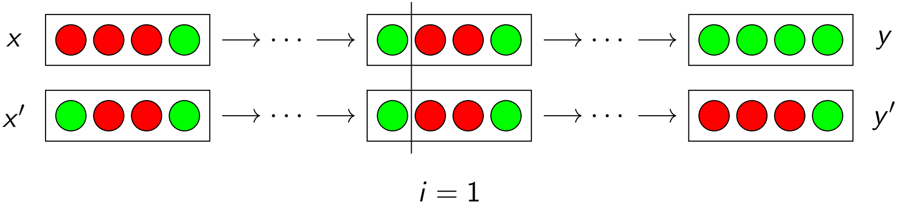
Theorem. $\omega(G_{R,M}) \leq \lfloor n/2 \rfloor$

$\kappa(N, S)$ when N is a bijective BAN



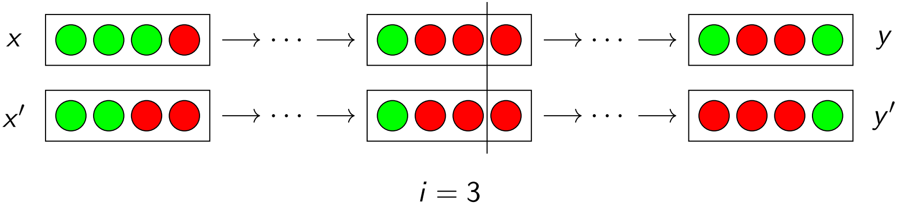
$i = 1$

$\kappa(N, S)$ when N is a bijective BAN



if $i \leq n/2$ then $x_{[n/2,n]} = x'_{[n/2,n]}$

$\kappa(N, S)$ when N is a bijective BAN



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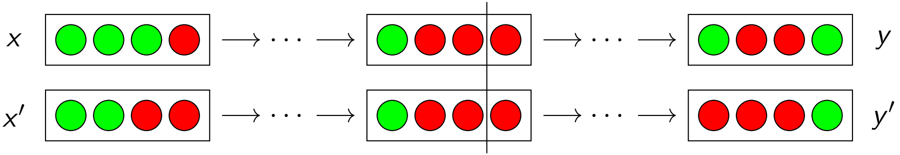
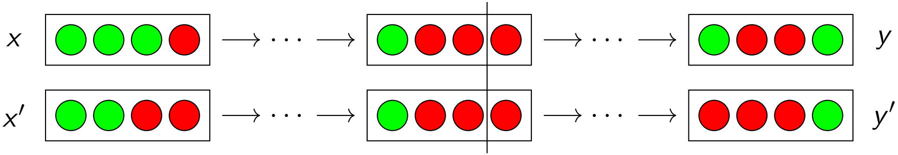


$i = 3$

if $i \leq n/2$ then $x_{[n/2,n]} = x'_{[n/2,n]}$

if $i \geq n/2$ then $y_{[0,n/2]} = y'_{[0,n/2]}$

$\kappa(N, S)$ when N is a bijective BAN



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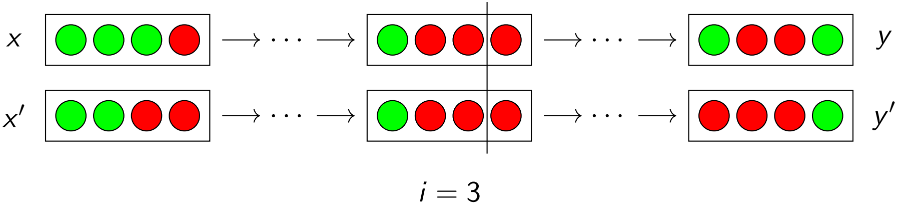
if $i \leq n/2$ then $x_{[n/2,n]} = x'_{[n/2,n]}$

if $i \geq n/2$ then $y_{[0,n/2]} = y'_{[0,n/2]}$

$d(x) \leq 2^{n/2+1}$

$\kappa(N, S)$ when N is a bijective BAN



if $i \leq n/2$ then $x_{[n/2,n]} = x'_{[n/2,n]}$

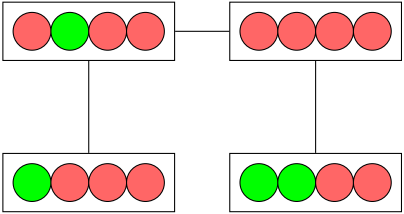
if $i \geq n/2$ then $y_{[0,n/2]} = y'_{[0,n/2]}$

$d(x) \leq 2^{n/2+1}$

Theorem. If N is bijective then $\kappa(N, S) \leq n/2 + 1$

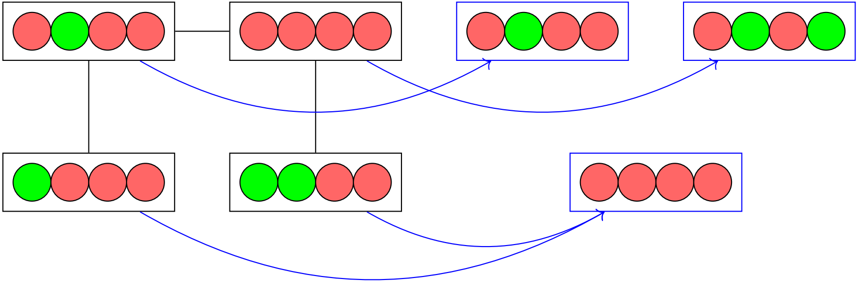
Factorized confusion graph G'

Confusion graph G :

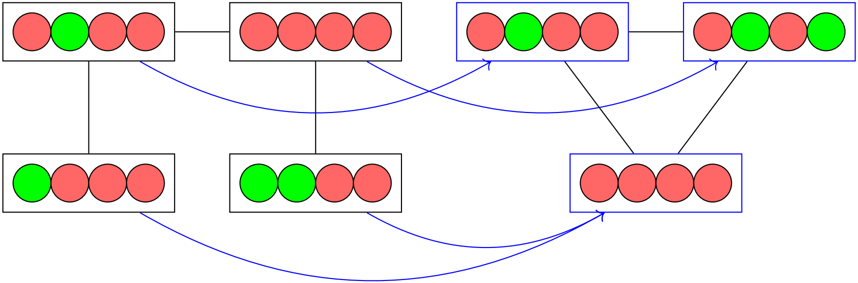


Factorized confusion graph G'

Confusion graph G : Factorized confusion graph G' :



Confusion graph G : Factorized confusion graph G' :



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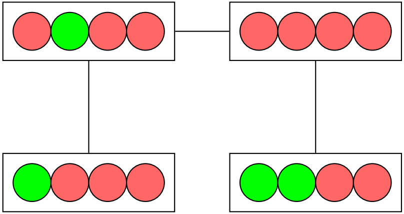

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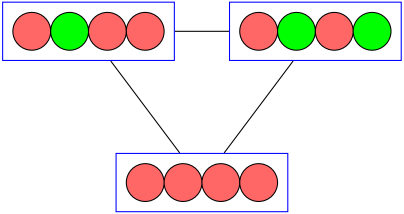
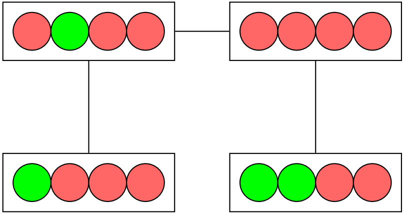


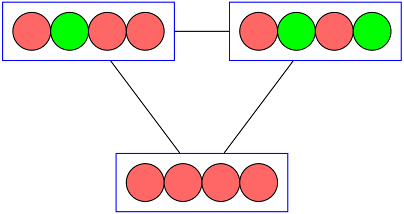
$$\chi(G') = 3$$

Factorized confusion graph G'

Confusion graph G : Factorized confusion graph G' :



$\chi(G) = 2$



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Factorized confusion graph G'

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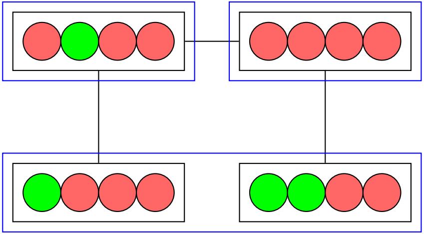
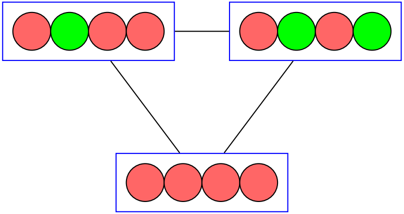


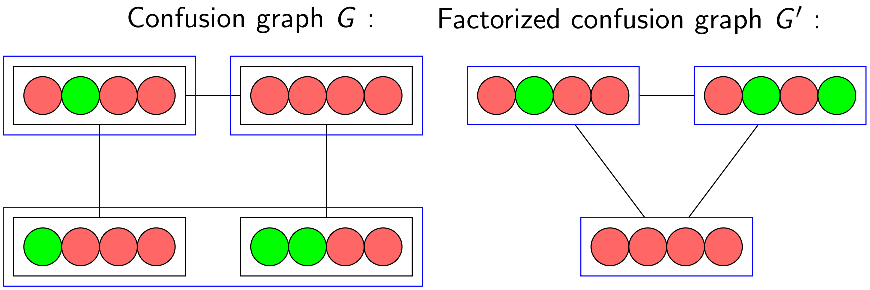
$\chi(G) = 2$

Factorized confusion graph G' :



$\chi(G') = 3$

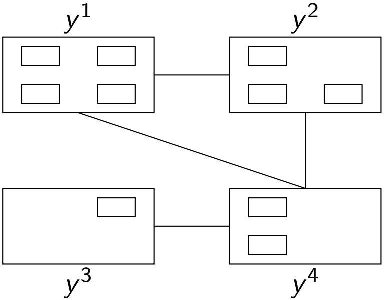
Factorized confusion graph G'



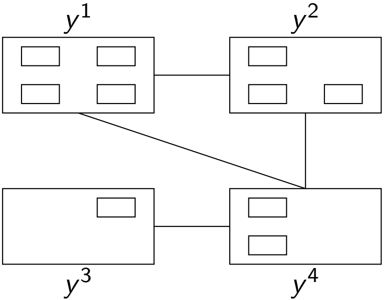
$\chi(G) = 2$ $\qquad$ $\chi(G') = 3$

Lemma. $\chi(G) \leq \chi(G')$

Upper bound for κ_n : 3n/4+1

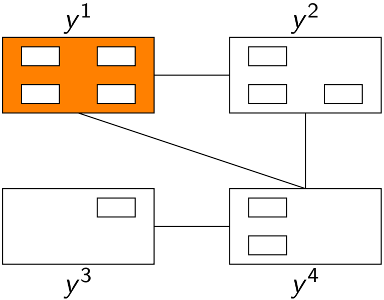


Upper bound for κ_n : 3n/4+1



We sort the images by decreasing size of Fiber :
$|A(y^1)| \leq |A(y^2)| \leq \cdots \leq |A(y^k)|$. We use a greedy coloration algorithm.

Upper bound for κ_n : 3n/4+1



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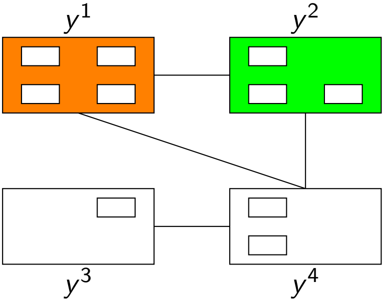

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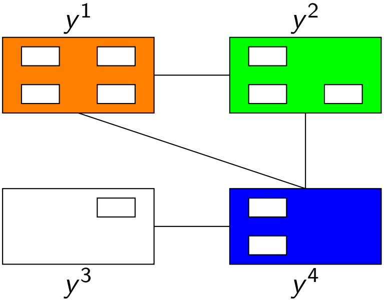

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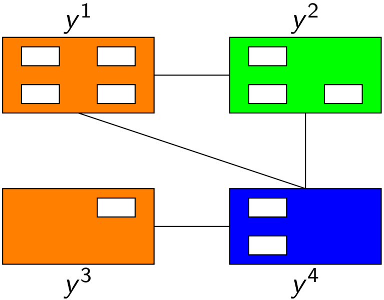

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Upper bound for κ_n : 3n/4+1

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Let k' such that color($y^{k'}$) =maxColor. We have :

- maxColor $\leq k'$.
- maxColor $\leq D(y^{k'})$

Upper bound for κ_n : 3n/4+1

We sort the images by decreasing size of Fiber : $|A(y^1)| \leq |A(y^2)| \leq \cdots \leq |A(y^k)|$. We use a greedy coloration algorithm.

Let k' such that color$(y^{k'})$ =maxColor. We have :

- maxColor $\leq k'$. But $|A(y^{k'})| \times k \leq 2^n$. Thus, maxColor $\leq k' \leq 2^n/|A(y^{k'})|$.
- maxColor $\leq D(y^{k'})$.

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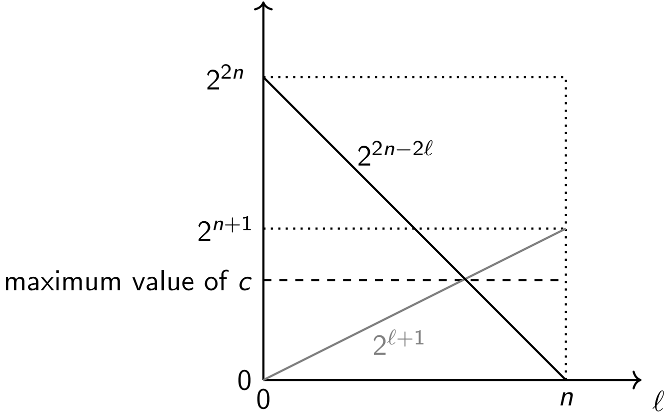
- maxColor $\leq D(y^{k'})$. But $D(y^{k'}) \leq |A(y^{k'})| \times 2^{n/2+1}$.

The maximum value is reach when $|A(y^{k'})| \times 2^{n/2+1} = 2^n\ |A(y^{k'})|$.
That is to say, when $|A(y^{k'})| = 2^{n/2+1}$.

Theorem. $\kappa_n \leq 2n/3 + 2$

Upper bound for κ_n : 2n/3+2



We can get a better upper bound if we sort the images by decreasing degree.

Theorem. $\kappa_n \leq 2n/3 + 2$

Conclusion and perspective

Principal results :

- $\kappa_n = \log(chi(G))$.
- $\lfloor n/2 \rfloor \leq \kappa_n \leq 2n/3 + 2$.
- $\omega(G_{R,M}) \leq \lfloor n/2 \rfloor$.
- If N is bijective then $\kappa(N, S) \leq n/2 + 1$.

Ongoing work :
Currently studying κ_n^- : like κ_n but with no imposed order for S'.
We have some results :

- $\kappa_n^- \leq \kappa_n$
- $\kappa_n^- \leq \tau$
- $\kappa_n^- \geq n/14$.