The benefits of sequent calculus

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Forewords

This talk is about:

sequent calculus / Curry-Howard / operational semantics

But also : *proofs, programs, type systems, safe computation/compilation, ...*

Sequent calculus provides wonderful tools!

Gives **principled answers** to problems such as:

- how to soundly compile xxx?
- how to prove normalization of xxx?
- how should control operators and xxx interact?
- deciding the equivalence of normal forms

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A fairy tale

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- how to soundly compile ×××?
- how to prove normalization of ×××?
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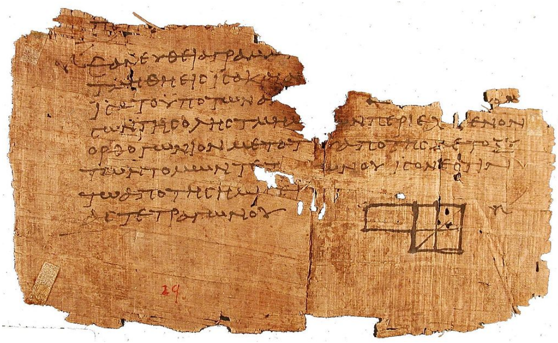
Proofs

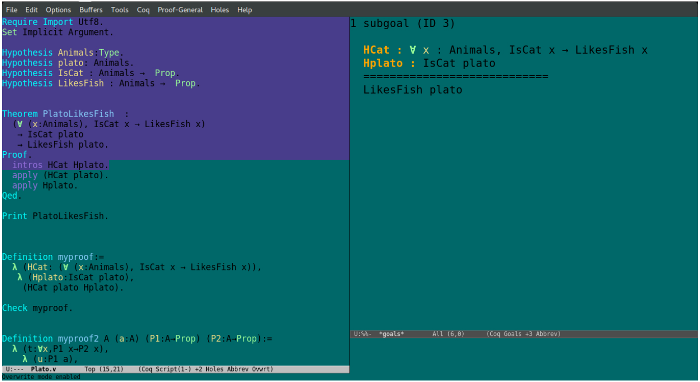
A bit of history, fast-tracked

Once upon a time...



Euclide's Elements

Once upon a time...



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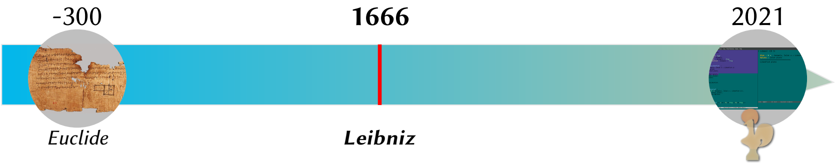
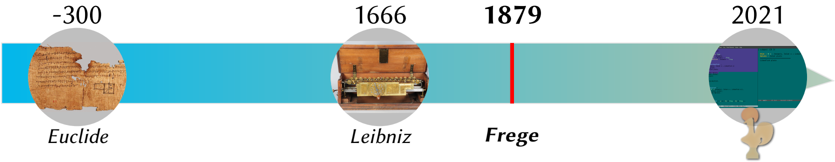


Leibniz's calculus ratiocinator

A crazy dream:

> *"when there are disputes among persons, we can simply say: Let us calculate, without further ado, to see who is right."*

Once upon a time...

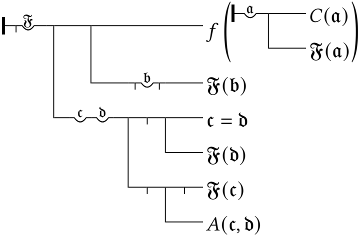


Frege's *Begriffsschrift*:

- formal notations
- quantifications ∀/∃
- distinction:

  x vs $'x'$

  signified *signifier*



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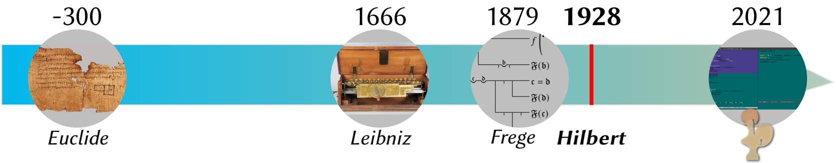
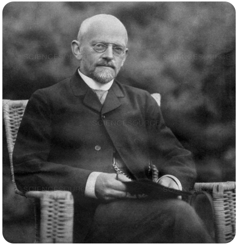




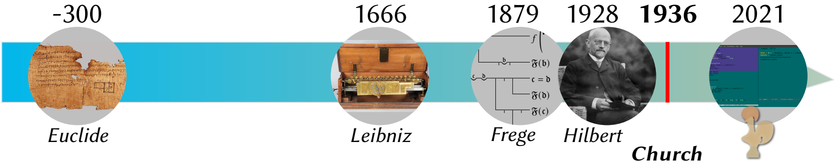
Hilbert

Entscheidungsproblem (with Ackermann):

> *To decide if a formula of first-order logic is a tautology.*

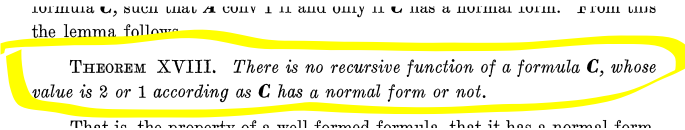
↬ "**to decide**" is meant by means of a procedure

Once upon a time...



Church

λ-calculus - first (negative) answer to the *Entscheidungsproblem* !



formula **C**, such that **A** conv 1 if and only if **C** has a normal form. From this the lemma follows.

THEOREM XVIII. *There is no recursive function of a formula* **C**, *whose value is 2 or 1 according as* **C** *has a normal form or not.*

That is, the property of a well-formed formula, that it has a normal form

A somewhat obvious observation

Deduction rules

$$\frac{A \in \Gamma}{\Gamma \vdash A}\ (\mathrm{Ax})$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\ (\Rightarrow_I)$$

$$\frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}\ (\Rightarrow_E)$$

Typing rules

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x : A}\ (\mathrm{Ax})$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \rightarrow B}\ (\rightarrow_I)$$

$$\frac{\Gamma \vdash t : A \rightarrow B \quad \Gamma \vdash u : A}{\Gamma \vdash t\,u : B}\ (\rightarrow_E)$$

Sequent, you said?

Sequent:

Hypotheses $\boxed{A_1, \ldots, A_n \vdash B}$ Conclusion

Remark:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}$$ is almost $$\frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \Rightarrow B}$$

"*à la Gentzen*" "*à la Prawitz*"

... a.k.a. natural deduction

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Gentzen's sequent calculus (1934)

Sequent:

Hypotheses $A_1, \ldots, A_n \vdash B_1, \ldots, B_p$ Conclusions

Identity rules *connect hypotheses/conclusions*

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma, A \vdash \Delta}{\Gamma \vdash \Delta}\ (\textsc{Cut}) \qquad \frac{}{A \vdash A}\ (\textsc{Ax})$$

Structural rules *weaken, contract, permute*

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\ (w_r) \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\ (c_r) \qquad \frac{\Gamma \vdash \sigma(\Delta)}{\Gamma \vdash \Delta}\ (\sigma_r) \qquad \ldots$$

Logical rules *left/right introduction of connectives*

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta}\ (\Rightarrow_r) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \Rightarrow B \vdash \Delta}\ (\Rightarrow_l) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ (\wedge_r)$$

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Gentzen's sequent calculus (1934)

Sequent:

Hypotheses $\boxed{A_1, \ldots, A_n \vdash B_1, \ldots, B_p}$ Conclusions

Proof-theoretic properties:

- cut elimination
- last rule
- subformula
- classical logic built-in
- ...

Gentzen's sequent calculus (1934)

Sequent:

Hypotheses $\boxed{A_1, \ldots, A_n \vdash B_1, \ldots, B_p}$ Conclusions

Identity rules *connect hypotheses/conclusions*

Symmetry

Logical rules *left/right introduction of connectives*

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What about the computational content?

The λ-calculus (1/3)

Syntax:

$$t, u \quad ::= \quad x \quad | \quad \lambda x.t \quad | \quad t\,u$$

(variables) $\quad x \mapsto f(x) \quad f\,2$

Reduction

$$(\lambda x.t)\,u \longrightarrow_\beta t[u/x]$$

+ contextual closure: $\quad C[t] \longrightarrow_\beta C[t'] \quad$ (if $t \longrightarrow_\beta t'$)

Examples:

$$(\lambda x.x)\,t \longrightarrow_\beta t$$

$$(\lambda x.\lambda y.y\,x)\,\bar{2}\,t \longrightarrow_\beta (\lambda y.y\,\bar{2})\,t \longrightarrow_\beta t\,\bar{2}$$

$$\omega = (\lambda x.x\,x)\,(\lambda x.x\,x) \longrightarrow_\beta (\lambda x.x\,x)\,(\lambda x.x\,x) \longrightarrow_\beta \dots$$

$$(\lambda x.\lambda y.y)\,\omega\,\bar{2} \longrightarrow_\beta\ ?$$

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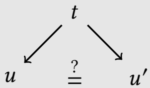
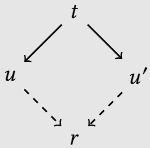
The λ-calculus (2/3)

Determinism:



Confluence:



Normalization:

$$t \longrightarrow t' \longrightarrow t'' \overset{?}{\dashrightarrow} V \not\rightarrow$$

The λ-calculus (3/3)

Evaluation strategy - *How to evaluate $(\lambda x.t)\, u$?*

$(\lambda x.t)u$

- call-by-name:
  substitute x by u to give $t[u/x]$;
- call-by-value:
  first reduce u, (try to) reach a value V,
  then substitute to give $t[V/x]$

$P(x,y) = 2x^2 + x + 1$ / *how to compute* $P(2+3,y), P(x,2+3)$?

Abstract machine - *How to implement the reduction?*

(Push)	$t\, u \star \pi$	$>$	$t \star u \cdot \pi$
(Grab)	$\lambda x\,.\,t \star u \cdot \pi$	$>$	$t[u/x] \star \pi$

Thm. The KAM implements the *weak head call-by-name reduction.*

The λ-calculus (3/3)

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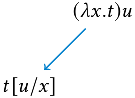

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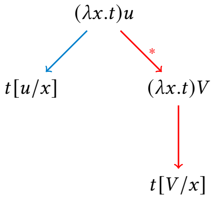

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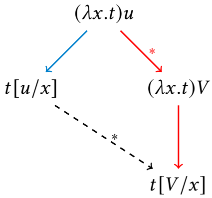

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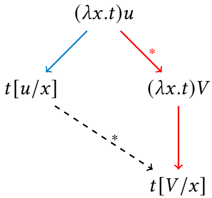

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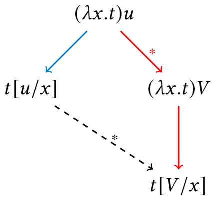

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program

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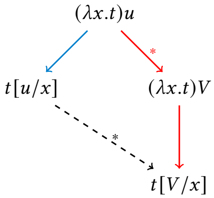

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program *arguments*

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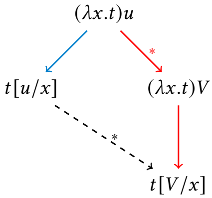

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Curien-Herbelin's duality of computation

Griffin (1990): classical logic ≅ control operator

(Push)	$t\,u \star \pi$	$\succ_1$	$t \star u \cdot \pi$
(Grab)	$\lambda x.t \star u \cdot \pi$	$\succ_1$	$t[u/x] \star \pi$
(Save)	$\boldsymbol{cc} \star t \cdot \pi$	$\succ_1$	$t \star \boldsymbol{k}_\pi \cdot \pi$
(Restore)	$\boldsymbol{k}_\pi \star t \cdot \pi'$	$\succ_1$	$t \star \pi$

Starting observation:

Computational duality:

Sequent calculus ≃ abstract machine-like calculus

Curien-Herbelin's duality of computation

Griffin (1990): classical logic ≅ control operator

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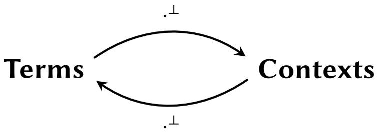

Sequent calculus ≅ abstract machine-like calculus

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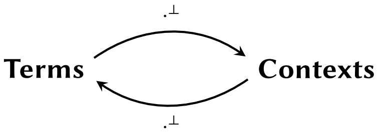

Sequent calculus ≅ *abstract machine-like calculus*

Abstract machine

Reduction

$$\langle t\ u \parallel e\rangle \quad \triangleright_{\text{abs}} \quad \langle t \parallel u \cdot e\rangle$$
$$\langle \lambda x.\, t \parallel u \cdot e\rangle \quad \triangleright_{\text{abs}} \quad \langle t\ [u/x] \parallel e\rangle$$

Syntax

$$c ::= \langle t \parallel e\rangle$$ commands

terms	$t, u ::=$	
variable	$\mid x, y, z$	
application	$\mid t\ u$	
λ-abstraction	$\mid \lambda x.\, t$	

$e, f ::=$		contexts
	$\mid \bullet$	empty
	$\mid t \cdot e$	application stack

Introducing μ

$$\langle t\ u \parallel e \rangle \triangleright_{\text{abs}} \langle t \parallel u \cdot e \rangle$$

This reduction **defines** $(t\ u)$:

It is the term that, when put against $\mid e\rangle$, reduces to $\langle t \parallel u \cdot e \rangle$.

Idea: introduce a more primitive syntax

$$\langle \mu\alpha.\, c \parallel e \rangle \triangleright_{\mu} c\,[e/\alpha]$$

$$t\ u \quad \triangleq \quad \mu\alpha.\, \langle t \parallel u \cdot \alpha \rangle$$

(actually the intuitionistic version $\mu\star.c$ is enough)

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(actually the intuitionistic version $\mu\star.c$ is enough)

Introducing $\tilde{\mu}$

A regular syntax?

$$c ::= \langle t \parallel e \rangle$$

$$t, u ::= \; \mid x, y \; \mid \lambda x.t \; \mid \mu\alpha.c$$

$$e, f ::= \; \mid \alpha, \beta \; \mid t \cdot e \; \mid \; ?$$

Reminder:

calculus and $\lambda\mu$-calculus. Our starting point was the observation that the call-by-value discipline manipulates input much in the same way as (the classical extension of) λ-calculus manipulates output. Computing MN in call-by-

Introducing $\tilde{\mu}$

A regular syntax?

$$c ::= \langle t \parallel e \rangle$$

$$t, u ::= \mid x, y \mid \lambda x.t \mid \mu\alpha.c \qquad e, f ::= \mid \alpha, \beta \mid t \cdot e \mid ?$$

Same idea, in the **dual situation**:

$$\langle (\lambda x.t)u \parallel e \rangle \triangleright_{\text{abs}} \langle \text{let } x = t \text{ in } u \parallel e \rangle \quad \triangleright_{\text{abs}} \quad \langle t \parallel \text{“let } x = \square \text{ in } \langle u \parallel e \rangle\text{''} \rangle$$

$$\langle t \parallel \tilde{\mu}x.c \rangle \quad \triangleright_{\tilde{\mu}} \quad c\,[t/x]$$

Introducing $\tilde{\mu}$

A regular syntax

$$c ::= \langle t \parallel e \rangle$$

$$t, u ::= \mid x, y \mid \lambda x.t \mid \mu\alpha.\, c \qquad e, f ::= \mid \alpha, \beta \mid t \cdot e \mid \tilde{\mu}x.\, c$$

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$$\langle t \parallel \tilde{\mu}x.c \rangle \quad \triangleright_{\tilde{\mu}} \quad c\,[t/x]$$

Curien-Herbelin's $\lambda\mu\tilde{\mu}$-calculus

Syntax:

$$
\begin{array}{lll}
t, u ::= & c ::= \langle t \,\|\, e \rangle & e, f ::= \\
\quad \mid x, y & & \quad \mid \alpha, \beta \\
\quad \mid \lambda x.t & & \quad \mid t \cdot e \\
\quad \mid \mu\alpha.\, c & & \quad \mid \tilde{\mu}x.\, c
\end{array}
$$

Reduction:

$$
\begin{array}{rcl}
\langle \lambda x.t \,\|\, u \cdot e \rangle & \to & \big\langle u \,\|\, \tilde{\mu}x.\langle t \,\|\, e \rangle \big\rangle \\
\langle t \,\|\, \tilde{\mu}x.c \rangle & \to & c[t/x] \\
\langle \mu\alpha.c \,\|\, e \rangle & \to & c[e/\alpha]
\end{array}
$$

Curien-Herbelin's $\lambda\mu\tilde{\mu}$-calculus

Syntax:

$$t, u ::= \mid x, y \mid \lambda x.t \mid \mu\alpha.\, c \qquad c ::= \langle t \parallel e \rangle \qquad e, f ::= \mid \alpha, \beta \mid t \cdot e \mid \tilde{\mu}x.\, c$$

Reduction:

$$\begin{aligned}
\langle \lambda x.t \parallel u \cdot e \rangle &\rightarrow \big\langle u \parallel \tilde{\mu}x.\langle t \parallel e \rangle \big\rangle \\
\langle t \parallel \tilde{\mu}x.c \rangle &\rightarrow c[t/x] \\
\langle \mu\alpha.c \parallel e \rangle &\rightarrow c[e/\alpha]
\end{aligned}$$

Critical pair:

$$\langle \mu\alpha.c \parallel \tilde{\mu}x.c' \rangle$$

$$\swarrow \qquad\qquad \searrow$$

$$c[\tilde{\mu}x.c'/\alpha] \qquad\qquad c'[\mu\alpha.c/x]$$

Curien-Herbelin's $\lambda\mu\tilde{\mu}$-calculus

Syntax:

$$t, u ::= \quad \text{Values} \left\{ \begin{array}{l} \mid x, y \\ \mid \lambda x.t \end{array} \right. \quad \mid \mu\alpha.\, c$$

$$c ::= \langle t \,\|\, e \rangle$$

$$e, f ::= \left. \begin{array}{l} \mid \alpha, \beta \\ \mid t \cdot e \end{array} \right\} \text{Co-values} \quad \mid \tilde{\mu}x.\, c$$

Reduction:

$$\begin{array}{rcll} \langle \lambda x.t \,\|\, u \cdot e \rangle & \to & \langle u \,\|\, \tilde{\mu}x.\langle t \,\|\, e \rangle \rangle & \\ \langle t \,\|\, \tilde{\mu}x.c \rangle & \to & c[t/x] & t \in \mathcal{V} \\ \langle \mu\alpha.c \,\|\, e \rangle & \to & c[e/\alpha] & e \in \mathcal{E} \end{array}$$

Critical pair:

$$\langle \mu\alpha.c \,\|\, \tilde{\mu}x.c' \rangle$$

CbV $\swarrow$ $c[\tilde{\mu}x.c'/\alpha]$

CbN $\searrow$ $c'[\mu\alpha.c/x]$

Curien-Herbelin's $\lambda\mu\tilde{\mu}$-calculus

Syntax:

$$t,u ::= \quad | \; x,y \quad | \; \lambda x.t \quad | \; \mu\alpha.c$$

Values { x,y ; $\lambda x.t$ }

$$c ::= \langle t \parallel e \rangle$$

$$e,f ::= \quad | \; \alpha,\beta \quad | \; t \cdot e \quad | \; \tilde{\mu}x.c$$

{ α,β ; $t \cdot e$ } Co-values

Reduction:

$$\begin{array}{rcll} \langle \lambda x.t \parallel u \cdot e \rangle & \to & \langle u \parallel \tilde{\mu}x.\langle t \parallel e \rangle \rangle & \\ \langle t \parallel \tilde{\mu}x.c \rangle & \to & c[t/x] & t \in \mathcal{V} \\ \langle \mu\alpha.c \parallel e \rangle & \to & c[e/\alpha] & e \in \mathcal{E} \end{array}$$

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$$\langle \mu\alpha.c \parallel \tilde{\mu}x.c' \rangle$$

CbV ↙ $c[\tilde{\mu}x.c'/\alpha]$

CbN ↘ $c'[\mu\alpha.c/x]$

Duality
(again and again)

Curien-Herbelin's $\lambda\mu\tilde{\mu}$-calculus

Syntax:

$$t, u ::= \quad \underbrace{\mid x, y \quad \mid \lambda x.t}_{\text{Values}} \quad \mid \mu\alpha.c$$

$$c ::= \langle t \parallel e \rangle$$

$$e, f ::= \quad \underbrace{\mid \alpha, \beta \quad \mid t \cdot e}_{\text{Co-values}} \quad \mid \tilde{\mu}x.c$$

Typing rules:

$$\frac{\Gamma \vdash t : A \mid \Delta \qquad \Gamma \mid e : A \vdash \Delta}{\langle t \parallel e \rangle : (\Gamma \vdash \Delta)}$$

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x : A \mid \Delta} \qquad \frac{\Gamma, x : A \vdash t : B \mid \Delta}{\Gamma \vdash \lambda x.t : A \to B \mid \Delta} \qquad \frac{c : (\Gamma \vdash \Delta, \alpha : A)}{\Gamma \vdash \mu\alpha.c : A \mid \Delta}$$

$$\frac{(\alpha : A) \in \Delta}{\Gamma \mid \alpha : A \vdash \Delta} \qquad \frac{\Gamma \vdash u : A \mid \Delta \qquad \Gamma \mid e : B \vdash \Delta}{\Gamma \mid u \cdot e : A \to B \vdash \Delta} \qquad \frac{c : (\Gamma, x : A \vdash \Delta)}{\Gamma \mid \tilde{\mu}x.c : A \vdash \Delta}$$

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Curien-Herbelin's $\lambda\mu\tilde{\mu}$-calculus

Syntax:

$$t, u ::= \quad \text{Values} \begin{cases} \mid x, y \\ \mid \lambda x.t \end{cases} \quad \mid \mu\alpha.\, c$$

$$c ::= \langle t \parallel e \rangle$$

$$e, f ::= \quad \begin{rcases} \mid \alpha, \beta \\ \mid t \cdot e \end{rcases} \text{Co-values} \quad \mid \tilde{\mu}x.\, c$$

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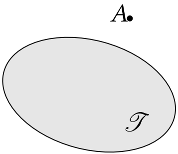
"Why should I care?"

"Why should I care?"

Because sequent calculus is well-behaved!

Extending Curry-Howard

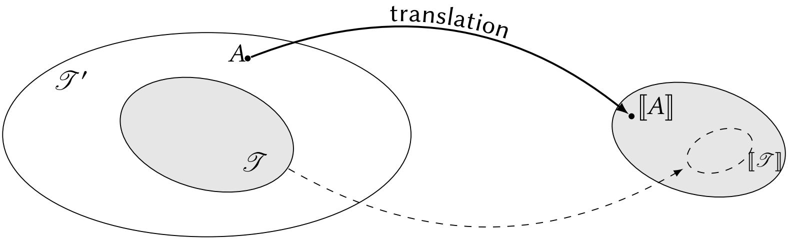


New axiom $\sim$ Programming instruction

$\Updownarrow$ $\Updownarrow$

Logical translation $\sim$ Program translation

Extending Curry-Howard



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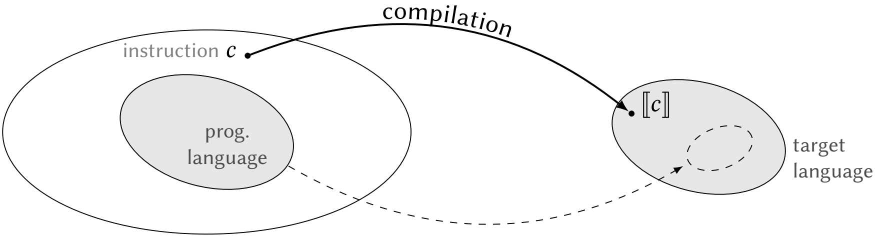
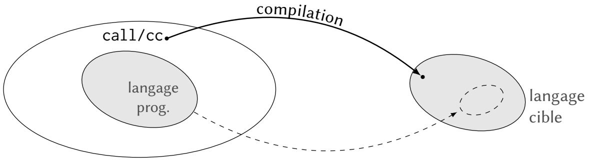


New axiom	$\sim$	Programming instruction
$\Updownarrow$		$\Updownarrow$
Logical translation	$\sim$	Program translation

Extending Curry-Howard

Classical logic = Intuitionistic logic + $A \lor \neg A$



New axiom		Programming instruction
$A \lor \neg A$	~	catch / throw
Excluded middle		Backtracking
⇕		⇕
Logical translation		Program translation
$A \mapsto \neg\neg A$	~	$2 \mapsto \texttt{fun}\ k \rightarrow k(2)$
Gödel negative translation		*Continuation-passing style* translation

Non constructive reasoning *Side effects*

Sequent calculus as IR

You just defined a wonderful calculus, and you are wondering:

Problem

How to define a continuation-passing style translation?

CPS translation:
$[\![\cdot]\!] : \mathit{source} \longrightarrow \lambda^{\text{something}}$

- preserving reduction
- preserving typing
- the type $[\![\bot]\!]$ is not inhabited

Typically: $[\![V]\!] \triangleq \lambda k.k\,V$
$[\![t]\!] \triangleq \lambda k.?$

Benefits:
If $\lambda^{\text{something}}$ is sound and normalizing:

1. If $[\![t]\!]$ normalizes, then t normalizes
2. If t is typed, then t normalizes
3. There is no term $\vdash t : \bot$

Sequent calculus as IR

You just defined a wonderful calculus, and you are wondering:

Problem

How to define a continuation-passing style translation?

Solution

Use sequent calculus!

Slogan:

A sequent calculus is a defunctionalization of CPS representations.

↬ as such it defines a good intermediate representation for compilation

Method: Danvy's semantics artifacts

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Method: Danvy's semantics artifacts

Semantics artifacts in action

Call-by-name $\lambda\mu\tilde{\mu}$-calculus:

$$\begin{array}{llll} \text{Terms} & t ::= V \mid \mu\alpha.c & \text{Contexts} & e ::= E \mid \tilde{\mu}x.c \\ \text{Values} & V ::= x \mid \lambda x.t & \text{Co-values} & E ::= \alpha \mid t \cdot e \end{array}$$

$$\text{Commands} \quad c ::= \langle t \,\|\, e \rangle$$

Reduction rules:

$$\begin{array}{lcl} \langle t \,\|\, \tilde{\mu}x.c \rangle & \to & c[t/x] \\ \langle \mu\alpha.c \,\|\, E \rangle & \to & c[E/\alpha] \\ \langle \lambda x.t \,\|\, u \cdot e \rangle & \to & \big\langle u \,\big\|\, \tilde{\mu}x.\langle t \,\|\, e \rangle \big\rangle \end{array}$$

Semantics artifacts in action

Terms	$t ::= V \mid \mu\alpha.c$	Contexts	$e ::= E \mid \tilde{\mu}x.c$
Values	$V ::= x \mid \lambda x.t$	Co-values	$E ::= \alpha \mid t \cdot e$

Commands $c ::= \langle t \| e \rangle$

Small steps

e	$\langle t \| \tilde{\mu}x.c \rangle_e$	$\rightsquigarrow$	$c_e[t/x]$
	$\langle t \| E \rangle_e$	$\rightsquigarrow$	$\langle t \| E \rangle_t$
t	$\langle \mu\alpha.c \| E \rangle_t$	$\rightsquigarrow$	$c_e[E/\alpha]$
	$\langle V \| E \rangle_t$	$\rightsquigarrow$	$\langle V \| E \rangle_E$
E	$\langle V \| u \cdot e \rangle_E$	$\rightsquigarrow$	$\langle V \| u \cdot e \rangle_V$
V	$\langle \lambda x.t \| u \cdot e \rangle_V$	$\rightsquigarrow$	$\langle u \| \tilde{\mu}x.\langle t \| e \rangle \rangle_e$

Semantics artifacts in action

Terms	$t ::= V \mid \mu\alpha.c$	Contexts	$e ::= E \mid \tilde{\mu}x.c$
Values	$V ::= x \mid \lambda x.t$	Co-values	$E ::= \alpha \mid t \cdot e$

Commands $c ::= \langle t \parallel e \rangle$

	Small steps			CPS
e	$\langle t \parallel \tilde{\mu}x.c \rangle_e$	$\rightsquigarrow$	$c_e[t/x]$	$[\![\tilde{\mu}x.c]\!]_e\, t \triangleq (\lambda x.[\![c]\!]_c)\, t$
	$\langle t \parallel E \rangle_e$	$\rightsquigarrow$	$\langle t \parallel E \rangle_t$	$[\![E]\!]_e\, t \triangleq t\, [\![E]\!]_E$
t	$\langle \mu\alpha.c \parallel E \rangle_t$	$\rightsquigarrow$	$c_e[E/\alpha]$	$[\![\mu\alpha.c]\!]_t\, E \triangleq (\lambda\alpha.[\![c]\!]_c)\, E$
	$\langle V \parallel E \rangle_t$	$\rightsquigarrow$	$\langle V \parallel E \rangle_E$	$[\![V]\!]_t\, E \triangleq E\, [\![V]\!]_V$
E	$\langle V \parallel u \cdot e \rangle_E$	$\rightsquigarrow$	$\langle V \parallel u \cdot e \rangle_V$	$[\![u \cdot e]\!]_E\, V \triangleq V\, [\![u]\!]_t\, [\![e]\!]_e$
V	$\langle \lambda x.t \parallel u \cdot e \rangle_V$	$\rightsquigarrow$	$\langle u \parallel \tilde{\mu}x.\langle t \parallel e \rangle \rangle_e$	$[\![\lambda x.t]\!]_V\, u\, e \triangleq (\lambda x.e\, [\![t]\!]_t)\, u$

Preservation

$$c \overset{1}{\rightsquigarrow} c' \quad \Rightarrow \quad [\![c]\!]_c \xrightarrow{+}_\beta [\![c']\!]_c$$

Semantics artifacts in action

Terms	$t ::= V \mid \mu\alpha.c$	Contexts	$e ::= E \mid \tilde{\mu}x.c$
Values	$V ::= x \mid \lambda x.t$	Co-values	$E ::= \alpha \mid t \cdot e$

Commands $c ::= \langle t \| e \rangle$

CPS

e:

$$[\![\tilde{\mu}x.c]\!]_e\, t \triangleq (\lambda x.[\![c]\!]_c)\, t$$

$$[\![E]\!]_e\, t \triangleq t\, [\![E]\!]_E$$

t:

$$[\![\mu\alpha.c]\!]_t\, E \triangleq (\lambda\alpha.[\![c]\!]_c)\, E$$

$$[\![V]\!]_t\, E \triangleq E\, [\![V]\!]_V$$

E:

$$[\![u \cdot e]\!]_E\, V \triangleq V\, [\![u]\!]_t\, [\![e]\!]_e$$

V:

$$[\![\lambda x.t]\!]_V\, u\, e \triangleq (\lambda x.e\, [\![t]\!]_t)\, u$$

Types translation

$$[\![A]\!]_e \triangleq [\![A]\!]_t \to \bot$$

$$[\![A]\!]_t \triangleq [\![A]\!]_E \to \bot$$

$$[\![A]\!]_E \triangleq [\![A]\!]_V \to \bot$$

$$[\![A \to B]\!]_V \triangleq [\![A]\!]_t \to [\![B]\!]_e \to \bot$$

Preservation

$$\Gamma \vdash t : A \mid \Delta \quad \Rightarrow \quad [\![\Gamma]\!]_t, [\![\Delta]\!]_E \vdash [\![t]\!]_t : [\![A]\!]_t$$

Semantics artifacts in action

Normalization

Typed commands of the call-by-name $\lambda\mu\tilde{\mu}$-calculus normalize.

Inhabitation

There is no simply-typed λ-term t such that $\vdash t : [\![\bot]\!]_t$.

Soundness

There is no proof t such that $\vdash t : \bot \mid$.

Normalization proofs

You just defined a wonderful calculus, but the CPS method is too complex:

Problem

How can I prove normalization?

Solution

Use sequent calculus + Krivine realizability!

Slogan:

A sequent calculus specifies the interactions of terms and contexts.

↬ as you will see, this helps a lot the definition of a realizability interpretation

Method: Danvy's semantics artifacts, again

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Method: Danvy's semantics artifacts, again

Realizability *à la* Krivine

Intuition

- falsity value $\|A\|$: contexts, opponent to A
- truth value $|A|$: terms, player of A
- pole $\perp\!\!\!\perp$: commands, referee

$$\langle t \| e \rangle > c_0 > \cdots > c_n \in \perp\!\!\!\perp ?$$

$\rightsquigarrow \perp\!\!\!\perp \subset \Lambda \star \Pi$ closed by anti-reduction

Truth value defined by **orthogonality** :

$$|A| = \|A\|^{\perp\!\!\!\perp} = \{t \in \Lambda : \forall e \in \|A\|, \langle t \| e \rangle \in \perp\!\!\!\perp\}$$

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Semantic artifacts, bis

Terms	$t ::= V \mid \mu\alpha.c$	Contexts	$e ::= E \mid \tilde{\mu}x.c$
Values	$V ::= x \mid \lambda x.t$	Co-values	$E ::= \alpha \mid t \cdot e$

Commands $c ::= \langle t \parallel e \rangle$

Small steps

$$
\begin{array}{llcl}
e & \langle t \parallel \tilde{\mu}x.c \rangle_e & \rightsquigarrow & c_e[t/x] \\
 & \langle t \parallel E \rangle_e & \rightsquigarrow & \langle t \parallel E \rangle_t \\
t & \langle \mu\alpha.c \parallel E \rangle_t & \rightsquigarrow & c_e[E/\alpha] \\
 & \langle V \parallel E \rangle_t & \rightsquigarrow & \langle V \parallel E \rangle_E \\
E & \langle V \parallel u \cdot e \rangle_E & \rightsquigarrow & \langle V \parallel u \cdot e \rangle_V \\
V & \langle \lambda x.t \parallel u \cdot e \rangle_V & \rightsquigarrow & \langle u \parallel \tilde{\mu}x.\langle t \parallel e \rangle \rangle_e
\end{array}
$$

Semantic artifacts, bis

Terms	$t ::= V \mid \mu\alpha.c$	Contexts	$e ::= E \mid \tilde\mu x.c$
Values	$V ::= x \mid \lambda x.t$	Co-values	$E ::= \alpha \mid t \cdot e$

Commands $c ::= \langle t \parallel e \rangle$

	Small steps			**Realizability**
e	$\langle t \parallel \tilde\mu x.c \rangle_e$	$\rightsquigarrow$	$c_e[t/x]$	$\|A\|_e \triangleq \lvert A\rvert_t{}^{\perp\!\!\!\perp}$
	$\langle t \parallel E \rangle_e$	$\rightsquigarrow$	$\langle t \parallel E \rangle_t$	
t	$\langle \mu\alpha.c \parallel E \rangle_t$	$\rightsquigarrow$	$c_e[E/\alpha]$	$\lvert A\rvert_t \triangleq \|A\|_E{}^{\perp\!\!\!\perp}$
	$\langle V \parallel E \rangle_t$	$\rightsquigarrow$	$\langle V \parallel E \rangle_E$	
E	$\langle V \parallel u \cdot e \rangle_E$	$\rightsquigarrow$	$\langle V \parallel u \cdot e \rangle_V$	$\|A \to B\|_E \triangleq \{u \cdot e : u \in \lvert A\rvert_t \wedge e \in \|B\|_e\}$
V	$\langle \lambda x.t \parallel u \cdot e \rangle_V$	$\rightsquigarrow$	$\langle u \parallel \tilde\mu x.\langle t \parallel e \rangle \rangle_e$	

Adequacy

1. $\cdot \vdash t : A \mid \cdot \quad \Rightarrow \quad t \in \lvert A\rvert_t$
2. $\cdot \mid e : A \vdash \cdot \quad \Rightarrow \quad e \in \|A\|_e$
3. $c : (\cdot \vdash \cdot) \quad \Rightarrow \quad c \in {\perp\!\!\!\perp}$

Consequences

Normalizing commands

$\perp\!\!\!\perp_{\Downarrow} \triangleq \{c : c \text{ normalizes}\}$ defines a valid pole.

Proof. If $c \to c'$ and c' normalizes, so does c. □

Normalization

For any command c, if $c : \Gamma \vdash \Delta$, then c normalizes.

Proof. By adequacy, any typed command c belongs to the pole $\perp\!\!\!\perp_{\Downarrow}$. □

Soundness

There is no proof t such that $\vdash t : \bot \mid$.

Proof. Otherwise, $t \in |\bot|_t = \Pi^{\perp\!\!\!\perp}$ for any pole, absurd ($\perp\!\!\!\perp \triangleq \emptyset$). □

Polarized sequent calculus

You added sums to your favorite λ-calculus, it broke all your proofs:

Problem

What can I do?

Solution

Use sequent calculus + polarities!

Negative polarity	Every expression is a value (CBN)
Positive polarity	Every context is a covalue (CBV)

Slogan:

Polarized $\lambda\mu\tilde{\mu}$ is a good, regular syntax for programs.

↬ a.k.a. system L, a great syntax for call-by-push-value

Method: see Munch-Maccagnoni & Scherer's paper (LICS'15)

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Take away

Sequent calculus:

- is more *regular* than natural deduction
- corresponds to *abstract-machine-like* calculi (*e.g.* $\lambda\mu\tilde{\mu}$-calculus)
- provides great insights on *operational semantics*

A **flexible** tool:

- can be decomposed with connectives of *linear logic*
- can be *polarized* (Munch-Maccagnoni's system L)
- supports *effectful* constructors
- ...

If you don't use it already,

What are you waiting for?

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