Introduction to Homology and Holes

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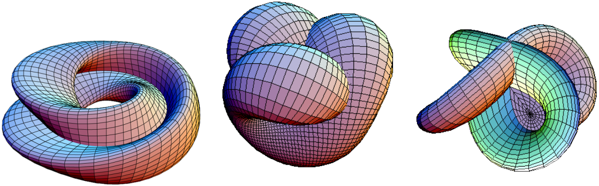
LABORATOIRE
D'INFORMATIQUE
& SYSTÈMES
UMR 7020

Topology



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1. Topology joke, by Henry Segerman.
2. Wikipedia.

Algebraic Topology

Topology

- spaces
- simplicial complexes
- shapes
- . . .

$\longleftrightarrow$

Algebra

- groups
- rings
- fields
- . . .

Subfields

Homotopy, homology, cohomology, knot theory...

Algebraic Topology

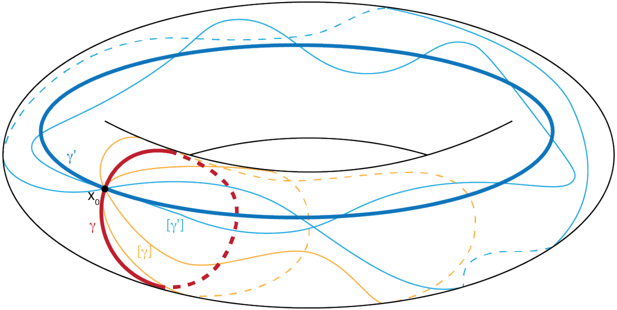

Figure – Illustration of homotopy on a torus.

homeomorphic $\Longrightarrow$ homotopic $\Longrightarrow$ homologuous

Algebraic Topology

Decidability in Algebraic Topology

- Are two groups isomorphic given their representations ? Undecidable[1].
- Are two triangulations homeomorphic ? Undecidable.
- Are two triangulations homotopic ? Undecidable.
- Are two triangulations homologuous ? **Decidable** !

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[1] P. S. Novikov, "Unsolvability of the conjugacy problem in the theory of groups"(1954)

Why computing topology ?

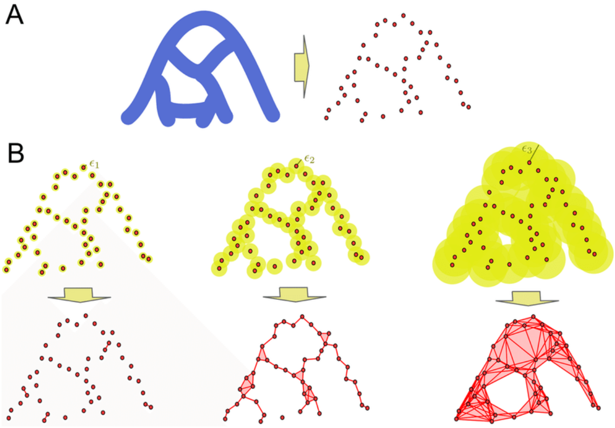

Figure – Topological data analysis.

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1. Camara, Pablo & Levine, Arnold & Rabadan, Raul. (2015). Inference of Ancestral Recombination Graphs through Topological Data Analysis.

Why computing topology ?

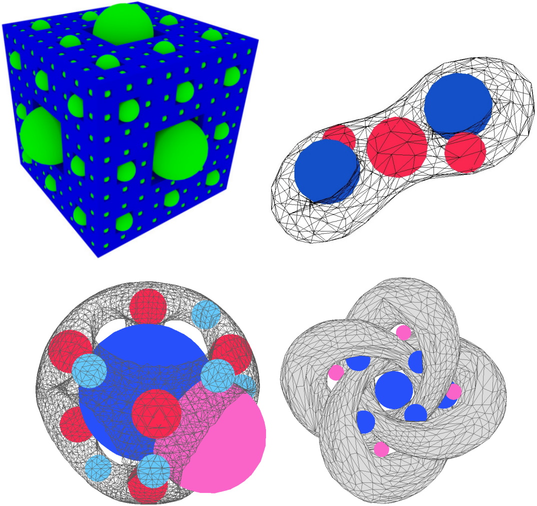
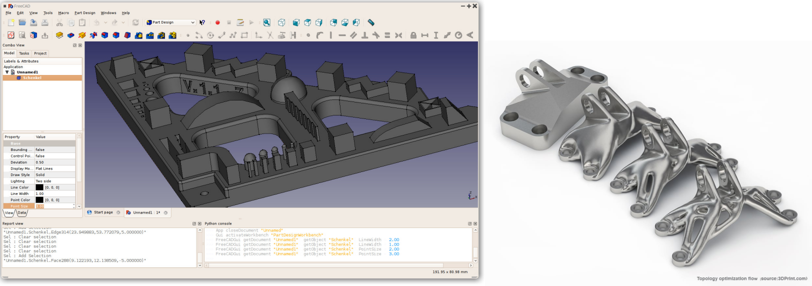


Figure – Holes measure, mainly useful for shape analysis or classification.

Why computing topology ?



Property	Value
Base	
Bounding ...	false
Control Poi...	false
Deviation	0.50
Display Mo...	Flat Lines
Draw Style	Solid
Lighting	Two side
Line Color	[0, 0, 0]
Line Width	1.00
Point Color	[0, 0, 0]
Point Size	3.00

```
"Unnamed1.Schenkel.Edge314(23.949883,53.772079,5.000000)"
Sel : Clear selection
Sel : Clear selection
Sel : Clear selection
Sel : Clear selection
Sel : Add Selection
"Unnamed1.Schenkel.Face288(9.122193,12.138509,-5.000000)"
```

```
>>> App.closeDocument("Unnamed")
>>> Gui.activateWorkbench("PartDesignWorkbench")
>>> FreeCADGui.getDocument("Unnamed1").getObject("Schenkel").LineWidth = 2.00
>>> FreeCADGui.getDocument("Unnamed1").getObject("Schenkel").LineWidth = 1.00
>>> FreeCADGui.getDocument("Unnamed1").getObject("Schenkel").PointSize = 2.00
>>> FreeCADGui.getDocument("Unnamed1").getObject("Schenkel").PointSize = 3.00
>>>
```

191.95 x 80.98 mm

Topology optimization flow (source:3DPrint.com)

Figure – Computer aided design and topological optimization.

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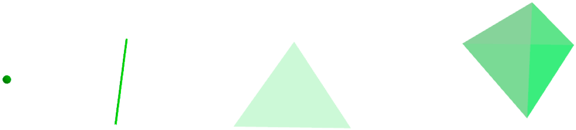
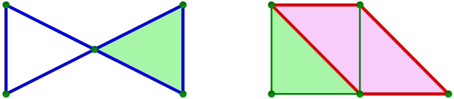
1. wiki.freecad.org
2. 3DPrint.com

Introduction to Homology and Holes

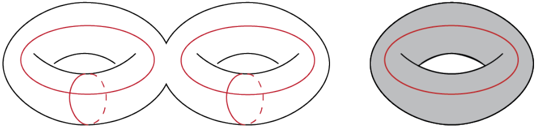
1. Simplicial and Chain Complexes



2. Cycles and Boundaries

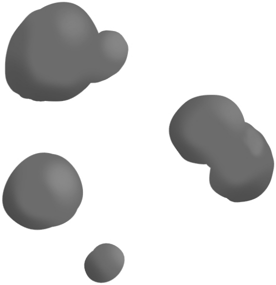


3. Holes and Homology



Holes and Dimension

- 0-holes : connected components

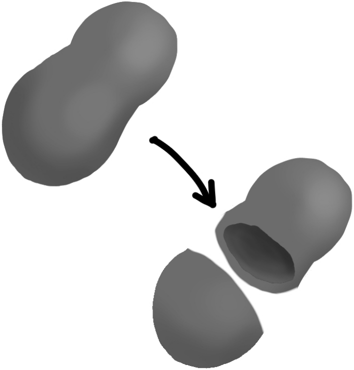


Holes and Dimension

- 0-holes : connected components
- 1-holes : tunnels or handles

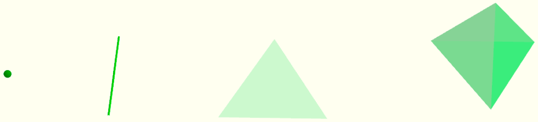
Holes and Dimension

- 0-holes : connected components
- 1-holes : tunnels or handles
- 2-holes : cavities



Simplicial complexes

Simplices



Simplicial complexes

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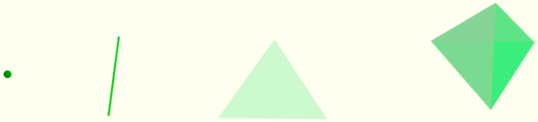
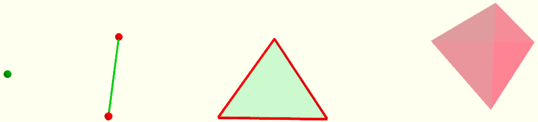


Boundary of simplices



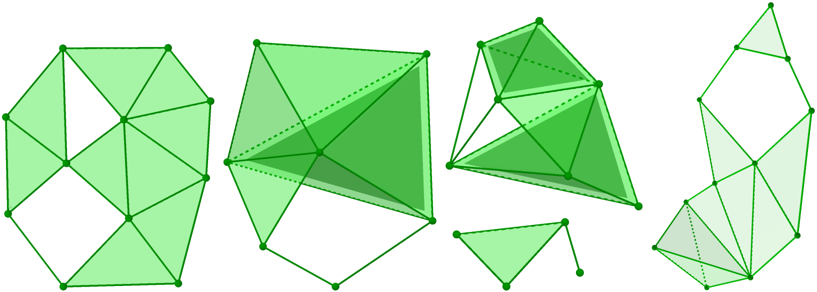
Simplicial complexes

Simplicial complex - Definition

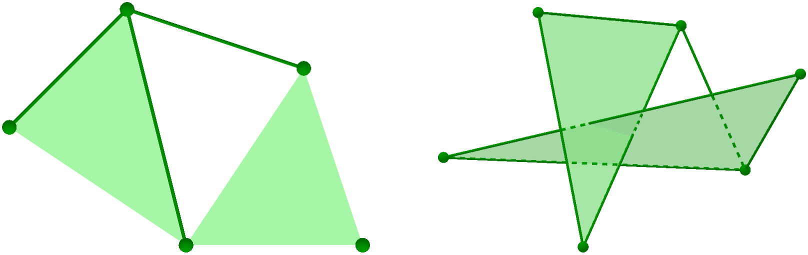
A simplicial complex K is a set of simplices satisfying the two following properties :

- the boundary of every simplex in K is also included in K.
- the intersection of two simplices of K is either empty, either exactly one common subface.

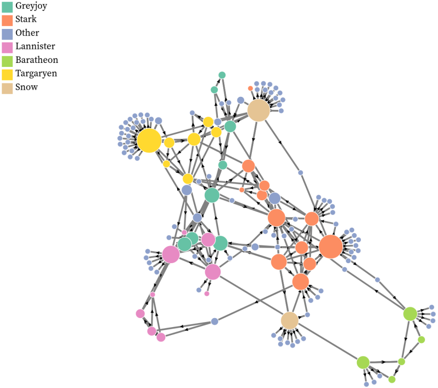
Examples of simplicial complexes

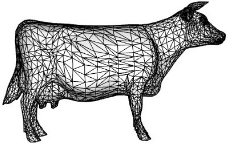


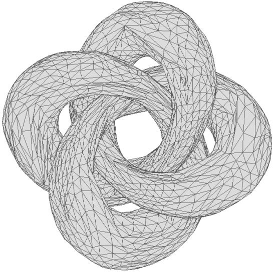
Counter examples of simplicial complexes



Examples of practical simplicial complexes







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a. Game of Thrones Relationship Graph, by Kumar, Martinez, Wong, Zhao.

Chain complex

$$K: \quad C_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_{q+1}} C_q \xrightarrow{\partial_q} C_{q-1} \xrightarrow{\partial_{q-1}} \dots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

Chain complex

$$K: \quad C_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_{q+1}} C_q \xrightarrow{\partial_q} C_{q-1} \xrightarrow{\partial_{q-1}} \dots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

C_q : q-chains

C_q is a vector space called the q-chains.

∂_q : boundary operator

∂_q is a linear map from C_q to C_{q-1} that satisfies $\partial_{q+1} \circ \partial_q = 0$.
It is called the q-boundary operator.

Chain complex of a simplicial complex

$$K : \quad C_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_{q+1}} C_q \xrightarrow{\partial_q} C_{q-1} \xrightarrow{\partial_{q-1}} \dots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

C_q : q-chains

C_q is a vector space called the q-chains.

C_q is the $\mathbb{Z}/2\mathbb{Z}$ vector space generated by the q-**simplices**.

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∂_q is a linear map from C_q to C_{q-1} that satisfies $\partial_{q+1} \circ \partial_q = 0$.
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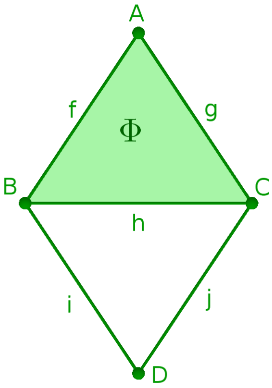
∂_q is the map generated by the boundary of the q-**simplices**.

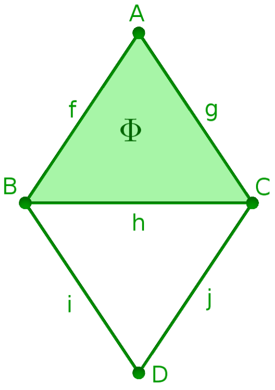
Concrete example



K : $\mathbb{Z}/2\mathbb{Z}$-chain complex

- $C_0 = span(A, B, C, D)$
- $C_1 = span(f, g, h, i, j)$
- $C_2 = span(\Phi)$

Concrete example



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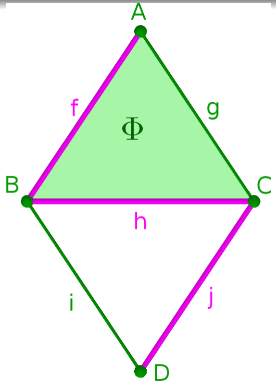
$$\partial_0 = \begin{array}{cccc} A & B & C & D \\ \hline 0 & 0 & 0 & 0 \end{array} \qquad \partial_2 = \begin{array}{cc} \Phi & \\ 1 & f \\ 1 & g \\ 1 & h \\ 0 & i \\ 0 & j \end{array}$$

$$\partial_1 = \begin{array}{cccccc} f & g & h & i & j & \\ 1 & 1 & 0 & 0 & 0 & A \\ 1 & 0 & 1 & 1 & 0 & B \\ 0 & 1 & 1 & 0 & 1 & C \\ 0 & 0 & 0 & 1 & 1 & D \end{array}$$

Concrete example



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$$\begin{aligned} x &= f + h + j \quad \in C_1 \\ \partial_1(x) &= \partial_1(f + h + j) \\ &= (A + B) + (B + C) \\ &\quad + (C + D) \end{aligned}$$

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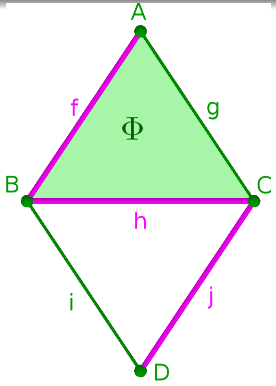

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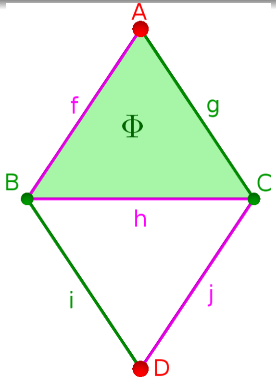
Concrete example



$$
\begin{aligned}
{\color{magenta}x} &= f + h + j \quad \in C_1 \\
{\color{red}\partial_1(x)} &= \partial_1(f + h + j) \\
&= (A + B) + (B + C) \\
&\quad + (C + D) \\
{\color{red}\partial_1(x)} &= A + D
\end{aligned}
$$

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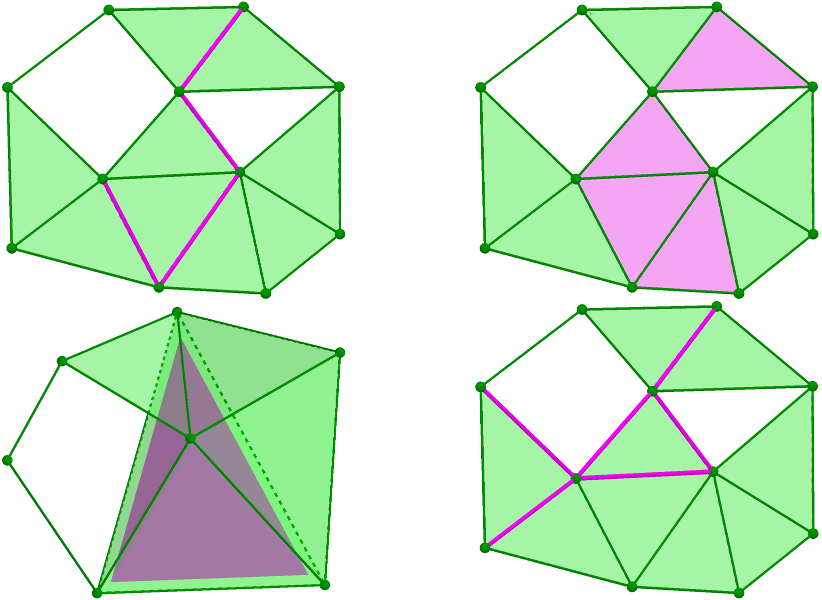
Boundary operator examples



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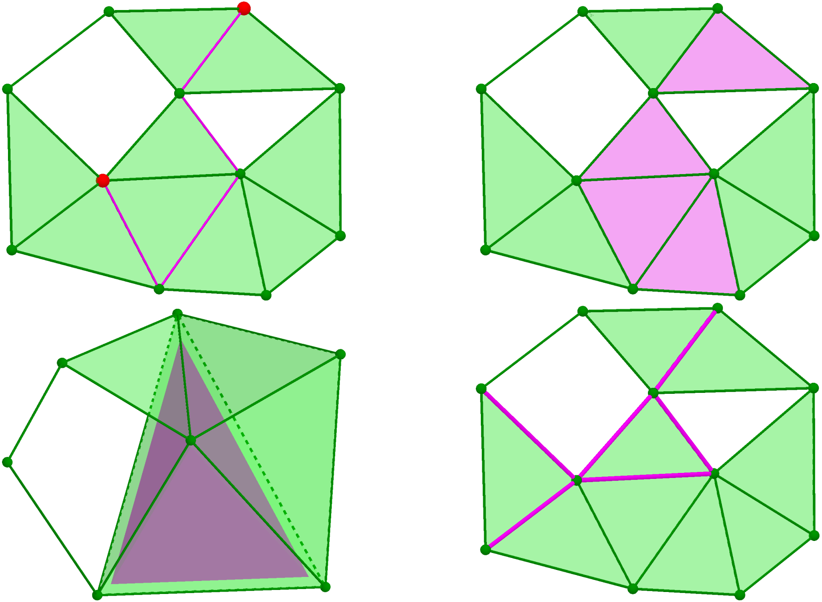

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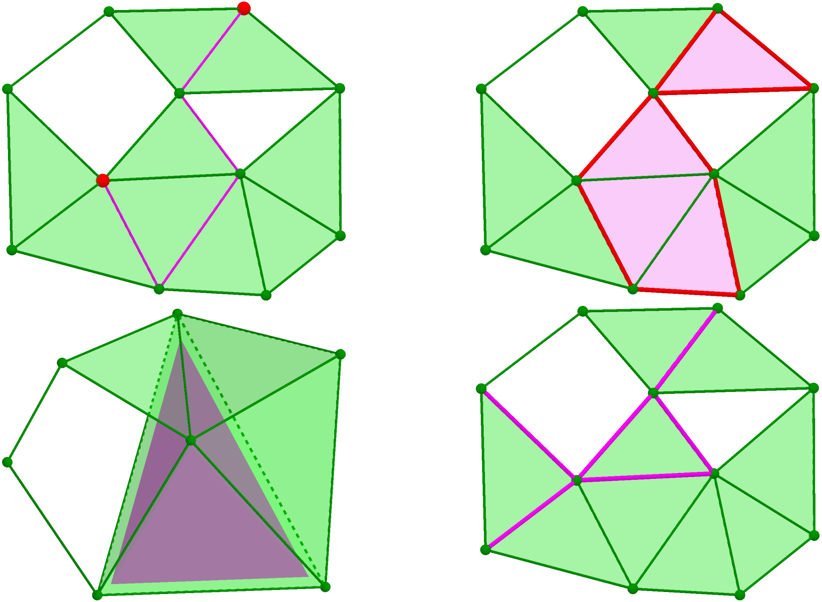

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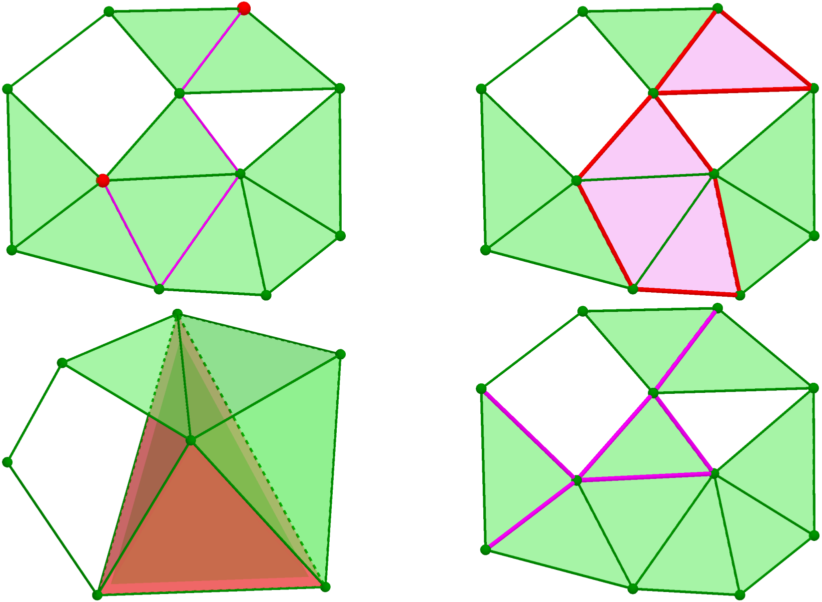

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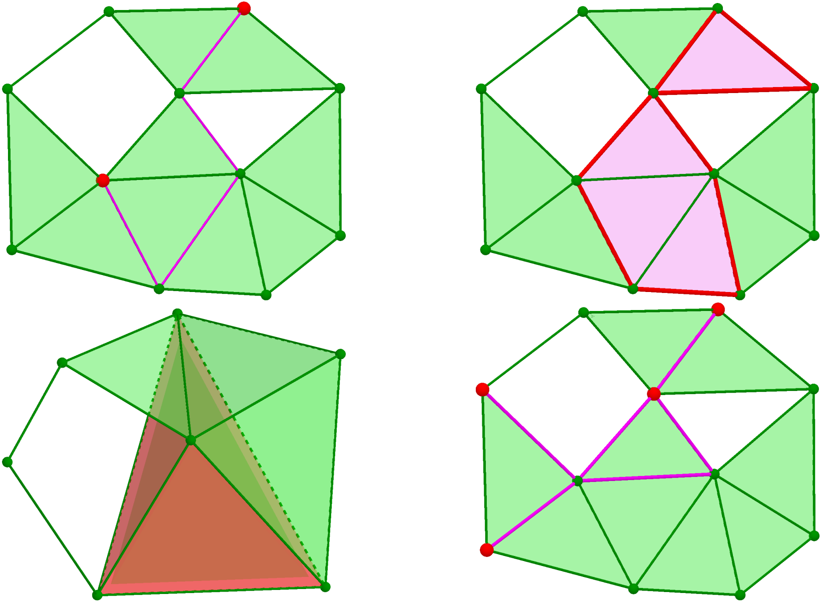

Boundaries

Boundary - Definition

A q-boundary is a q-chain that is the boundary of a $(q+1)$-chain.

$$q\text{-boundaries} = \operatorname{im}(\partial_{q+1})$$

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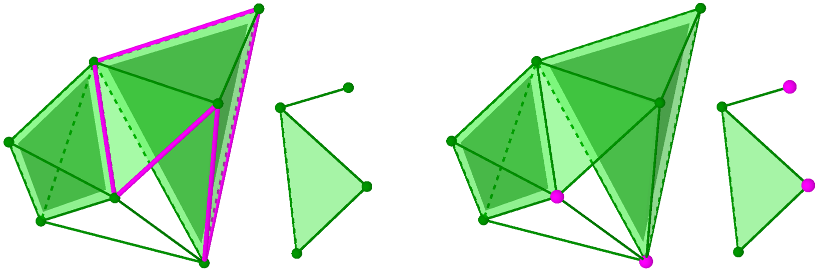

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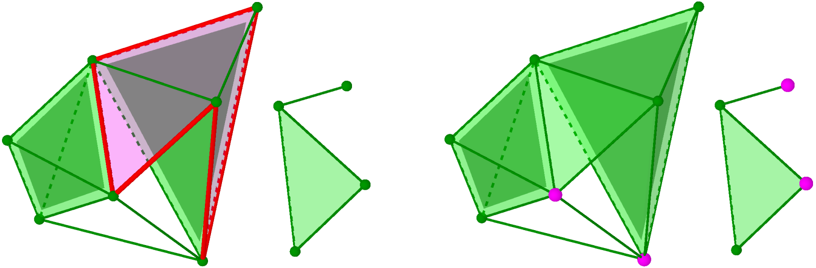

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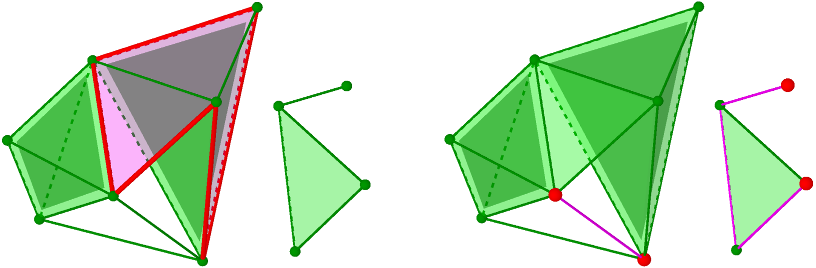
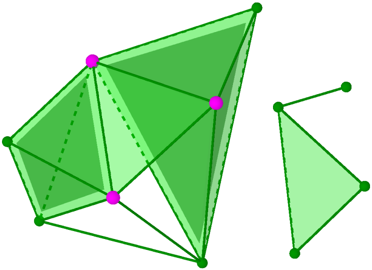
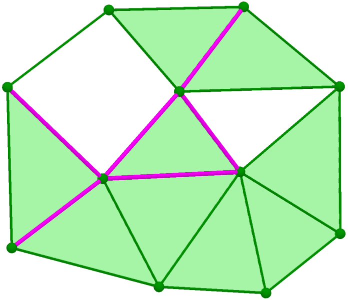


Non boundaries





Cycles

Cycle - Definition

A q-cycle is a q-chain whose boundary is null.

$$q\text{-cycles} = \ker(\partial_q)$$

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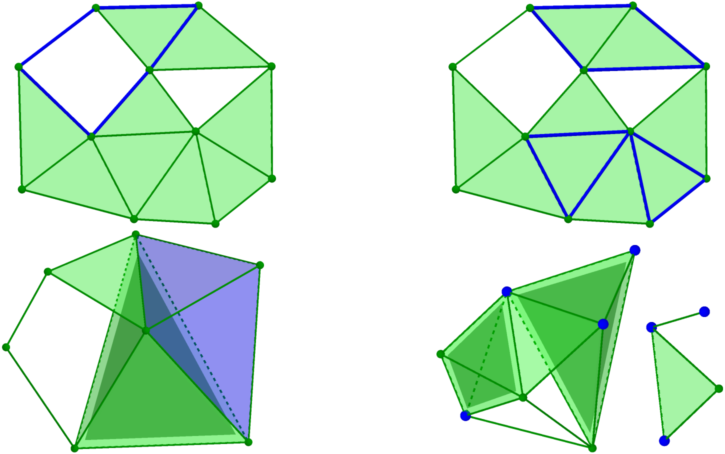

Cycles and Boundaries

Proposition

A boundary is a cycle.

Cycles and Boundaries

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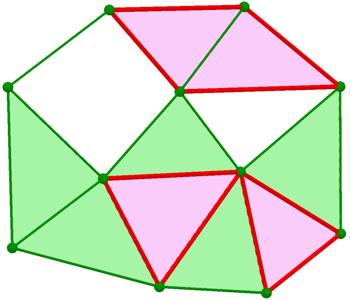
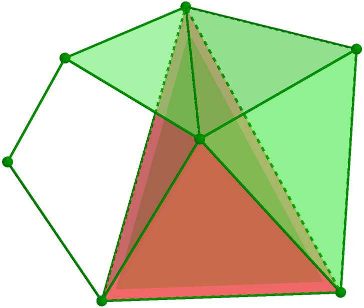
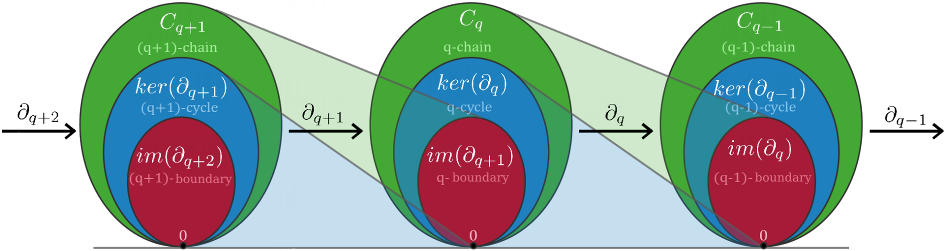
A boundary is a cycle.

$\partial_{q+1} \circ \partial_q = 0$, "a boundary has no boundary", $\mathrm{im}(\partial_{q+1}) \subseteq \ker(\partial_q)$.





Cycles and Boundaries : Summary



Holes and Homology

Hole - Intuitive definition

A q-hole is a q-cycle that is not a q-boundary.

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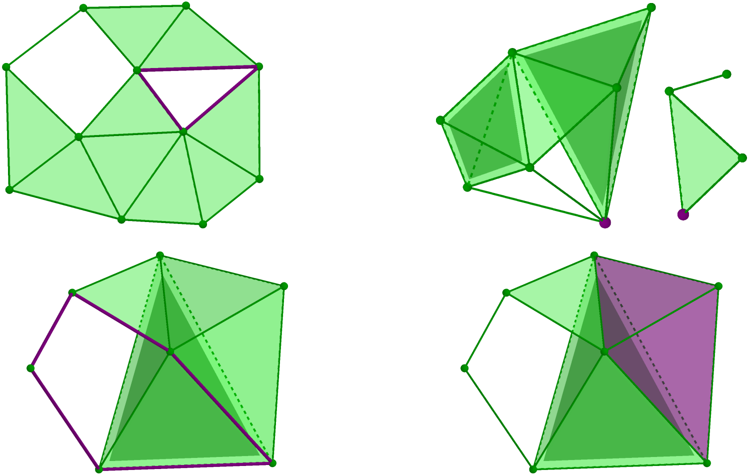

Holes and Homology

Hole - Equivalence

Two *q*-holes are equivalent iff their difference is a *q*-boundary.

$$x \overset{q}{\sim} x \Longleftrightarrow x - y \in \mathrm{im}(\partial_{q+1})$$

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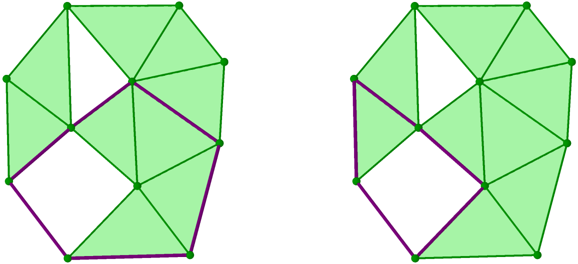

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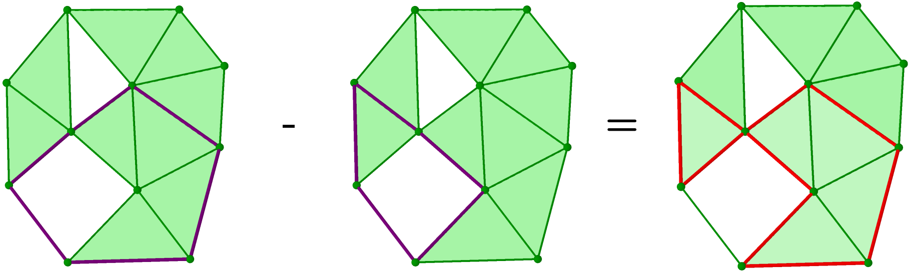

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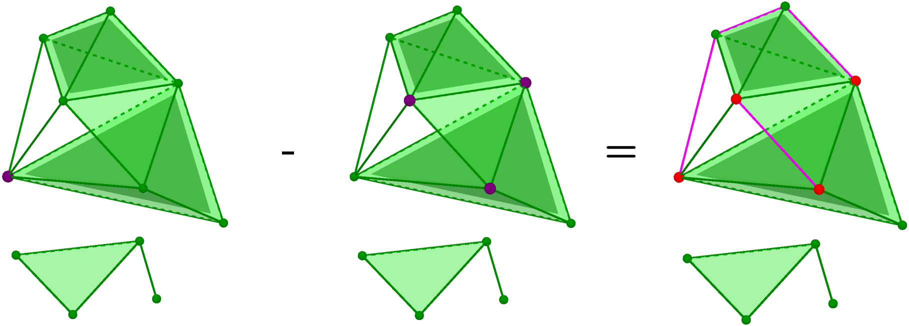

Holes and Homology

Homology group - Definition

The equivalence classes of $\overset{q}{\sim}$ form a group structure, called the *q*-**homology group** :

$$\mathrm{H_q(K)} = \frac{\ker(\partial_q)}{\overset{q}{\sim}}$$

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Homology group - Definition

The equivalence classes of $\overset{q}{\sim}$ form a group structure, called the q-**homology group** :

$$\mathrm{H_q(K)} = \frac{\ker(\partial_q)}{\overset{q}{\sim}} \qquad = \frac{\ker(\partial_q)}{\mathrm{im}(\partial_{q+1})}$$

Betti numbers - Proposition

There exist a number β_q such that $\mathrm{H_q(K)} \approx (\mathbb{Z}/2\mathbb{Z})^{\beta_q}$.

β_q is called the **Betti number** of dimension q and intuitively represent the number of holes of dimension q.

Holes and Homology

Clarification

$\mathsf{H_q(K)} \approx (\mathbb{Z}/2\mathbb{Z})^{\beta_q}$: there are β_q holes and 2^{β_q} equivalence classes in $\mathsf{H_q(K)}$. Each equivalence class represents a subset of holes.

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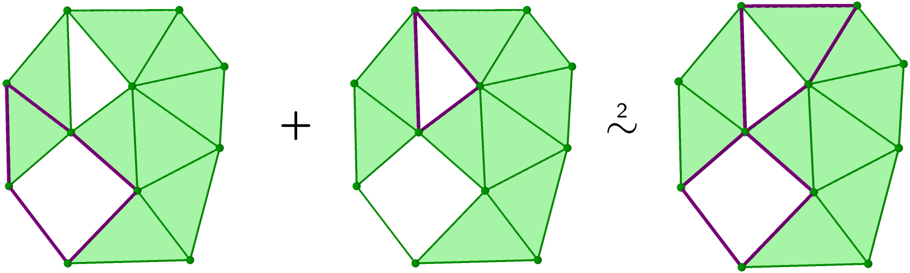
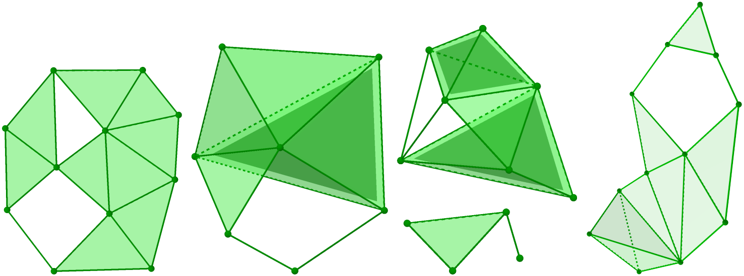


$$H_1(K) \approx (\mathbb{Z}/2\mathbb{Z})^2$$

Conclusion

Starting from a combinatorial/geometric structure (**simplicial complex**), we built an algebraic structure (**chain complex**) that allowed us to intuitively define **holes** and formally grasp **homology groups**.



To go further : Computing homology

Three approaches for computational homology :

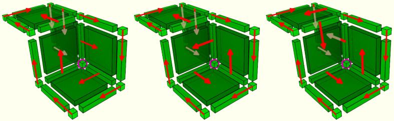
Effective approach

Computation of reductions

$$\begin{tikzcd}
\cdots \arrow[r] & C_{q+1} \arrow[r, shift left, "\partial_{q+1}"] \arrow[d, shift left, "f_{q+1}"] & C_q \arrow[l, shift left, "h_q"] \arrow[r] \arrow[d, shift left, "f_q"] & \cdots \\
\cdots \arrow[r] & C'_{q+1} \arrow[r, "0"] \arrow[u, shift left, "g_{q+1}"] & C'_q \arrow[r] \arrow[u, shift left, "g_q"] & \cdots
\end{tikzcd}$$

Combinatorial approach

Discrete Morse Theory



Algebraic approach

Smith Normal Form

$$\partial = P \begin{pmatrix} \alpha_1 & 0 & 0 & \cdots & 0 \\ & \ddots & & & \\ 0 & & 0 & \cdots & 0 \\ 0 & 0 & \alpha_r & & 0 \\ \vdots & & & 0 & \vdots \\ 0 & & \cdots & & 0 \end{pmatrix} Q$$

where $\alpha_i \mid \alpha_{i+1}$.

To go further : Homology over a ring

Chain complex with a ring

If use $\mathbb{Z}$ instead of $\mathbb{Z}/2\mathbb{Z}$, C_q is not anymore a vector space but a $\mathbb{Z}$-**module**. Weird things happen...

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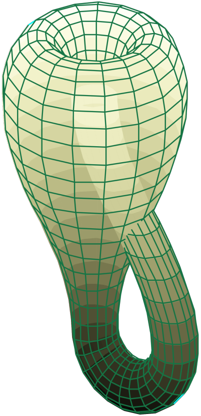
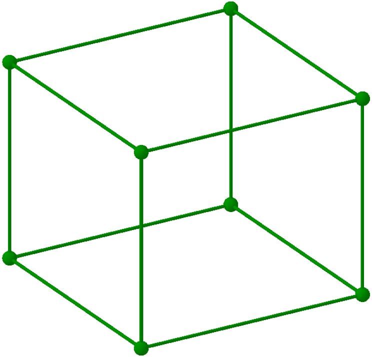
Holes and torsion

$$H_q(K) \approx \underbrace{\mathbb{Z}^{\beta_q}}_{\textbf{holes}} \times \underbrace{\frac{\mathbb{Z}}{\alpha_1\mathbb{Z}} \times \frac{\mathbb{Z}}{\alpha_2\mathbb{Z}} \times \cdots \times \frac{\mathbb{Z}}{\alpha_m\mathbb{Z}}}_{\textbf{torsion}}$$



$H_1(K) \approx \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

The location of a hole : where intuition struggles



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