Boundary rigidity of finite CAT(0) cube complexes

Jérémie Chalopin and Victor Chepoi, LIS, Marseille, France

Master 2 IMD, Luminy, November 2023

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Based on the paper:

- J. Chalopin and V. Chepoi, Boundary rigidity of finite CAT(0) cube complexes, arXiv:2310.04223, 2023.

Problem's formulation

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- *Boundary ∂X of X:* the downward closure of all non-maximal cells of X such that each of them is a facet of a unique cell of X.
- *1-Skeleton of X:* the graph $G = G(X)$ with 0-cells as vertices and 1-cells as edges and endowed with the standard graph-distance d_G.
- *Boundary rigidity of X:* X can be reconstructed from the pairwise distances (computed in G) between all vertices belonging to ∂X.

Motivation and History

- *Origins:* Riemannian geometry, where it is conjectured (Michel, 1981/82) that any Riemannian manifold (M, g) is boundary rigid, i.e., its metric d_g is determined up to isometry by its boundary distance function.

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- *Discrete version:* I. Benjamini asked if any plane triangulation in which all inner vertices have degrees ≥ 6 is boundary rigid. Confirmed by Haslegrave in 2023.
- *Partial results for CAT(0) cube complexes:* 2-dimensional and embedded in $\mathbb{R}^3$ 3-dimensional CAT(0) cube compelxes (Haslegrave et al., 2023).

CAT(0) spaces

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- *Comparison triangle: for* $\Delta(x_1, x_2, x_3)$ is a triangle $\Delta(x'_1, x'_2, x'_3)$ in $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(x'_i, x'_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

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- *Comparison axiom:* If y is a point on the side of $\Delta(x_1, x_2, x_3)$ with vertices x_1 and x_2 and y' is the unique point on the line segment $[x'_1, x'_2]$ of the comparison triangle $\Delta(x'_1, x'_2, x'_3)$ such that $d_{\mathbb{E}^2}(x'_i, y') = d(x_i, y)$ for $i = 1, 2$, then $d(x_3, y) \leq d_{\mathbb{E}^2}(x'_3, y')$.

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- *CAT(0) space:* A geodesic metric space (X, d) in which all geodesic triangles satisfy the comparison axiom.

CAT(0) cube complexes

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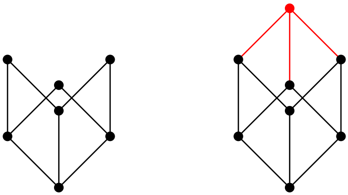
- *Cube condition:* any three d-cubes, pairwise intersecting in $(d-1)$-cubes and all three intersecting in a $(d-2)$-cube, belong to a $(d+1)$-cube.



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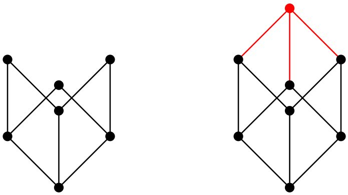


Theorem (Gromov, 1987)

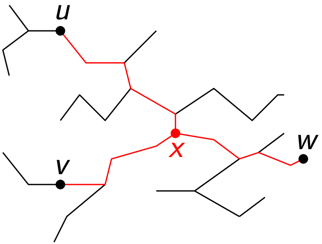
A cube complex X endowed with the ℓ_2-metric is CAT(0) iff X is simply connected and X satisfies the cube condition.

Median graphs

- In a graph G, the interval $I(u, v)$ between two vertices u and v is

$$I(u, v) = \{x : d(u, x) + d(x, v) = d(x, v).\}$$

- A graph is median if for all u, v, w, there exists a unique $x \in I(u, v) \cap I(v, w) \cap I(u, w)$.



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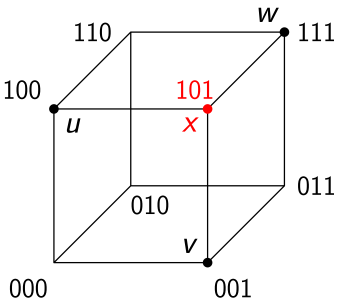

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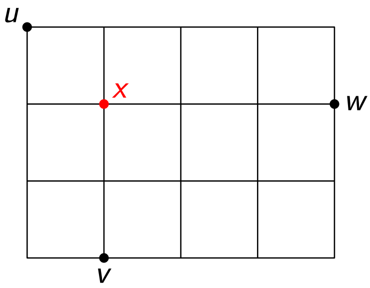

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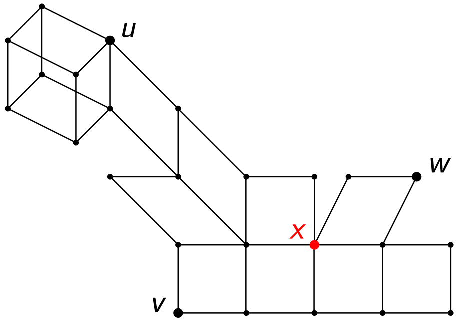


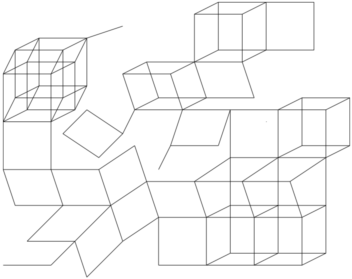
CAT(0) cube complexes and median graphs

Theorem (C, 1998, Roller, 1998)

A cube complex X is CAT(0) iff it 1-skeleton is a median graph.

Theorem (C, 1998)

A graph G is a median graph if and only if its cube complex $X_{cube}(G)$ is simply connected and G satisfies the 3-cube condition. Furthermore, if X is a CAT(0) cube complex, then $X = X_{cube}(G(X))$.



Facts about median graphs

- *Quadrangle condition:* For any u, v, w, z such that $v, w \sim z$ and $d(u, v) = d(u, w) = d(u, z) - 1 = k$, there is a unique vertex $x \sim v, w$ such that $d(u, x) = k - 1$;

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- *Cubes a gated:* Cubes of median graphs are gated;
- *Downward cube property:* For any basepoint z and any vertex v, there exists a unique cube $C(v)$ containing all neighbors $\Lambda(v)$ of v in $I(v,z)$. The vertex $\overline{v}$ opposite to v in $C(v)$ is the gate of z in the cube $C(v)$.

Corner peeling

Definition

- A *corner* of a graph G is a vertex v of G such that v and all its neighbors in G belong to a unique cube of G. Note that any corner belongs to the boundary ∂G.

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- A *corner peeling* of $G = (V, E)$ is a total order $v_1, \ldots, v_n$ of V such that v_i is a corner of the subgraph $G_i = G[v_1, \ldots, v_i]$ induced by the first i vertices of this order.

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- A *monotone corner peeling (mcp)* of G with respect to z is a corner peeling $v_1 = z, v_2, \ldots, v_n$ such that $d(z, v_1) \leq d(z, v_2) \leq \ldots \leq d(z, v_n)$.

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Proposition

For any basepoint z of G, any ordering $v_1 = z, v_2, \ldots, v_n$ such that $d(z, v_1) \leq d(z, v_2) \leq \ldots \leq d(z, v_n)$ is a mcp*. Furthermore, $C(v_i)$ is the unique cube of G_i containing v_i and the neighbors of v_i in G_i and the vertex $\overline{v_i}$ opposite to v_i in $C(v_i)$ is the gate of z in C_i.*

Mcp and lemmas about boundaries

Notations: Let $v_1 = z, v_2, \dots, v_n$ be a mcp of G. Denote by ∂G_i the boundary of the cube complex $X_i = X_{cube}(G_i)$ restricted to G_i, $i = n, \dots, 1$. Let $C_i = C(v_i)$ the unique cube of G_i containing v_i and $\Lambda(v_i)$ be the set of all neighbors of v_i in G_i. Denote also by $u_i = \overline{v_i}$ the opposite of v_i in C_i.

Lemma

All vertices of the cube C_i except eventually u_i belong to the boundary ∂G_i of G_i.

Set $S(G_n) = \partial G_n = \partial G$ and $S(G_{i-1}) = S(G_i) \setminus \{v_i\} \cup \{u_i\}$, $i = n-1, \dots 2$. We call $S(G_i)$ the *extended boundary* of G_i.

Lemma

For any $i = n, \dots 2$, we have $\partial G_{i-1} \subseteq \partial G_i \cup \{u_i\}$ and $\partial G_i \subseteq S(G_i)$.

Proof of the lemma

Lemma

For any $i = n, \dots 2$, we have $\partial G_{i-1} \subseteq \partial G_i \cup \{u_i\}$ and $\partial G_i \subseteq S(G_i)$.

Proof: Inclusion $\partial G_{i-1} \subseteq \partial G_i \cup \{u_i\}$.

(1) Let $x \in \partial G_{i-1} \setminus \partial G_i$.
(2) $x \in \partial G_{i-1} \Rightarrow \exists\, C \in X_i$ s.t. $x \in C$ and C is a facet of unique $C' \in X_i$.
(3) $X_{i-1} \subset X_i$ and $x \notin \partial X_i \Rightarrow C$ is a facet of yet another cube C'' of X_i.
(4) $C'' \in X_i \setminus X_{i-1} \Rightarrow v_i \in C''$.
(5) All cubes of X_i containing v_i are included in $C_i \Rightarrow x \in C_i$.
(6) $C_i \setminus \{u_i\} \subset \partial X_i$ and $x \notin \partial G_i \Rightarrow x = u_i$.

Inclusion $\partial G_i \subseteq S(G_i)$. By induction on $i = n, \dots, 1$. For $i = n$, $S(G_n) = \partial G_n$. Suppose the assertion holds for G_i and consider G_{i-1}. Since $v_i \notin G_i$, the first inclusion and the induction assumption yield

$$\partial G_{i-1} \subseteq \partial G_i \setminus \{v_i\} \cup \{u_i\} \subseteq S(G_i) \setminus \{v_i\} \cup \{u_i\} = S(G_{i-1}).$$

The reconstruction algorithm, I

Goal: Reconstruct a median graph G and its cube complex $X = X_{cube}(G)$ from the pairwise distances between the vertices of the boundary ∂G.

Variables:

- Pick an arbitrary vertex $z \in \partial G$ as a basepoint.
- During the algorithm, the reconstructor knows a set S of vertices (that is initially ∂G) as well as the distance matrix D of S.
- The reconstructor constructs a graph Γ that is initially the subgraph of G induced by ∂G and will ultimately coincide with G.
- To analyze the algorithm, we consider the values S_i of the set S, D_i of the distance matrix D, and Γ_i of the graph Γ at the beginning of the ith step of the algorithm, and at each step, we decrease the values of i.
- For the analysis of the algorithm, we also consider a graph G_i (unknown to the algorithm), where $G_n = G$.

The reconstruction algorithm, II

Step n: The input consists of the set $S_n = \partial G$ and its distance matrix D_n. The graph Γ_n is computed from D_n.

Step i:

1. The reconstructor picks a vertex v_i of S_i furthest from z;
2. The reconstructor removes v_i from S_i and eventually adds to S_i (if it is not already in S_i) the vertex u_i opposite to v_i in the unique cube C_i of G containing v_i and its neighbors in S_i. The resulting set is denoted by S_{i-1}.
3. From D_i, we compute the distance matrix D_{i-1} of S_{i-1} by computing the distances from u_i to the vertices of $S_{i-1} = S_i \setminus \{v_i\} \cup \{u_i\}$. These distances are easily computed since C_i is gated and $C_i \setminus \{u_i\} \subset S_i$.
4. If $u_i \in S_i$, we set $\Gamma_{i-1} = \Gamma_i$, otherwise Γ_{i-1} is Γ_i plus u_i and the edges between u_i and its neighbors in $S_{i-1} \cup \{v_i\}$ (detected via D_{i-1}).

Endstep: The algorithm ends when S_i becomes empty.

Correctness, I: the invariants

Let G_i be the subgraph of G obtained from G by removing the vertices $v_n, \dots v_{i+1}$. Note that G_i is not known to the reconstructor. Suppose that the removed vertices $v_n, \dots, v_{i+1}$ and the eventually added vertices $u_n, \dots, u_{i+1}$ satisfy the following inductive properties:

1. $d(z, v_n) \geq \dots \geq d(z, v_{i+1}) \geq d(z, v)$ for any vertex of v of G_i,
2. each vertex v_j with $n \geq j \geq i+1$ is a corner of the graph G_j,
3. for each $n \geq j \geq i+1$, either all neighbors of v_j in G_j are in S_j, or u_j is the unique neighbor of v_j in G_j, and $u_j \in S_{j-1}$,
4. S_i coincides with the extended boundary $S(G_i)$ of G_i, D_i is the distance matrix of $S(G_i)$ in G, and $\Gamma_i = G[\bigcup_{n \geq j \geq i} S_j]$.

Correctness, II: v_i is a corner of G_i

Lemma

Let v_i be a vertex of S_i maximizing $d(z, v_i)$. Then $d(z, v_i) \geq d(z, v)$ for any vertex v of G_i and thus v_i is a corner of G_i.

Proof: (1) Suppose $\exists u$ in G_i s.t. $d(z, v_i) < d(z, u)$ and wlog u maximizes $d(z, u)$ among vertices of G_i.

(2) Since $d(z, v_n) \geq \ldots \geq d(z, v_{i+1}) \geq d(z, v)$ for any vertex of v of G_i by invariant (1), from Proposition there exists a mcp of G starting with $v_n, \ldots, v_{i+1}, u$.

(3) Thus u is a corner of G_i, i.e. $u \in \partial G_i$. Since $\partial G_i \subseteq S(G_i)$ by Lemma and $S(G_i) = S_i$ by invariant (4), $u \in S_i$, contradicting the choice of v_i.

(4) Hence v_i is a vertex of G_i maximizing $d(z, v_i)$ and a corner of G_i.

Correctness, III: the invariants hold after step i

Invariants (1) and (2) follow from previous Lemma and the definition of v_i.

Invariant (3) follows from the definition of u_i and lemmas about boundaries.

Invariant (4): (a) Since $S_i = S(G_i)$, and by the definitions of v_i and u_i, we have $S_{i-1} = S_i \setminus \{v_i\} \cup \{u_i\} = S(G_i) \setminus \{v_i\} \cup \{u_i\} = S(G_{i-1})$.

(b) Since the distances from u_i to all vertices of S_{i-1} have been correctly computed, by induction hypothesis, D_{i-1} is the distance matrix of S_{i-1} that coincides with $S(G_{i-1})$.

(c) If $u_i \in S_i$, then $\Gamma_{i-1} = \Gamma_i = G[\bigcup_{n \geq j \geq i} S_j] = G[\bigcup_{n \geq j \geq i-1} S_j]$.
If $u_i \notin S_i$, then $V(\Gamma_{i-1}) = V(\Gamma_i) \cup \{u_i\} = \bigcup_{n \geq j \geq i-1} S_j$.
Now, pick any edge $u_i w$ of G with $w \in V(\Gamma_{i-1})$. If $w \in S_{i-1} \cup \{v_i\}$, then the edge wu_i is in $E(\Gamma_{i-1})$. Otherwise, $w = v_j$ with $j > i$. However, since $u_i \notin S_j$, this implies by invariant (3) that $u_i \in S_{j-1}$ and thus in S_i, a contradiction. Therefore, Γ_{i-1} is the subgraph of G induced by $\bigcup_{n \geq j \geq i-1} S_j$.

The main result

Lemma

The graph Γ_0 returned by the reconstructor is isomorphic G.

Proof: By invariant (1) of the algorithm, z is the last vertex removed from S. By the last lemma, when z is considered by the algorithm, all vertices of G have been already processed. This implies, that each vertex $x \in V(G)$ belongs to some S_i and thus to $V(\Gamma_0)$, establishing $V(\Gamma_0) = V(G)$. By invariant (4), Γ_0 is an induced subgraph of G and is thus isomorphic to G. From this lemma and the bijection between X and $X_{cube}(G(X))$, we obtain:

Theorem

Any finite CAT(0) cube complex is boundary rigid.

Thank you!