Introduction to Homology and Holes

Yann-Situ Gazull

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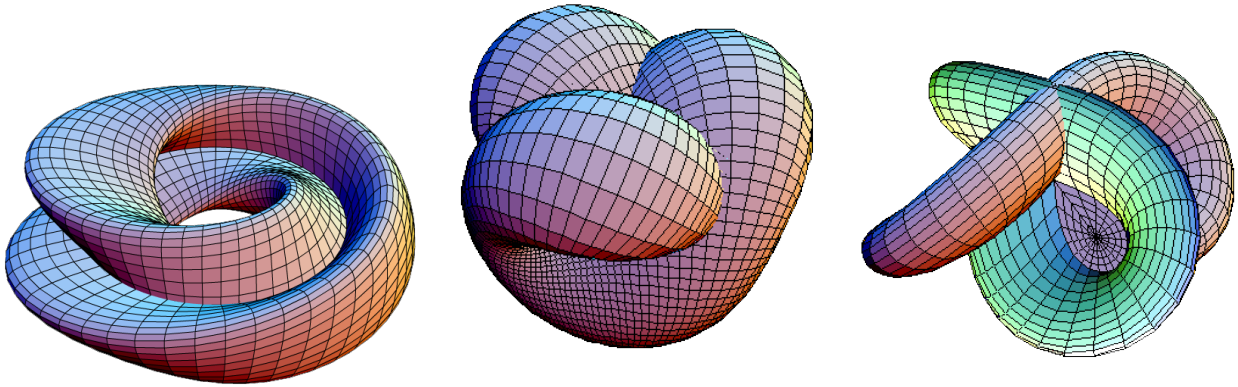
Topology



1. Topology joke, by Henry Segerman.
2. Wikipedia.

Algebraic Topology

Topology

- spaces
- simplicial complexes
- shapes
- ...

$\longleftrightarrow$

Algebra

- groups
- rings
- fields
- ...

Subfields

Homotopy, homology, cohomology, knot theory...

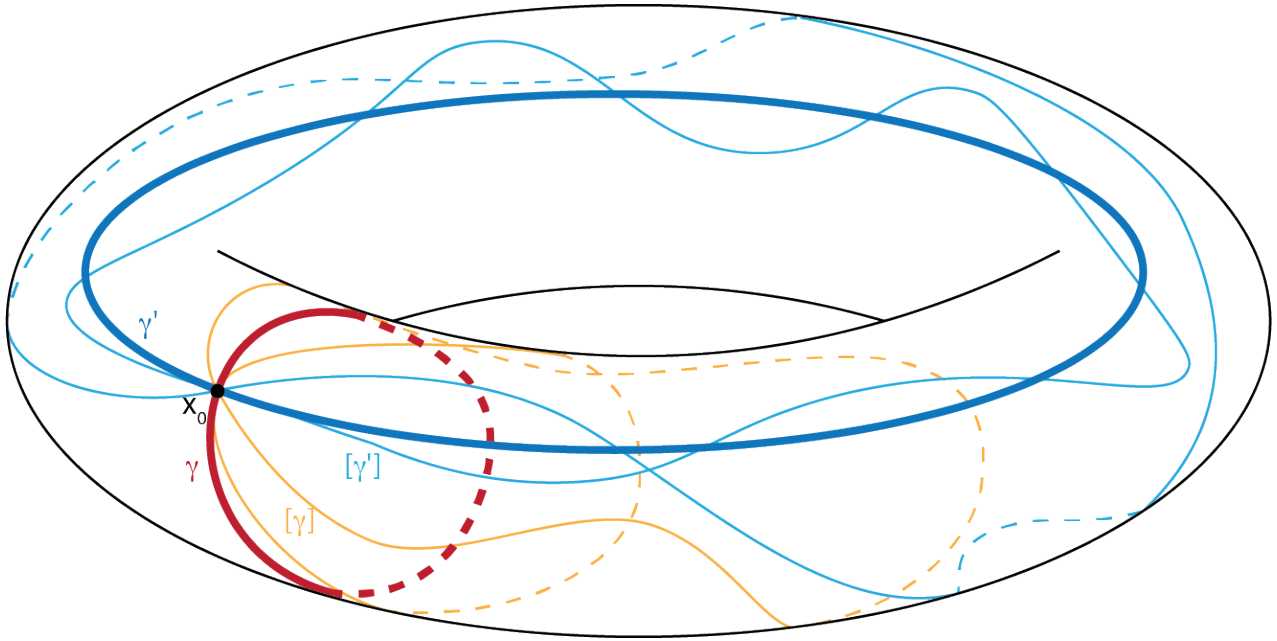


Figure – Illustration of homotopy on a torus.

$$\text{homeomorphic} \implies \text{homotopic} \implies \text{homologuous}$$

Decidability in Algebraic Topology

- Are two groups isomorphic given their representations ? Undecidable [1].
- Are two triangulations homeomorphic ? Undecidable.
- Are two triangulations homotopic ? Undecidable.
- Are two triangulations homologuous ? **Decidable** !

1. P. S. Novikov, "Unsolvability of the conjugacy problem in the theory of groups"(1954)

Why computing topology ?

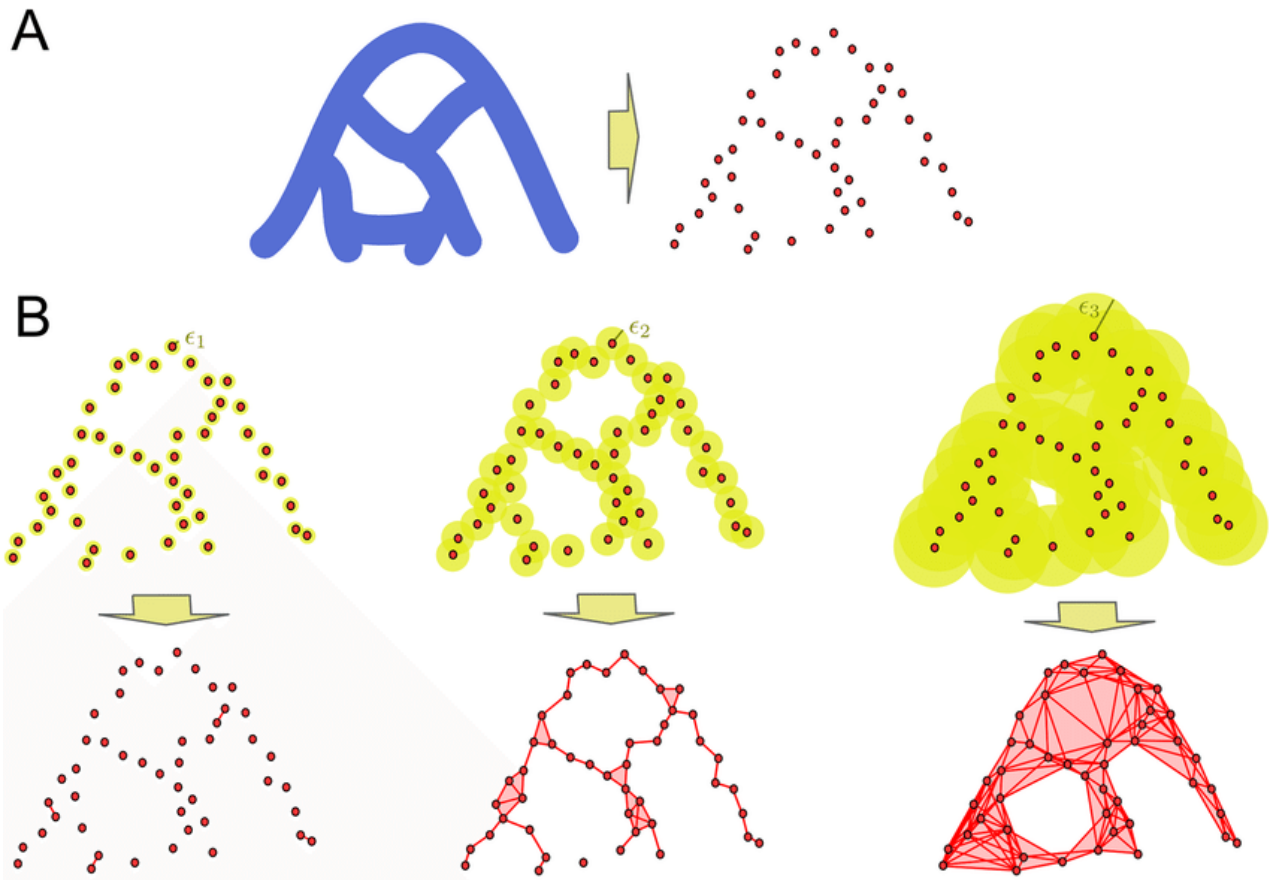


Figure – Topological data analysis.

1. Camara, Pablo & Levine, Arnold & Rabadan, Raul. (2015). Inference of Ancestral Recombination Graphs through Topological Data Analysis.

Why computing topology ?

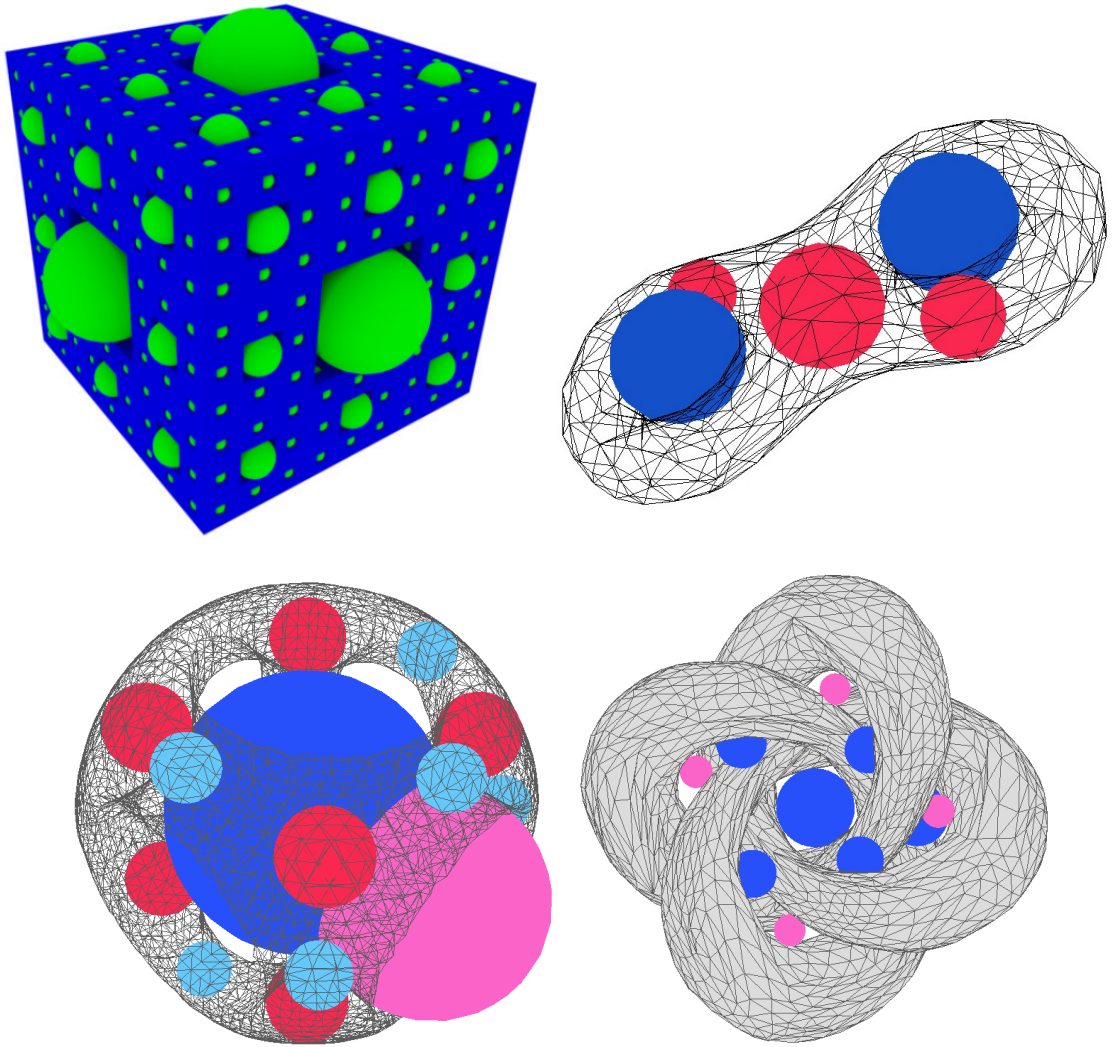


Figure – Holes measure, mainly useful for shape analysis or classification.

Why computing topology ?

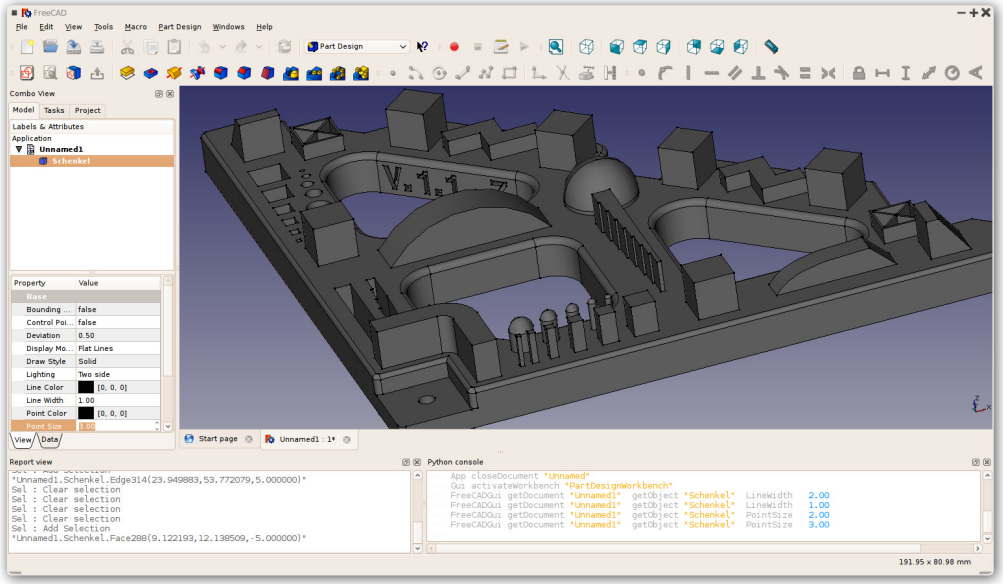


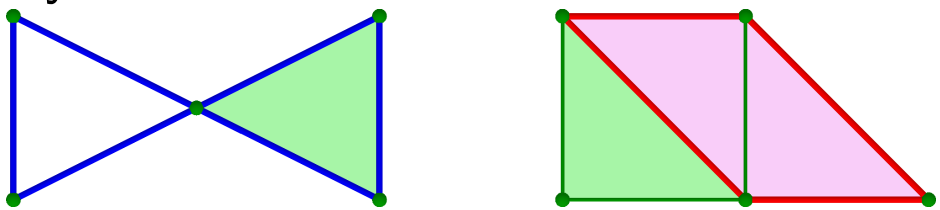
Figure – Computer aided design and topological optimization.

1. wiki.freecad.org
2. 3DPrint.com

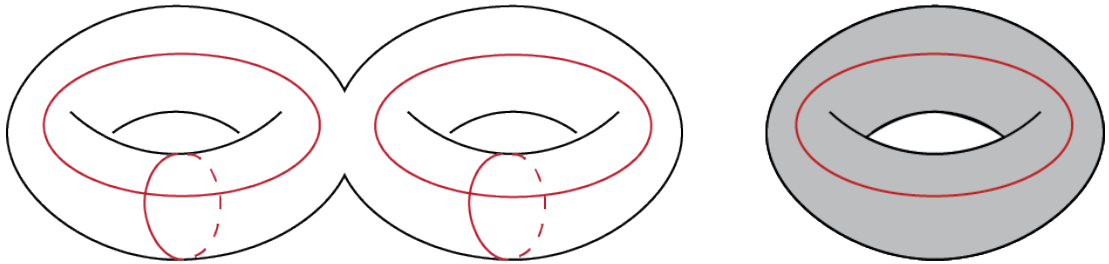
Introduction to Homology and Holes

1. Simplicial and Chain Complexes

2. Cycles and Boundaries

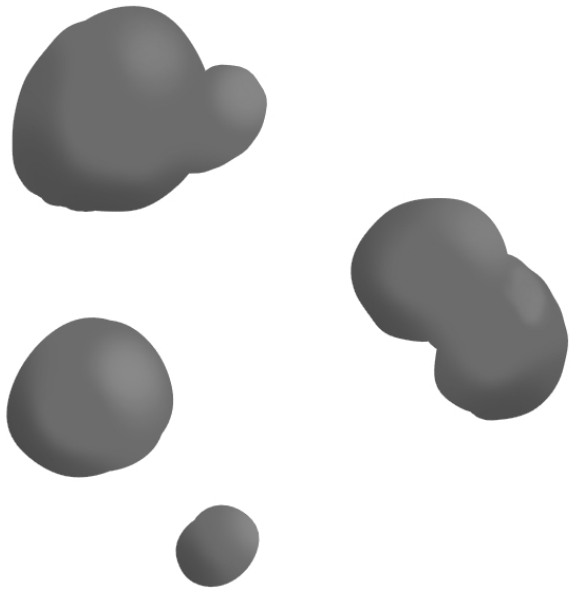


3. Holes and Homology



Holes and Dimension

- 0-holes : connected components

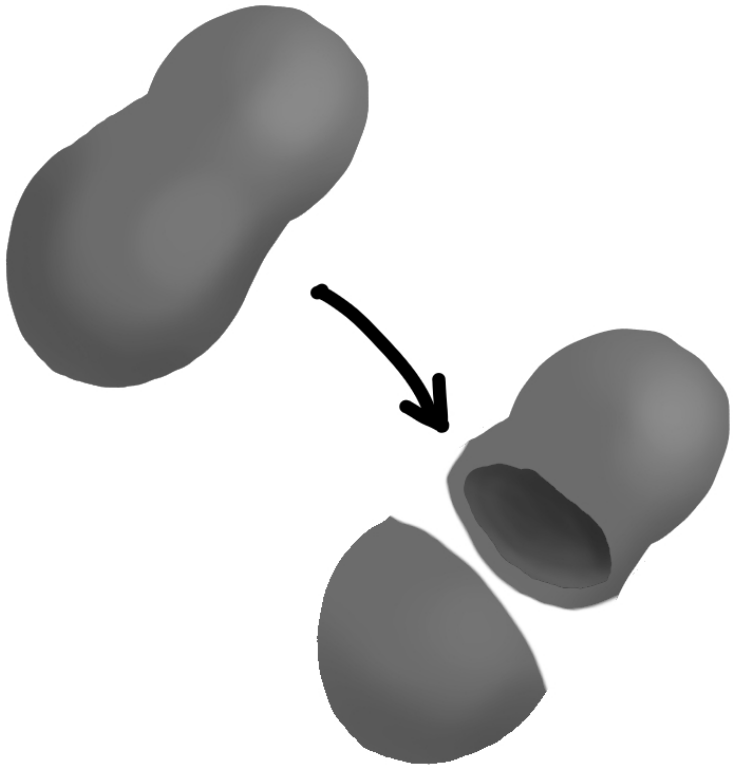


Holes and Dimension

- 0-holes : connected components
- 1-holes : tunnels or handles

Holes and Dimension

- 0-holes : connected components
- 1-holes : tunnels or handles
- 2-holes : cavities



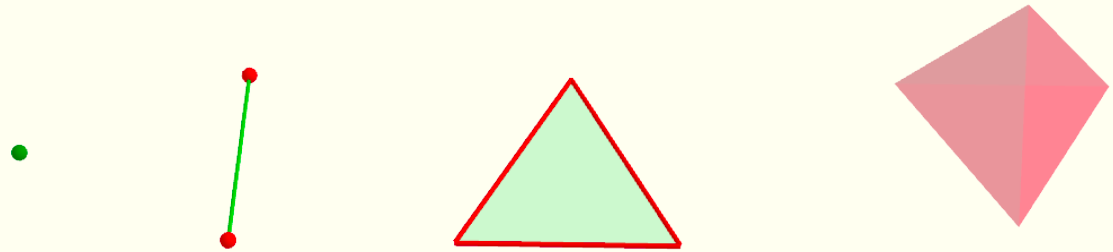
Simplicial complexes

Simplices

Simplicial complexes

Simplices

Boundary of simplices



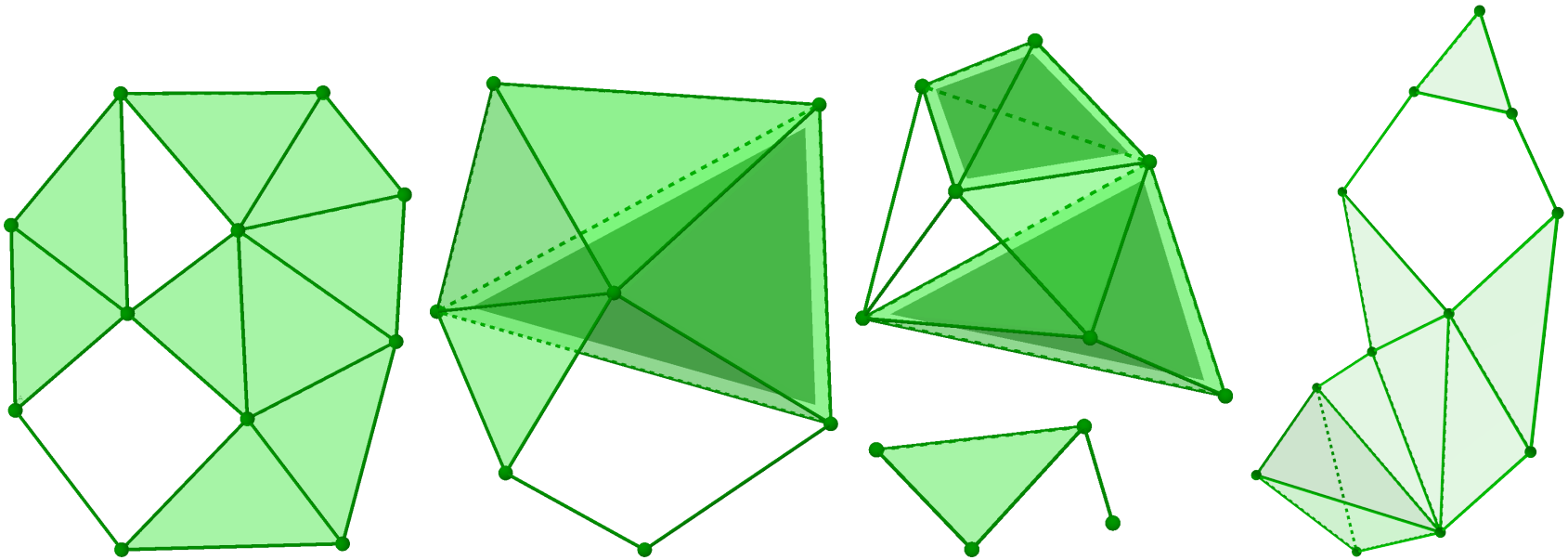
Simplicial complexes

Simplicial complex - Definition

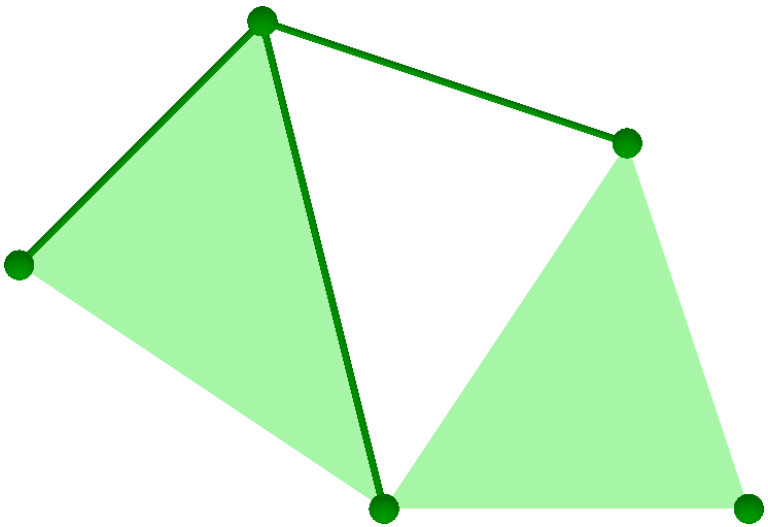
A simplicial complex K is a set of simplices satisfying the two following properties :

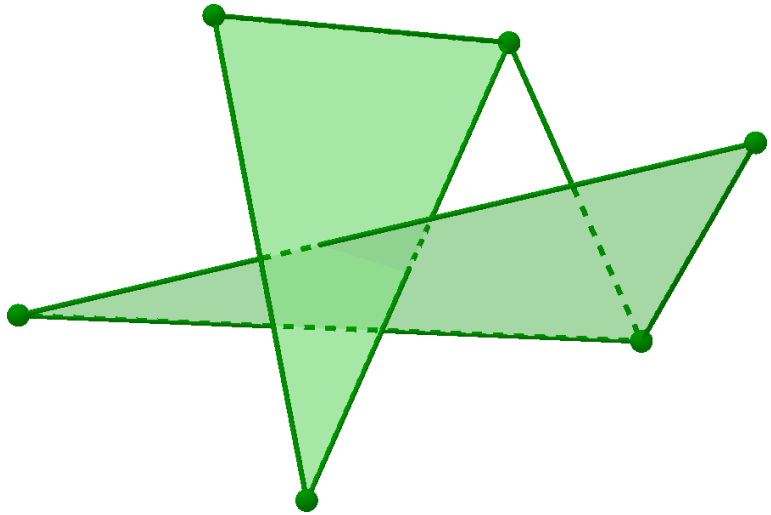
- the boundary of every simplex in K is also included in K.
- the intersection of two simplices of K is either empty, either exactly one common subface.

Examples of simplicial complexes

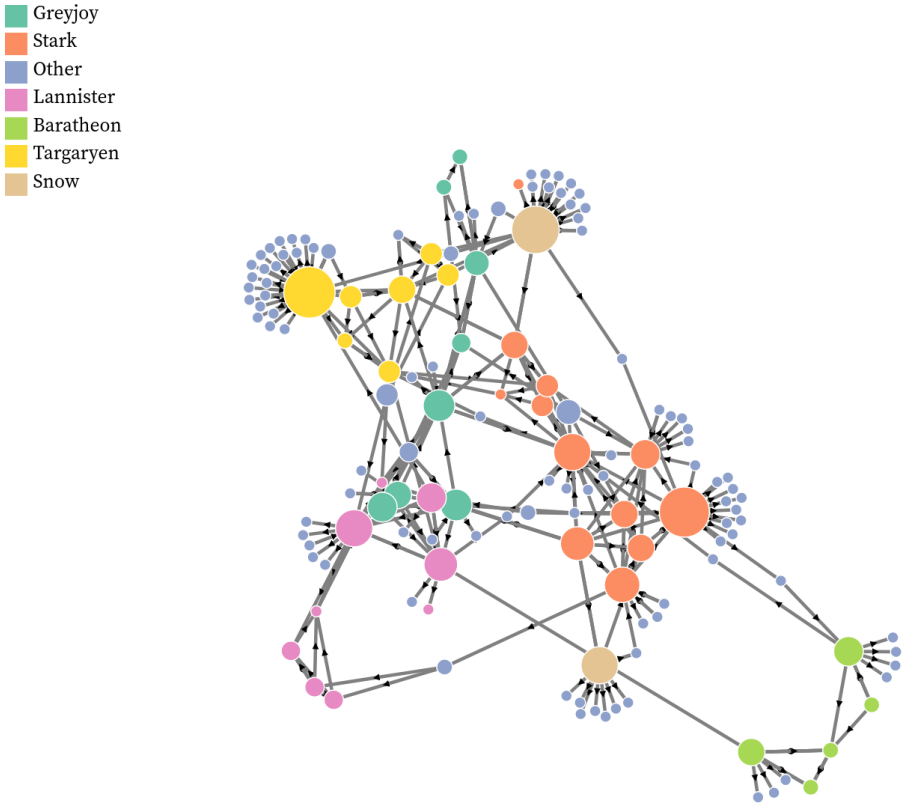


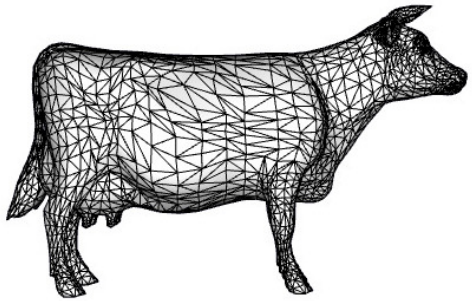
Counter examples of simplicial complexes

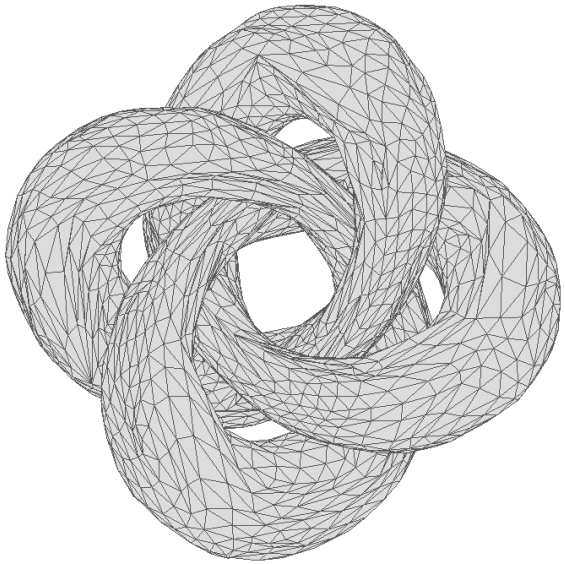




Examples of practical simplicial complexes







a. Game of Thrones Relationship Graph, by Kumar, Martinez, Wong, Zhao.

Chain complex

$$K : \quad C_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_{q+1}} C_q \xrightarrow{\partial_q} C_{q-1} \xrightarrow{\partial_{q-1}} \dots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

Chain complex

$$K: \quad C_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_{q+1}} C_q \xrightarrow{\partial_q} C_{q-1} \xrightarrow{\partial_{q-1}} \dots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

C_q : q-chains

C_q is a vector space called the q-chains.

∂_q : boundary operator

∂_q is a linear map from C_q to C_{q-1} that satisfies $\partial_{q+1} \circ \partial_q = 0$.
It is called the q-boundary operator.

Chain complex of a simplicial complex

$$K: \quad C_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_{q+1}} C_q \xrightarrow{\partial_q} C_{q-1} \xrightarrow{\partial_{q-1}} \dots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

C_q : q-chains

C_q is a vector space called the q-chains.

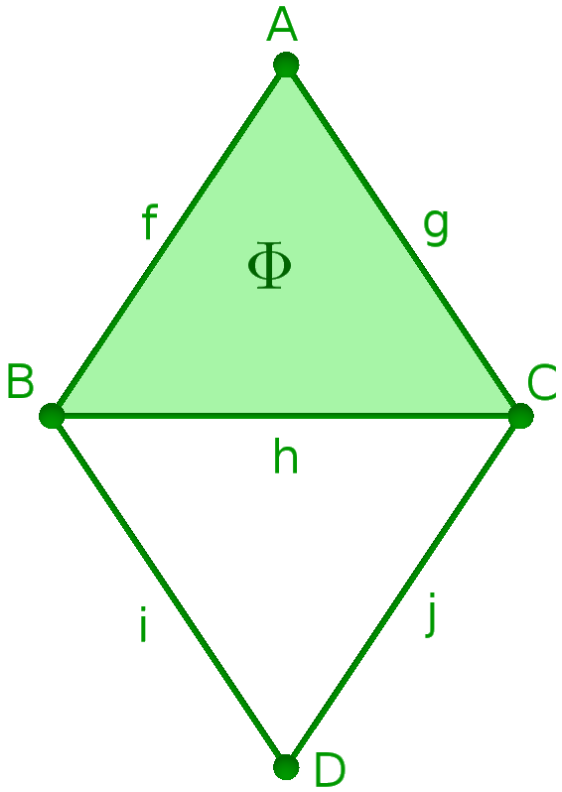
C_q is the $\mathbb{Z}/2\mathbb{Z}$ vector space generated by the q-**simplices**.

∂_q : boundary operator

∂_q is a linear map from C_q to C_{q-1} that satisfies $\partial_{q+1} \circ \partial_q = 0$.
It is called the q-boundary operator.

∂_q is the map generated by the boundary of the q-**simplices**.

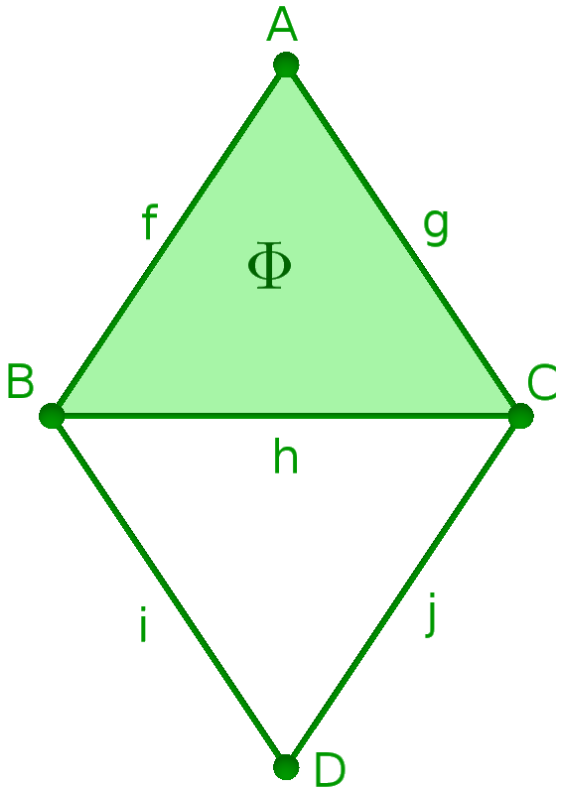
Concrete example



K : $\mathbb{Z}/2\mathbb{Z}$-chain complex

- $C_0 = span(A, B, C, D)$
- $C_1 = span(f, g, h, i, j)$
- $C_2 = span(\Phi)$

Concrete example



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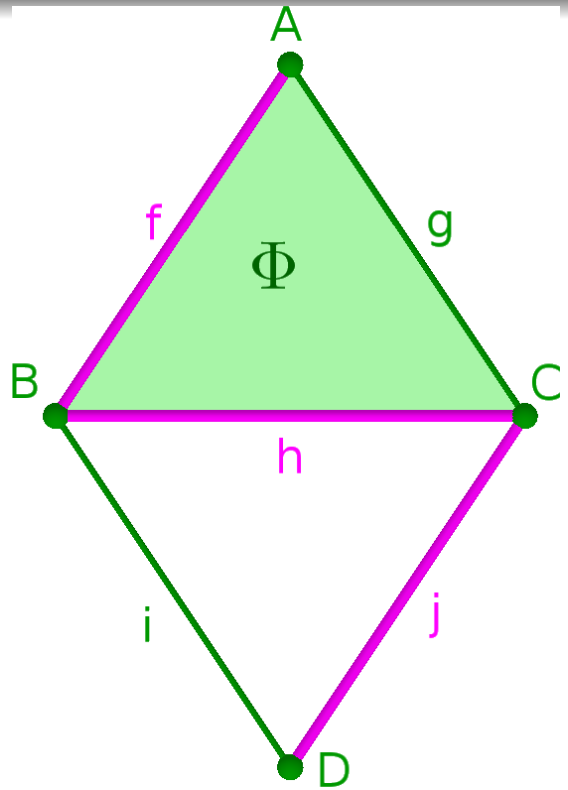
$$\partial_0 = \begin{matrix} A & B & C & D \end{matrix} \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} \qquad \partial_2 = \begin{matrix} \Phi \end{matrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} f \\ g \\ h \\ i \\ j \end{matrix}$$

$$\partial_1 = \begin{matrix} f & g & h & i & j \end{matrix} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{matrix} A \\ B \\ C \\ D \end{matrix}$$

Concrete example



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- $C_2 = span(\Phi)$

$$x = f + h + j \quad \in C_1$$

$$\begin{aligned}\partial_1(x) &= \partial_1(f + h + j) \\ &= (A + B) + (B + C) \\ &\quad + (C + D)\end{aligned}$$

$$\partial_0 = \begin{matrix} A & B & C & D \end{matrix} \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} \qquad \partial_2 = \begin{matrix} \Phi \\ \end{matrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} f \\ g \\ h \\ i \\ j \end{matrix}$$

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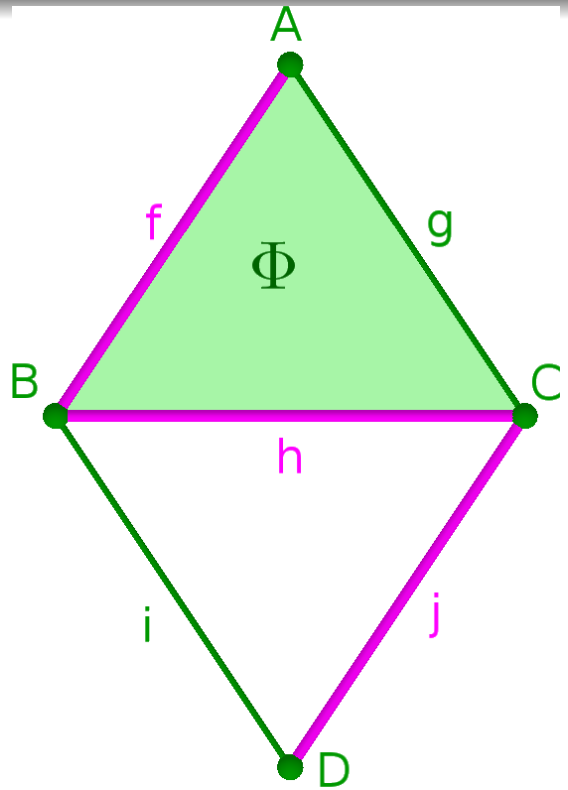

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$$\begin{aligned} x &= f + h + j \quad \in C_1 \\ \partial_1(x) &= \partial_1(f + h + j) \\ &= (A + B) + (B + C) \\ &\quad + (C + D) \\ &= A + 2B + 2C + D \end{aligned}$$

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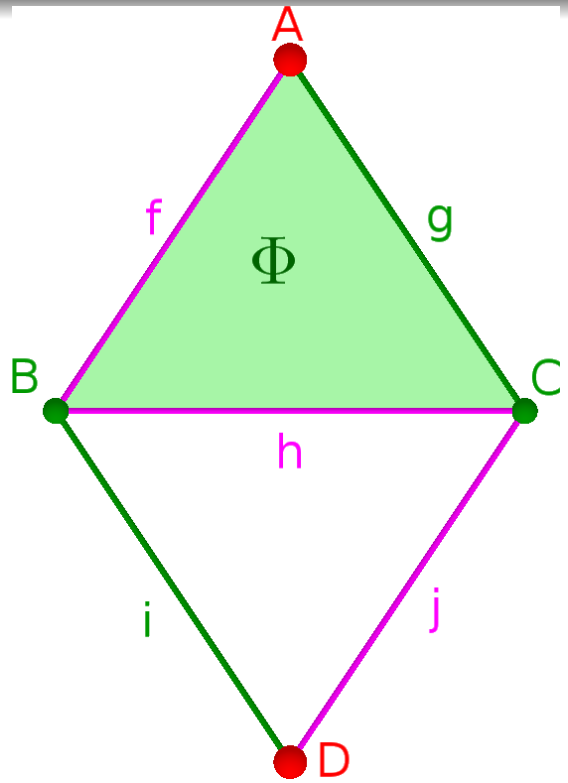
Concrete example



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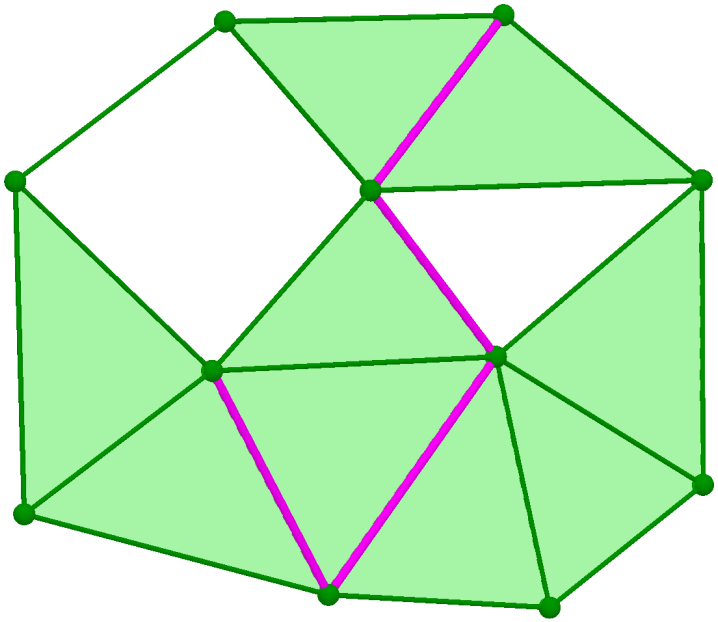
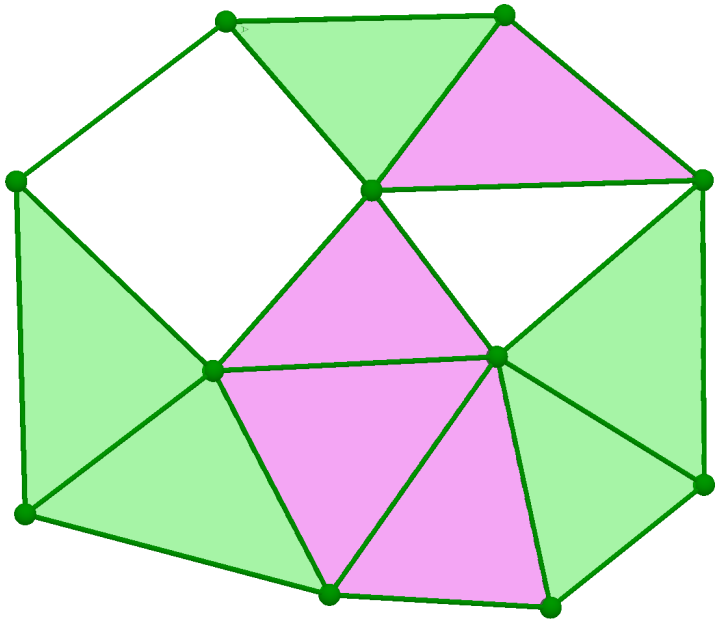
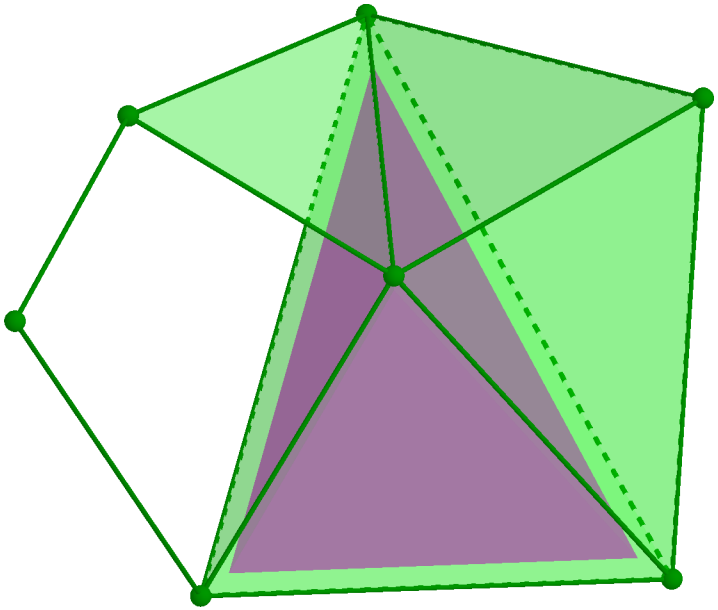
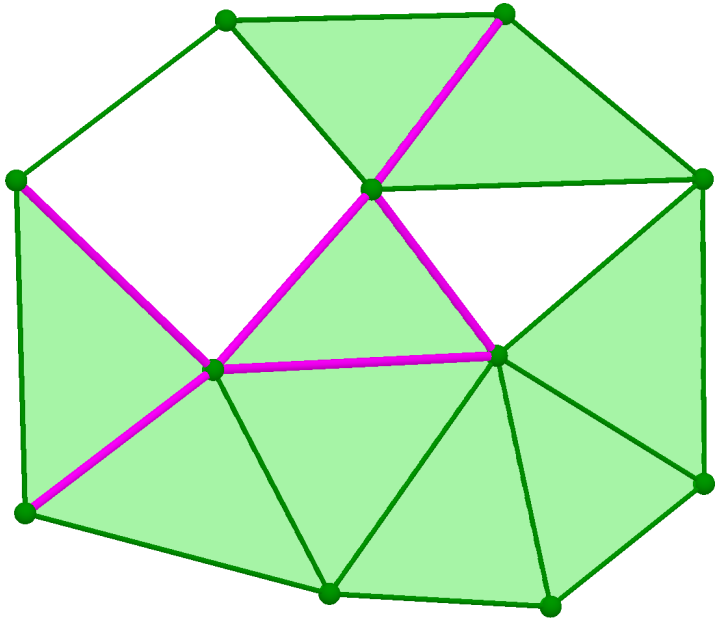
Boundary operator examples









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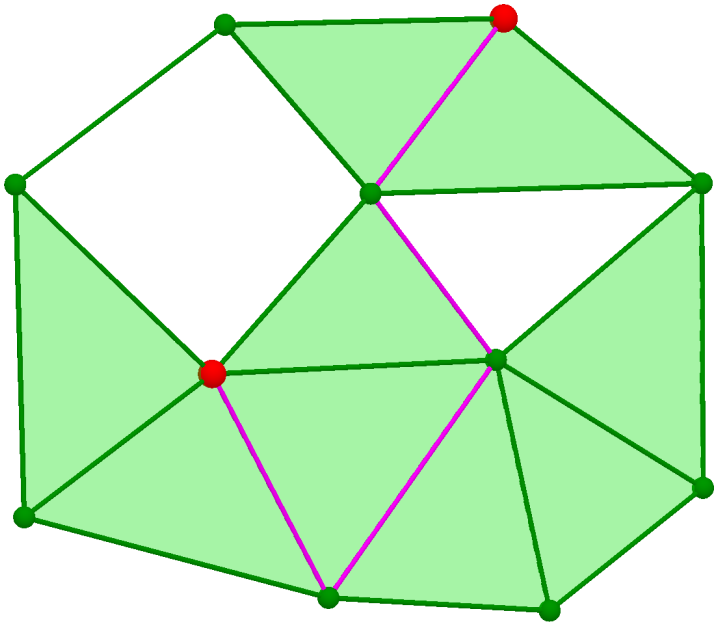

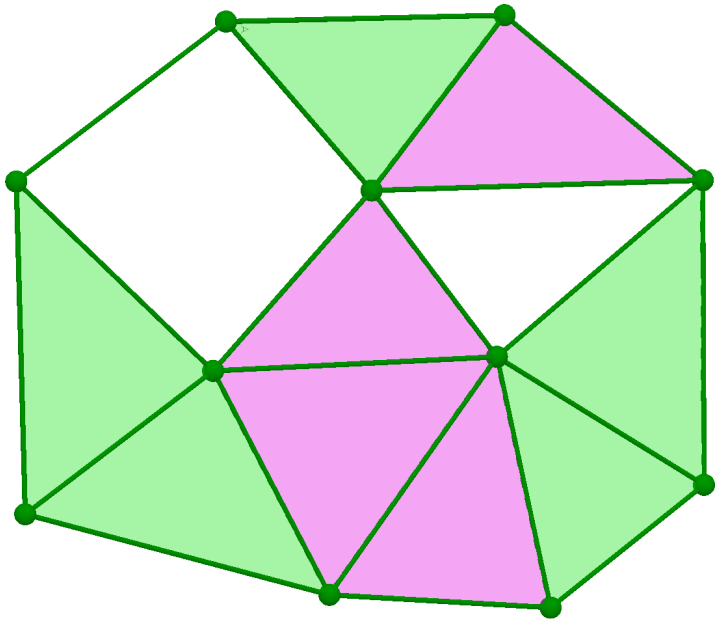

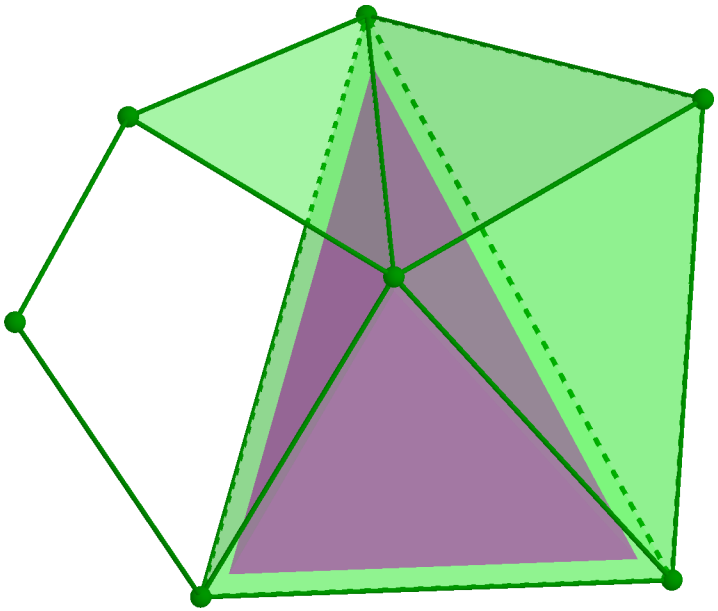

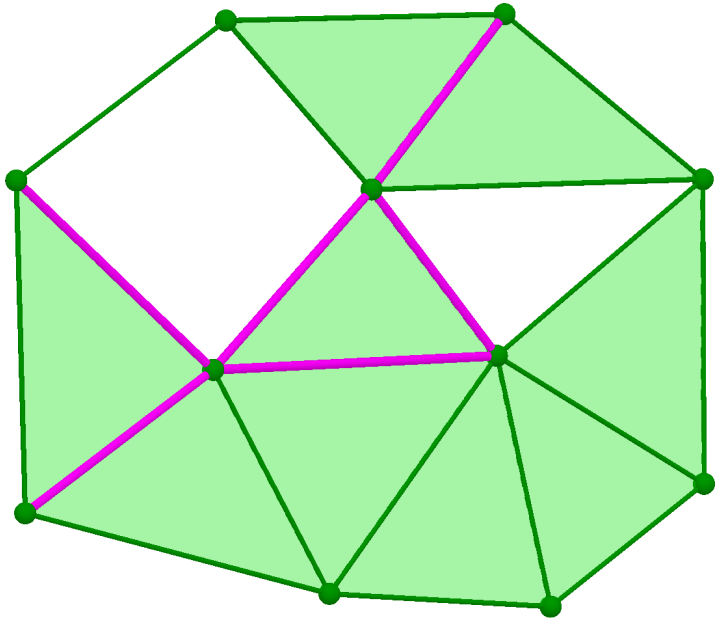

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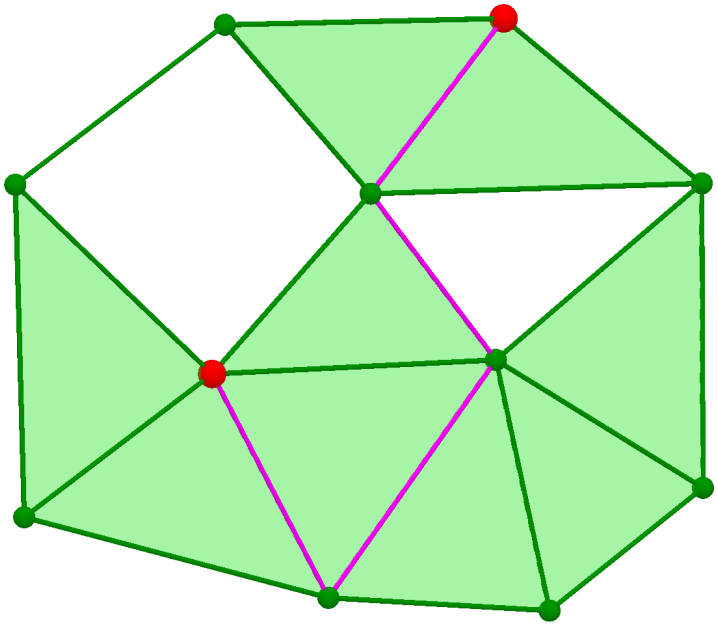

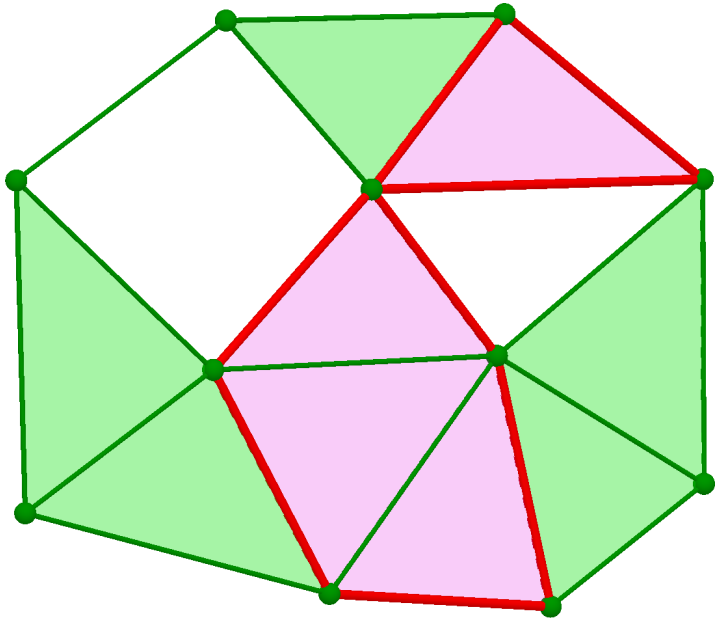

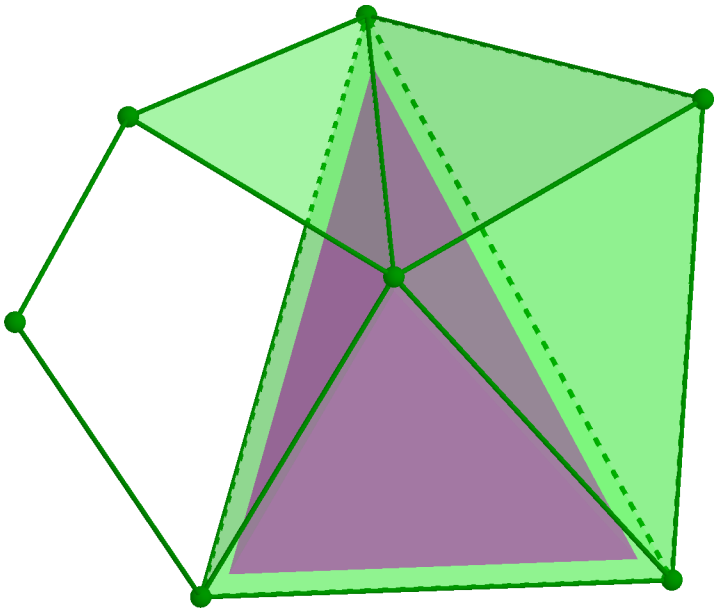

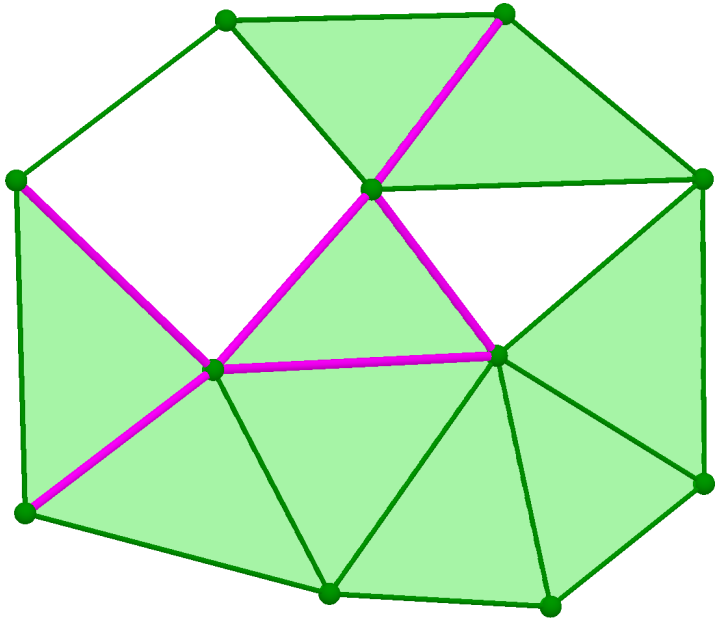

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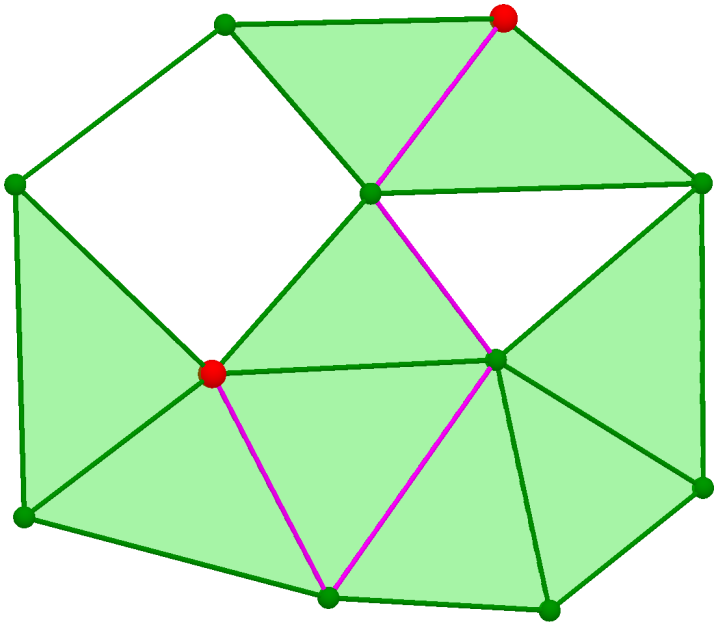

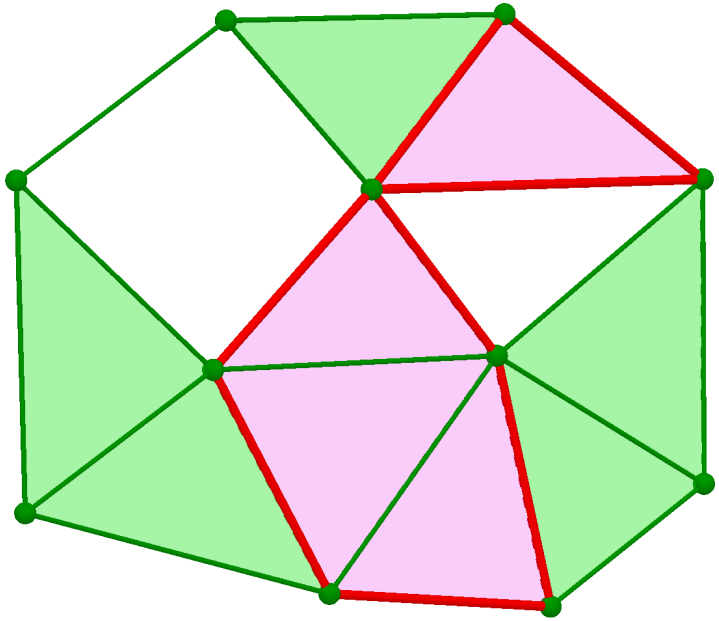

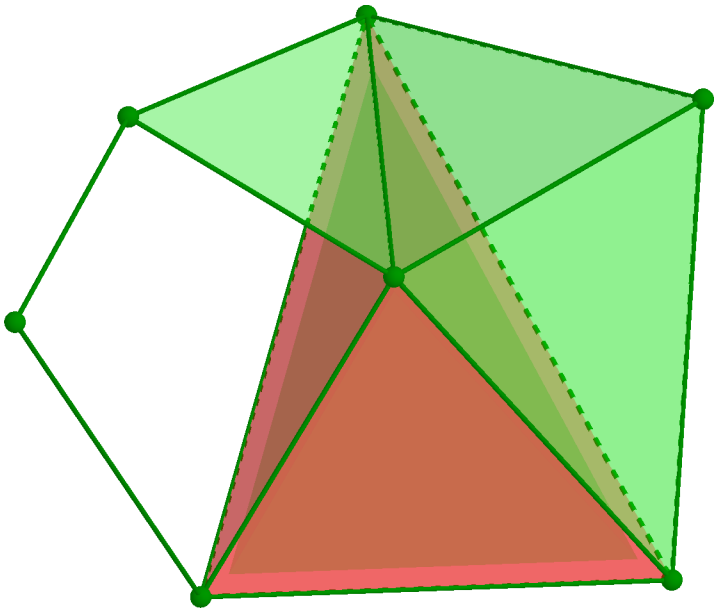

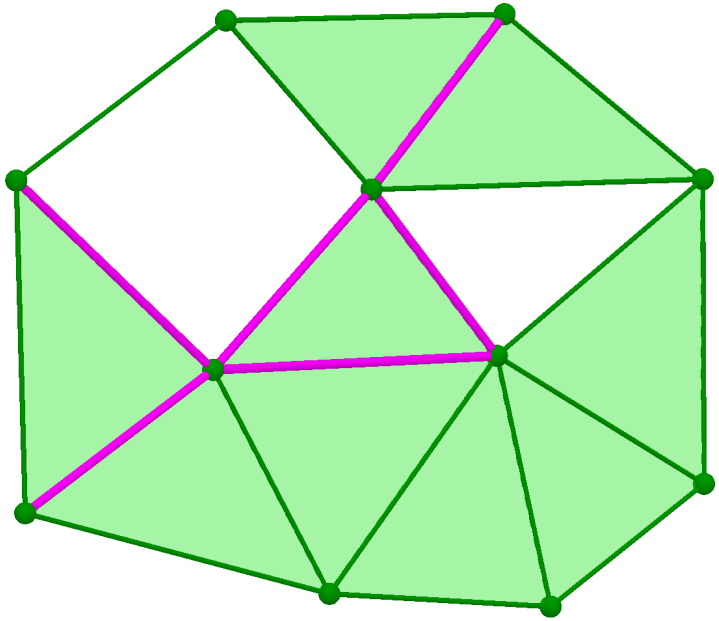

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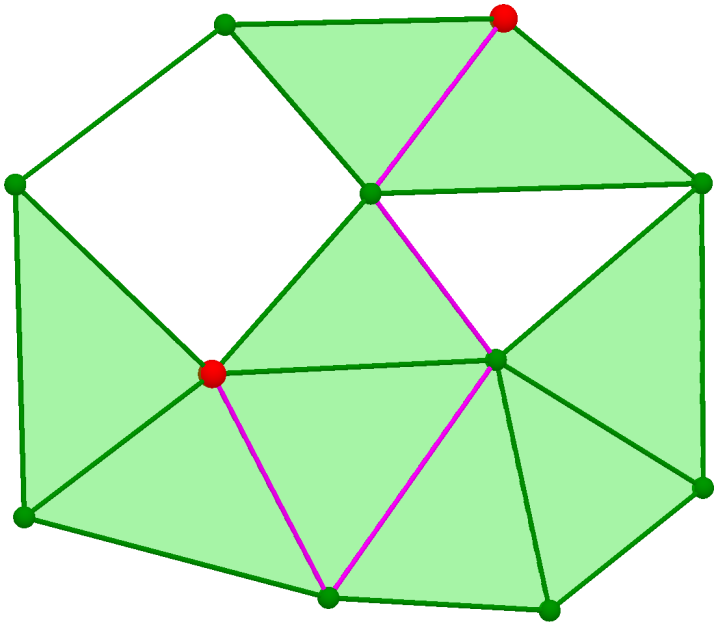

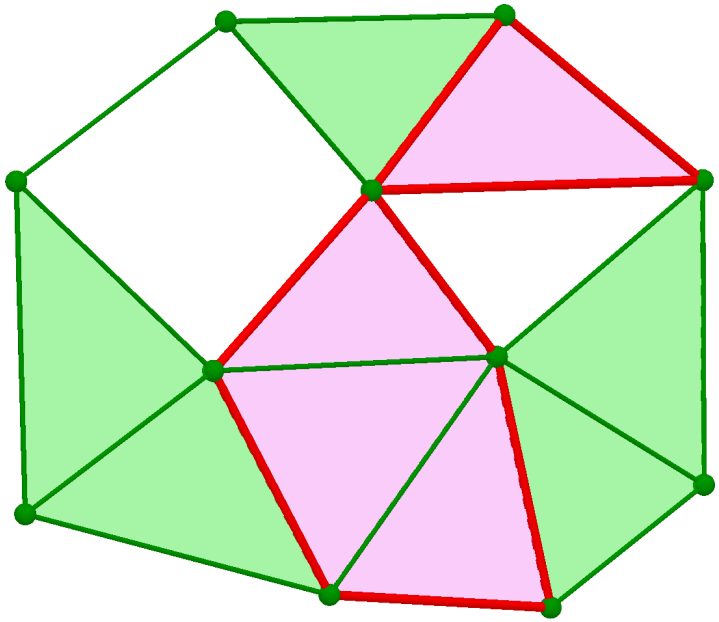

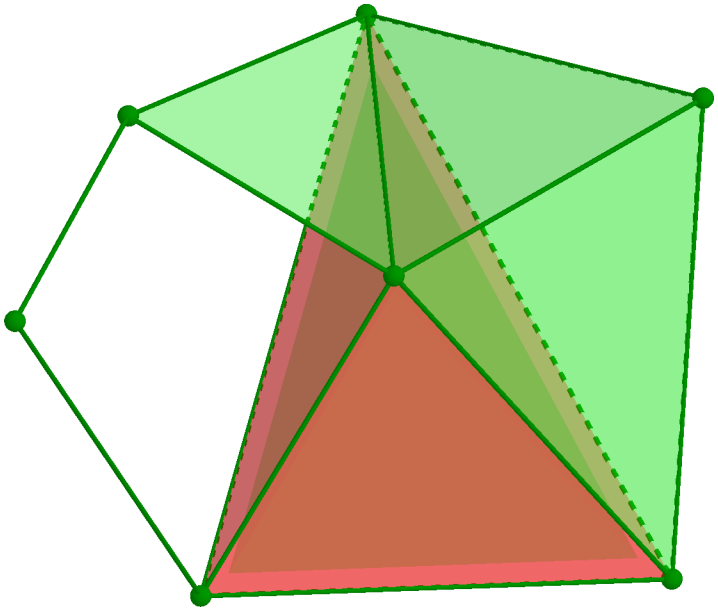

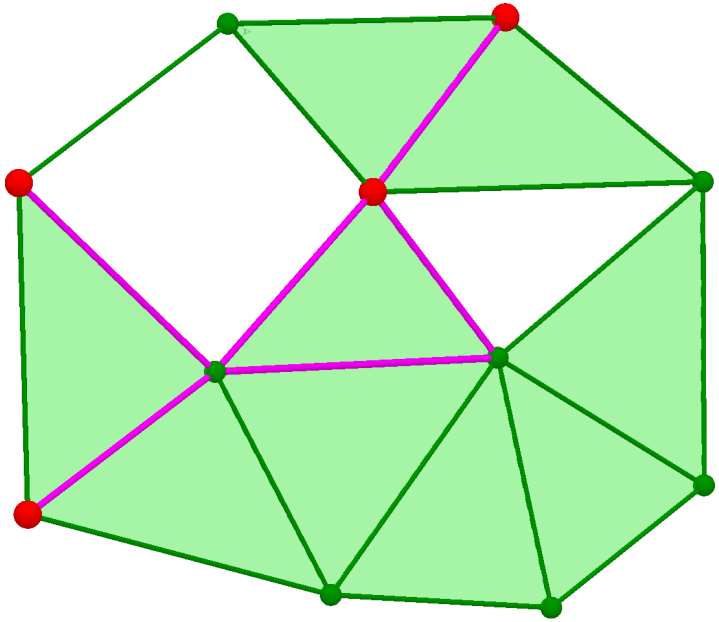

Boundary - Definition

A q-boundary is a q-chain that is the boundary of a $(q+1)$-chain.

$$q\text{-boundaries} = \text{im}(\partial_{q+1})$$

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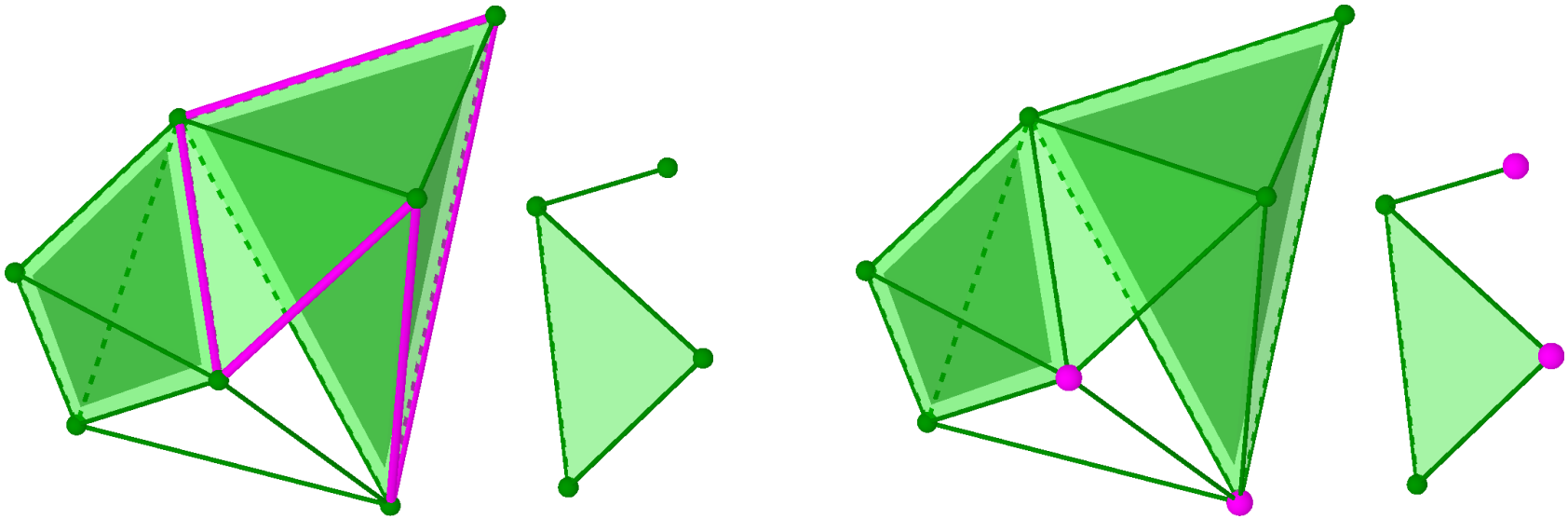

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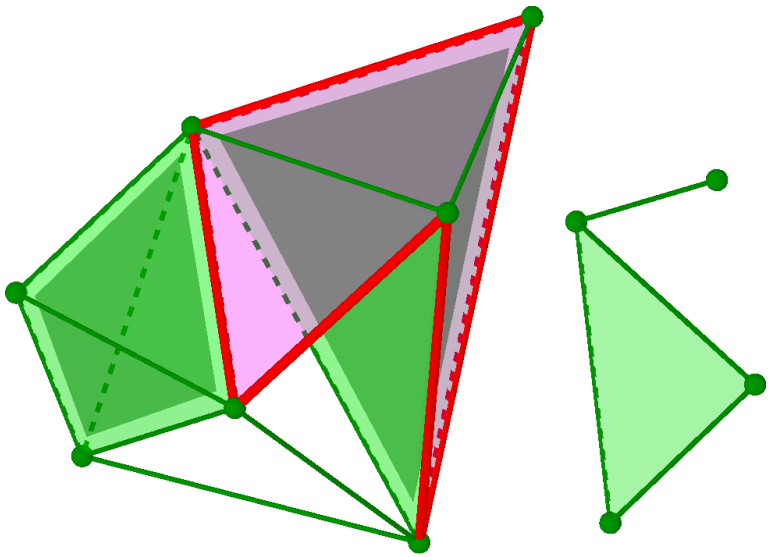

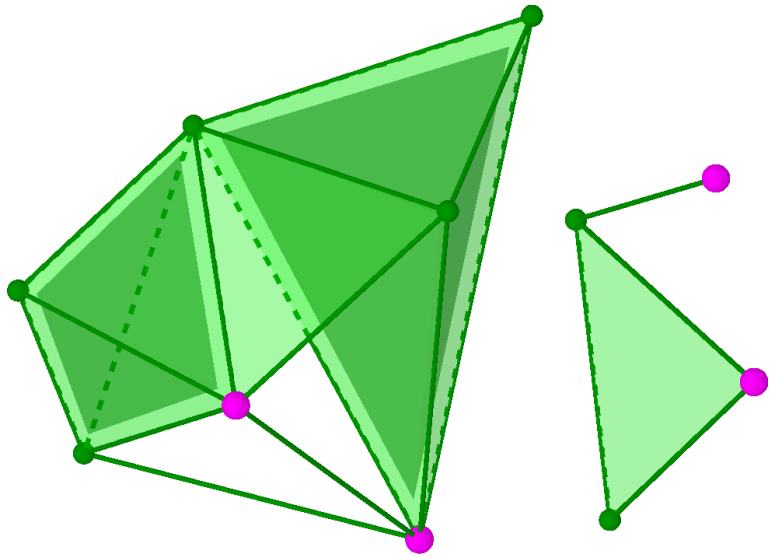

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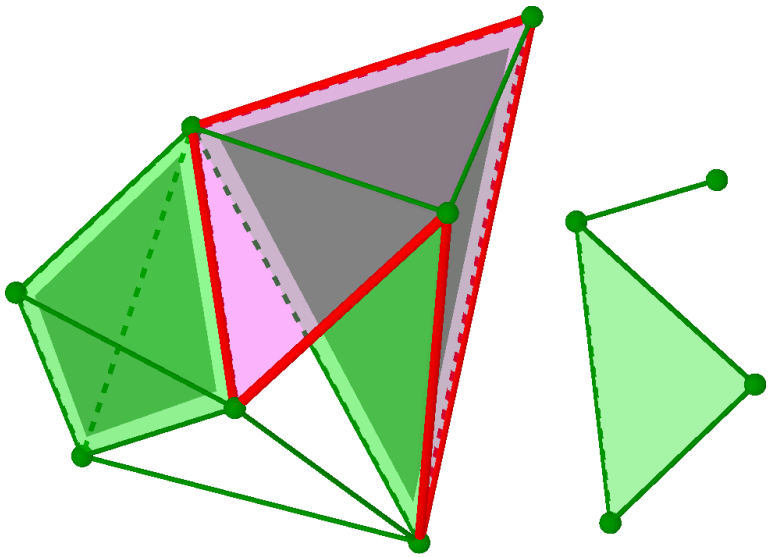
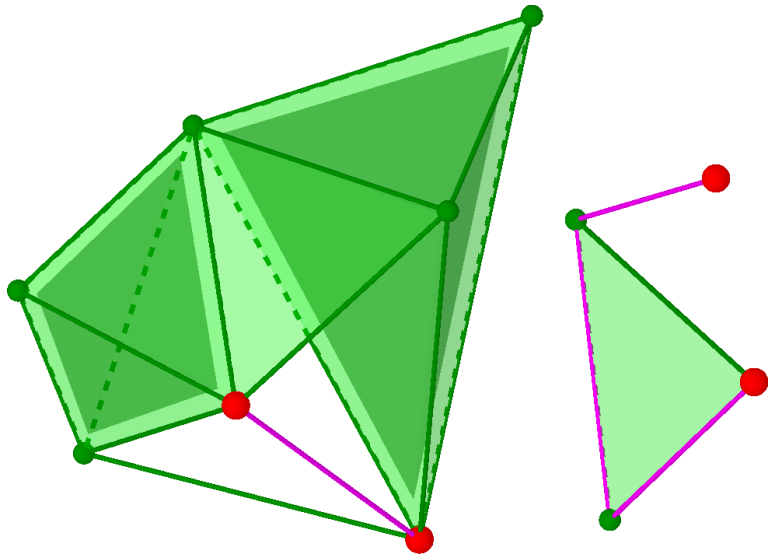
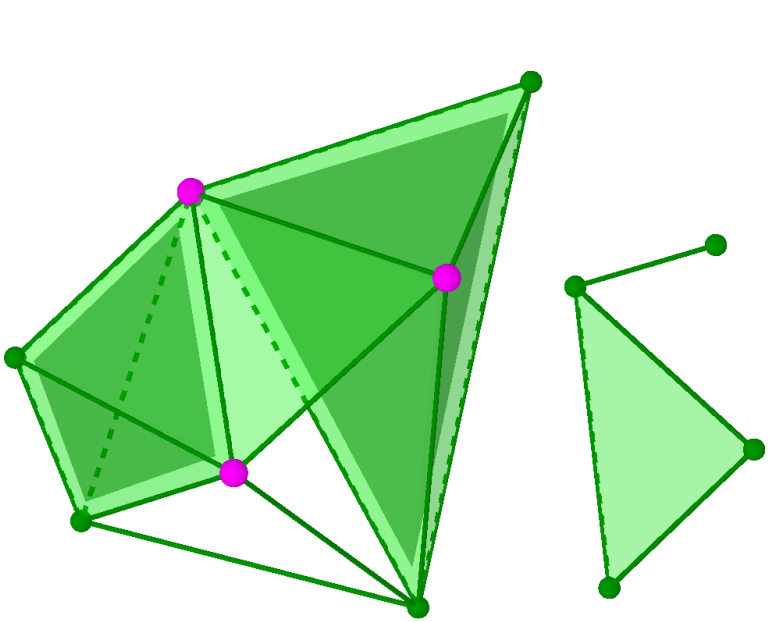
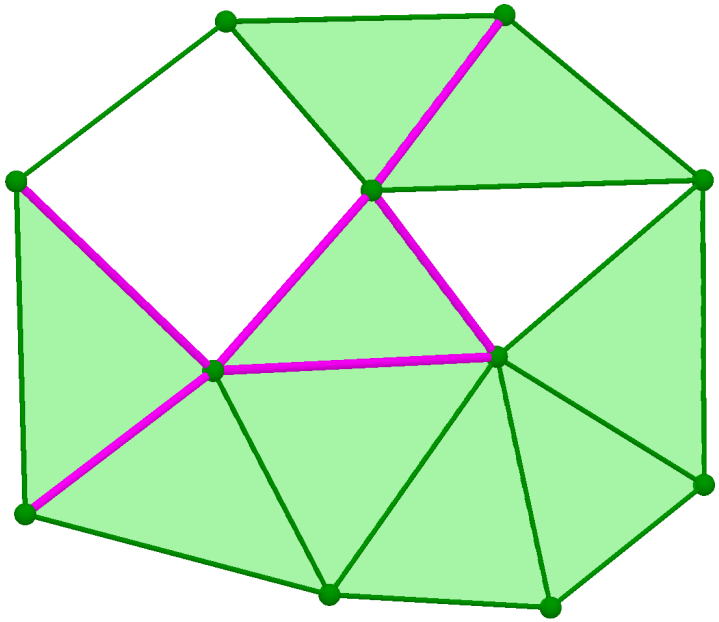




Non boundaries





Cycle - Definition

A q-cycle is a q-chain whose boundary is null.

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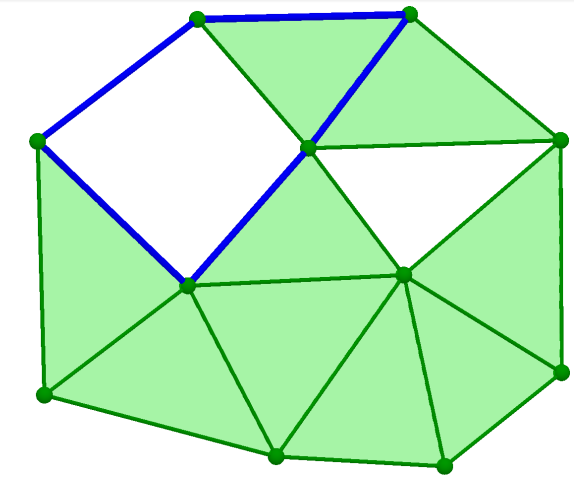

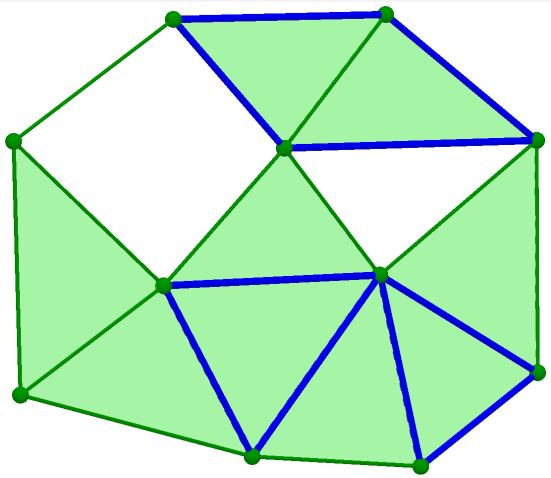

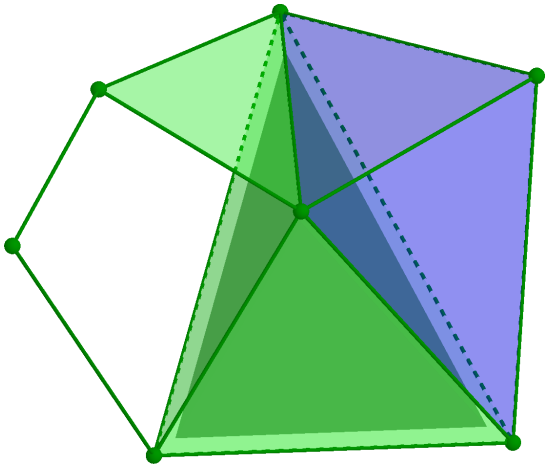

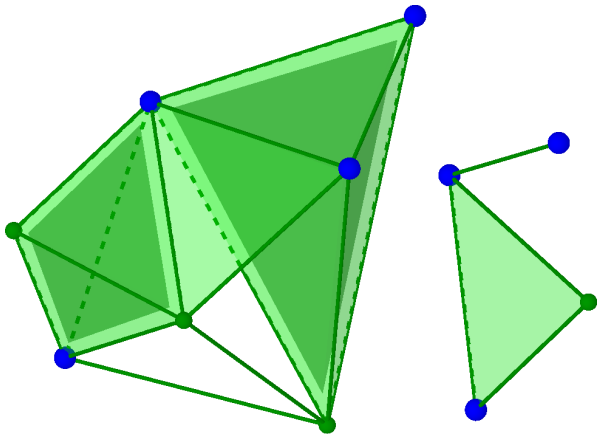

Cycles and Boundaries

Proposition

A boundary is a cycle.

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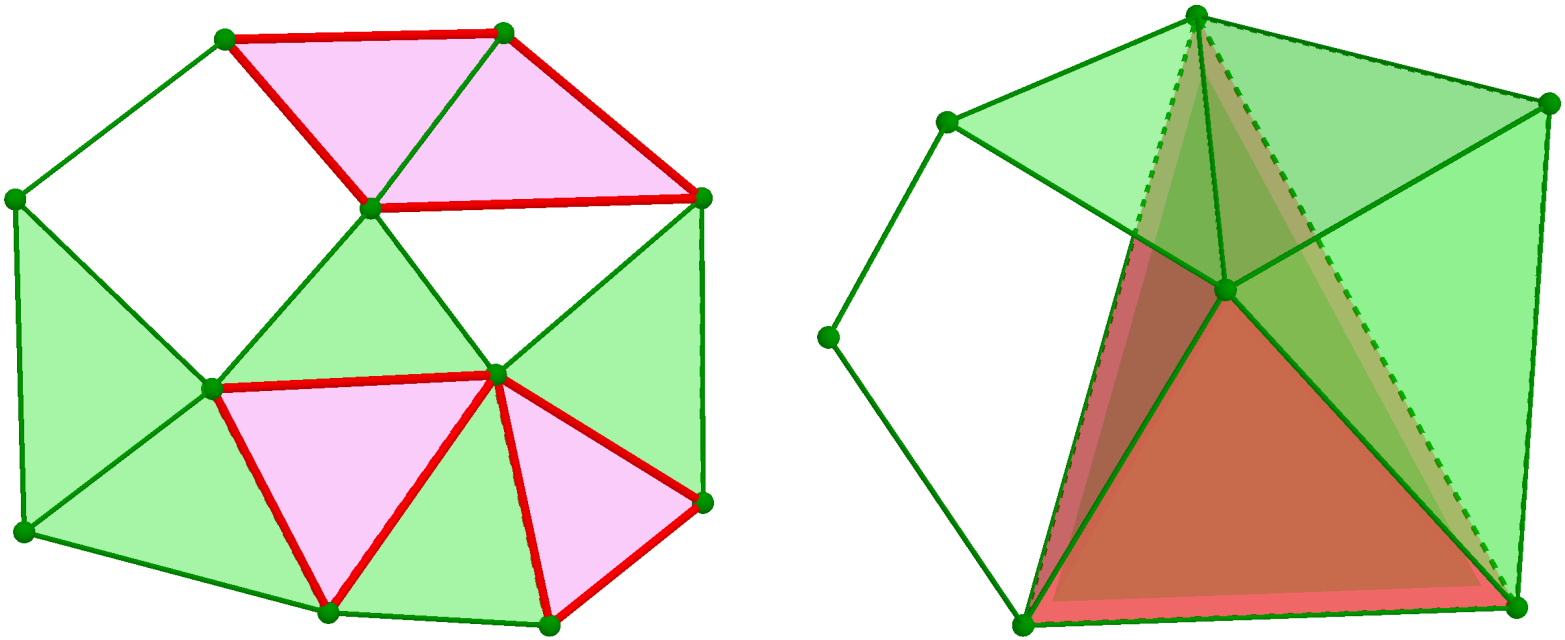
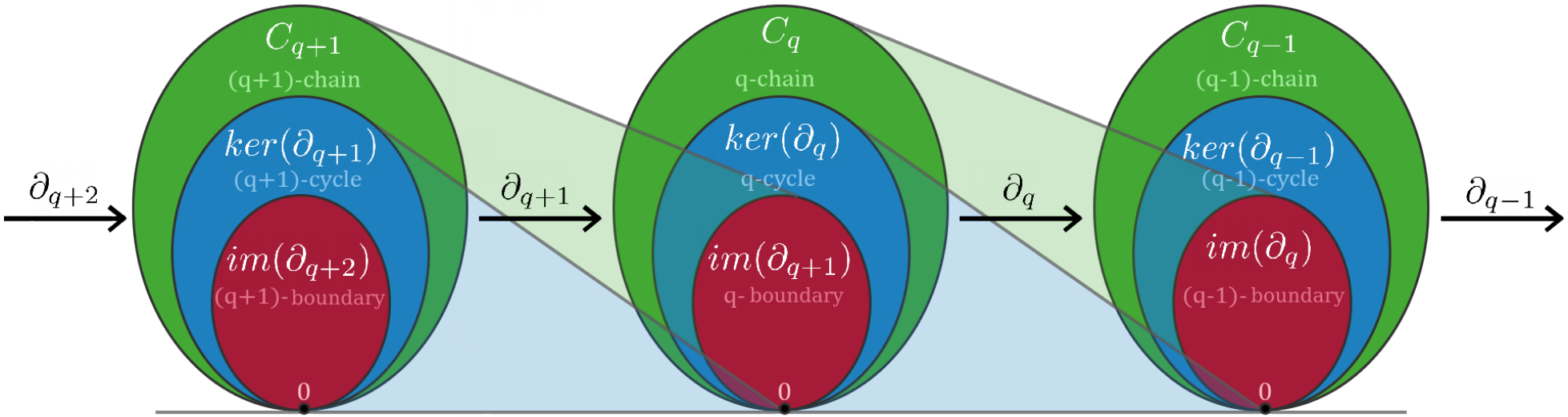
$\partial_{q+1} \circ \partial_q = 0$, "a boundary has no boundary", $\text{im}(\partial_{q+1}) \subseteq \ker(\partial_q)$.



Cycles and Boundaries : Summary



Hole - Intuitive definition

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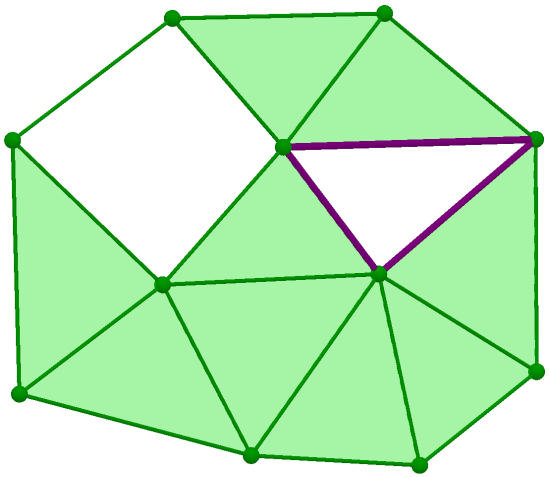

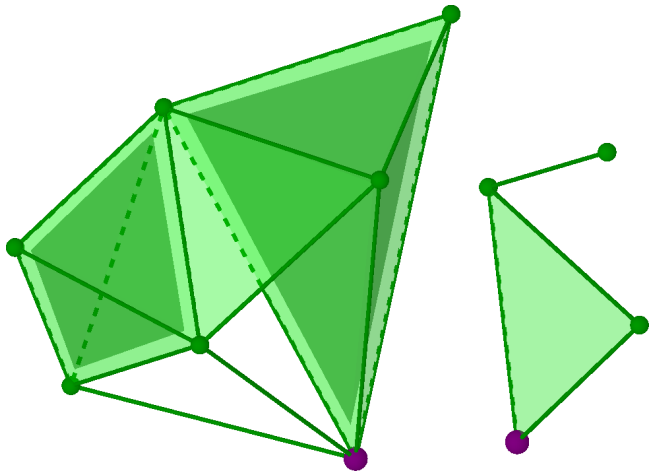

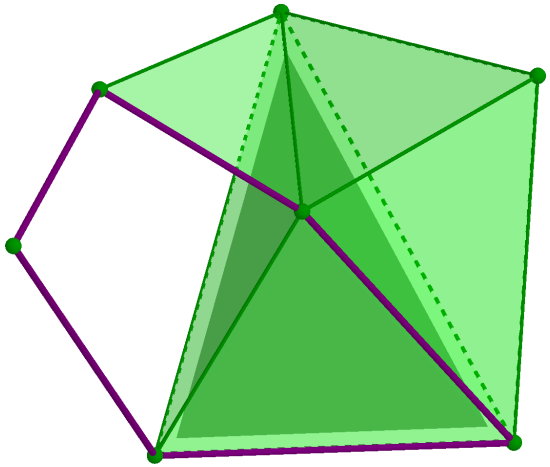

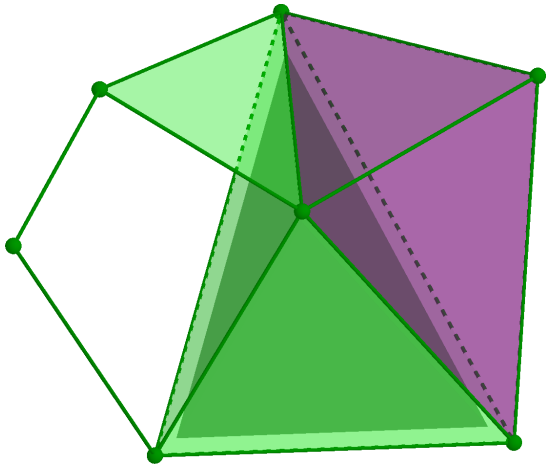

Hole - Equivalence

Two q-holes are equivalent iff their difference is a q-boundary.

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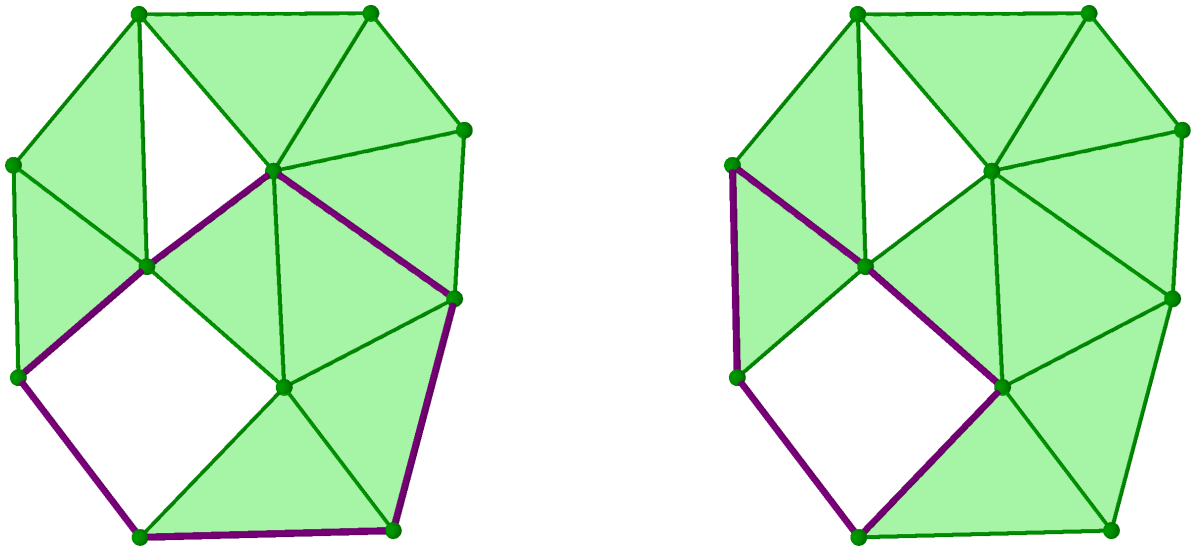

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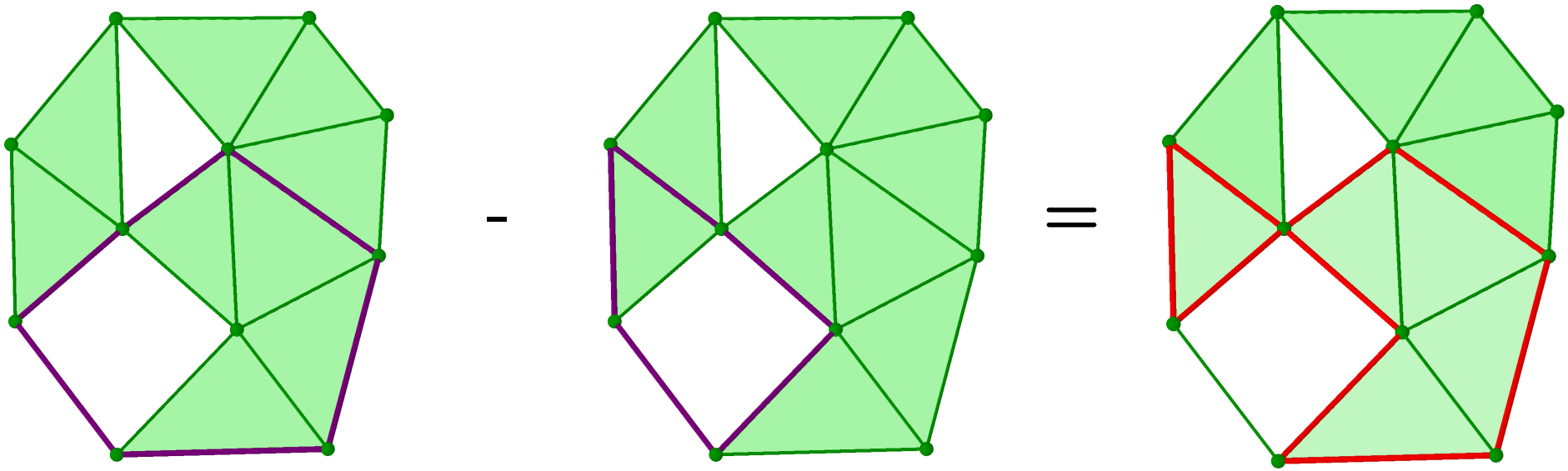

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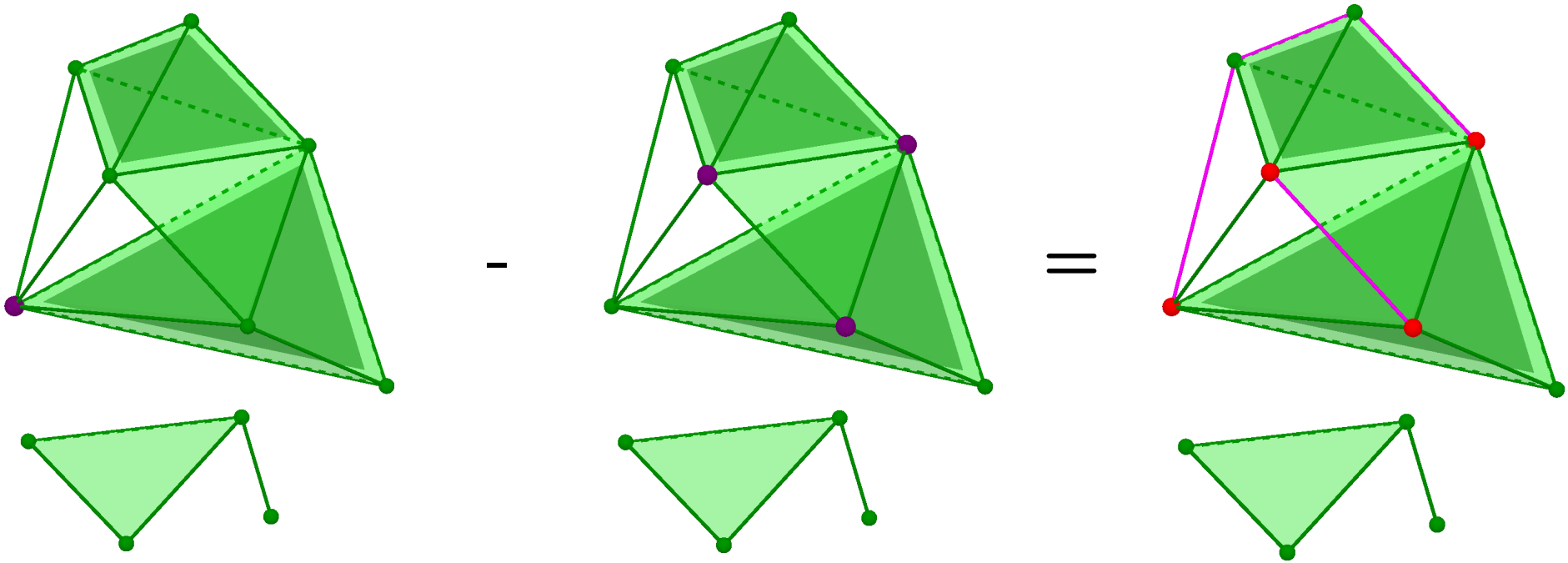

Homology group - Definition

The equivalence classes of $\overset{q}{\sim}$ form a group structure, called the q-**homology group** :

$$\mathrm{H_q(K)} = \frac{\ker(\partial_q)}{\underset{q}{\sim}}$$

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Betti numbers - Proposition

There exist a number β_q such that $H_q(K) \approx (\mathbb{Z}/2\mathbb{Z})^{\beta_q}$.

β_q is called the **Betti number** of dimension q and intuitively represent the number of holes of dimension q.

Clarification

$H_q(K) \approx (\mathbb{Z}/2\mathbb{Z})^{\beta_q}$: there are β_q holes and 2^{β_q} equivalence classes in $H_q(K)$. Each equivalence class represents a subset of holes.

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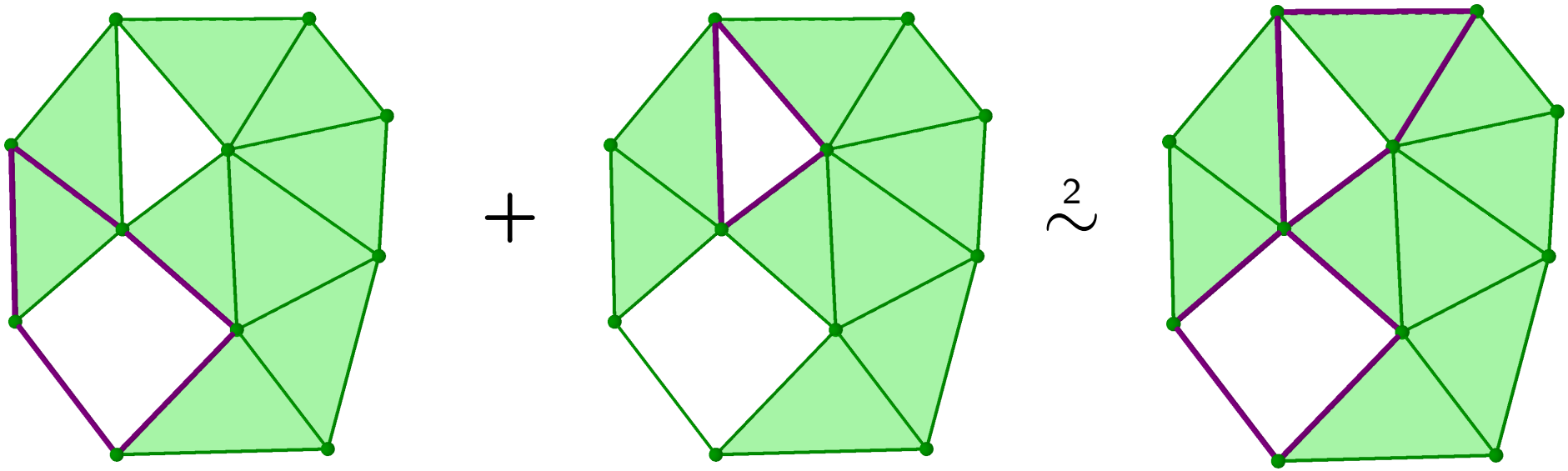
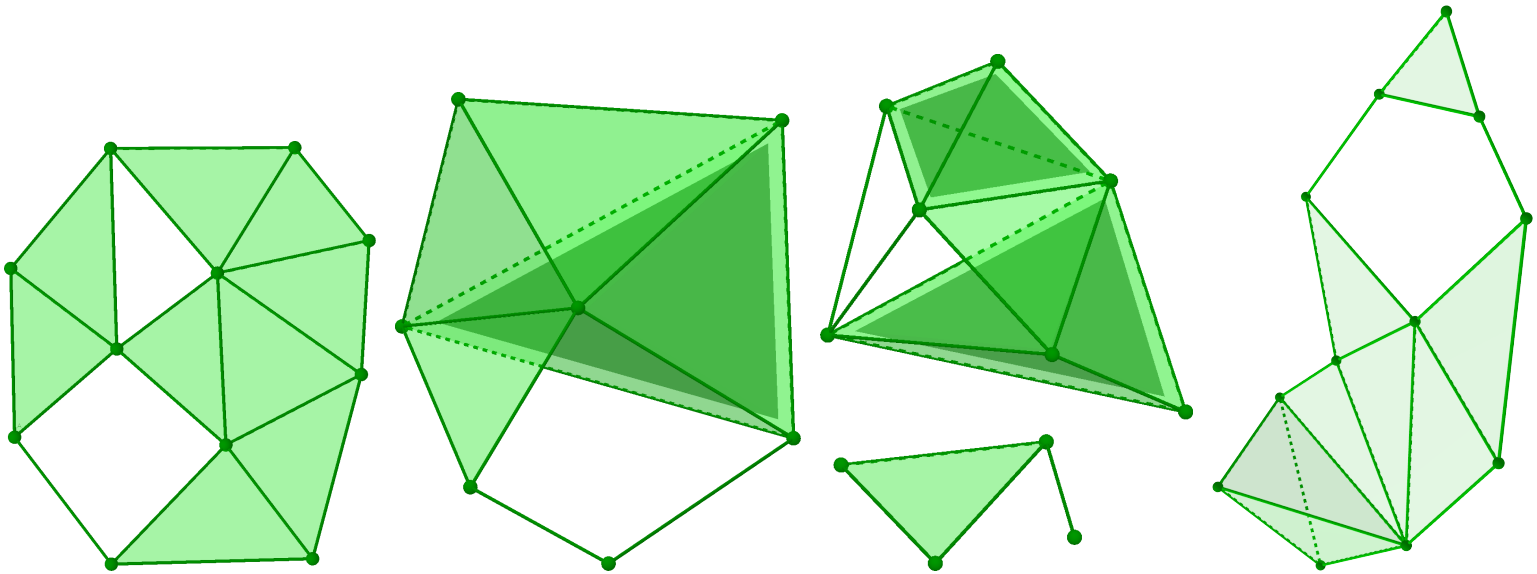


$$H_1(K) \approx (\mathbb{Z}/2\mathbb{Z})^2$$

Conclusion

Starting from a combinatorial/geometric structure (**simplicial complex**), we built an algebraic structure (**chain complex**) that allowed us to intuitively define **holes** and formally grasp **homology groups**.



Three approaches for computational homology :

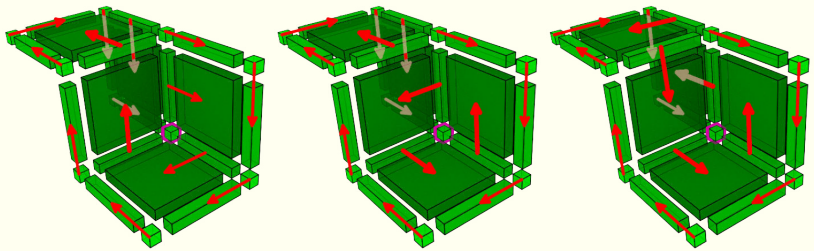
Effective approach

Computation of reductions

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & C_{q+1} & \underset{h_q}{\overset{\partial_{q+1}}{\rightleftarrows}} & C_q & \longrightarrow & \cdots \\
 & & \Big\uparrow{\scriptstyle g_{q+1}} \Big\downarrow{\scriptstyle f_{q+1}} & & \Big\uparrow{\scriptstyle g_q} \Big\downarrow{\scriptstyle f_q} & & \\
\cdots & \longrightarrow & C'_{q+1} & \xrightarrow{0} & C'_q & \longrightarrow & \cdots
\end{array}$$

Combinatorial approach

Discrete Morse Theory



Algebraic approach

Smith Normal Form

$$\partial = P \begin{pmatrix} \alpha_1 & 0 & 0 & \cdots & 0 \\ & \ddots & & & \\ 0 & & \alpha_r & & 0 \\ \vdots & & & 0 & \vdots \\ 0 & & \cdots & & 0 \end{pmatrix} Q$$

where $\alpha_i \mid \alpha_{i+1}$.

To go further : Homology over a ring

Chain complex with a ring

If use $\mathbb{Z}$ instead of $\mathbb{Z}/2\mathbb{Z}$, C_q is not anymore a vector space but a $\mathbb{Z}$-**module**. Weird things happen...

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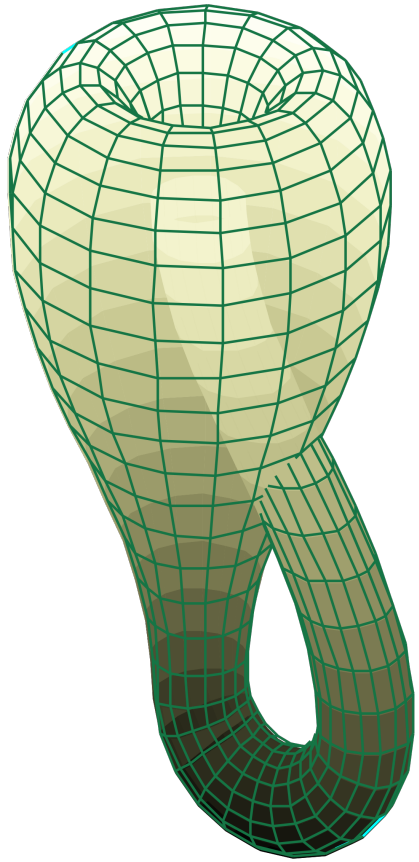
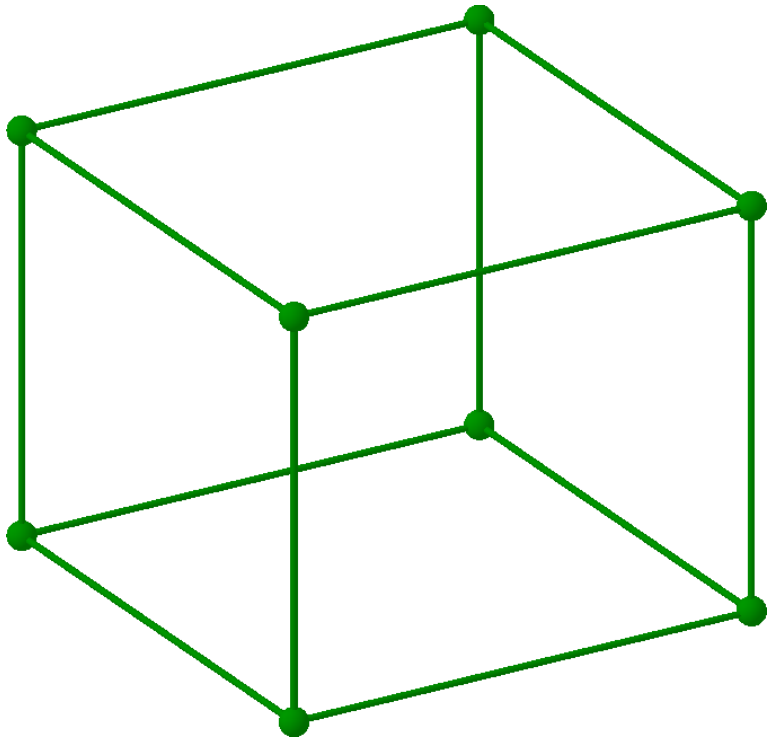
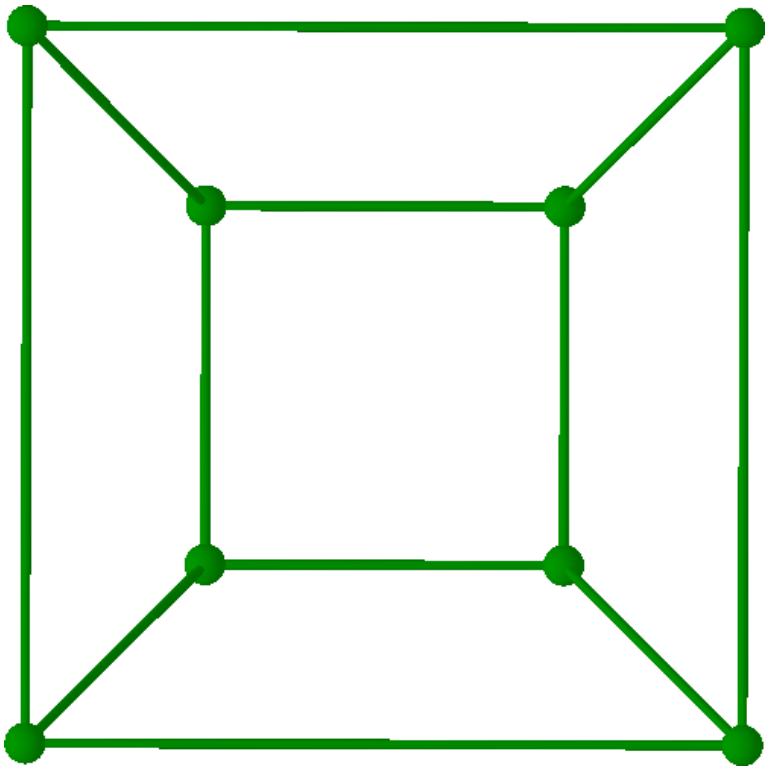
Holes and torsion

$$H_q(K) \approx \underbrace{\mathbb{Z}^{\beta_q}}_{\textbf{holes}} \times \underbrace{\frac{\mathbb{Z}}{\alpha_1\mathbb{Z}} \times \frac{\mathbb{Z}}{\alpha_2\mathbb{Z}} \times \cdots \times \frac{\mathbb{Z}}{\alpha_m\mathbb{Z}}}_{\textbf{torsion}}$$



$$H_1(K) \approx \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

The location of a hole : where intuition struggles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