On concurrent games with payoff

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Abstract

The paper considers an extension of concurrent games with a payoff, *i.e.* a numerical value resulting from the interaction of two players. We extend a recent determinacy result on concurrent games [5] to a *value theorem*, *i.e.* a value that both players can get arbitrarily close to, whatever the behaviour of their opponent. This value is not reached in general, *i.e.* there is not always an optimal strategy for one of the players (there is for finite games). However when they exist, we show that optimal strategies are closed under composition, which opens up the possibility of computing optimal strategies for complex games compositionally from optimal strategies for their component games.

Keywords: Concurrent Games, Game Semantics, Determinacy.

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1 Introduction

Games are a well-established tool in mathematics, economics, logic, and of course computer science: in the latter, two-player games in particular are very widely used to model situations where an agent (*e.g.* a program) interacts with its environment (*e.g.* the user, the operating system). For instance, researchers in *game semantics* [9] have managed to build very precise (*fully abstract* [1,8]) models of higher-order programming languages with various computational effects. Another particularly rich line of work has been the application of game-theoretic tools for algorithmic and verification purposes: one expresses a desirable property of a system as a game, and reduces the satisfaction of this property to the existence of a "good" strategy for this game. Here, the meaning of "good" can be either qualitative (positions are winning or losing, with each player wanting to reach a winning position) or quantitative

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(positions have a given *payoff*, with both players trying to maximize their payoff). For these purposes, one generally wants the games considered to be *determined*: qualitatively, this means that one of the players necessarily has a winning strategy, and quantitatively that the game has a well-defined *value* that well-chosen strategies can reach or get arbitrarily close to. For this reason, the classes of games considered for these purposes generally enjoy *determinacy*: the most well-known such result is Martin's famous theorem [12] stating that for sequential, tree-like games whose winning positions form a *Borel set*, one of the players must have a winning strategy. It is well-known that Martin's theorem generalizes to the quantitative setting if the game is *zero-sum*, *i.e.* in each position the payoffs of the two players sum to zero. In the last decade, there has been a growing interest in extensions of these games with *concurrency*. One very successful definition of (turn-based) concurrent games has been proposed by Henzinger, de Alfaro *et. al.* [3,4,7]: their games are based on Blackwell games [13], where at any point, the next state is decided by a function of parallel choices of both players. In these games, the *pure strategy* determinacy result of sequential games is weakened into a *mixed strategy* determinacy, where strategies are allowed to make probabilistic choices.

However in semantics, models of concurrent processes generally allow a more liberal, non turn-based form of concurrency. Starting with the work of Petri, many have come to advocate a view of concurrency based on *partial orders*, specifying the causal dependency between events—see [16] for an early summary of Petri's work and its relation with domain theory. Following this approach, several notions of concurrent games have been proposed as a basis for denotational semantics: in terms of closure operators [2] or asynchronous transition systems [15]. Recently, Winskel and Rideau introduced a more general setting for concurrent games [17]. It is based on the notion of *event structure* [18], a partial order of causal dependency on events with a consistency relation expressing nondeterministic choice. In the present paper, it is this framework that we will refer to as *concurrent games*. We showed in [5] that in this setting qualitative determinacy is satisfied for well-founded games meeting a structural condition called *race-freedom* expressing that moves of one player do not directly interfere with moves of the other. Here, we consider an extension of concurrent games with *zero-sum payoff*, and show a generalization of the qualitative determinacy result of [5] to a quantitative one. As the reader will see this is not a trivial exercise and requires a much finer analysis than for the qualitative case.

Note that we obtain *pure strategy* determinacy – our strategies do not make probabilistic choices, although they can act non-deterministically. There is an apparent contradiction with the line of work based on Blackwell games mentioned above, since they only have mixed strategy determinacy. This is due to a crucial difference between the two settings: in our games, no *progess* assumption is made and strategies can legitimately choose *not to play*, possibly resulting in a deadlock if both strategies choose to do so. We argue that this is a desirable property, since very often in computer science we have to deal with systems that might not terminate. However from the game theory perspective, this implies that Blackwell games are *not* instances of our zero-sum concurrent games. (They *do* fit into our general framework however—see [20].)

We also investigate quantitative features with respect to the *compositional* structure of concurrent games. In sequential games, strategies can be composed using a form of parallel composition and a hiding operation to make internal play invisible. This fact (first remarked on by Conway and emphasised by Joyal [10] in his analysis of Conway's work [6]) is seldom used in economics and algorithmics. However, it is at the very heart of game semantics, the compositional analysis of programs and programming languages in terms of games and strategies. Our concurrent games are compositional; in fact, the main result of [17] was to define and characterise strategies for which composition behaves well (*i.e.* is associative, and has identities). Not only is compositionality a prerequisite for building denotational models of programming languages (as they organize naturally as *categories*, see *e.g.* [11]), but it is also a very successful general approach for proving properties of complex programs. Adapting the earlier work on concurrent strategies, we show here that optimal strategies are stable under composition, thus building a bicategory of optimal strategies. This is a significant step towards a compositional analysis of optimal strategies: instead of modeling complex behaviours as payoff functions and then computing values and optimal strategies, construct complex optimal strategies by composition from elementary ones. Extensions with payoff should also prove useful for purely semantic purposes: pay-off is a powerful notion that allows us to express familiar *winning strategies*—as strategies of positive value—as well as more arcane game-theoretic notions, such as *well-bracketing* [14].

In Section 2, we recall the framework of concurrent games originally presented in [17]. In Section 3, we show how to enrich these concurrent games with payoff and introduce the notion of value of games and strategies. In Section 4, we prove the main result of our paper, the value theorem. Finally in Section 5, we investigate the compositional aspects of payoff games; in particular we show that optimal strategies are stable under composition and form a bicategory.

2 Preliminaries

2.1 *Event structures*

An *event structure* comprises $(E, \leq, \mathrm{Con})$, consisting of a set E, of *events* which are partially ordered by $\leq$, the *causal dependency relation*, and a nonempty *consistency relation* Con consisting of finite subsets of E, which satisfy

$$\{e' \mid e' \leq e\} \text{ is finite for all } e \in E,$$
$$\{e\} \in \mathrm{Con} \text{ for all } e \in E,$$
$$Y \subseteq X \in \mathrm{Con} \Rightarrow Y \in \mathrm{Con}, \text{ and}$$
$$X \in \mathrm{Con} \ \& \ e \leq e' \in X \Rightarrow X \cup \{e\} \in \mathrm{Con}$$

The *configurations*, $\mathcal{C}^\infty(E)$, of an event structure E consist of those subsets $x \subseteq E$ which are

Consistent: $\forall X \subseteq x. \ X$ is finite $\Rightarrow X \in \mathrm{Con}$, and

Down-closed: $\forall e, e'. \ e' \leq e \in x \Rightarrow e' \in x.$

Often we shall be concerned with just the finite configurations of an event structure. We write $\mathcal{C}(E)$ for the *finite* configurations of an event structure E.

Two events which are both consistent and incomparable w.r.t. causal dependency in an event structure are regarded as *concurrent*. In games the relation of *immediate* dependency $e \rightarrowtail e'$, meaning e and e' are distinct with $e \leq e'$ and no event in between, will play an important role. For $X \subseteq E$ we write $[X]$ for $\{e \in E \mid \exists e' \in X.\ e \leq e'\}$, the down-closure of X; note if $X \in \mathrm{Con}$, then $[X] \in \mathrm{Con}$ is a configuration and in particular each event e is associated with a *prime* configuration $[e]$.

Notation 1 Let E be an event structure. We use $x{-}{\subset}y$ to mean y covers x in $\mathcal{C}^{\infty}(E)$, *i.e.* $x \subset y$ in $\mathcal{C}^{\infty}(E)$ with nothing strictly in between, and $x{-}\!\!\overset{e}{\subset} y$ to mean $x \cup \{e\} = y$ for $x, y \in \mathcal{C}^{\infty}(E)$ and event $e \notin x$. We use $x{-}\!\!\overset{e}{\subset}$, expressing that event e is enabled at configuration x, when $x{-}\!\!\overset{e}{\subset} y$ for some y.
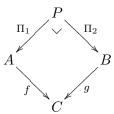
Definition 2.1 Let E and E' be event structures. A *(partial) map* of event structures $f : E \to E'$ is a partial function on events $f : E \rightharpoonup E'$ such that for all $x \in \mathcal{C}^{\infty}(E)$ its direct image $fx \in \mathcal{C}^{\infty}(E')$ and $\forall e_1, e_2 \in x,\ f(e_1) = f(e_2)$ (with both defined) $\Rightarrow e_1 = e_2$. (It is sufficient to verify the above conditions just for finite configurations $x \in \mathcal{C}(E)$.)

Maps of event structures compose as partial functions, with identity maps given by identity functions. We will say the map is *total* if the function f is total; then f restricts to a bijection $x \cong fx$ for $x \in \mathcal{C}^{\infty}(E)$.

Definition 2.2 We define the following two process operations.

- **Products.** The category of event structures has *products* $A \times B$ with projections Π_1 to A and Π_2 to B. The effect is to introduce arbitrary synchronisations between events of A and events of B in the manner of process algebra.

- **Restriction.** The restriction of an event structure E to a subset of events R, written $E \restriction R$, is the event structure with events $E' = \{e \in E \mid [e] \subseteq R\}$ and causal dependency and consistency induced by E.

Using these two operations, we can obtain a notion of **synchronized composition**. Synchronized compositions play a central role in process algebra, in such seminal work as Milner's CCS and Hoare's CSP. Synchronized compositions of event structures A and B are obtained as restrictions $A \times B \restriction R$. We obtain *pullbacks* as a special case. Let $f : A \to C$ and $g : B \to C$ be maps of event structures. Defining P to be $A \times B \restriction \{p \in A \times B \mid f\Pi_1(p) = g\Pi_2(p) \text{ with both defined}\}$, we obtain a pullback square

$$
\begin{array}{ccc}
 & P & \\
{\scriptstyle \Pi_1}\swarrow & \vee & \searrow{\scriptstyle \Pi_2} \\
A & & B \\
{\scriptstyle f}\searrow & & \swarrow{\scriptstyle g} \\
 & C &
\end{array}
$$

in the category of event structures. When f and g are total the same construction gives the pullback in the category of event structures with *total* maps.

Some technology—for instance that of *stable families* [19]—is needed to construct and analyze products, synchronized compositions and pullbacks. For the purposes of this paper the following characterization of the pullback of *total* maps will be sufficient. By a *finitary* partial order is meant a partial order where every element dominates only finitely many elements. (The characterization follows from the construction of pullbacks in event structures from those in stable families [19].)

Proposition 2.3 . *Let* P, Π_1, Π_2 *be the pullback of total maps* $f : A \to C$ *and* $g : B \to C$ *in the category of event structures. Configurations* $\mathcal{C}^\infty(P)$ *correspond to the composite bijections*

$$\theta : x \cong fx = gy \cong y$$

between configurations $x \in \mathcal{C}^\infty(A)$ *and* $y \in \mathcal{C}^\infty(B)$ *s.t.* $fx = fy$ *for which the transitive relation generated on* θ *by taking*

$$(a, b) \leq (a', b') \text{ if } a \leq_A a' \text{ or } b \leq_B b'$$

is a finitary partial order—we call such θ secured bijections. *The correspondence takes* $z \in \mathcal{C}^\infty(P)$ *to the secured bijection got as the composite* $\theta_z : \Pi_1 z \cong f\Pi_1 z = g\Pi_2 z \cong \Pi_2 z$ *and respects inclusion. A fortiori,* $\Pi_1 z = \pi_1 \theta_z$ *and* $\Pi_2 z = \pi_2 \theta_z$, *where* π_1 *and* π_2 *are projections to the first and second components of the pairs in* θ_z.

Definition 2.4 Let $(E, \leq, \mathrm{Con})$ be an event structure. Let $V \subseteq E$ be a subset of 'visible' events. Define the *projection* of E on V, to be $E{\downarrow}V =_{\mathrm{def}} (V, \leq_V, \mathrm{Con}_V)$, where $v \leq_V v'$ iff $v \leq v'$ & $v, v' \in V$ and $X \in \mathrm{Con}_V$ iff $X \in \mathrm{Con}$ & $X \subseteq V$.

2.2   Concurrent strategies

2.2.1   Event structures with polarity

Both a game and a strategy in a game are to be represented using event structures with polarity, which comprise (E, pol) where E is an event structure with a polarity function $pol : E \to \{+, -\}$ ascribing a polarity $+$ (Player) or $-$ (Opponent) to its events. The events correspond to (occurrences of) moves. Maps of event structures with polarity are maps of event structures which preserve polarities.

Definition 2.5 We define the following two simple operations on event structures with polarity.

- **Dual.** The *dual*, $E^\perp$, of an event structure with polarity E comprises the same underlying event structure E but with a reversal of polarities.

- **Simple parallel composition.** Let A and B be event structures with polarity. The operation $A \| B$ simply juxtaposes disjoint copies of A and B, maintaining their causal dependency and specifying a finite subset of events as consistent if it restricts to consistent subsets of A and B. Polarities are unchanged.

The constructions previously introduced for event structures generalize directly in the presence of polarities.
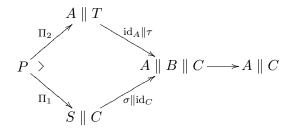
2.2.2   Pre-strategies

Let A be an event structure with polarity, thought of as a game; its events stand for the possible occurrences of moves of Player and Opponent and its causal dependency

and consistency relations stand for the constraints imposed by the game. A pre-strategy represents a nondeterministic play of the game—all its moves are moves allowed by the game and obey the constraints of the game; the concept will later be refined to that of *strategy*. A *pre-strategy in* A is defined to be a total map $\sigma : S \to A$ from an event structure with polarity S. Two pre-strategies $\sigma : S \to A$ and $\tau : T \to A$ in A will be essentially the same when they are isomorphic, *i.e.* there is an isomorphism $\theta : S \cong T$ such that $\sigma = \tau\theta$; then we write $\sigma \cong \tau$.

Let A and B be event structures with polarity. Following Joyal [10], a *pre-strategy from* A *to* B is a pre-strategy in $A^{\perp}\|B$, so a total map $\sigma : S \to A^{\perp}\|B$. We write $\sigma : A \nrightarrow B$ to express that σ is a pre-strategy from A to B. Note that a pre-strategy σ *in* a game A, *e.g.* $\sigma : S \to A$, coincides with a pre-strategy *from* the empty game $\emptyset$ *to* the game A, *i.e.* $\sigma : \emptyset \nrightarrow A$.

2.2.3 Composing pre-strategies

We present the composition of pre-strategies via pullbacks. Given two pre-strategies $\sigma : S \to A^{\perp}\|B$ and $\tau : T \to B^{\perp}\|C$, ignoring polarities we can consider the maps on the underlying event structures, *viz.* $\sigma : S \to A\|B$ and $\tau : T \to B\|C$. Viewed this way we can form the pullback in the category of event structures as shown below

where the map $A\|B\|C \to A\|C$ is undefined on B and acts as identity on A and C. The partial map from P to $A\|C$ given by the diagram above (either way round the pullback square) factors as the composition of the partial map $P \to P \downarrow V$, where V is the set of events of P at which the map $P \to A\|C$ is defined, and a total map $P \downarrow V \to A\|C$; the partial map acts as identity on V and is undefined otherwise, and the total map as f. The resulting total map gives us the composition $\tau \odot \sigma : P \downarrow V \to A^{\perp}\|C$ once we reinstate polarities.

Identities w.r.t. composition are given by copy-cat strategies. Let A be an event structure with polarity. The copy-cat strategy from A to A is an instance of a pre-strategy, so a total map $\gamma_A : \mathbb{C}_A \to A^{\perp}\|A$. It describes a concurrent strategy based on the idea that Player moves, of positive polarity, always copy previous corresponding moves of Opponent, of negative polarity. For $c \in A^{\perp}\|A$ we use $\bar{c}$ to mean the corresponding copy of c, of opposite polarity, in the alternative component. Define $\mathbb{C}_A$ to comprise the event structure with polarity $A^{\perp}\|A$ together with the extra causal dependencies generated by $\bar{c} \leq_{\mathbb{C}_A} c$ for all events c with $pol_{A^{\perp}\|A}(c) = +$. A finite subset of $\mathbb{C}_A$ is consistent if its down-closure w.r.t. $\leq_{\mathbb{C}_A}$ is consistent in $A^{\perp}\|A$. The *copy-cat* pre-strategy $\gamma_A : A \nrightarrow A$ is defined to be the map $\gamma_A : \mathbb{C}_A \to A^{\perp}\|A$ where γ_A acts as the identity function on the common set of events.

2.2.4   Strategies

The main result of [17] is that two conditions on pre-strategies, *receptivity* and *innocence*, are necessary and sufficient for copy-cat to behave as identity w.r.t. the composition of pre-strategies. Receptivity ensures an openness to all possible moves of Opponent. Innocence restricts the behaviour of Player: Player may only introduce new relations of immediate causality of the form $\ominus \rightarrowtail \oplus$ beyond those imposed by the game.

Definition 2.6 • Receptivity. A pre-strategy σ is *receptive* iff

$$\sigma x \stackrel{a}{\longrightarrow\!\subset} \ \& \ pol_A(a) = - \ \Rightarrow \ \exists! s \in S. \ x \stackrel{s}{\longrightarrow\!\subset} \ \& \ \sigma(s) = a \,.$$
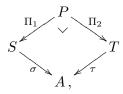
• **Innocence.** A pre-strategy σ is *innocent* when it is both
  $+$-*innocent:* if $s \rightarrowtail s' \ \& \ pol(s) = +$ then $\sigma(s) \rightarrowtail \sigma(s')$, and
  $-$-*innocent:* if $s \rightarrowtail s' \ \& \ pol(s') = -$ then $\sigma(s) \rightarrowtail \sigma(s')$.
  A **strategy** is a receptive and innocent pre-strategy.

Theorem 2.7 (from [17]) *Let* $\sigma : A \nrightarrow B$ *be pre-strategy. Copy-cat behaves as identity w.r.t. composition, i.e.* $\sigma \circ \gamma_A \cong \sigma$ *and* $\gamma_B \circ \sigma \cong \sigma$, *iff* σ *is receptive and innocent. Copy-cat pre-strategies* $\gamma_A : A \nrightarrow A$ *are receptive and innocent.*

Theorem 2.7 motivated the definition of a *strategy* as a pre-strategy which is receptive and innocent. In fact, we obtain a bicategory, **Games**, in which the objects are event structures with polarity—the games, the arrows from A to B are strategies $\sigma : A \nrightarrow B$ and the 2-cells are maps of spans. The vertical composition of 2-cells is the usual composition of maps of spans. Horizontal composition is given by the composition of strategies $\odot$ (which extends to a functor on 2-cells via the universality of pullback).

2.2.5   Interaction

In this paper, we will be particularly interested in the results of the interaction between a strategy $\sigma : S \rightarrow A$ and a counter-strategy $\tau : T \rightarrow A^\perp$ in order to determine the resulting payoff. Unlike the composition $\tau \odot \sigma$ where the interaction of σ and τ are hidden, it is the status of the configurations in $\mathcal{C}^\infty(A)$ their full interaction induces which decides the resulting payoff. Ignoring polarities, we have total maps of event structures $\sigma : S \rightarrow A$ and $\tau : T \rightarrow A$. Form their pullback,

to obtain the event structure P resulting from the interaction of σ and τ. Because σ or τ may be nondeterministic there could be several maximal configurations in $\mathcal{C}^\infty(P)$. Define the set of results of the interaction of σ and τ to be

$$\langle \sigma, \tau \rangle =_{\text{def}} \{\sigma\Pi_1 z \mid z \text{ is maximal in } \mathcal{C}^\infty(P)\} \,.$$

For any two strategies $\sigma : S \rightarrow A$ and $\tau : T \rightarrow A^\perp$, maximal configurations

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$\mathcal{C}^\infty(P)$ correspond to maximal secured bijections—see Proposition 2.3. The set of such maximal secured bijections, the possible interactions of σ and τ, is denoted by $[\sigma, \tau]$. In particular, for any $x \in \mathcal{C}^\infty(A)$ we have $x \in \langle \sigma, \tau \rangle$ iff there exists a *witness* $\theta \in [\sigma, \tau]$ such that $x = \sigma \pi_1 \theta$. We explore a little further the nature of such witnesses. Say that x is +-**maximal** if for all $x \overset{a}{-\!\!-\!\!\subset}$, then $pol(a) = -$. Because σ and τ are receptive, maximality of $z \in \mathcal{C}^\infty(P)$ amounts to +-maximality of $\Pi_1 z$ and $\Pi_2 z$. Accordingly, finding a witness $\theta \in [\sigma, \tau]$ amounts to finding θ a secured bijection for which $\pi_1 \theta$ and $\pi_2 \theta$ are +-maximal.

3 Concurrent games with payoff

We begin the core of the paper, the treatment of payoff in concurrent games. $\overline{\mathbb{R}}$ denotes $\mathbb{R} \cup \{-\infty, +\infty\}$, the reals extended with a minimum and maximum.

Definition 3.1 A **concurrent game with payoff** is a triple $(A, \kappa_A^+, \kappa_A^-)$, where A is a concurrent game and $\kappa_A^+, \kappa_A^- : \mathcal{C}^\infty(A) \to \overline{\mathbb{R}}$ are **payoff functions**.

Throughout this paper, we will only consider **zero-sum** concurrent games, *i.e.* for all $z \in \mathcal{C}^\infty(A)$, $\kappa_A^-(z) = -\kappa_A^+(z)$. It follows that our games with payoff will be described by a concurrent game and its payoff function $\kappa_A = \kappa_A^+ : \mathcal{C}^\infty(A) \to \overline{\mathbb{R}}$. We extend the usual constructions on concurrent games to games with payoff.

Definition 3.2 • **Dual.** If A is a concurrent game with payoff, then the payoff function on $A^\perp$ is defined by $\kappa_{A^\perp}(x) = -\kappa_A(x)$, for $x \in \mathcal{C}^\infty(A^\perp)$.

• **Parallel composition.** If A, B are concurrent games with payoff, then the payoff function on $A \parallel B$ is defined by $\kappa_{A\parallel B}(x) = \kappa_A(x_1) + \kappa_A(x_2)$, where $x_1 \in \mathcal{C}^\infty(A)$ is the projection of x on A and $x_2 \in \mathcal{C}^\infty(B)$ is the projection of x on B.

We now turn to the definitions leading to the value of a game. Since games and strategies are nondeterministic, these definitions come in two variants: the *optimistic* describing the outcome of a game if all the nondeterministic choices are in favour of Player, and the *pessimistic* describing the dual case, when all of those choices are in favour of Opponent. One of the main result of the paper will be that for race-free well-founded games (to be defined below), the two corresponding notions of value coincide.

Definition 3.3 We define the **optimistic** ($\uparrow$) and **pessimistic** ($\downarrow$) **results** of an interaction, and **values** of a strategy and of a game, as follows. Here, σ is a strategy on A and τ is a counter-strategy (a strategy on $A^\perp$), and the notation $\sigma : A$ signifies a strategy $\sigma : S \to A$.

$$r^\uparrow(\sigma, \tau) = \sup_{x \in \langle \sigma, \tau \rangle} \kappa_A(x) \qquad\qquad r^\downarrow(\sigma, \tau) = \inf_{x \in \langle \sigma, \tau \rangle} \kappa_A(x)$$

$$v^\uparrow(\sigma) = \inf_{\tau : A^\perp} r^\uparrow(\sigma, \tau) \qquad\qquad v^\downarrow(\sigma) = \inf_{\tau : A^\perp} r^\downarrow(\sigma, \tau)$$

$$v^\uparrow(A) = \sup_{\sigma : A} v^\uparrow(\sigma) \qquad\qquad v^\downarrow(A) = \sup_{\sigma : A} v^\downarrow(\sigma)$$

We say that a game A **has a value** if $v^\uparrow(A) = v^\downarrow(A) = -v^\downarrow(A^\perp) = -v^\uparrow(A^\perp)$: the optimistic and pessimistic values coincide, and commute with $(-)^\perp$. The commutation with $(-)^\perp$ is a form of minimax property, since the order of quantification

on strategies is reversed in $v(A)$ and $-v(A^\perp)$, whereas the coincidence of the optimistic and pessimistic value deals with nondeterminism. Note that not all games have a value:

Example 3.4 Take the game $A = \ominus \leadsto \oplus$ with two events of opposite polarities conflicting with each other, along with $\kappa(\emptyset) = 0$, $\kappa(\{\oplus\}) = 1$ and $\kappa(\{\ominus\}) = -2$. Then it is easy to prove that $v^\uparrow(A) = 1$, $v^\downarrow(A) = -2$, $v^\uparrow(A^\perp) = 2$ and $v^\downarrow(A^\perp) = -1$.

Example 3.5 The example above suggests a simple relationship between $v^\downarrow(A)$ and $v^\uparrow(A^\perp)$ but this is not always the case. For example, consider the infinite game A comprising the event structure with polarity

$$\ominus \qquad \oplus_1 \longrightarrow\!\!\!\triangleright \oplus_2 \longrightarrow\!\!\!\triangleright \oplus_3 \longrightarrow\!\!\triangleright \cdots \longrightarrow\!\!\!\triangleright \oplus_n \longrightarrow\!\!\triangleright \cdots$$

where $\kappa(\emptyset) = 0$, $\kappa(\{\oplus_1, \ldots, \oplus_n\}) = -n$, $\kappa(\{\oplus_1, \ldots, \oplus_n\} \cup \{\ominus\}) = n$, $\kappa(\{\oplus_1, \ldots\}) = +\infty$ and $\kappa(\{\oplus_1, \ldots\} \cup \{\ominus\}) = -\infty$. Then, one can establish the following values:

$$v^\downarrow(A) = 0 \qquad\qquad v^\downarrow(A^\perp) = -\infty$$
$$v^\uparrow(A) = +\infty \qquad\qquad v^\uparrow(A^\perp) = -\infty$$

These values are a bit subtle to establish, so we detail the proofs in the Appendix.

The first example features a *race*, where both players compete for the same resource, whereas the second example is not *well-founded*: the game allows infinite configurations. These brings us to the two following definitions, that will be crucial to get the value theorem.

Definition 3.6 An event structure S is **well-founded** if all its configurations are finite. A game A is **race-free** if for all $x \in \mathcal{C}(A)$ such that $x \overset{a}{\longrightarrow\!\!\!\subset}$ and $x \overset{a'}{\longrightarrow\!\!\!\subset}$ with $pol(a) = -$ and $pol(a') = +$, we have $x \cup \{a, a'\} \in \mathcal{C}(A)$.

Definition 3.7 Let A be a concurrent game with payoff, and $x \in \mathcal{C}^\infty(A)$. Let A/x be the **residual of A after x**, comprising

- events, $\{a \in A \setminus x \mid x \cup [a]_A \in \mathcal{C}^\infty(A)\}$,
- consistency relation, $X \in \mathrm{Con}_{A/x} \Leftrightarrow X \subseteq_f A/x \ \& \ x \cup [X]_A \in \mathcal{C}^\infty(A)$,
- causal dependency, the restriction of that on A.

Define $\kappa_{A/x} : \mathcal{C}^\infty(A/x) \to \overline{\mathbb{R}}$ by taking $\kappa_{A/x}(y) = \kappa_A(x \cup y)$. Finally, define $(A, \kappa_A)/x = (A/x, \kappa_{A/x})$. When x is a singleton $\{a\}$, we shall generally write A/a instead of $A/\{a\}$. Finally, we will often write $v^\uparrow(x)$ (resp. $v^\downarrow(x)$) for $v^\uparrow(A/x)$ (resp. $v^\downarrow(A/x)$) when A is understood from the context.

4 The value theorem

In this section, we prove the value theorem on concurrent games. The proof proceeds in two steps. First, we exhibit key constructions on strategies and the study the results of their interactions. This analysis will allow us to characterize the values of all positions of the game. Exploiting well-foundedness of the game, we will deduce the sought-for value theorem.

4.1 Constructions on strategies

In "glueing" strategies together it is helpful to assume that all the initial negative moves of the strategies are exactly the same, and indeed coincide with the initial negative moves of the game:

Lemma 4.1 *Let $\sigma : S \to A$ be a strategy. Then there exists a strategy $\sigma' : S' \to A$ with $\sigma \cong \sigma'$, for which*

$$\forall s' \in S'. \ pol_{S'}[s']_{S'} = \{-\} \Rightarrow \sigma'(s') = s' . \tag{†}$$

Proof. As a consequence of receptivity and negative innocence [17], whenever $\emptyset \subseteq^- y$ in $\mathcal{C}(A)$ there is a *unique* $x \in \mathcal{C}(S)$ so that $\emptyset \subseteq^- x$ & $\sigma x = y$. Consequently, the map σ induces an order isomorphism w.r.t. inclusion between configurations $x \in \mathcal{C}(S)$ where $\emptyset \subseteq^- x$ and $y \in \mathcal{C}(A)$ where $\emptyset \subseteq^- y$. The order isomorphism restricts to an order isomorphism between prime configurations. It follows that σ is bijective between events $s \in S_-$ and events $a \in A_-$. This bijection extends to a bijective renaming of events of S to those of S'. □

Henceforth we will assume all strategies satisfy the property (†). In particular, its adoption facilitates the definition of a 'sum' of strategies in a game.

Proposition 4.2 *Let $\sigma_i : S_i \to A$, for $i \in I$, be strategies (assumed to satisfy (†)). W.l.o.g. we may assume that whenever indices $i, j \in I$ are distinct then so are those events of S_i and S_j which causally depend on a positive event (otherwise we could tag such events by their respective indices). Define S to be the event structure with events $\bigcup_{i \in I} S_i$, causal dependency $s \leq_S e'$ iff $s \leq_{S_i} e'$, for some $i \in I$, and consistency $X \in \mathrm{Con}_S$ iff $X \in \mathrm{Con}_{S_i}$, for some $i \in I$. Defining $[\![_{i \in I} \sigma_i(s) = \sigma_i(s)$ if $s \in S_i$ yields a strategy $[\![_{i \in I} \sigma_i : S \to A$.*

Proof. *Pre-strategy.* Follows from the observation that for any $x \in \mathcal{C}(\bigcup_{i \in I} S_i)$ there is $i \in I$ such that $x \in \mathcal{C}(S_i)$. Therefore preservation of configuration and local injectivity directly follow from those properties for the σ_is.

Receptivity. Trivial, since (†) is preserved by union and implies receptivity.

Innocence. For any $s_1, s_2 \in \bigcup_{i \in I} S_i$, if $s_1 \rightarrowtail s_2$ then there is $i \in I$ such that $s_1, s_2 \in S_i$ and $s_1 \rightarrowtail s_2$ in S_i as well. Therefore if $pol(s_1) \neq -$ or $pol(s_2) \neq +$ then by innocence of σ_i we have $\sigma_i(s_1) \rightarrowtail \sigma_i(s_2)$ as well, therefore $([\![_{i \in I} \sigma_i)(s_1) \rightarrowtail ([\![_{i \in I} \sigma_i)(s_2)$ and $[\![_{i \in I} \sigma_i$ is innocent. □

The next construction takes a strategy σ on a game A/a, where a is a minimal positive event of game A, and creates a strategy on A that starts by playing a, then resumes as σ.

Proposition 4.3 *Suppose A is a race-free game such that $\emptyset \overset{a}{—\!\subset}$ with $pol(a) = +$. Then for any strategy $\sigma : S \to A/a$, where w.l.o.g. $a \notin S$, there is a strategy $\mathrm{play}_a(\sigma) : S' \to A$: the event structure S' comprises*

- *events, $S \cup \{a\}$,*

- *causal dependency, that on S extended by $a \leq_{S'} s$, for $s \in S$, whenever $a \leq_A \sigma(s)$,*

- *with consistency, $X \in \mathrm{Con}_{S'}$ iff $X \cap S \in \mathrm{Con}_S$,*

and $\mathrm{play}_a(\sigma)(s) = \sigma(s)$, for $s \in S$, with $\mathrm{play}_a(\sigma)(a) = a$.

Proof. It is easy to check that S' is an event structure and that $\text{play}_a(\sigma)$ is a total map of event structures which preserves polarity. Innocence is inherited from σ. That it is receptive follows from the race-freedom of A: Let $x \in \mathcal{C}(S')$ and $\text{play}_a(\sigma)x \xrightarrow{a'} \subset$ where $a' \in A$ and $pol(a') = -$. If $a \in x$ then receptivity condition for $\text{play}_a(\sigma)$ follows directly from that of σ. If $a \notin x$ then $x \in \mathcal{C}(S)$ and $\text{play}_a(\sigma)x = \sigma x$. From the race-fredom of A we deduce that $\sigma x \xrightarrow{a'} \subset$ in A/a. Again the receptivity condition for $\text{play}_a(\sigma)$ follows from that of σ. $\qquad\square$

Given a strategy on σ on a residual game A/a, where a is an initial negative event of A, we can create a strategy on A that awaits a, then resumes as σ.

Proposition 4.4 *Suppose A is a game such that $\emptyset \xrightarrow{a} \subset$ with $pol(a) = -$. Then for any strategy $\sigma : S \to A/a$, where w.l.o.g. $a \notin S$, there is a strategy $\text{wait}_a(\sigma) : S' \to A$: the event structure S' comprises*

- *events, $S \cup A_-$, where $A_- =_{\text{def}} \{a' \in A \mid pol_A[a']_A \subseteq \{-\}\}$,*
- *causal dependency, that on S and A_- extended by $a \leq_{S'} s$, for $s \in S$, whenever $a \leq_A \sigma(s)$ or $pol(s) = +$,*
- *with consistency, $X \in \text{Con}_{S'}$ iff $X \cap S \in \text{Con}_S$ & $\text{wait}_a(\sigma)X \in \text{Con}_A$,*

where $\text{wait}_a(\sigma)(s')$ is defined to be $\sigma(s')$ if $s' \in S$, otherwise s'.

Proof. By innocence, the causal dependencies on S and A_- agree where they overlap. As $a \notin S$, by assumption, we obtain a partial order $\leq_{S'}$ from the definition above. It is routine to check that S' is an event structure.

Observe that if $\sigma(s) \in A_-$ then $s \in S_-$, for all $s \in S$: otherwise there would be a maximal positive event on which s causally depended, contradicting $-$-innocence of σ.

In checking that $\text{wait}_a(\sigma)$, clearly a total function, is a map of event structures it is straightforward to show that the image of a configuration $x \in \mathcal{C}(S')$ is down-closed in A. By definition $\text{wait}_a(\sigma)$ preserves consistency, so $\text{wait}_a(\sigma)x$ is also consistent and in $\mathcal{C}(A)$. Suppose now $s_1, s_2 \in x$ with $\text{wait}_a(\sigma)(s_1) = \text{wait}_a(\sigma)(s_2)$. If both $s_1, s_2 \in S$ then $\sigma(s_1) = \sigma(s_2)$ so $s_1 = s_2$ as σ is map of event structures. Otherwise, either $s_1 \notin S$ or $s_2 \notin S$. If both $s_1 \notin S$ and $s_2 \notin S$, then $s_1 = s_2$, directly from the definition of $\text{wait}_a(\sigma)$. Otherwise, w.lo.g. suppose $s_1 \in S$ and $s_2 \notin S$. Then $\sigma(s_1) = s_2$ and $s_2 \in A_-$. By the observation above, $s_1 \in S_-$. But σ is assumed to satisfy (†), so $\sigma(s_1) = s_1 = s_2$. The function $\text{wait}_a(\sigma)$ is indeed a map of event structures.

The map $\text{wait}_a(\sigma)$ clearly preserves polarity. The construction preserves the innocence inherited from σ. We show receptivity. Suppose $x \in \mathcal{C}(S)$ and $\text{wait}_a(\sigma)x \xrightarrow{a'} \subset$ in A where a' has negative polarity. Consider first the case when $a' \in A_-$. Then it can be checked that $x \cup \{a'\} \in \mathcal{C}(S')$. This yields $x \xrightarrow{a'} \subset$ with $\text{wait}_a(\sigma)(a') = a'$. To show uniquesness, assume $\text{wait}_a(\sigma)(s') = a'$. If $s' \notin S$ we obtain $\text{wait}_a(\sigma)(s') = s' = a'$. If on the other hand, $s' \in S$ we obtain $\text{wait}_a(\sigma)(s') = \sigma(s') = a' \in A_-$. By the observation, $s' \in S_-$ and $\sigma(s') = s'$ as σ satisfies (†), and again $s' = a'$.

In the case where $a' \notin A_-$ there must be $a_1 \leq_A a'$ with $pol(a_1) = +$. Hence there

is $s_1 \in x$, with $pol(s_1) = +$, such that $\sigma(s_1) = a_1$. From the causal dependency of S' we must have $a \in x$. It follows that $x \setminus \{a\} \in \mathcal{C}(S)$ and $\sigma(x \setminus \{a\}) \overset{a'}{-\!\!\!-\!\subset}$ in A/a, whereupon receptivity of σ ensures the required receptivity condition for $\mathrm{wait}_a(\sigma)$. $\square$

We will make use later of the following extension of the notion of residual from games to strategies:

Definition 4.5 Let $\sigma : S \to A$ be a strategy and $x \in \mathcal{C}^\infty(S)$. Define the function $\sigma/x : S/x \to A/\sigma x$ to be the restriction of σ. In the case where x is a singleton $\{s\}$, we shall generally write σ/s instead of $\sigma/\{s\}$.

Proposition 4.6 *For $\sigma : S \to A$ a strategy and $x \in \mathcal{C}^\infty(S)$, the function $\sigma/s : S/s \to A/\sigma(s)$ is a strategy.*

Proof. A straightforward check. $\square$

4.2 Interactions between the constructed strategies

Let A be a game with payoff κ_A and $\sigma : S \to A$ and $\tau : T \to A^\perp$ be strategies. The set of values resulting from their interaction is given by $\{\kappa_A x \mid x \in \langle \sigma, \tau \rangle\}$, which we generally write as $\kappa\langle \sigma, \tau \rangle$ when the game is clear from the context. We use $\langle \sigma, \tau \rangle^+ =_{\mathrm{def}} \{x \in \langle \sigma, \tau \rangle \mid + \in pol\ x\}$ for the configurations in $\langle \sigma, \tau \rangle$ containing events of positive polarity. In this subsection, we analyse the possible interactions between the strategies constructed above, and prove a sequence of key lemmas.

Lemma 4.7 *Let A be a well-founded race-free game with payoff. Let $\sigma : S \to A$ and $\tau : T \to A^\perp$ be strategies. Then:*

$$\kappa\langle \sigma, \tau \rangle = \{-v \mid v \in \kappa\langle \tau, \sigma \rangle\}$$

Proof. Straightforward by definition of $\kappa_{A^\perp}$. $\square$

Lemma 4.8 *If the game A is well-founded and race-free,*

$$\kappa\langle \underset{i \in I}{\|}\ \sigma_i, \tau \rangle \subseteq \bigcup_{i \in I} \kappa\langle \sigma_i, \tau \rangle \qquad \kappa\langle \underset{i \in I}{\|}\ \sigma_i, \tau \rangle^+ = \bigcup_{i \in I} \kappa\langle \sigma_i, \tau \rangle^+$$

Proof. First, we prove that $\kappa\langle \|_{i \in I} \sigma_i, \tau \rangle \subseteq \bigcup_{i \in I} \kappa\langle \sigma_i, \tau \rangle$. Take $y \in \langle \|_{i \in I} \sigma_i, \tau \rangle$. Necessarily, there is a secured bijection $\theta \in [\|_{i \in I} \sigma_i, \tau]$ such that $\sigma \pi_1 \theta = y$. By definition of $\|_{i \in I} \sigma_i$, there is $i \in I$ such that $\pi_1 \theta \in S_i$. It follows that $\theta \in [\sigma_i, \tau]$, therefore $y \in \langle \sigma_i, \tau \rangle$ as well.

Likewise if $y \in \langle \sigma_i, \tau \rangle$ with a positive event, take its witness $\theta \in [\sigma_i, \tau]$. We have $\theta \in [\|_{i \in I} \sigma_i, \tau]$: in particular maximality follows from that of θ in $[\sigma_i, \tau]$. Indeed, since y has a +-event, this event is only consistent with events in S_i, hence any extension of θ must be compatible with S_i. $\square$

Lemma 4.9 *If A is race-free and well-founded, then,*

$$\kappa\langle \mathrm{play}_a(\sigma), \tau \rangle = \kappa\langle \sigma, \tau/a \rangle$$

Proof. First we prove that $\kappa\langle \mathrm{play}_a(\sigma), \tau \rangle \subseteq \kappa\langle \sigma, \tau/a \rangle$. Take $y \in \langle \mathrm{play}_a(\sigma), \tau \rangle$ and its witness $\theta \in [\mathrm{play}_a(\sigma), \tau]$ such that $y = \mathrm{play}_a(\sigma)\pi_1 \theta$.

We first prove that $a \in y$. Indeed, suppose $a \notin y$. Then, we have $\pi_1\theta \in \mathcal{C}(S')$ (with $S' = S \cup \{a\}$ and $\text{play}_a(\sigma) : S' \to A$, by definition of $\text{play}_a(\sigma)$) with $a \notin \pi_1\theta$. By definition of $\text{play}_a(\sigma)$ that immediatly implies that $\pi_1\theta \in \mathcal{C}(S)$, and $\text{play}_a(\sigma)\pi_1\theta = \sigma\pi_1\theta$. But since $\sigma : S \to A/a$ we have $y = \text{play}_a(\sigma)\pi_1\theta \in \mathcal{C}(A/a)$, so $y \cup \{a\} \in \text{Con}_A$ and $y \overset{a}{\relbar\joinrel\subset}$, since a is minimal in A. This implies that $\pi_2\theta \overset{a}{\relbar\joinrel\subset}$ by receptivity, and we also have $\pi_1\theta \overset{a}{\relbar\joinrel\subset}$ by definition of S'. It is immediate that

$$\theta' = \theta \cup \{(a,a)\}$$

is a secured bijection extending θ between $y \cup \{a\}$ and $\pi_2\theta \cup \{a\}$, contradicting the maximality of θ.

Therefore, $a \in y$, so we have $(a,a) \in \theta$. Set $\theta' = \theta \setminus \{(a,a)\}$. First, we note that $\pi_1\theta' = y \setminus \{a\} \in \mathcal{C}(S)$. Indeed it is down-closed ($\leq_{S'}$ being conservative over $\leq_S$ for events distinct from a), and consistent (since $y \setminus \{a\} = y \cap S \in \text{Con}_S$ by definition of $\text{Con}_{S'}$). Likewise we have $\pi_2\theta' = \pi_2\theta \setminus \{a\} \in \mathcal{C}(T/a)$ (with $\tau : T \to A^\perp$), by definition of T/a. Moreover θ' is still a secured bijection. Finally it is maximal. Indeed, suppose $\theta' \cup \{(s,t)\}$ is a valid extension of θ'. Then $\theta \cup \{(s,t)\}$ would also be a valid extension of θ, by definition of S' and T/a. But that contradicts maximality of θ, so $\theta' \in [\sigma, \tau/a]$. Since $\kappa_{A/a}(\sigma\pi_1\theta') = \kappa_A y$, we have proved the inequality.

We now turn to the other inequality. Take $y \in \kappa\langle\sigma, \tau/a\rangle$ along with its witness $\theta \in [\sigma, \tau/a]$. Then setting $\theta' = \theta \cup \{(a,a)\}$ it is direct by definition of S' and T/a that we have $\pi_1\theta' \in \mathcal{C}(S')$ and $\pi_2\theta' \in \mathcal{C}(T)$. It is a bijection, and is maximal since any extension (s,t) would also be a valid extension for θ. Therefore $\theta' \in [\text{play}_a(\sigma), \tau]$ with $\kappa_A(\text{play}_a(\sigma)\pi_1\theta') = \kappa_{A/a}\sigma\pi_1 z = \kappa_{A/a}(y)$. □

Lemma 4.10 *We have the following* equalities *between strategies:*

$$\text{play}_a(\sigma)/a = \sigma \qquad\qquad \text{wait}_a(\sigma)/a = \sigma$$

Proof. Trivial. □

Lemma 4.11 *If A is well-founded and race-free, then,*

$$\kappa\langle\text{wait}_a(\sigma), \tau\rangle \supseteq \bigcup_{t:\tau(t)=a} \kappa\langle\sigma, \tau/t\rangle \qquad \kappa\langle\text{wait}_a(\sigma), \tau\rangle^+ = \bigcup_{t:\tau(t)=a} \kappa\langle\sigma, \tau/t\rangle^+$$

Proof. We start with the left-to-right inclusion. Take $y \in \langle\text{wait}_a(\sigma), \tau\rangle$ (supposed to have positive events) along with its witness $\theta \in [\text{wait}_a(\sigma), \tau]$. Since y has positive events it must contain a, because positive events in S' are set to depend on a. Therefore there is some $t \in T$ such that $\tau(t) = a$ and $(a,t) \in \theta$. Defining $\theta' = \theta \setminus (a,t)$, we still have a secured bijection and we have $\pi_1\theta' = \pi_1\theta \setminus \{a\} \in \mathcal{C}(S)$ by definition of S' (with $\text{wait}_a(\sigma) : S' \to A$), and likewise $\pi_2\theta' = \pi_2\theta \setminus \{t\} \in \mathcal{C}(T/t)$. Moreover θ' is maximal: any extension (s',t') would still hold on θ. Indeed suppose that $\theta' \cup \{(s',t')\}$ is a valid extension of θ'. Then we have $\pi_1(\theta' \cup \{s'\}) \in \mathcal{C}(S)$, so $\pi_1(\theta' \cup \{s',a\}) \in \mathcal{C}(S')$: it is down-closed because $\pi_1(\theta' \cup \{s'\})$ is down-closed in S and by definition of S', and consistent because $(\pi_1(\theta' \cup \{s',a\})) \cap S = \pi_1(\theta' \cup \{s'\}) \in \text{Con}_S$ and $\text{wait}_a(\sigma)(\pi_1(\theta' \cup \{s',a\})) = \sigma(\pi_1(\theta' \cup \{s'\})) \cup \{a\} \in \mathcal{C}(A)$ (since $\sigma(\pi_1(\theta' \cup \{s'\})) \in \mathcal{C}(A/a)$). Likewise, $\pi_2(\theta' \cup \{t'\}) \in \mathcal{C}(T/t)$ so obviously $\pi_2(\theta' \cup \{t',t\}) \in \mathcal{C}(T)$. We have established that $\theta' \in [\sigma, \tau/t]$, with $\kappa_{A/a}(\pi_1\theta') = \kappa_A(\pi_1\theta)$.

Reciprocally take $t \in T$ such that $\tau(t) = a$, and $y \in \langle \sigma, \tau/t \rangle$ with its witness $\theta \in [\sigma, \tau/t]$. Then it is direct to prove that $\theta' = \theta \cup \{(a, t)\} \in [\text{wait}_a(\sigma), \tau]$, and $\kappa((\text{wait}_a(\sigma))\pi_1\theta') = \kappa(y)$ by definition of κ on A/a. Moreover if y had a positive event, so has $(\text{wait}_a(\sigma))\pi_1\theta'$. □

From this follow two important corollaries. Firstly, if a is an initial positive event of A we have $\kappa\langle \text{play}_a(\sigma), \text{wait}_a(\tau) \rangle = \kappa\langle \sigma, \tau \rangle$; two strategies, one playing a move and the other waiting for the move, synchronise. This immediately follows from the Lemmas 4.9 and 4.10. Secondly, the following additional construction will be crucial. For $(e_i)_{i \in I}$ the family of negative minimal events of A and strategies $\sigma_i : S_i \to A/e_i$, we define $\text{case}_{i \in I}\sigma_i =_{\text{def}} \|_{i \in I} \text{wait}_{e_i}\sigma_i$. Roughly, this strategy waits for an input e_i and then proceeds as σ_i; though the full story is subtle as two distinct events e_i and e_j may be consistent with each other and the strategies σ_i and σ_j overlap. Then, we have:

Corollary 4.12 *Setting* $\text{case}_{i \in I}\sigma_i = \|_{i \in I} \text{wait}_{a_i}(\sigma_i)$*, and if* $\tau : T \to A^\perp$ *is such that* T *has a minimal +-event, then.*

$$\kappa\langle \text{case}_{i \in I}\sigma_i, \tau \rangle \subseteq \bigcup_{i \in I, t:\tau(t)=a_i} \kappa\langle \sigma_i, \tau/t \rangle \qquad \kappa\langle \text{case}_{i \in I}\sigma_i, \tau \rangle^+ = \bigcup_{i \in I, t:\tau(t)=a_i} \kappa\langle \sigma_i, \tau/t \rangle^+$$

If T *has no +-minimal event, then* $\kappa\langle \text{case}_{i \in I}\sigma_i, \tau \rangle = \{\kappa(\emptyset)\}$.

Proof. We apply the following reasoning, putting all the previous lemmas together:

$$\kappa\langle \text{case}_{i \in I}\sigma_i, \tau \rangle = \kappa\langle \|_{i \in I} \text{wait}_{a_i}(\sigma_i), \tau \rangle$$
$$\subseteq \bigcup_{i \in I} \kappa\langle \text{wait}_{a_i}(\sigma_i), \tau \rangle$$
$$\subseteq \bigcup_{i \in I} \left(\bigcup_{t:\tau(t)=a_i} \kappa\langle \sigma_i, \tau/t \rangle \right)$$

All these inclusions become equalities when restricted to configurations with a positive event. □

In the lemmas above and this corollary, in all the cases where we have inclusions instead of equalities this is by necessity. For instance with the case construction above, a configuration in $\langle \sigma_i, \tau \rangle$, by definition a maximal configuration of the pullback of σ_i and τ, although it reappears as a configuration of the pullback of $\text{case}_{i \in I}\sigma_i$ and τ, may no longer be maximal so fail to contribute to $\langle \text{case}_{i \in I}\sigma_i, \tau \rangle$.

4.3 Values of these constructions

Lemma 4.13 *For any race-free well-founded game A, a a minimal +-event of A and $\sigma : S \to A/a$ a strategy, we have:*

$$v^\uparrow(\text{play}_a(\sigma)) = v^\uparrow(\sigma) \qquad v^\downarrow(\text{play}_a(\sigma)) = v^\downarrow(\sigma)$$

Proof. We first prove $v^\uparrow(\text{play}_a(\sigma)) \leq v^\uparrow(\sigma)$. Let $\tau : T \to A^\perp/a$, and $k \in \kappa\langle \text{play}_a\sigma, \text{wait}_a\tau \rangle$. By Lemma 4.9, we have $k \in \kappa\langle \sigma, (\text{wait}_a\tau)/a \rangle$. By Lemma 4.10, we have $k \in \kappa\langle \sigma, \tau \rangle$. Let us now prove $v^\uparrow(\sigma) \leq v^\uparrow(\text{play}_a(\sigma))$. Let $\tau : T \to A^\perp$, and $k \in \kappa\langle \sigma, \tau/a \rangle$. Then by Lemma 4.9 we also have $k \in \kappa\langle \text{play}_a\sigma, \tau \rangle$.

We prove $v^\downarrow(\text{play}_a\sigma) \leq v^\downarrow(\sigma)$. Let $\tau : T \to A^\perp/a$ and $k \in \kappa\langle \sigma, \tau \rangle$. By Lemma 4.10, we have $k \in \kappa\langle \sigma, (\text{wait}_a\tau)/a \rangle$. By Lemma 4.9, we also have

$k \in \kappa\langle \text{play}_a\sigma, \text{wait}_a\tau\rangle$. We prove the converse inequality. Take $\tau : T \to A^\perp$, and $k \in \kappa\langle\text{play}_a\sigma, \tau\rangle$. By Lemma 4.9 we also have $k \in \kappa\langle\sigma, \tau/a\rangle$. $\qquad\Box$

Lemma 4.14 *For any race-free well-founded game A, a strategy $\sigma : S \to A$ and s a minimal event in S, we have:*

$$v^\downarrow(\sigma) \leq v^\downarrow(\sigma/s)$$

Proof. Let $\tau : T \to A^\perp/\sigma s$, and $k \in \kappa\langle\sigma/s, \tau\rangle$. If s has positive polarity, then by Lemmas 4.11 and 4.7, we also have $k \in \kappa\langle\sigma, \text{wait}_{\sigma s}\tau\rangle$. Likewise if s has negative polarity, then by Lemmas 4.9 and 4.7 we also have $k \in \kappa\langle\sigma, \text{play}_{\sigma s}\tau\rangle$. $\qquad\Box$

Lemma 4.15 *Suppose A is race-free and well-founded and $\sigma : S \to A$ is a strategy with a minimal $+$-event. Let $(f_j)_{j\in J}$ be the family of minimal $+$-events of A. Then, $v^\downarrow(\sigma) \leq \sup_{j\in J} v^\downarrow(\{f_j\})$ and $v^\uparrow(\sigma) \leq \sup_{j\in J} v^\uparrow(\{f_j\})$.*

Proof. *Pessimistic case.* Let s be the minimal $+$-event of S. Necessarily there must be $j_0 \in J$ such that $\sigma(s) = f_{j_0}$. By Lemma 4.14, we have $v^\downarrow(\sigma) \leq v^\downarrow(\sigma/s)$ as needed.

Optimistic case. If we had the optimistic inequality $v^\uparrow(\sigma) \leq v^\uparrow(\sigma/s)$ the proof would be just as straightforward, unfortunately it is not true in general since σ could have another conflicting minimal $+$-event. Therefore in the optimistic case we cannot directly pick a $j \in J$ and we are required to think of the problem more globally, involving classical reasoning.

Suppose that the inequality is false, *i.e.* $\sup_{j\in J} v^\uparrow(\{f_j\}) < v^\uparrow(\sigma)$. This implies that there is $\alpha \in \mathbb{R}$ such that $\sup_{j\in J} v^\uparrow(\{f_j\}) < \alpha$ and $v^\uparrow(\sigma) > \alpha$. The first inequality implies $\forall j \in J,\ \forall\sigma : A/f_j,\ \exists\tau' : A^\perp/f_j,\ \forall k \in \kappa\langle\sigma', \tau'\rangle,\ k < \alpha$, which is easily shown to imply

(1) $\forall(\sigma_m)_{m\in M},\ \exists(\tau_a)_{a\in\sigma M},\ \forall m \in M,\ \forall k \in \kappa\langle\sigma_m, \tau_{\sigma m}\rangle,\ k < \alpha$

where M is the set of positive minimal events in S, and for all $m \in M$, $\sigma_m : A/\sigma m$ and $\tau_a : A/a$. Applying this property to the family of strategies obtained by $\sigma_m = \sigma/m$, we get a family of counter-strategies $(\tau_a)_{a\in\sigma M}$. We extend this family to J by setting τ_j to be the empty strategy (closed under receptivity) whenever $e_j \notin \sigma M$. Thus, we get a family $(\tau_j)_{j\in J}$. Similarly, the second inequality implies that:

$$\forall\tau : A^\perp,\ \exists k \in \kappa\langle\sigma, \tau\rangle,\ k > \alpha.$$

Applied to $\tau = \text{case}_{j\in J}\tau_j$, we get $k \in \kappa\langle\sigma, \text{case}_{j\in J}\tau_j\rangle$ such that $k > \alpha$. By Corollary 4.12 (and Lemma 4.7), there is $m_0 \in M$ such that we also have $k \in \kappa\langle\sigma/m_0, \tau_{\sigma m_0}\rangle$. However, applying (1) to m_0 also shows that $k < \alpha$—a contradiction. Hence, the required inequality is true. $\qquad\Box$

Lemma 4.16 *Let A be a race-free well-founded game and $(e_i)_{i\in I}$ the family of its negative minimal events. Then,*

$$\min(\kappa(\emptyset), \inf_{i\in I}\sup_{\sigma:A/e_i} v^\downarrow(\sigma)) \leq v^\downarrow(A)$$

$$\min(\kappa(\emptyset), \inf_{i\in I}\sup_{\sigma:A/e_i} v^\uparrow(\sigma)) \leq v^\uparrow(A)$$

Proof. For as long as possible, we do not distinguish the optimistic and pessimistic cases. If the inequality is false, then there is a real $\alpha \in \mathbb{R}$ such that $\min(\kappa(\emptyset), \inf_{i\in I}\sup_{\sigma:A/e_i} v(\sigma)) > \alpha > v(A)$, which in turn implies:

(2) $$\kappa(\emptyset) > \alpha$$

(3) $\forall i \in I,\ \exists \sigma_i : A/e_i,\ \forall \tau : A^\perp/e_i,\ r(\sigma_i, \tau) > \alpha$

(4) $$\forall \sigma : A,\ \exists \tau : A^\perp,\ r(\sigma, \tau) < \alpha$$

In particular, (3) gives a family $(\sigma_i)_{i \in I}$. Instantiating (4) to $\mathrm{case}_{i \in I}\sigma_i$, we get $\tau : T \to A^\perp$ such that $r(\mathrm{case}_{i \in I}\sigma_i, \tau) < \alpha$. Reformulating the inequalities above with this data, we have:

(5) $$\kappa(\emptyset) > \alpha$$

(6) $\forall i \in I,\ \forall t,\ \tau(t) = e_i \Rightarrow r(\sigma_i, \tau/t) > \alpha$

(7) $$r(\mathrm{case}_{i \in I}, \tau) < \alpha$$

Pessimistic case. Since $r^\downarrow(\mathrm{case}_{i \in I}\sigma_i, \tau) < \alpha$, there must be $y \in \langle \mathrm{case}_{i \in I}\sigma_i, \tau \rangle$ such that $\kappa(y) < \alpha$. If T has no minimal +-event, then necessarily we have $y = \emptyset$ (since all the +-events of $\mathrm{case}_{i \in I}\sigma_i : S' \to A$ depend on a negative event), therefore $\kappa_A(y) = \kappa_A(\emptyset) > \alpha$—a contradiction. Therefore, T has a minimal +-event. Then by Corollary 4.12, there is a minimal +-event $t \in T$ and $\tau(t) = e_{i_0}$ and $k \in \kappa\langle \sigma_{i_0}, \tau/t \rangle$ such that $k = \kappa(y) < \alpha$. But this is absurd by (6), so we have found a contradiction. *Optimistic case.* By (7) instantiated on the optimistic case we have that for all $k \in \kappa\langle \mathrm{case}_{i \in I}, \tau \rangle,\ k < \alpha$. Take $y \in \langle \mathrm{case}_{i \in I}, \tau \rangle$ ($\langle \mathrm{case}_{i \in I}, \tau \rangle$ is non-empty by Zorn's lemma). As above, y cannot be empty as that would cause a contradiction, and T must have a minimal +-event. By Corollary 4.12, there is a minimal +-event $t \in T$ and $\tau(t) = e_{i_0}$ and $k \in \kappa\langle \sigma_{i_0}, \tau/t \rangle$ such that $k = \kappa(y) < \alpha$, contradicting (6). □

4.4 Value theorem

Let A be a fixed well-founded and race-free game.

Lemma 4.17 *Let $x \in \mathcal{C}(A)$. Let $(e_i)_{i \in I}$ be the family of extensions of x of negative polarity, and $(f_j)_{j \in J}$ be the family of extensions of x of positive polarity. Then,*

$$v^\uparrow(x) = \max(\min(\kappa(x), \inf_{i \in I} v^\uparrow(x \cup \{e_i\})), \sup_{j \in J} v^\uparrow(x \cup \{f_j\}))$$

$$v^\downarrow(x) = \max(\min(\kappa(x), \inf_{i \in I} v^\downarrow(x \cup \{e_i\})), \sup_{j \in J} v^\downarrow(x \cup \{f_j\}))$$

Proof. The reasoning is the same in the optimistic and pessimistic cases, so we do not distinguish them.

$\leq$. Let $\sigma : S \to A/x$ be a strategy. If there is a minimal event $s \in S$ with $pol(s) = +$, then $v(x) \leq v(\sigma) \leq \sup_{j \in J} v(x \cup \{f_j\})$ by Lemma 4.15. Otherwise, there is no such minimal $s \in S$. Then $v(\sigma) \leq \kappa(x)$. Indeed, letting $\tau : T \to A/x$ be the empty strategy closed by receptivity, we have $\langle \sigma, \tau \rangle = \{\emptyset\}$ and $r(\sigma, \tau) = \kappa(x)$. Similarly taking $i_0 \in I$, by Lemma 4.14 we have $v(\sigma) \leq v(\sigma/e_{i_0})$, and therefore $v(\sigma) \leq \inf_{i \in I} v(x \cup \{e_i\})$.

$\geq$. Let us prove that $\sup_{j \in J} v(x \cup \{f_j\}) \leq v(x)$, taking $j_0 \in J$ and $\sigma : A/(x \cup \{f_{j_0}\})$. Then by Lemma 4.13 we have $v(\mathrm{play}_{f_{j_0}} \sigma) = v(\sigma)$, so $v(\sigma) \leq v(x)$. Finally, by Lemma 4.16 we have as needed $\min(\kappa(x), \inf_{i \in I} v(x \cup \{e_i\})) \leq v(x)$. □

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Theorem 4.18 *If A is well-founded and race-free then A has a value, i.e. we have:*

$$v^{\uparrow}(A) = v^{\downarrow}(A) \qquad\qquad v(A) = -v(A^{\perp})$$

*(Note that the second equality only makes sense because by the first, we can talk in a non-ambiguous way of the **value** $v(A)$ of a game A.)*

Proof. Let $x \in \mathcal{C}(A)$ be maximal such that $v(A/x) \neq -v(A^{\perp}/x)$. Let $(e_i)_{i \in I}$ be the family of negative extensions of x and $(f_j)_{j \in J}$ its family of positive extensions. Then,

$$v(A/x) = \max(\min(\kappa_A(x), \inf_{i \in I} v(A/(x \cup \{e_i\}))), \sup_{j \in J} v(A/(x \cup \{f_j\})))$$

$$= \max(\min(-\kappa_{A^{\perp}}(x), \inf_{i \in I} -v(A^{\perp}/(x \cup \{e_i\}))), \sup_{j \in J} -v(A^{\perp}/(x \cup \{f_j\})))$$

$$= -\min(\max(\kappa_{A^{\perp}}(x), \sup_{i \in I} v(A^{\perp}/(x \cup \{e_i\}))), \inf_{j \in J} v(A^{\perp}/(x \cup \{f_j\})))$$

$$= -\max(\min(\kappa_{A^{\perp}}(x), \inf_{j \in J} v(A^{\perp}/(x \cup \{f_j\}))),$$

$$\min(\sup_{i \in I} v(A^{\perp}/(x \cup \{e_i\})), \inf_{j \in J} v(A^{\perp}/(x \cup \{f_j\})))))$$

But for all $i_0 \in I$, $v(A^{\perp}/(x \cup \{e_{i_0}\})) \leq v(A^{\perp}/x)$. Indeed take $\sigma : S \to A^{\perp}/(x \cup \{e_{i_0}\})$, take $\tau : T \to A/x$, then by Lemma 4.9 we have $\kappa\langle\sigma, \tau/e_{i_0}\rangle = \kappa\langle\text{play}_{e_{i_0}} \sigma, \tau\rangle$. Likewise for all $j_0 \in J$ we have $v(A^{\perp}/x) \leq v(A^{\perp}/(x \cup \{f_{j_0}\}))$. Indeed take $\sigma : S \to A^{\perp}/x$ and $\tau : T \to A/(x \cup \{f_{j_0}\})$, then by Lemma 4.9 (and Lemma 4.7) we have $\kappa\langle\sigma, \text{play}_{f_{j_0}} \tau\rangle = \kappa\langle\sigma/f_{j_0}, \tau\rangle$. Therefore $\sup_{i \in I} v(A^{\perp}/(x \cup \{e_i\})) \leq \inf_{j \in J} v(A^{\perp}/(x \cup \{f_j\}))$, and:

$$v(A/x) = -\max(\min(\kappa_{A^{\perp}}(x), \inf_{j \in J} v(A^{\perp}/(x \cup \{f_j\}))), \sup_{i \in I} v(A^{\perp}/(x \cup \{e_i\})))$$

$$= -v(A^{\perp}/x)$$

—a contradiction. Therefore there is no such maximal x and the property is true for the empty configuration, thus $v(A) = -v(A^{\perp})$ since A is well-founded. $\qquad\square$

We say that a strategy $\sigma : S \to A$ is **optimal** when its *pessimistic value* is equal to the value of the game. Note that it also implies that the optimistic value is equal to the value of the game, since for all $\sigma : S \to A$ we must have $v^{\downarrow}(\sigma) \leq v^{\uparrow}(\sigma) \leq v(A)$. It also follows that for optimal strategies, the pessimistic and optimistic values coincide. When σ is optimal, we will therefore sometimes just write $v(\sigma)$ for its value.

Example 4.19 Any well-founded race-free game has a value. However this value is not necessarily reached: there are games without optimal strategies. Consider the game A with events $\{\oplus_i \mid i \in \mathbb{N}\}$, pairwise inconsistent, with $\kappa(\emptyset) = 0$ and $\kappa(\{\oplus_i\}) = i$. Its value is $+\infty$ since each positive natural number can be reached, but no strategy σ satisfies $v^{\downarrow}(\sigma) = +\infty$ (though the strategy that plays a nondeterministic choice of natural number satisfies $v^{\uparrow}(\sigma) = +\infty$).

5   Compositionality of optimal strategies

Finally we study how payoff relates to the composition of strategies. We hope that thinking compositionally about values and optimal strategies can be helpful

in computing values and optimal strategies for complex games from smaller ones. There are two main kinds of composition of strategies. The first is the categorical composition $\tau \odot \sigma$ of $\sigma : S \to A^\perp \parallel B$ and $\tau : T \to B^\perp \parallel C$. The second is parallel composition $\sigma \parallel \tau : S \parallel T \to A \parallel B$.

We start this section with the following characterisation of the pessimistic value of strategies.

Proposition 5.1 *Let A be a game and $\sigma : S \to A$ a strategy. Then,*
$$v^\downarrow(\sigma) = \inf\{\kappa(\sigma x) \mid x \in \mathcal{C}(S) \ \text{+-maximal}\}.$$

Proof. $\leq$. It suffices to show:
$$\forall x \in \mathcal{C}^\infty(S) \ \text{+-maximal}, \ \exists \tau : T \to A^\perp, \ \exists y \in \langle \sigma, \tau \rangle, \ \kappa(y) \leq \kappa(\sigma x)$$

Thus, let $x \in \mathcal{C}^\infty(S)$ be +-maximal. Set $T = (\sigma x)^\perp$ with $\tau : T \to A^\perp$ acting as the identity on events. τ is obviously innocent but not necessarily receptive, consider its closure $\tau' : T' \to A^\perp$ under receptivity. Define
$$\theta = \{(s, \sigma s) \mid s \in x\}.$$

Because σ is a map of event structures it follows that θ is a secured bijection. It is maximal since x is +-maximal σx is +-maximal in T by construction of T. Additionally we have $\pi_1 \theta = x \in \mathcal{C}^\infty(S)$ and $\pi_2 \theta = \sigma x \in \mathcal{C}^\infty(T')$, and both are +-maximal by the hypothesis on x and construction of T'. Therefore $\theta \in [\sigma, \tau']$. It follows that $\sigma \pi_1 \theta = \sigma x \in \langle \sigma, \tau' \rangle$.

$\geq$. It suffices to show that for all $\tau : T \to A^\perp/x$ and $y \in \langle \sigma, \tau \rangle$ there exists a +-maximal $x \in \mathcal{C}^\infty(S)$ such that $\kappa(\sigma x) \leq \kappa(y)$. Such a $y \in \langle \sigma, \tau \rangle$ necessarily has a witness $\theta \in [\sigma, \tau]$. Necessarily $x = \pi_1 z$ is +-maximal and $\kappa(\sigma x) = \kappa(y)$. $\quad\square$

From this, we get:

Proposition 5.2 *For strategies $\sigma : S \to A^\perp \parallel B$ and $\tau : T \to B^\perp \parallel C$, we have $v^\downarrow(\tau \odot \sigma) \geq v^\downarrow(\tau) + v^\downarrow(\sigma)$. Likewise for $\sigma : S \to A$ and $\tau : T \to B$, we have $v^\downarrow(\sigma \parallel \tau) = v^\downarrow(\sigma) + v^\downarrow(\tau)$.*

For categorical composition, $v^\downarrow(\tau \odot \sigma) \leq v^\downarrow(\tau) + v^\downarrow(\sigma)$ does not hold in general, and neither do the two inequalities in the optimistic case. However, the situation is different for *optimal strategies*. To establish this, we first note:

Proposition 5.3 *For race-free, well-founded A and B, $v(A \parallel B) = v(A) + v(B)$.*

Proof. By the value theorem, it does not matter whether we work on the optimistic or pessimistic cases. For simplicity, let us pick the pessimistic one. Firstly, we prove that $v(A \parallel B) \geq v(A) + v(B)$. Indeed, let $\sigma : S \to A$ and $\tau : T \to B$ be strategies. Then, as needed we have $v^\downarrow(\sigma \parallel \tau) \geq v^\downarrow(\sigma) + v^\downarrow(\tau)$ by Proposition 5.2.

Moreover, this inequality also holds for $A^\perp$ and $B^\perp$, therefore $v(A^\perp \parallel B^\perp) \geq v(A^\perp) + v(B^\perp)$, from which it follows that $v(A \parallel B) \leq v(A) + v(B)$ by the value theorem and the definition of the dual of games with payoff. $\quad\square$

Theorem 5.4 *If $\sigma : S \to A^\perp \parallel B$ and $\tau : T \to B^\perp \parallel C$ are optimal strategies, so is $\tau \odot \sigma$. Moreover copycat is optimal, therefore there is a bicategory of concurrent games with payoff and optimal strategies.*

Proof. Suppose σ and τ optimal. We reason as follows:

$$v^{\downarrow}(\tau \odot \sigma) \geq v^{\downarrow}(\sigma) + v^{\downarrow}(\tau) = v(A^{\perp} \parallel B) + v(B^{\perp} \parallel C) = v(A^{\perp} \parallel C)$$

This implies that $v^{\downarrow}(\tau \odot \sigma) = v(A^{\perp} \parallel C)$, since a strict inequality would contradict the definition of $v(A^{\perp} \parallel C)$.

Any +-maximal $x \in \mathcal{C}(A^{\perp} \parallel A)$ has the form $y \cup \overline{y}$, where $y \in \mathcal{C}(A)$. Moreover, $\kappa_{A^{\perp} \parallel A}(x) = \kappa_A(y) - \kappa_A(y) = 0$, therefore we have $v^{\downarrow}(\gamma_A) = 0$. However we also have $v(A^{\perp} \parallel A) = v(A) - v(A) = 0$, therefore copycat is optimal. $\square$

We finish this section by remarking that from the theorem above it follows that when σ and τ are optimal, we have $v(\tau \odot \sigma) = v(\sigma) + v(\tau)$, since both sides are forced by optimality to coincide with the value of the game.

6 Conclusion

We have proved a value theorem for race-free well-founded concurrent games. Note that this theorem is not an equilibrium theorem since the value is not always reached. However it is always reached in finite games. In fact our constructions on strategies give an algorithm to compute the value and optimal strategies for finite games. In future we plan to investigate the existence and computation of equilibria in the non-zero-sum case. This will require the extension of our framework to deal with probabilistic strategies, and should allow us to formulate a better connection with the concurrent games of [3,7].

We proved that optimal strategies are stable under composition, forming a bi-category. This compositional structure is worth investigating further. In other work, we have developed an extension of concurrent games *with symmetry*, where events can be duplicated and hence form the basis for a *cartesian closed category* of concurrent strategies. We plan to investigate extensions of the present development in the presence of symmetry, thus providing the basis for a *concurrent programming language* based on the simply-typed λ-calculus and concurrent operations on strategies, for which typable terms describe optimal strategies.

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The values for the game of Example 3.5:

$$v^\downarrow(A) = 0 \qquad\qquad v^\downarrow(A^\perp) = -\infty$$
$$v^\uparrow(A) = +\infty \qquad\qquad v^\uparrow(A^\perp) = -\infty\,.$$

Proof. $v^\downarrow(A)$. Let $\sigma : S \to A$ be an arbitrary strategy. By Proposition 5.1, its pessimistic value is $\inf\{\kappa(\sigma x) \mid x \in \mathcal{C}(S)$ +-maximal$\}$, therefore it suffices to examine the +-maximal configurations of S of worst payoff. If S admits an infinite configuration x, then σx contains all the +-events. By receptivity there is $x \subseteq x' \in \mathcal{C}^\infty(S)$ such that $\sigma x' = A$ of payoff $-\infty$. Now, suppose S has no infinite configuration. If S has a +-maximal configuration x containing i +-events but no $-$-event, its payoff is $\kappa_A(\sigma x) = -i$. If S has such a configuration for arbitrarily large i then the value of σ is $-\infty$, otherwise it is $-i$ for i the largest such that S has a +-maximal configuration containing i +-events and no $-$-event. Taking the supremum over all strategies σ, we obtain a pessimistic value of 0, reached by the empty strategy (closed under receptivity).

$\quad v^\downarrow(A^\perp)$. Let $\sigma : S \to A^\perp$ be an arbitrary strategy. If all its +-maximal configurations contain no +-event, then the configuration x containing all the $-$-events is +-maximal and has value $\kappa_{A^\perp}(\sigma x) = -\infty$. If there is a +-maximal configuration $x \in \mathcal{C}^\infty(S)$ containing a +-event s, then there is i such that s depends on the first i $-$-events. This means that any +-maximal configuration containing at least i $-$-events must contain a +-event as well. So for any $j \geq i$, there is a +-maximal configuration $y \in \mathcal{C}^\infty(S)$ containing the first j $-$-events and a +-event, and hence has payoff $\kappa_{A^\perp}(\sigma y) = -j$. This is for arbitrary j, so σ has pessimistic value $-\infty$. Any σ has a pessimistic value of $-\infty$, so $A^\perp$ has a pessimistic value of $-\infty$.

$\quad v^\uparrow(A)$. This case is a bit more subtle. Intuitively, one might think that Player cannot ensure a payoff higher than 0, since he cannot control when -or if - Opponent will play the $-$-event. However, using nondeterminism Player can do better and apply intuitively a *family* of deterministic sub-strategies. The definition of the optimistic value will ensure that the most favorable such sub-strategy is always selected in an interaction.

Define S as having the following events and causality:

$S = \{\ominus,$

$\quad \oplus_{2,1}$

$\quad \oplus_{3,1} \rightarrowtail \oplus_{3,2}$

$\quad \oplus_{4,1} \rightarrowtail \oplus_{4,2} \rightarrowtail \oplus_{4,3}$

$\quad \ldots,$

$\quad \oplus_{\omega,1} \rightarrowtail \ldots \rightarrowtail \oplus_{\omega,n} \rightarrowtail \ldots\}$

with consistent subsets the finite subsets of $\{\ominus\} \cup \{\oplus_{\alpha,i} \mid i < \alpha\}$, for $\alpha \in \mathbb{N} \cup \{\omega\}$. Take $\sigma : S \to A$ the obvious morphism mapping $\ominus$ to $\ominus$ and $\oplus_{\alpha,i}$ to $\oplus_i$, σ is a strategy. Now, take an arbitrary $\tau : T \to A^\perp$. Two cases are possible. If T does not contain any +-event, then by receptivity of τ there is an interaction

$$\theta = \{(\oplus_{\omega,1}, \ominus_1), \ldots, (\oplus_{\omega,n}, \ominus_n), \ldots\} \in [\sigma, \tau]$$

which is such that $\sigma\pi_1\theta = \{\oplus_1, \ldots, \oplus_n, \ldots\}$ of payoff $+\infty$, so $r^\uparrow(\sigma, \tau) = +\infty$. If T does contain a +-event t, say it depends on (the unique events in T corresponding

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to) $\ominus_1, \ldots, \ominus_i$. For each $j \geq i$, there is a $+$-maximal configuration

$$x = \{\ominus, \oplus_{j+1,1}, \oplus_{j+1,2}, \ldots, \oplus_{j+1,j}\}$$

containing exactly j $+$-events and the $-$-event, and this yields an interaction

$$\theta = \{(\ominus, t), (\oplus_{j+1,1}, \ominus_1), \ldots, (\oplus_{j+1,j}, \ominus_i)\} \in [\sigma, \tau]$$

such that $\pi_1 \theta = x$. This configuration x has payoff $\kappa_A(\sigma x) = j$ and this holds for any j, so $r^\uparrow(\sigma, \tau) = +\infty$. Taking the infimum over all strategies $\tau : T \to A^\perp$ we get that $v^\uparrow(\sigma) = +\infty$, so $v^\uparrow(A) = +\infty$.

$v^\uparrow(A^\perp)$. Take an arbitrary strategy $\sigma : S \to A^\perp$. If it never plays any $+$-event, then its optimistic value is $-\infty$: this value is reached against the strategy $\tau : T \to A$ that plays all the $+$-events. If there is a configuration $x \in \mathcal{C}^\infty(S)$ with a $+$-event, say this $+$-event depends on the first i $-$-events. For $j \geq i$, σ has an optimistic value of $-j$ against the strategy τ_j that plays the first j $+$-events. So, the optimistic value of σ is $-\infty$ as well, yielding a value for the game of $v^\uparrow(A^\perp) = -\infty$. $\qquad \square$