A Tale of Additives and Concurrency in Game Semantics

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Abstract

Twenty years ago, Abramsky and Melliès published their famous paper, *Concurrent Games and Full Completeness*. In that paper, they advocated the switch to a *truly concurrent* canvas to address the issue known as the *Blass problem*, diagnosed as an excess of sequentiality. Their model, *concurrent games*, was the first of a family of *positional* or *causal* game semantics which has since then shown merits far beyond the full completeness problem for Linear Logic.

In this paper, we tell and revisit the story of models of MALL in game semantics, in the modern clothes of concurrent games on event structures, from Blass games to Melliès' approach to fully complete models of Linear Logic.

1 Introduction

Game semantics in its modern form arose in the early 90s, driven by the problem of *full abstraction for PCF* [55]. The idea to represent formulas as games and validity as the existence of a winning strategy was not new, going back to at least the work of Lorenzen and Lorenz in the 60s. But in the 80s and early 90s, the scientific landscape was rich in developments hinting at a dynamic semantics for programs and proofs. In 1982, Berry and Curien introduced *sequential algorithms* [11], attempting to capture higher-order sequentiality by presenting programs as functions along with a specific order, or "algorithms" to compute them, prefiguring strategies[1]. In 1989, Girard introduced the *Geometry of Interaction (GoI)* [37], a model of Linear Logic [36] representing proofs as operators on Hilbert spaces with an interactive form of composition[2]. In 1991, Coquand gave a game semantics of classical arithmetic, interpreting proofs as strategies with the ability to *backtrack*[3][31]. In 1992, Blass gave a games model for full propositional Linear Logic [13].

[1]It appeared later that sequential algorithms are *indeed* a game semantics, as they admit a linear decomposition into the category of simple games via the Curien-Lamarche exponential.

[2]It appeared later that these operators represent a "history-free skeleton" in terms of Abramsky-Jagadeesan games [2] or Abramsky-Jagadeesan-Malacaria (AJM) games [3], informing links between GoI and game semantics [9].

[3]Backtracking prefigures the *pointers* of Hyland-Ong games, with a composition mechanism prefiguring innocent interaction.

Perhaps the first paper on *game semantics*, taken with a modern understanding of the term, is Abramsky and Jagadeesan's 1992 paper on *full completeness* for Multiplicative Linear Logic (MLL) with the MIX rule [2]. More than just novel techniques, the paper introduces a change in perspective: while earlier games models were interested in capturing validity as the existence of a winning strategy for a game, Abramsky and Jagadeesan aim to capture *proofs* as strategies, such that all strategies correspond to proofs, making the model a useful tool to reason on proof identity[4]. The Cut rule corresponds to *composition* of strategies – and as we expect two proofs differing only with respect to associativity of Cut to be the same, it becomes crucial for games and strategies to form a category[5]. In contrast, Blass games, while interpreting full propositional Linear Logic, do not form a category: composition of strategy fails to be associative, a phenomenon now known as the *Blass problem*[6], reviewed in Section 2.2.

Following this early history, the first decade of game semantics was intertwinned with Linear Logic. Hyland and Ong extended the Abramsky-Jagadeesan model to get rid of MIX [42]. Further fragments of Linear Logic were addressed: *e.g.* classical Linear Logic by Baillot, Danos, Ehrhard and Regnier [10], the intuitionistic fragment was modeled by Lamarche [46], McCusker [49] and Abramsky, Jagadeesan and Malacaria [3]. Despite this, a proper treatment (the established "gold standard" now being *full completeness*) of additives in classical Linear Logic remained long elusive. This finally came in the 1999 paper by Abramsky and Melliès, "*Concurrent games and full completeness*" [5].

To construct a game semantics of Multiplicative Additive Linear Logic (MALL), it seems reasonable to start with Blass games, and attempt to understand and sidestep the Blass problem. In [1], Abramsky diagnoses the non-associativity as caused by an excessive sequentiality. In that view, and although it is not immediately clear in what sense MALL is intrinsically concurrent, it makes sense to move to a concurrent framework for games. Abramsky and Melliès introduce *concurrent games* [5] to that end. In this paper we shall however argue – following intuitions by Melliès ultimately leading to his fully complete model of full propositional linear logic [50] – that the reason why concurrent games manage to achieve full completeness is not quite that they are concurrent, but rather that they are *causal*, or *positional*. This will be discussed at length in the course of the paper.

This *positional/causal* (we shall see that the two notions are related) aspect is far from anecdotal. In that respect, Abramsky and Melliès' model is the first of a growing family of game semantics questioning the premise that strategies should simply be the aggregation of totally ordered, chronological execution traces. This family includes Melliès and Mimram's *asynchronous games* [53], Faggian and Piccolo's strategies as partial orders [34], Rideau and Winskel's non-deterministic extension to *concurrent games on event structures* [59], Sakayori and Tsukada's framework [60] using DAG-like structures as plays. This family

[4] To our knowledge the first paper examining what should be a model for *proofs* is Girard's [38]. This change of focus is also in line with a wealth of developments at the same time on the *Curry-Howard correspondence*, moving the focus from mere provability to proofs and their computational content.

[5] It seems that Joyal should be attributed the very first category of games and strategies (of Conway games [30]), in a paper in French in the *Gazette mathématique du Québec* [45].

[6] Note however that Blass never claimed composition to be associative in his model.

of games is behind numerous recent developments in game semantics. While they are typically not the best fit for full abstraction results (as the causal information they record is unobservable), they offer numerous advantages with respect to traditional models: for instance, they allow to extend conservatively traditional notions such as *innocence* to parallel evaluation of programs [20]. They support elegant quantitative extensions, for instance to probabilistic [18] or even quantum [26] effects. The causal analysis they provide may be leveraged, for instance to keep track of execution time in a concurrent language [7] or to collect witnesses for quantifier instantiations in first-order proofs [6]. They give close connections with session types and process algebra [23]. Finally, they throw a new light on the relationship between static and dynamic denotational models, including in the presence of quantitative effects [18]. In many of these achievements, the same *causality* and *positionality* that – as is our view – permitted Abramsky and Melliès' full completeness result for MALL come into play in a crucial way.

In this paper, the phrase *concurrent games* will refer to this entire family of games. We will adopt more specific phrases to refer to precise technical frameworks. The purpose of this paper is to give a modern account of fully complete games models for MALL, putting the historical approach in perspective with recent developments in this family of concurrent games. The paper has a few original contributions: most notably, the account given of the link between concurrent games via event structures and via closure operators is new. But mostly, the paper assembles and presents in a uniform technical setting results appearing in various earlier papers, notably by Abramsky, Melliès, Mimram and Tabareau [5, 50, 53, 54] – we nonetheless hope that it will be helpful in making more accessible a nice line of research on which few researchers have a complete view.

The paper is organized as follows. In Section 2 we present the model of Abramsky and Melliès [5] and its historical context. We describe MALL, Blass' model and the Blass problem, and introduce concurrent games via closure operators. In Section 3 we introduce concurrent games on event structures, at first as an alternative way to formulate Abramsky and Melliès' interpretation. We detail the connection between the two concurrent games framework. After a discussion on the quotient involved in Abramsky and Melliès' construction, in Section 4 we construct a fully complete model for (a fragment of) MALL, following Melliès' methodology for his fully complete model of full propositional linear logic [50].

2 The Blass problem and concurrent games

In this section we introduce MALL, then review the Blass problem and concurrent games.

2.1 Multiplicative Additive Linear Logic

We consider here the multiplicative additive fragment of Linear Logic with units, but *without* propositional atoms. Atoms are not a fundamental obstacle: all the results presented here could be extended for instance as in [5] by representing formulas with atoms as functors of mixed variance and proofs as dinatural transformations – Cuvillier has recently

$$\frac{}{\vdash A^\perp, A}\ \text{Ax} \quad \frac{}{\vdash 1}\ 1 \quad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\ \perp \quad \frac{}{\vdash \Gamma, \top}\ \top \quad \frac{\vdash \Gamma, A \qquad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}\ \otimes \quad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \parr$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\ \& \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\ \oplus_l \quad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\ \oplus_r \quad \frac{\vdash \Gamma, A \qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\ \textsc{Cut}$$

Figure 1: Rules of MALL

proposed an alternative relying on nominal sets [32]. We chose to omit them simply to keep the paper as simple as possible.

Formulas of MALL are generated by the following grammar.

$$A, B ::= 1 \mid \perp \mid 0 \mid \top \mid A \otimes B \mid A \parr B \mid A \oplus B \mid A \,\&\, B$$

We call $1, \perp, \otimes, \parr$ the **multiplicative** connectives, and $0, \top, \oplus$ and $\&$ the **additive** connectives. Each formula A has a **dual** $A^\perp$, defined by De Morgan duality between 1 and $\perp$, 0 and $\top$, $\otimes$ and $\parr$ and $\oplus$ and $\&$. We consider one-sided **sequents**, of the form $\vdash \Gamma$ where $\Gamma = A_1, \ldots, A_n$ is a list of formulas. We give the rules in Figure 1. In addition to these, we consider that there is an explicit exchange rule allowing us to reorder formulas in a sequent, coping with the fact that sequents are lists rather than multisets. We will however, keep applications of this rule silent throughout this paper.

The fragment with only multiplicative connectives is known as Multiplicative Linear Logic (MLL). It is well-known that a (categorical) model of MLL is a $\star$-autonomous category, *i.e.* symmetric monoidal closed category $\mathcal{C}$ with a *dualizing object* $\perp$, such that for all object A, the canonical map $A \to (A \multimap \perp) \multimap \perp$ is an isomorphism. It follows from this structure that $\mathcal{C}$ is self-dual: the *negation* $(-) \multimap \perp : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ is an equivalence; so in particular it has products if and only if it has coproducts. A model of MALL is a $\star$-autonomous category which is additionally cartesian – so in particular, it has both products and coproducts.

2.2 The Blass problem

Constructing a games-based self-dual category with products and coproducts, is really difficult. It is well-known among game semanticists that the behaviour of additive connectives strongly depends on the *polarity* of the games considered. All these notions will be made precise later on, but say – for the moment informally – that a game is *positive* if Player always start, and that a game is *negative* if Opponent always starts. It is part of the folklore of game semantics that categories of negative games support products (and naturally apply to model Call-By-Name languages), while categories of positive games support coproducts (and naturally apply to model Call-By-Value languages)[7]. This reading matches the natural game-theoretic reading of the additive connectives of Linear Logic: in $A \oplus B$ we have A or B but we, the proof, choose – which is positive; while in $A \,\&\, B$ we have A or B but the environment chooses – which is negative.

But this follows the implicit premise that formulas should be interpreted into a single model with fixed polarity, positive or negative. It would make sense instead to have some formulas give positive games, and some others give negative games; and this is indeed how Blass games proceed. We now recall Blass games and the Blass problem, following closely the presentation of Abramsky in [1]. Formally, Blass games are trees

$$A ::= \prod_{i \in I} A_i \mid \coprod_{j \in J} A_j$$

where I, J are finite sets. A game $\coprod_{j \in J} A_j$ is positive, and a game $\prod_{i \in I} A_i$ is negative. A strategy (for Player) on $\prod_{i \in I} A_i$ is the data of a strategy (for Player) on A_i, for all $i \in I$. Likewise, a strategy (for Player) on $\coprod_{j \in J} A_j$ is the data of some $j_0 \in J$, and a strategy (for Player) on A_{j_0}. For now, the only assumption the definition of Blass games makes is that games should be sequential: at each point, it is one of the players' turn to play. There is no general assumption as to which player starts the game, and the two might not alternate.

Setting up the interpretation of MALL formulas into Blass games, it is clear from the discussion that we should have $[\![A \,\&\, B]\!] = [\![A]\!] \sqcap [\![B]\!]$ and $[\![A \oplus B]\!] = [\![A]\!] \sqcup [\![B]\!]$. But if we are to form a category of Blass games and strategies, then it follows that $\otimes$, as a left adjoint, should preserve coproducts, hence a $\otimes$ involving at least one positive game should be positive. By duality, a $⅋$ involving at least one negative game should be negative. The only case left is the $\otimes$ of two negative games, which is defined as

$$A \otimes B = \prod_{i \in I} (A_i \otimes B) \sqcap \prod_{j \in J} (A \otimes B_j)$$

for $A = \prod_{i \in I} A_i$ and $B = \prod_{j \in J} B_j$, saying that if A and B are negative, Opponent first picks a component of the tensor and makes a move in that component. So a tensor of negative games is negative – in fact in exposing the Blass problem we will not refer to this specific definition but only to the fact that the tensor of negative games is negative, which is hard to avoid (as otherwise the tensor would only ever yield positive games). The definition for the $⅋$ of two positive games is dual.

At this point it looks like there is no obstacle to form a category Blass, with Blass games as objects and, as morphisms from A to B, strategies on $A^\perp ⅋ B$. But here comes the "Blass problem". Assume we want to compose (with games annotated with their polarity):

$$\sigma : (A_-)^\perp ⅋ B_+ \qquad \tau : (B_+)^\perp ⅋ C_- \qquad \delta : (C_-)^\perp ⅋ D_+$$

[7]It is possible to add the missing connective formally, for instance one can add coproducts freely via *Fam construction* [4] in a category of negative games so as to have both products and coproducts, but this does not provide a model of MALL as we still lack self-duality.

$$
\left[\!\!\left[\dfrac{\dfrac{\dfrac{}{\vdash \bot, 1}\,\mathrm{Ax}}{\vdash 1 \oplus \bot, 1}\,\oplus_r \quad \dfrac{\varpi}{\vdash \bot, 1 \,\&\, 1}}{\vdash 1 \oplus \bot, 1 \,\&\, 1}\,\textsc{Cut} \quad \dfrac{\dfrac{}{\vdash \bot \oplus \bot, 1 \,\&\, 1}\,\mathrm{Ax}}{\vdash \bot \oplus \bot, 1 \oplus (1 \,\&\, 1)}\,\oplus_r}{\vdash 1 \oplus \bot, 1 \oplus (1 \,\&\, 1)}\,\textsc{Cut} \right]\!\!\right] \neq \left[\!\!\left[\dfrac{\dfrac{\dfrac{}{\vdash \bot, 1}\,\mathrm{Ax}}{\vdash 1 \oplus \bot, 1}\,\oplus_r \quad \dfrac{\dfrac{\varpi}{\vdash \bot, 1 \,\&\, 1} \quad \dfrac{\dfrac{}{\vdash \bot \oplus \bot, 1 \,\&\, 1}\,\mathrm{Ax}}{\vdash \bot \oplus \bot, 1 \oplus (1 \,\&\, 1)}\,\oplus_r}{\vdash \bot, 1 \oplus (1 \,\&\, 1)}\,\textsc{Cut}}{\vdash 1 \oplus \bot, 1 \oplus (1 \,\&\, 1)}\,\textsc{Cut} \right]\!\!\right]
$$

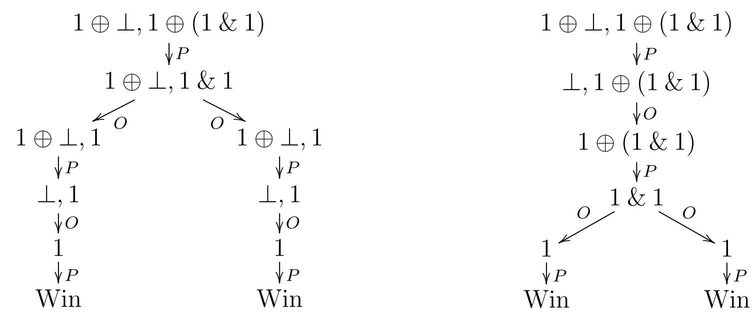


Figure 2: The concrete impact on the Blass problem on the interpretation

where σ wants to play immediately on the left, and δ wants to play immediately on the right. Both strategies want to perform immediately some visible action, so in principle there is no reason to make them wait. Indeed $(A_-)^\perp \parr D_+$ is positive: it is Player's time to play on either or A or D, and the moves offered by both σ and δ may apply; but since Blass games are sequential, only one of them will be able to play immediately. The situation being symmetric, it is clear that something is going to unfold differently between the two associations. And indeed, imagine we first form $\tau \circ \sigma : (A_-)^\perp \parr C_-$, and consider

$$\tau \circ \sigma : (A_-)^\perp \parr C_- \qquad \delta : (C_-)^\perp \parr D_+ \,.$$

The game $(A_-)^\perp \parr C_-$ is now negative, so σ is not able to play on the left, leaving δ to win the race playing on D. Symmetrically, in $(\delta \circ \tau) \circ \sigma$, σ starts playing on the left.

This happens very concretely in the interpretation of proofs: in Figure 2 we show two proofs differing only with the order of cuts, and the corresponding strategies, which differ because of the Blass problem – the strategies perform the same actions, but not in the same order. Here ϖ is the proof of $\vdash \bot, 1 \,\&\, 1$ obtained with a & rule followed by axioms.

The Blass problem is sometimes mistakenly quoted as expressing that composition is not associative in a non-polarized setting, *i.e.* unless one fixes the ambient polarity of games to be positive or negative. The author has heard some people explicitely avoiding non-polarized settings, for "fear of the Blass problem". But these people should rest in peace: associativity of composition is actually quite robust and does not need at all polarization. In fact the very first category of games and strategies, Joyal's category of Conway games, assumes no general polarization hypothesis – in a Conway game both players can have available actions in the same state, and we will see further examples later on in this paper. Instead, the Blass problem is a consequence of the very specific way in which we have set up the traffic lights so that to always give priority to coproducts on tensors and products on pars, which in turn was required to get the right behaviour for $\oplus$ and &.

Abramsky analyses the Blass problem as an excess of sequentiality [1]. And indeed, if above we authorized both σ and δ to play *concurrently*, the non-associativity would be resolved. We next review Abramsky and Melliès' *concurrent games via closure operators* [5], and observe how they resolve the non-associativity phenomenon.

2.3 Concurrent games

Concurrent games via closure operators [5] were motivated as a way around the Blass problem. They depart from the sequential substrate of earlier games models. In particular, concurrent strategies may play several moves simultaneously.

Firstly, *games* are replaced by *domains* with elements thought of as *positions*. More specifically, we consider games to be **dI-domains** [12], *i.e.* directed-complete, bounded-complete partial orders satisfying two further axioms *"d"* and *"I"* that we shall not need to repeat here. If D is a dI-domain, $D^\top$ denotes its extension with a top element $\top$ – it then follows that $D^\top$ is a complete lattice.

Secondly, strategies are *continuous, stable closure operators*[8] – $f : D \to D'$ between dI-domains is **stable** [12] iff for $x, y \in D$, if x, y are bounded then $f(x \wedge y) = f(x) \wedge f(y)$.

Definition 1. *A **closure-strategy** on dI-domain D, written $\sigma : D$, is a **continuous stable closure operator** on $D^\top$,* i.e. *a monotone and continuous function $\sigma : D^\top \to D^\top$ which is* (i) ***extensive*** *(for all $x \in D^\top$, $x \leq \sigma(x)$),* (ii) ***idempotent*** *(for all $x \in D^\top$, $\sigma(\sigma(x)) = \sigma(x)$), and* (iii) ***stable***[9] *(there is a stable function $f : D \to D$ such that for all $x \in D$ such that $\sigma(x) \neq \top$, $\sigma(x) = x \vee f(x)$).*

Intuitively, given a position $x \in D$, $\sigma(x)$ is the new position obtained by adding all moves that σ is prepared to play in position x. The first axiom, $x \leq \sigma(x)$, means intuitively that σ may only add new moves to those already present. The second axiom, $\sigma(\sigma(x)) = \sigma(x)$, formalizes the idea that as applying σ saturates the current position with all moves available with the current knowledge, any $\sigma(x)$ must be a fixpoint. The $\top$ element is meant to capture positions on which σ is undefined: $\sigma(x) = \top$ means that σ has no well-defined behaviour on x. It has to be a top element by monotonicity.

One may wonder in what sense it is legitimate to call this a game semantics. There are no polarities in the definition, no Player, no Opponent. In fact there are no moves, only positions. Moves can be captured indirectly as pairs $x, y \in D$ such that $x < y$ with no position in between (for which we write $x \longrightarrow\!\!\!\!\!\subset y$), but even then such a move has no well-defined notion of polarity. Nevertheless, any Blass game A induces a dI-domain D_A by starting with the partial order of finite branches which is then completed, adding the missing infinite branches. Any (sequential) strategy on A yields a closure-strategy which, for any finite branch, extends it with the moves it is prepared to play.

[8] Some additional conditions appear in the course of [5], omitted here as they play no role.

[9] Condition *(iii)* implies condition *(i)*, however we state conditions *(i)* and *(ii)* because together they define a *closure operator*, a standard notion independently of stability.

Furthermore, domains and concurrent strategies may be organized as a category. If D_1, D_2 are domains, then their *tensor* is defined simply as

$$D_1 \otimes D_2 = D_1 \times D_2$$

the cartesian product. A concurrent strategy *from* D_1 *to* D_2 is $\sigma : D_1 \otimes D_2$, written $\sigma : D_1 \twoheadrightarrow D_2$. To define composition, we first define *closed interaction*. If $\sigma : D$ and $\tau : D$ are two closure operators on the same domain, we may define, following [5]:

$$\langle \sigma \mid \tau \rangle = Y\,(\sigma \circ \tau) = Y\,(\tau \circ \sigma) \in D$$

obtained by playing alternatively σ and τ until reaching a fixpoint. Given $\sigma : D_1 \twoheadrightarrow D_2$, $\tau : D_2 \twoheadrightarrow D_3$, and $(x, z) \in D_1 \otimes D_3$ we first compute $y \in D_2$ that they agree to reach with

$$y = \langle \pi_2 \circ \sigma(x, -) \mid \pi_1 \circ \tau(-, z) \rangle \in D_2$$

and define $(\tau \odot \sigma)(x, z) = (\pi_1 \circ \sigma(x, y), \pi_2 \circ \tau(y, z)) \in D_1 \otimes D_3$; this defines a concurrent strategy. Composition is associative, and for any domain D, there is a strategy $\mathit{æ}_D : D \twoheadrightarrow D$ defined as $\mathit{æ}_D(x, y) = (x \vee y, x \vee y)$ serving as identity. Moreover:

Proposition 2. *There is a compact closed category* Clos *having as objects dI-domains and as morphisms from* D_1 *to* D_2 *the closure-strategies* $\sigma : D_1 \twoheadrightarrow D_2$.

Recall that a compact closed category is a degenerate model of MLL where $\otimes = ⅋$ [29]. Here, we furthermore have a trivial duality $D^* = D$, similarly to the relational model. Nevertheless, this lets us interpret the multiplicative connectives of MLL. We may extend this to the additives as well by setting

$$[\![A \mathbin{\&} B]\!] = [\![A \oplus B]\!] = ([\![A]\!] + [\![B]\!])_\perp$$

the lifted sum of $[\![A]\!]$ and $[\![B]\!]$. There are associated constructions on strategies for the introduction rules, omitted for now. Altogether this gives an interpretation of MALL in Clos, which does "solve the Blass problem" in the sense that composition is associative: so, for instance, the two proofs of Figure 2 have the same interpretation, a closure-strategy which intuitively starts playing the two competing Player moves of Figure 2 *in parallel*.

We postpone for now the concrete description of this interpretation and of its properties. Indeed, before we do that, we will see that the interpretation of MALL in Clos factors through the more concrete games formalism of *concurrent games on event structures* (where, for instance, the two proofs of Figure 2 will be both interpreted by the one parallel strategy of Figure 3). In the next section we give an introduction to concurrent games on event structures and a formal link with closure-strategies. Then we revisit the interpretation above, and discuss its properties.

3 Games on event structures and closure operators

Concurrent games via closure operators are inherently *positional*: points in the dI-domain interpreting a formula correspond to *positions* in the corresponding game. In contrast, concurrent games on event structures are more fine-grained: games focus on *individual observable events* rather than positions.

3.1 Games and domains

Games and constructions. To start our concrete reconstruction of the interpretation of MALL of the previous section, we will first aim to represent the dI-domains interpreting formulas as explicit domains of positions, *i.e.* to explicitly have points of the domains be sets of moves. For that it is natural to start with the definition of *event structures*, in light of the fact that their domains of configurations are exactly dI-domains [62]. For simplicity, we will work here with event structures with binary conflict.

Definition 3. *An* ***event structure*** *is a tuple* $E = \langle |E|, \leq_E, \#_E \rangle$ *where* $|E|$ *is a set of* ***events****,* $\leq_E$ *is a partial order called* ***causality****, and* $\#_E$ *is an irreflexive symmetric binary relation called* ***conflict****. These data must moreover satisfy the following additional axioms.*

finite causes:	*for all* $e \in |E|$*, the set* $[e]_E = \{e' \in E \mid e' \leq_E e\}$ *is finite,*
conflict inheritance:	*if* $e_1 \#_E e_2$ *and* $e_2 \leq_E e_2'$*, then* $e_1 \#_E e_2'$*.*

If $e_1, e_2 \in |E|$, we say that e_1 **immediately causes** e_2, written $e_1 \rightarrowtriangle_E e_2$, iff $e_1 <_E e_2$ and if $e_1 \leq_E e \leq_E e_2$, then $e_1 = e$ or $e_2 = e$. Events e_1 and e_2 are in **minimal conflict**, written $e_1 \rightsquigarrow_E e_2$, if $e_1 \#_E e_2$, for all $e_1' <_E e_1$ we have $\neg(e_1' \#_E e_2)$, and symmetrically.

The **configurations** of E are those $x \subseteq |E|$ that are down-closed for $\leq_E$, and pairwise compatible, *i.e.* for all $e_1, e_2 \in x$ we have $\neg(e_1 \#_E e_2)$. We write $\mathscr{C}(E)$ for the set of finite configurations of an event structure E, and $\mathscr{C}^\infty(E)$ for possibly infinite configurations.

Proposition 4. *For any event structure* E*,* $\mathscr{C}^\infty(E)$*, ordered by* $\subseteq$*, forms a dI-domain.*

Hence, in our attempt to recover concurrent games in the sense of the previous section as explicit *positions*, *i.e.* sets of events/moves, it is sensible to define games simply as event structures. But we also want moves to be explicitely Player or Opponent moves, so, following [59], we define games as event structures with an additional *polarity* annotation. From now on, in this paper, by *game* we will mean the following.

Definition 5. *A* ***game*** *is a tuple* $\langle |A|, \leq_A, \#_A, \mathrm{pol}_A \rangle$ *where* $\langle |A|, \leq_A, \#_A \rangle$ *is an event structure, and* $\mathrm{pol}_A : |A| \to \{-, +\}$ *provides, for each event* $a \in |A|$*, a polarity indicating whether it is a Player move (*$\mathrm{pol}_A(a) = +$*), or an Opponent move (*$\mathrm{pol}_A(a) = -$*).*

We additionally require that games are ***race-free****,* i.e. *that for all* $a_1, a_2 \in |A|$*, if* $a_1 \rightsquigarrow_A a_2$ *then* $\mathrm{pol}_A(a_1) = \mathrm{pol}_A(a_2)$*.*

Games support a number of constructions. The **empty game** $\emptyset$ has no events. If A is a game, its **dual** $A^\perp$ is A with polarities reversed. If A and B are games, their **simple parallel composition** $A \parallel B$ is the game with events the tagged disjoint union $(\{1\} \times |A|) \cup (\{2\} \times |B|)$, with causal order, conflict, and polarities simply inherited from A and B. Their **sum** $A + B$ has same components as $A \parallel B$, with additional conflicts all $(1, a) \#_{A+B} (2, b)$ for $a \in |A|, b \in |B|$. If A is a game, its **down-shift** $\downarrow A$ has events $|A| \uplus \{\updownarrow\}$ (where by $\uplus$ we mean $|A| \cup \{\updownarrow\}$ with the implicit assumption that $\updownarrow \notin |A|$), causal order that of A plus $\updownarrow \leq_{\downarrow A} a$ for all $a \in |\downarrow A|$, conflict the same as in A, and polarities those of A plus $\mathrm{pol}_{\downarrow A}(\updownarrow) = +$. The **up-shift** $\uparrow A$ is defined in the same way, with $\mathrm{pol}_{\uparrow A}(\updownarrow) = -$.

We introduce now some notations for configurations of these compound games. Configurations of $A \parallel B$ have the form $(\{1\} \times x_A) \cup (\{2\} \times x_B)$ where $x_A \in \mathscr{C}(A)$ and $x_B \in \mathscr{C}(B)$, also written $x_A \parallel x_B$. Configurations of $\downarrow A$ are either empty, or $\{\updownarrow\} \cup x_A$ with $x_A \in \mathscr{C}(A)$, also written $\updownarrow x_A$. Configurations of A and $A^\perp$ are the same: A and $A^\perp$ have the same underlying set of events. In particular, the polarity of an event is not an intrinsic property of that event, but depends of the ambiant game within which that polarity is taken. Finally, all of the above applies to both finite and possibly infinite configurations.

Interpretation of MALL formulas. All units (multiplicative and additive) are interpreted by the empty game $\emptyset$. For other constructors:

$$\begin{array}{rcl} [\![A \otimes B]\!] & = & [\![A]\!] \parallel [\![B]\!] \\ [\![A \parr B]\!] & = & [\![A]\!] \parallel [\![B]\!] \end{array} \qquad \begin{array}{rcl} [\![A \oplus B]\!] & = & \downarrow [\![A]\!] + \downarrow [\![B]\!] \\ [\![A \,\&\, B]\!] & = & \uparrow [\![A]\!] + \uparrow [\![B]\!] \,. \end{array}$$

Some expected laws from linear logic obviously do not hold under this interpretation. For instance we have $[\![A \oplus 0]\!] \neq [\![A]\!]$ – associativity of $\oplus$ and & also fail. This shows clearly already at this point that some additional work will have to be done in order to get full completeness. Notice that this is already true of the interpretation of the previous section, of which this is a direct refinement, in the following sense:

Proposition 6. *For any* MALL *formula A, we have $[\![A]\!]_{\mathsf{Clos}} \cong \mathscr{C}^\infty([\![A]\!]_{\mathsf{Games}})$, where $[\![-]\!]_{\mathsf{Clos}}$ is the interpretation of the previous section, while $[\![-]\!]_{\mathsf{Games}}$ is the one introduced just above.*

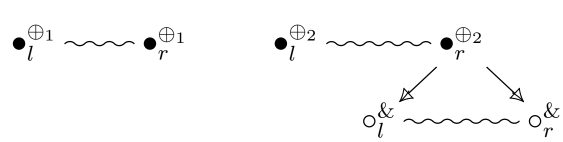
Proof. For units, $\mathscr{C}^\infty(\emptyset)$ is the singleton domain, which matches the interpretation of units in [5]. It is direct from the definition that for the other constructors we have

$$\begin{array}{rcccl} & & \mathscr{C}^\infty(A \parallel B) & \cong & \mathscr{C}^\infty(A) \times \mathscr{C}^\infty(B) \\ \mathscr{C}^\infty(\downarrow A + \downarrow B) & = & \mathscr{C}^\infty(\uparrow A + \uparrow B) & \cong & (\mathscr{C}^\infty(A) + \mathscr{C}^\infty(B))_\perp \end{array}$$

from which the property announced follows by induction. □

The interpretation of multiplicatives remains degenerate. The interpretation of additives $\oplus$ and & yields events of distinct polarity, a distinction that is forgotten when considering the associated domain of configurations.

As an example, we show below the interpretation as a game of the sequent $\vdash 1 \oplus_1 \perp, 1 \oplus_2 (1 \,\&\, 1)$ of Figure 2, where the two occurrences of $\oplus$ have been labeled for disambiguation. In this diagram and others to come, we take the convention that we label with $\circ$ moves of negative polarity, and with $\bullet$ moves with positive polarity.



Each event corresponds to selecting one component of an additive connective. Occurrences of $\oplus$ correspond to positive/Player events, occurrences of & to negative/Opponent events, and causal dependency corresponds to the nesting of additive connectives. Multiplicative connectives, being interpreted as juxtaposition, do not contribute events. By Proposition 6, the domain of configurations of the sequent matches its interpretation in Clos. The closure operator interpreting either of the proofs of Figure 2, when applied to any configuration containing $\bullet_l^{\oplus_1}$ or $\bullet_l^{\oplus_2}$, returns $\top$. Applied to any other configuration, it adds both $\bullet_r^{\oplus_1}$ and $\bullet_r^{\oplus_2}$, effectively playing them simultaneously, illustrating the resolution of the the Blass problem.

We shall now give a corresponding notion of *strategy*.

3.2 Deterministic concurrent strategies

In the past two decades, besides closure-strategies, multiple alternative ways to set up concurrent strategies have appeared. Melliès and Mimram [53] define concurrent strategies as certain sets of plays subject to stability conditions. Faggian and Piccolo [34] define them as partial orders enriching the causality of the game. Rideau and Winskel [59] define them as event structures labeled by the game. Castellan and Clairambault [17] define them as *rigid families*, *i.e.* prefix-closed sets of partial orders [58]. Finally, Castellan, Clairambault and Winskel [21] define them simply as certain sets of configurations of the game. These settings differ in expressivity, but for causally deterministic strategies such as those obtained by interpreting MALL, those are all equivalent.

In this paper we will exploit these last two presentations of concurrent strategies.

3.2.1 Strategies as rigid families

Whereas closure-strategies are inherently *positional*, strategies as rigid families offer a *causal* presentation of concurrent strategies. Following [17], we first recall:

Definition 7. *If A is a game, a* ***courteous augmentation*** *on A is a finite event structure $\mathrm{q} = \langle |\mathrm{q}|, \leq_{\mathrm{q}}, \emptyset\rangle$ with no conflict such that $\mathscr{C}(\mathrm{q}) \subseteq \mathscr{C}(A)$, and satisfying* ***courtesy****: if $a_1 \rightarrowtriangle_{\mathrm{q}} a_2$, then either $\mathrm{pol}_A(a_1) = -$ and $\mathrm{pol}_A(a_2) = +$, or $a_1 \rightarrowtriangle_A a_2$.*

We write $\mathrm{aug}(A)$ *the set of* courteous augmentations *on A.*

Courteous augmentations on A will provide the notion of *state* for our strategies, which includes the *causal history* behind the actions. Courtesy expresses that a strategy may only condition the appearance of Player moves to the prior appearance of certain Opponent moves. A strategy can obviously not delay an Opponent move until after a Player move if that is not already forced by the game, but neither can it force an order between its own moves if that order is not forced by the game. This may be understood as expressing a deep *asynchrony* property: a program sending two packets on the network has no guarantee that they will arrive in the same order if that is not controled by the protocol, so it makes no sense to impose that order in the first place. Courtesy is necessary in order for strategies to behave well with respect to the asynchronous copycat [59].

There is a natural ordering on courteous augmentations. We say that $\mathrm{q} \in \mathrm{aug}(A)$ **rigidly embeds** into $\mathrm{q}' \in \mathrm{aug}(A)$, or is a **prefix** of q', if $\mathscr{C}(\mathrm{q}) \subseteq \mathscr{C}(\mathrm{q}')$, and the inclusion

Figure 3: One strategy on $[\![\vdash 1 \oplus \bot, 1 \oplus (1 \,\&\, 1)]\!]$ for the two proofs of Figure 2

preserves causality: if $a_1 \leq_{\mathrm{q}} a_2$, then $a_1 \leq_{\mathrm{q}'} a_2$ as well. We write $\mathrm{q} \hookrightarrow \mathrm{q}'$. It follows that for $a_1, a_2 \in |\mathrm{q}|$, we have $a_1 \leq_{\mathrm{q}} a_2$ iff $a_1 \leq_{\mathrm{q}'} a_2$. A **courteous rigid family** on A is a non-empty subset of $\mathrm{aug}(A)$ closed under prefix. We can finally define:

Definition 8. *A **strategy** $\sigma : A$ is a* courteous rigid family *on A which is additionally:*

(i) Receptive*: if $\mathrm{q} \in \sigma$ and $a^- \notin |\mathrm{q}|$ such that $|\mathrm{q}| \cup \{a^-\} \in \mathscr{C}(A)$, then there is a (necessarily unique) $\mathrm{q} \hookrightarrow \mathrm{q}'$ such that $|\mathrm{q}'| = |\mathrm{q}| \cup \{a^-\}$.*

(ii) Deterministic*: if $X \subseteq \sigma$ is a finite set of augmentations such that $\cup\{|\mathrm{q}|_- \mid \mathrm{q} \in X\}$ is compatible, then X has a supremum $\vee X$ in σ with respect to $\hookrightarrow$.*

Here, $|\mathrm{q}|_-$ comprises the events of q of negative polarity. Without determinism, this is exactly the notion of strategy from [17]. For finite games, strategies are entirely characterized by their maximal augmentations. For instance, Figure 3 displays the two maximal augmentations of the strategy arising as the interpretation of the two proofs of Figure 2 – they are augmentations of the game for $\vdash 1 \oplus \bot, 1 \oplus (1 \& 1)$ presented in the previous section. Instead of annotating events to point out which syntactic construct they correspond to, we adopt the usual convention in game semantics and picture this association by drawing events below the corresponding component of the formula.

Observe that the two augmentations both admit as prefix the two Player moves that both proofs of Figure 2 are prepared to make unconditionally. They only differ with respect to the two incompatible resolutions of the & that Opponent may make. There are only two Player moves, whereas there are three in Figure 2; this is because unlike in Blass games, we have interpreted units as the empty game, following [5].

Let us see how to *compose* strategies. Composition relies on a (partial) composition of *courteous augmentations*. If $\mathrm{q} \in \mathrm{aug}(A^\perp \parallel B)$ and $\mathrm{q}' \in \mathrm{aug}(B^\perp \parallel C)$, they are **causally compatible** if $|\mathrm{q}| = x_A \parallel x_B$, $|\mathrm{q}'| = x_B \parallel x_C$, and q, q' induce no causal loop, *i.e.*

$$(\leq_{\mathrm{q}} \| \leq_C) \cup (\leq_A \| \leq_{\mathrm{q}'}) \text{ is acyclic,}$$

where $\leq_{\mathrm{q}} \| \leq_C$ and $\leq_A \| \leq_{\mathrm{q}'}$ denote partial orders on $x_A \parallel x_B \parallel x_C$ in the obvious way.

If two augmentations do induce a causal loop, that means that their interaction *deadlocks*: they impose incompatible constraints as to the order following which moves should be played. In contrast, if q and q′ are *causally compatible*, then the transitive closure

$$\leq_{\mathrm{q}' \circledast \mathrm{q}} = ((\leq_{\mathrm{q}} \| \leq_C) \cup (\leq_A \| \leq_{\mathrm{q}'}))^*$$

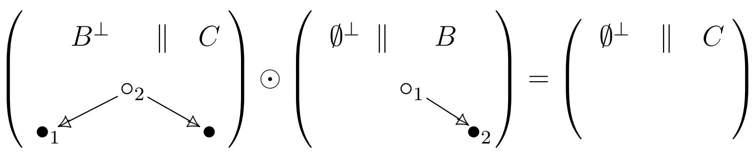


Figure 4: Composition of deadlocking strategies

is a partial order, and $\mathrm{q}' \circledast \mathrm{q} = \langle x_A \parallel x_B \parallel x_C, \leq_{\mathrm{q}'\circledast\mathrm{q}} \rangle$ is the **interaction** of q and q′. Their **composition** is then $\mathrm{q}' \odot \mathrm{q} = \langle x_A \parallel x_C, \leq_{\mathrm{q}'\odot\mathrm{q}} \rangle$ where $\leq_{\mathrm{q}'\odot\mathrm{q}}$ is $\leq_{\mathrm{q}'\circledast\mathrm{q}}$ restricted to $x_A \parallel x_C$; then we have $\mathrm{q}' \odot \mathrm{q} \in \mathrm{aug}(A^\perp \parallel C)$. From this, we can compose strategies via

$$\tau \odot \sigma = \{\mathrm{q}' \odot \mathrm{q} \mid \mathrm{q} \in \sigma,\ \mathrm{q}' \in \tau \text{ are causally compatible}\}$$

for $\sigma : A^\perp \parallel B$ and $\tau : B^\perp \parallel C$ – it is a strategy on $A^\perp \parallel C$ as required.

It is worth emphasizing and illustrating the *causal compatibility* condition in the definition of $\mathrm{q}' \circledast \mathrm{q}$. Consider a game B with two incomparable but compatible events $\circ_1$ and $\bullet_2$, with $\mathrm{pol}(\circ_1) = -$ and $\mathrm{pol}(\bullet_2) = +$; and a game C with unique event $\bullet$, of positive polarity. Consider two strategies $\sigma : \emptyset^\perp \parallel B$ and $\tau : B^\perp \parallel C$, generated each by one maximal augmentation, represented in Figure 4. Although these two maximal augmentations play the same events on B, they are not causally compatible: the order on B induced by their union is *cyclic*, with $\circ_1 \leq \circ_2 \leq \circ_1$. In fact, the only compatible augmentation of σ and τ is empty, which entails that, as in Figure 4, their composition will be restricted to the empty augmentation on $A^\perp \parallel C$. Their composition *deadlocks*, as they impose incompatible constraints on the order of events. This deadlocking mechanism is a fundamental aspect present, implicitely or explicitely, in almost all games models.

With respect to this notion of composition, we have:

Proposition 9. *There is a compact closed category* Games *with games as objects, and as morphisms from A to B the strategies $\sigma : A^\perp \parallel B$.*

To prove this proposition we must define a number of other constructions on strategies, including *e.g.* copycat strategies and the functorial action of $\parallel$. Those may be defined directly on strategies as rigid families, as is done *e.g.* in [17]. Instead we will describe them in an alternative description of strategies as sets of configurations in the next subsection.

3.2.2 Strategies as sets of configurations

We now give a different, *positional*, presentation of the *same* deterministic concurrent strategies. We are aware that it is a lot to ask to the reader to digest not only *one*, but *two* definitions for a games model. But it is a distinctive feature of deterministic concurrent strategies that they may be described in these different ways. Each representation has distinct advantages: the *causal* description above relates to the inductive structure of terms. It highlights the causal flavour of traditional game semantical notions such as *P-views* [41], and accordingly supports a simple notion of innocence (see Section 4.3.1). In contrast, the *positional* presentation of strategies as sets of configurations that we are about to present emphasises their relationship with relational-like models. This connection will be extensively used in the remainder of the paper.

If X is a set of sets $x, y \in X$, and $y = x \cup \{a\}$ for $a \notin x$, we write $x \stackrel{a}{\longrightarrow\!\!\!\!\subset}$, or $x \longrightarrow\!\!\!\!\subset y$. In that case we say that x *extends to* y *within* X. We write $X\uparrow^-$ if X is *negatively compatible*, meaning that $\{a \in \cup X \mid \mathrm{pol}(a) = -\}$ is compatible. Finally, if $\sigma : A$ is a strategy on A, we write $\mathscr{C}(\sigma) = \{\mathscr{C}(\mathrm{q}) \mid \mathrm{q} \in \sigma\}$ for its **configurations**.

Proposition 10. *For any game A, there is a* 1*-to-*1 *correspondence between strategies $\sigma : A$ and sets of finite configurations $\mathcal{S} \subseteq \mathscr{C}(A)$ satisfying:*

(i) *for any $X \subseteq \mathcal{S}$, if $X\uparrow^-$ then $\cup X \in \mathcal{S}$ and $\cap X \in \mathcal{S}$,*
(ii) *if $a_1, a_2 \in x \in \mathcal{S}$, there exists $y \subseteq x$ such that $y \in \mathcal{S}$ and $a_1 \in y \Leftrightarrow \neg(a_2 \in y)$,*
(iii) *if $x \stackrel{a_1}{\longrightarrow\!\!\!\!\subset} \stackrel{a_2}{\longrightarrow\!\!\!\!\subset}$ in $\mathcal{S}$ with $x \stackrel{a_2}{\longrightarrow\!\!\!\!\subset}$ in $\mathscr{C}(A)$ but not in $\mathcal{S}$,* $\mathrm{pol}(a_1) = -$ *and* $\mathrm{pol}(a_2) = +$,
(iv) *if $x \in \mathcal{S}$ and $x \stackrel{a^-}{\longrightarrow\!\!\!\!\subset}$ in $\mathscr{C}(A)$, then $x \stackrel{a^-}{\longrightarrow\!\!\!\!\subset}$ in $\mathcal{S}$.*

Proof. Let $\sigma : A$ be a strategy. First, conditions *(i)-(iv)* may be directly verified on $\mathscr{C}(\sigma)$.

Reciprocally, for each $x \in \mathcal{S}$ we construct a partial order $\mathrm{q}_x \in \mathrm{aug}(A)$ as $(x, \leq_x)$ where $a_1 \leq_x a_2$ iff for all $y \subseteq x$ in $\mathcal{S}$, if $a_2 \in y$ then $a_1 \in y$ also. We refer the reader to [61] for properties of this partial order, which is used to link *prime event structures* and *stable families.* In particular, we have $\mathscr{C}(\mathrm{q}_x) \subseteq \mathcal{S}$, and if $x \subseteq y$, then $\mathrm{q}_x \hookrightarrow \mathrm{q}_y$. Moreover, by *(iii)* it follows that q_x is courteous, and by *(iv)* it follows that $\{\mathrm{q}_x \mid x \in \mathcal{S}\}$ is receptive.

It is direct to verify that these constructions are inverses of each other. □

In particular, strategies are determined by their configurations: if $\sigma, \tau : A$ are such that $\mathscr{C}(\sigma) = \mathscr{C}(\tau)$, then $\sigma = \tau$. But this also lets us *define* deterministic strategies simply via sets of configurations. Indeed, for instance, we may define the *copycat strategy* via

$$\mathscr{C}(\text{æ}_A) = \{x_A \parallel y_A \in \mathscr{C}(A^\perp \parallel A) \mid y_A \supseteq^- x_A \cap y_A \subseteq^+ x_A\}$$

where $\subseteq^+, \subseteq^-$ mean inclusion where the elements added have the polarity indicated, and polarity is always taken to be in A (not $A^\perp$). In other words, those are identity pairs $x_A \parallel x_A \in \mathscr{C}(A^\perp \parallel A)$ closed under receptivity on both sides. Likewise, it is convenient to define the functorial action of $\parallel$ on strategies via its action on sets of configurations, as

$$\mathscr{C}(\sigma_1 \parallel \sigma_2) = \{(x_{A_1} \parallel x_{A_2}) \parallel (x_{B_1} \parallel x_{B_2}) \mid x_{A_1} \parallel x_{B_1} \in \mathscr{C}(\sigma_1) \wedge x_{A_2} \parallel x_{B_2} \in \mathscr{C}(\sigma_2)\}$$

for $\sigma_1 : A_1^\perp \parallel B_1$ and $\sigma_2 : A_2^\perp \parallel B_2$. Other structural components of the compact closed structure of Games may be defined similarly. Observe that these definitions are very close to those of the corresponding constructions in the *relational model.* Accordingly, composition of strategies viewed as sets of configurations is fairly close to relational composition:

Proposition 11. *Let $\sigma : A^\perp \parallel B$ and $\tau : B^\perp \parallel C$ be two strategies. Then, $\mathscr{C}(\tau \odot \sigma)$ comprises exactly the pairs $x_A \parallel x_C \in \mathscr{C}(A^\perp \parallel C)$ such that there exists $x_A \parallel x_B \parallel x_C$ such that $x_A \parallel x_B \in \mathscr{C}(\sigma)$ and $x_B \parallel x_C \in \mathscr{C}(\tau)$ which is additionally* reachable, *i.e. there is*

$$x_A^0 \parallel x_B^0 \parallel x_C^0 \;\longrightarrow\!\!\!\!\subset\; x_A^1 \parallel x_B^1 \parallel x_C^1 \;\longrightarrow\!\!\!\!\subset\; \dots \;\longrightarrow\!\!\!\!\subset\; x_A^n \parallel x_B^n \parallel x_C^n$$

such that x_A^0, x_B^0, x_C^0 are empty, $x_A^n = x_A, x_B^n = x_B$ and $x_C^n = x_C$, and for all $0 \leq i \leq n$, we have $x_A^i \parallel x_B^i \in \mathscr{C}(\sigma)$ and $x_B^i \parallel x_C^i \in \mathscr{C}(\tau)$.

Proof. If $x \in \mathscr{C}(\tau \odot \sigma)$, then there are $\mathrm{q} \in \sigma$, $\mathrm{q}' \in \tau$ such that $x = |\mathrm{q}' \odot \mathrm{q}|$. By definition, $\mathrm{q}' \odot \mathrm{q}$ is the restriction on A, C of $\mathrm{q}' \circledast \mathrm{q}$ with $|\mathrm{q}' \circledast \mathrm{q}| = x_A \parallel x_B \parallel x_C$. But then, any linearization of the partial order $\leq_{\mathrm{q}' \circledast \mathrm{q}}$ on $x_A \parallel x_B \parallel x_C$ yields a chain as required.

Reciprocally, if $x_A \parallel x_B \parallel x_C$ is such that $x_A \parallel x_B \in \mathscr{C}(\sigma)$, $x_B \parallel x_C \in \mathscr{C}(\tau)$ with a chain as above, then that chain is a linearization of the transitive closure of

$$(\leq_{\mathrm{q}_{x_A \parallel x_B}} \parallel \leq_C) \cup (\leq_A \parallel \leq_{\mathrm{q}_{x_B \parallel x_C}})$$

which is therefore acyclic, making $\mathrm{q}_{x_A \parallel x_B} \in \sigma$ and $\mathrm{q}_{x_B \parallel x_C} \in \tau$ causally compatible. We then have $|\mathrm{q}_{x_B \parallel x_C} \odot \mathrm{q}_{x_A \parallel x_B}| = x_A \parallel x_C$ by construction. □

The reachability condition corresponds to the *causal compatibility* requirement in the definition of interaction of augmentations. Coming back to Figure 4, we have $\mathscr{C}(\sigma) = \{\emptyset, \{\circ_1\}, \{\circ_1, \bullet_2\}\}$ and $\mathscr{C}(\tau) = \{\emptyset, \{\circ_2\}, \{\circ_2, \bullet_1\}, \{\circ_2, \bullet\}, \{\circ_2, \bullet_1, \bullet\}\}$. Then, the set

$$\emptyset \parallel \{\circ_1, \circ_2\} \parallel \{\circ\} \in \mathscr{C}(\emptyset \parallel B \parallel C)$$

is a candidate to be a configuration of the interaction as $\emptyset \parallel \{\circ_1, \bullet_2\} \in \mathscr{C}(\sigma)$ and $\{\bullet_1, \circ_2\} \parallel \{\bullet\} \in \mathscr{C}(\tau)$. However, it is rejected by the *reachability* condition, although it would be present in a purely relational composition of the strategies.

This presentation of composition highlights the proximity of deterministic concurrent strategies with relational semantics, but also the fundamental difference between the two models: namely, *reachability*, and the ability of the composition of strategies to deadlock.

3.3 Strategies and closure operators

Now, we link deterministic concurrent strategies and strategies as closure operators.

3.3.1 From strategies to closure operators

For $\sigma : A$, the developments in the previous section yield a set of finite configurations $\mathscr{C}(\sigma)$. From this, we may define the *(potentially) infinite configurations* $\mathscr{C}^\infty(\sigma)$ of σ as the unions of directed sets of finite configurations. Potentially infinite configurations are partially ordered by inclusion, and we write $x \subseteq^+ y$ or $x \subseteq^- y$ as for finite configurations.

In defining a closure operator, we will use that any compatible set of negative events enables a unique $+$-maximal possibly-infinite configuration in $\mathscr{C}^\infty(\sigma)$. In the sequel, for $x \in \mathscr{C}^\infty(A)$ we write x_- for its set of negative events and x_+ for its set of positive events.

Lemma 12. *Let $\sigma : A$ be a strategy, and $x \in \mathscr{C}^\infty(A)$. Then, defining the set*

$$\overline{x}^\sigma = \cup\{y \in \mathscr{C}(\sigma) \mid y_- \subseteq x_-\},$$

we have $\overline{x}^\sigma \in \mathscr{C}^\infty(\sigma)$.

Proof. All finite subsets of $Y = \{y \in \mathscr{C}(\sigma) \mid y_- \subseteq x_-\}$ are negatively compatible, hence by *(1)* of Proposition 10, have a union in $\mathscr{C}(\sigma)$. Therefore, $\cup Y \in \mathscr{C}^\infty(\sigma)$; we set $\overline{x}^\sigma = \cup Y$. □

If $x \in \mathscr{C}^\infty(A)$, $\overline{x}^\sigma$ is obtained by playing all moves that σ is prepared to play with the *negative moves* already present in x. It *is not necessarily the case* that $x \subseteq \overline{x}^\sigma$; indeed x might contain positive moves that σ is not prepared to play with the negative moves in x. In fact, it is not necessarily the case that $x \cup \overline{x}^\sigma \in \mathscr{C}^\infty(A)$.

Proposition 13. *Let $\sigma : A$ be a strategy. Then, the function*

$$\begin{array}{rcll} \mathsf{C}(\sigma) & : & \mathscr{C}^\infty(A) & \to \mathscr{C}^\infty(A)^\top \\ & & x & \mapsto \begin{cases} x \cup \overline{x}^\sigma & \textit{if } x \cup \overline{x}^\sigma \textit{ is compatible} \\ \top & \textit{otherwise,} \end{cases} \end{array}$$

extended to $\mathsf{C}(\sigma) : \mathscr{C}^\infty(A)^\top \to \mathscr{C}^\infty(A)^\top$ with $\mathsf{C}(\sigma)(\top) = \top$, is a closure-strategy.

Proof. By construction, $\mathsf{C}(\sigma) : \mathscr{C}^\infty(A)^\top \to \mathscr{C}^\infty(A)^\top$ is extensive, monotone, and idempotent. Continuity is longer but essentially direct, exploiting the axiom of finite causes for A. Stability is simply stability of $\overline{(-)}^\sigma$, which is obvious from the definition. □

When applied to some $x \in \mathscr{C}^\infty(A)$, the closure operator associated to σ will add to x all the positive moves whose causal dependencies in σ appear in x. This is done regardless of the fact that there may be moves in x that σ will never play, but if the effect of adding these events yields an incompatible set, then the result is $\top$ instead. This differs from the two other transformations from concurrent strategies to closure-strategies appearing in the literature that we are aware of: in writing this paper we observed that they both suffer from some pathologies (for instance they both fail continuity), see Appendix A.

3.3.2 On the functoriality of the transformation

We now investigate whether the transformation from concurrent strategy to closure-strategy is functorial. We first make a key observation on closure-strategies: their composition may be presented relationally. Although this fact does not appear in [5], it was known by Melliès in the 00s when working on *asynchronous games*. To our knowledge its only appearance in a published source is in Mimram's PhD thesis [56].

If $\sigma : D$ is a closure-strategy, write $\mathsf{fix}(\sigma)$ for its set of **fixpoints**, *i.e.* those $x \in D$ such that $\sigma(x) = x$. Closure operators on complete lattices are determined by their fixpoints; in particular from $X = \mathsf{fix}(\sigma)$ one can recover σ as $\sigma(x) = \wedge\{y \in X \mid x \leq_D y\}$ for $x \in D$. Perhaps surprisingly in view of the interactive definition of composition, we have:

Proposition 14. *Let $\sigma : D_1 \mathrel{+\!\!\!\!\to} D_2$ and $\tau : D_2 \mathrel{+\!\!\!\!\to} D_3$ be closure-strategies. Then:*

$$\mathsf{fix}(\tau \odot \sigma) = \mathsf{fix}(\tau) \circ \mathsf{fix}(\sigma)$$

where $\circ$ is relational composition.

Proof. $\subseteq$. If $(x,z) \in \mathsf{fix}(\tau \odot \sigma)$ then by definition there is $y \in D_2$ such that $y = \pi_2(\sigma(x,y))$ and $y = \pi_1(\tau(y,z))$, and with $x = \pi_1(\sigma(x,y))$ and $z = \pi_2(\tau(y,z))$. In particular, $(x,y) \in \mathsf{fix}(\sigma)$ and $(y,z) \in \mathsf{fix}(\tau)$, so $(x,z) \in \mathsf{fix}(\tau) \circ \mathsf{fix}(\sigma)$.

$\supseteq$. If $(x,y) \in \mathsf{fix}(\sigma)$ and $(y,z) \in \mathsf{fix}(\tau)$, then compute

$$y' = \langle \pi_2 \circ \sigma(x,-) \mid \pi_1 \circ \tau(-,z) \rangle = \bigvee_{n \in \mathbb{N}} ((\pi_2 \circ \sigma(x,-)) \circ (\pi_1 \circ \tau(-,z)))^n(\bot) \in D_2\,.$$

Since $\sigma(x,y) = (x,y)$ and $\tau(y,z) = (y,z)$ with both monotone, it follows by induction on n that for all $n \in \mathbb{N}$ the n-th approximant $y_n \in D_2$ is below y; hence $y' \leq y$. But then we have $\pi_1(\sigma(x,y')) = x$ since σ is monotone and increasing, and likewise $\pi_2(\tau(y',z)) = z$; therefore $(\tau \odot \sigma)(x,z) = (x,z)$ as required. $\square$

Hence composition of closure-strategies can be presented purely relationally – one can observe notably the absence of a *reachability* condition as in Proposition 11. This seems in stark contrast with the original interactive flavour of composition of closure-strategies. In light of our previous discussion on reachability, this gives the impression that composition of closure-strategies fails to take *deadlocks* into account and eliminate causal loops.

To put some light on this issue, it is informative to look at the deadlocking composition of Figure 4 through the lens of closure-strategies. We first fix some notations. If $\sigma : A^\perp \parallel B$ is a strategy from A to B, $\mathsf{C}(\sigma)$ is a closure-strategy on $\mathscr{C}^\infty(A^\perp \parallel B) \cong \mathscr{C}^\infty(A) \times \mathscr{C}^\infty(B)$ – we still write $\mathsf{C}(\sigma)$ for the corresponding closure-strategy on the latter domain.

Considering the closure-strategies coming from the strategies of Figure 4, we have $\mathsf{C}(\tau)(\emptyset \parallel \{\bullet\}) = (\emptyset \parallel \{\bullet\})$ so $\emptyset \parallel \{\bullet\}$ is a fixpoint of $\mathsf{C}(\tau)$ even though τ will never play $\bullet$ on its own. Likewise, $\emptyset \parallel \emptyset \in \mathsf{fix}(\mathsf{C}(\sigma))$; hence $\emptyset \parallel \{\bullet\} \in \mathsf{fix}(\mathsf{C}(\tau) \odot \mathsf{C}(\sigma))$. Although this seems to vindicate the view that deadlocks are not satisfactorily taken into account by composition, this is misleading. Instead, we argue that it is inaccurate to think of fixpoints as stopping states of a strategy: not because they are not stopping, but because they might not be *states*, in the sense that they may not be reachable through a normal interactive computation. Indeed, in this example we also have $\emptyset \parallel \emptyset \in \mathsf{fix}(\mathsf{C}(\tau) \odot \mathsf{C}(\sigma))$ – in particular applying $\mathsf{C}(\tau) \odot \mathsf{C}(\sigma)$ on $\emptyset \parallel \emptyset$ does *not* add $\bullet$; so the deadlock *is* accurately represented. The configuration $\emptyset \parallel \{\bullet\}$ is a fixpoint for $\mathsf{C}(\tau) \odot \mathsf{C}(\sigma)$, but not a *reachable* one.

In fact, we have:

Proposition 15. *For any two strategies $\sigma : A^\perp \parallel B$ and $\tau : B^\perp \parallel C$, we have*

$$\mathsf{C}(\tau \odot \sigma) = \mathsf{C}(\tau) \odot \mathsf{C}(\sigma)\,.$$

Proof. To save space we only detail the right-to-left inclusion, which is the most surprising in light of the relational nature of composition of closure-strategies. Take $(x,z) \in \mathsf{fix}(\mathsf{C}(\tau) \odot \mathsf{C}(\sigma))$. By Proposition 14, there are $(x,y) \in \mathsf{fix}(\mathsf{C}(\sigma))$ and $(y,z) \in \mathsf{fix}(\mathsf{C}(\tau))$. Take $x' \parallel z' \in \mathscr{C}(\tau \odot \sigma)$ such that $(x' \parallel z')_- \subseteq x \parallel z$. There is some $y' \in \mathscr{C}(B)$ such that $x' \parallel y' \in \mathscr{C}(\sigma)$ and $y' \parallel z' \in \mathscr{C}(\tau)$, and which is reachable in the sense that there is a covering chain

$$x'_0 \parallel y'_0 \parallel z'_0 \longrightarrow\!\!\!\!\!\subset \ \dots \ \longrightarrow\!\!\!\!\!\subset x'_n \parallel y'_n \parallel z'_n$$

such that $x'_0 \parallel y'_0 \parallel z'_0 = \emptyset$, $x'_n \parallel y'_n \parallel z'_n = x' \parallel y' \parallel z'$, and for all $0 \leq i \leq n$ we have $x'_i \parallel y'_i \in \mathscr{C}(\sigma)$ and $y'_i \parallel z'_i \in \mathscr{C}(\tau)$. By induction on i, using $x \parallel y \in \mathsf{fix}(\mathsf{C}(\sigma))$ and $y \parallel z \in \mathsf{fix}(\mathsf{C}(\tau))$, we have $y'_i \subseteq y$, hence $y' \subseteq y$. Hence, $(x' \parallel y')_- \subseteq x \parallel y$, so $x' \parallel y' \subseteq x \parallel y$ since $x \parallel y \in \mathsf{fix}(\mathsf{C}(\sigma))$. Likewise, $y' \parallel z' \subseteq y \parallel z$. Therefore, $x' \parallel z' \subseteq x \parallel z$ and $(x, z) \in \mathsf{fix}(\mathsf{C}(\tau \odot \sigma))$ as required. □

However, for a game A it almost never holds that $\mathsf{C}(\text{æ}_A) = \text{æ}_{\mathscr{C}^\infty(A)}$. For instance, consider the game A having only one positive move $\bullet$. Then $A^\perp \parallel A$ has two moves, one negative move written $\circ$ and one positive still written $\bullet$. Then, $\text{æ}_{\mathscr{C}^\infty(A)}(\emptyset, \{\bullet\}) = (\{\circ\}, \{\bullet\})$, whereas $\mathsf{C}(\text{æ}_A)(\emptyset, \{\bullet\}) = (\emptyset, \{\bullet\})$. The identity in Clos adds missing negative dependencies (as it must, because it must be defined on arbitrary domains, with therefore no access to polarity information). In contrast, applying closure operators imported from strategies only adds positive moves. For this reason, it is tempting, instead of the transformation from strategies to closure-strategies presented above, to adopt one adding the missing negative dependencies to reachable positive events, as does the identity in Clos. But as presented in Appendix A.1 this leads to issues, notably non-stability and non-continuity of the corresponding closure operators – besides, then, $\mathsf{C}(-)$ would not give a functor either: identities would be preserved, but not composition.

Instead, we moderate this mismatch on identities by remarking that although $\mathsf{C}(\text{æ}_A)$ and $\text{æ}_{\mathscr{C}^\infty(A)}$ do not coincide, *they have the same reachable configurations*. The set of **reachable** fixpoints of a closure-strategy $\sigma : \mathscr{C}^\infty(A)$ is the smallest subset of $\mathscr{C}^\infty(A)$ containing $\sigma(\emptyset)$, and such that if $x \in \mathsf{fix}(\sigma)$ is reachable and $x \subseteq^- y$, then $\sigma(y)$ is reachable. Write $\mathsf{reach}(\sigma)$ for the set of reachable fixpoints of σ. Then, $\sigma, \sigma' : \mathscr{C}^\infty(A)$ are **reachable-equivalent** if $\mathsf{reach}(\sigma) = \mathsf{reach}(\sigma')$, written $\sigma \approx \sigma'$. Then, it is direct to prove that $\mathsf{C}(\text{æ}_A) \approx \text{æ}_{\mathscr{C}^\infty(A)}$.

Computing reachable configurations of compositions only uses reachable configurations:

Lemma 16. *Let $\sigma : \mathscr{C}^\infty(A^\perp \parallel B)$ and $\tau : \mathscr{C}^\infty(B^\perp \parallel C)$ be two closure-strategies. Using silently the order-isomorphism $\mathscr{C}^\infty(A^\perp \parallel B) \cong \mathscr{C}^\infty(A) \times \mathscr{C}^\infty(B)$, we regard them as morphisms from A to B and from B to C respectively.*

Then, for any $x_A \parallel x_C \in \mathsf{fix}(\tau \odot \sigma)$, we have $x_A \parallel x_C \in \mathsf{reach}(\tau \odot \sigma)$ iff there is a chain

$$x_A^0 \parallel x_B^0 \parallel x_C^0 \subseteq \ldots \subseteq x_A^n \parallel x_B^n \parallel x_C^n$$

where $x_A^0 \parallel x_B^0 \parallel x_C^0 = \emptyset$, $x_A^n \parallel x_C^n = x_A \parallel x_C$, and where each $x_A^i \parallel x_B^i \parallel x_C^i \longrightarrow\!\!\!\!\subset x_A^{i+1} \parallel x_B^{i+1} \parallel x_C^{i+1}$ is obtained by (1) $x_A^i \parallel x_B^i \in \mathsf{reach}(\sigma)$, $x_B^i \parallel x_C^i \in \mathsf{reach}(\tau)$, $x_B^i = x_B^{i+1}$ and $x_A^i \parallel x_C^i \subseteq^- x_A^{i+1} \parallel x_C^{i+1}$; or (2) $x_C^i = x_C^{i+1}$ and $x_A^{i+1} \parallel x_B^{i+1} = \sigma(x_A^i \parallel x_B^i)$; or (3) $x_A^i = x_A^{i+1}$ and $x_B^{i+1} \parallel x_C^{i+1} = \tau(x_B^i \parallel x_C^i)$. It follows that $x_A \parallel x_B \in \mathsf{reach}(\sigma)$ and $x_B \parallel x_C \in \mathsf{reach}(\tau)$.

Proof. Direct verification. □

In particular, a direct consequence of this lemma is that $\mathsf{reach}(\tau \odot \sigma) \subseteq \mathsf{reach}(\tau) \circ \mathsf{reach}(\sigma)$ where $\circ$ is relational composition. But unlike the case for all fixpoints in Proposition 14, the converse does not hold for *reachable* fixpoints.

Finally, we deduce:

Proposition 17. *Consider $\sigma, \sigma' : \mathscr{C}^\infty(A^\perp \parallel B)$ and $\tau, \tau' : \mathscr{C}^\infty(B^\perp \parallel C)$ satisfying $\sigma \approx \sigma'$ and $\tau \approx \tau'$, regarded as morphisms from A to B and from B to C in* Clos. *Then, we have*

$$\tau \odot \sigma \approx \tau' \odot \sigma' .$$

Proof. Straightforward from Lemma 16 as reachable fixpoints of $\tau \odot \sigma$ and $\tau' \odot \sigma'$ are reduced to chains formed from reachable fixpoints of σ/σ' and τ/τ', which are the same. □

From all the developments above, we may conclude:

Theorem 18. *There is a strong compact closed functor*

$$\mathsf{C}(-) : \mathsf{Games} \to \mathsf{Clos}/{\approx}$$

where $\mathsf{Clos}/{\approx}$ *has as morphisms closure operators up to* $\approx$.

It might seem that the quotient $\approx$ may create a mismatch between the two, but it is in fact much milder than the extensional collapse used in [5] to obtain full completeness – we will introduce it in the next subsection.

3.4 Extensional collapse

The interpretation of MALL formulas as games was given in Section 3.1. A context $\Gamma = A_1, \ldots, A_n$ is interpreted as a tensor $[\![\Gamma]\!] = \otimes_{1 \leq i \leq n} [\![A_i]\!]$ and a proof of $\vdash \Gamma$ as a strategy on $[\![\Gamma]\!]$. The interpretation of MLL rules proceeds as is standard in a compact closed category. A proof starting with an introduction rule for $\oplus$ will have the corresponding positive move as minimal, otherwise playing as the sub-proof – for $\sigma : A$, we write $\mathrm{in}_l(\sigma) : A \oplus B$. A proof starting with an introduction rule for & will delay positive moves until it receives one of the Opponent moves coming from the &; it then proceeds as the corresponding sub-proof – for $\sigma_A : A$ and $\sigma_B : B$, we write $\langle \sigma_A, \sigma_B \rangle : A \,\&\, B$. We have not been able to formally verify that this interpretation is compatible (through $\mathsf{C}(-)$) with that of [5] up to $\approx$, as the details of the interpretation do not appear in [5]. Nevertheless we believe this to be the case, and our interpretation seems compatible with informal descriptions in [5]. Thus it is informative to look at the interpretation of some proofs in our model as representations of their interpretation with closure-strategies.

In Figure 5, we display two proofs, along with one typical maximal augmentation in their respective interpretations in Games. In the proofs, we omit the $\vdash$ symbol and we color the units to track the specific rules used. For each proof we only display one maximal augmentation, corresponding to the one complete branch of the proof where the left component of & is always selected. The two proofs are convertible using standard commuting conversions. Despite this, they are distinguished by the semantics in Games and Clos. The maximal augmentations pictured show the phenomenon: if Opponent always selects the left component of & then the two proofs perform the same actions, but *not in the same order*. This phenomenon was of course noticed in [5], where the authors say:

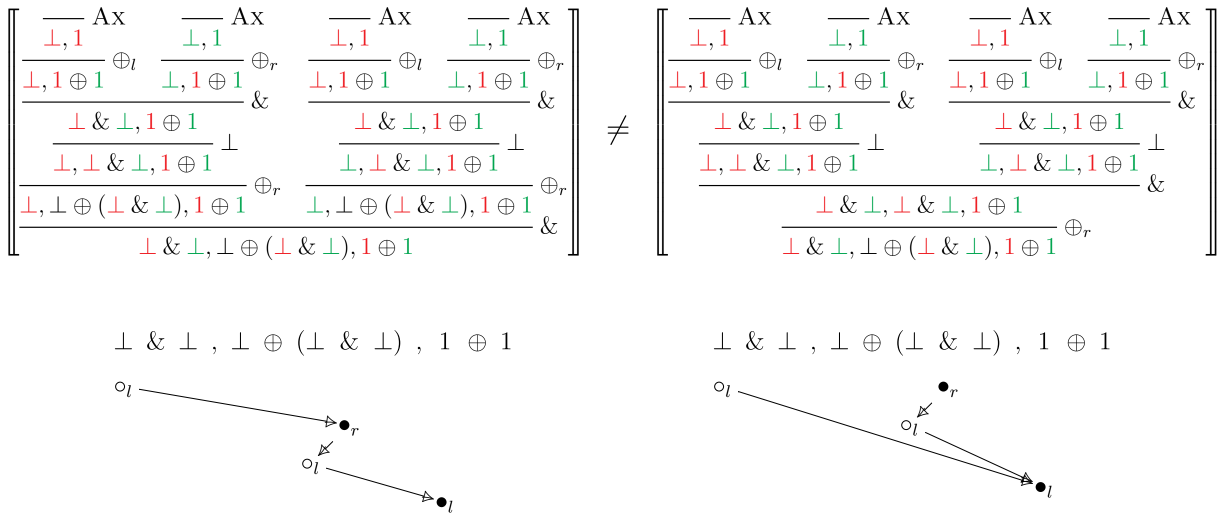


Figure 5: Two distinct strategies for two equivalent proofs

> "To motivate the passage to the extensional category, note that Clos only has weak products and coproducts. Indeed, the lifted sum which we used to model the additives is non-associative, and we need to quotient out the behaviour at the partial elements in order to obtain the required structure."

Indeed, they quotient the model using partial equivalence relations (*pers*, satisfying transitivity and symmetry but not reflexivity), a standard methodology to construct models of linear logic [44]. Concretely, for every formula A they build (by induction on A) a per $\sim_A$ on strategies on A. It has two effects: firstly, it identifies strategies with the same extensional behaviour, even though they might be intensionally distinct. With respect to the interpretation of MALL above, this quotients out the intensional behaviour caused by the lifts in the interpretation of additive connectives. In particular, the two proofs of Figure 5 are identified. Secondly, it cuts out those strategies that can taste intensional information: the new model restricts to strategies that are self-equivalent, which for morphisms from A to B essentially amounts to sending $\sim_A$-equivalent strategies to $\sim_B$-equivalent strategies.

Sometimes, a miracle occurs after this "cutting out" process: only definable elements remain and the new model is fully complete – and indeed Abramsky and Melliès show that this is the case for closure-strategies. This is by all means not a general fact: for instance, the same construction applied to the relational model does not yield full completeness[10]. Performing this construction on an intensional canvas such as game semantics helps, in that morphisms in the new model are equivalence classes of concrete strategies. Representatives have intensional behaviour that can be tracked down to reconstruct a proof.

However, something remains puzzling. There seems to be a tension between definability,

[10] A similar construction applied on *hypercoherences*, which build on the relational model, does yield full completeness [14] – note that there are links between hypercoherences and game semantics [51].

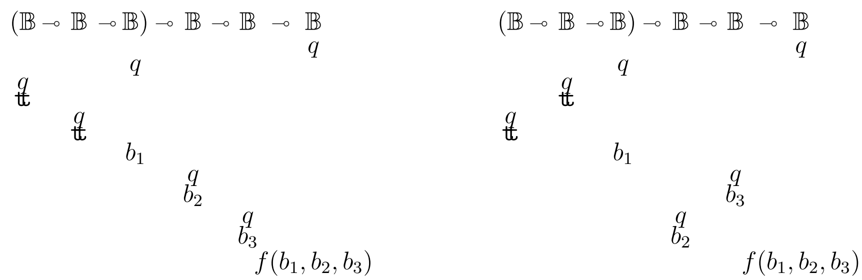


Figure 6: Example of extensionally correct yet undefinable behaviour

which is facilitated by more *intensionality*; and validating all required equations, which is facilitated by more *extensionality*. The solution of [5] is to first build an interpretation failing some equations but with a tractable intensional description of proofs, and then quotient and cut it down by extensional collapse. But then, why did we need concurrent games to do that? After all, there are plenty of simpler intensional not-quite-models around, the obvious one being Blass games. Composition is not associative in Blass games, which was the original motivation for concurrent games. But that cannot be the end of the story: non-associativity means that the two equivalent proofs of Figure 2 are interpreted by two strategies "doing the same actions but not in the same order". Moving to Clos and Games solved the Blass problem and made those two equivalent, but with the cost that the two proofs of Figure 5 now give rise to two strategies "doing the same actions but not in the same order" – but those two were interpreted with the same strategy in Blass games! So have we just moved the problem around?

As it turns out, the conceptual advance offered by concurrent games is much greater than merely solving the Blass problem. While the extensional collapse may be applied to a sequential games substrate (such as Blass games) in order to get a model of MALL, this will usually land us far from full completeness. Figure 6 illustrates this in a basic game semantics setting (*e.g.* simple games [43]), though we do not see why the phenomenon would not occur as well in Blass games. The strategy pictured with its two maximal plays acts like $\lambda gxy.\, f(g\, \mathtt{tt}\, \mathtt{tt}, x, y)$, except it calls x and y in the same order than the argument g used to call its arguments. This strategy is undefinable (it is not *innocent*), yet it survives the extensional collapse. Such a behaviour cannot be expressed with closure-strategies, because it is not *positional*: after the first 7 moves, the strategy acts differently in the two plays, although the set of moves that have been played is the same. So it appears that the ability of concurrent games to give a fully complete model to MALL is not due to concurrency *per se*[11]. Instead, and as investigated in depth by Melliès in *asynchronous games*, the key conceptual advance of concurrent games is that they are *positional / causal*.

[11] Indeed the Blass problem could easily be solved by a non-polarized version of Ghica and Murawski's concurrent games [35], but there is no reason why it would not suffer from phenomena as in Figure 6.

4 Full Completeness via MALLP

We have seen that Abramsky and Melliès' full completeness result rests on two ingredients: *(1)* an unsound intensional model, whose dynamics can be tracked down to guide definability; and *(2)* a quotient which restores the necessary equations between proofs. The definability process of [5] is challenging, as strategies are far from sequential. Rather than a sequent proof, the argument reconstructs from the action of the strategy an MALL proof structure in the sense of Girard [40], which is proved correct.

In this paper we do not review this argument. Instead we adopt a different route, following later work by Melliès [50]. Since it seems a quotient is required anyway, why not add much more intensional information, with enough dynamic content as to make definability straightforward, yielding directly a sequent proof by induction? This will mechanically break more of the expected MALL laws, but those will be reinstated by quotient anyway. Likewise, this added sequentiality should not prevent us from quotienting, provided the model is phrased in a *positional* setting.

In the remainder of this paper we build a fully complete model of MALL following that route. Rather than directly giving a sequential interpretation of MALL, we first interpret a *polarized* variant. It is obtained by annotating formulas with new constructors marking additional observable computation steps, and constraining proofs making their dynamic sequential (MALL formulas will later on be interpreted by first *polarizing* them, then interpreting the obtained formula). We will first build a fully complete model of polarized MALL, then perform the quotient and deduce full completeness for MALL.

The first fully complete games model for a polarized version of MALL was by Girard, in the framework of Ludics [39]. In his thesis, Laurent introduced a more symmetric presentation of Girard's system, called MALLP [48], which we adopt in this paper. The developments in this section and the next are strongly inspired from Melliès' in *asynchronous games*, rephrased in the game semantics language presented in Section 3.

4.1 Polarized Multiplicative Additive Linear Logic

Linear Logic is inherently *non-polarized*: $(1 \oplus 1) \otimes (1 \,\&\, 1)$ is interpreted (following the previous section) by the game $\bullet_1 \sim \bullet_2 \quad \circ_1 \sim \circ_2$ with both players having available events, reflecting the fact that the formula does not carry explicit information as to which side of the tensor is to be resolved first, if any. *Polarized Multiplicative Additive Linear Logic (*MALLP*)*, introduced by Laurent [48], starts with the same connectives, but restricts formulas so as to follow a strict *polarity* discipline ensuring among other things that execution is sequential. New unary connectives, *shifts*, are used to transport between polarities.

The **formulas** of MALLP are as follows

$$\begin{array}{rcl} P, Q & ::= & 0 \mid 1 \mid P \otimes Q \mid P \oplus Q \mid {\downarrow} M \\ M, N & ::= & \top \mid \bot \mid M \parr N \mid M \,\&\, N \mid {\uparrow} P \end{array}$$

where P, Q are called **positive** and M, N are called **negative**. There is a clear duality between the two, defined as $0^\perp = \top$, $1^\perp = \bot$, $({\downarrow} M)^\perp = {\uparrow} M^\perp$, $(P \otimes Q)^\perp = P^\perp \parr Q^\perp$,

$$\frac{\vdash \Gamma, P \qquad \vdash \Delta, Q}{\vdash \Gamma, \Delta, P \otimes Q}\otimes \qquad \frac{\vdash \Gamma, M, N, [P]}{\vdash \Gamma, M \parr N, [P]}\parr \qquad \frac{}{\vdash 1}1 \qquad \frac{\vdash \Gamma, [P]}{\vdash \Gamma, \bot, [P]}\bot \qquad \frac{\vdash \Gamma, M}{\vdash \Gamma, {\downarrow} M}\downarrow$$

$$\frac{\vdash \Gamma, P}{\vdash \Gamma, {\uparrow} P}\uparrow \qquad \frac{}{\vdash P^\perp, P}\text{Ax} \qquad \frac{\vdash \Gamma, P \qquad \vdash \Delta, P^\perp, [Q]}{\vdash \Gamma, \Delta, [Q]}\textsc{Cut} \qquad \frac{\vdash \Gamma, P}{\vdash \Gamma, P \oplus Q}\oplus_l \qquad \frac{\vdash \Gamma, Q}{\vdash \Gamma, P \oplus Q}\oplus_r$$

$$\frac{\vdash \Gamma, M, [P] \qquad \vdash \Gamma, N, [P]}{\vdash \Gamma, M \,\&\, N, [P]}\& \qquad \frac{}{\vdash \Gamma, \top, [P]}\top$$

Figure 7: Rules of MALLP

and $(P \oplus Q)^\perp = P^\perp \,\&\, Q^\perp$; and vice versa. There are two kinds of sequents: those of the form $\vdash \Gamma$ and those of the form $\vdash \Gamma, P$, where in both cases all formulas in Γ are assumed negative. Following [54] we write $\vdash \Gamma, [P]$ any of the two cases. We show the rules in Figure 7. As before we consider exchange rules present, though not written explicitely.

MALLP is a refinement of MALL, in the sense that given a MALLP proof, erasing the shifts and the corresponding deduction rules yields an MALL proof. In fact MALL proofs obtained from MALLP are *focused*[12] proofs: indeed, in a focused proof, at any given time in proof construction we focus on at most one positive formula. The only positive rules used must apply to this positive formula, until we reach a negative formula. This process is faithfully reflected by the presence of at most one positive formula in a MALLP sequent. In fact, the focusing property of Linear Logic, first noticed by Andreoli [8], can be proved through a translation of MALL in MALLP.

4.2 Interpretation of MALLP

The polarity of formulas is directly reflected in the accompanying games. A game A is **positive** (resp. **negative**) if all its minimal events are positive (resp. negative). In general games may be neither negative nor positive, as in the example above interpreting $(1 \oplus 1) \otimes (1 \,\&\, 1)$. In contrast, games interpreting MALLP formulas will always have a clear polarity. As a matter of fact, their shape will be even more restricted.

Definition 19. *A finite game A is an **arena** if:* (1) *all its minimal events share the same polarity and conflict with each other;* (2) *causal dependency is* alternating *(if $a_1 \rightarrowtriangle_A a_2$ then $\mathrm{pol}_A(a_1) \neq \mathrm{pol}_A(a_2)$) and tree-shaped (if $a_1, a_1' \leq_A a_2$ then either $a_1 \leq_A a_1'$ or $a_1' \leq_A a_1$); and* (3) *conflict is* local *in the sense that if $a_1 \sim\!\sim a_2$, then either they are both minimal or they share the same (necessarily unique) immediate predecessor.*

[12] More precisely, *weakly focused* in the sense of [47].

By definition, an arena is either negative or positive. We denote negative arenas by M, N and positive arenas by P, Q. Every positive arena may be written as

$$P = \sum_{i \in I} \downarrow N_i$$

with I finite, and where each N_i is a negative game (which might not be an arena). This lets us define the **tensor** of positive arenas as

$$P \otimes Q = \sum_{(i,j) \in I \times J} \downarrow (N_i \parallel M_j)$$

where $P = \sum_{i \in I} \downarrow N_i$ and $Q = \sum_{j \in J} \downarrow M_j$. We also define their **sum** $P \oplus Q$ simply as $P + Q$ (note that this use of the notation $P \oplus Q$ is incompatible with that in Section 3.1 – from now on, all uses of $\oplus$ refer to the present definition). The arena 1 consists of only one move, which is positive; and 0 is the empty arena. Negative arenas and their constructions are defined dually. Altogether, this gives us an interpretation $[\![-]\!]$ of formulas as arenas.

To interpret proofs, we flesh out the categorical structure relative to these constructions. We preface this with a few remarks. Firstly, MALLP may be presented as a 2-sided sequent calculus with at most one formula on the right: $\vdash N_1, \ldots, N_n, [P]$ is represented simply as $N_1^\perp, \ldots, N_n^\perp \vdash P$. All formulas involved are then positive. Positive MALLP formulas are uniquely written with the positive connectives $0, 1, \otimes$ and $\oplus$; along with *negation* $\neg$ defined as $\neg P = \downarrow(P^\perp)$. The resulting system is known as Multiplicative Additive Tensorial Logic [54]. To interpret MALLP we construct a *dialogue category with coproducts* [54] which matches the Tensorial Logic presentation of MALLP, but the two are completely equivalent.

We start with the category Arenas, whose objects are positive arenas, and morphisms from P to Q are strategies $\sigma : P^\perp \parallel Q$ which are **negative**, in the sense that each $x \in \mathscr{C}(\sigma)$ must contain at least one negative event, and **thunkable**, in the sense that for each $x \in \mathscr{C}(\sigma)$, if x contains one positive event, then it contains one in Q. So morphisms in Arenas first wait for an Opponent input on the left; then immediately play on the right. Let us write $\sigma : P \overset{\mathrm{Ar}}{\twoheadrightarrow} Q$ to denote that σ is a morphism from P to Q in Arenas.

The construction $\oplus$ yields a coproduct in Arenas, let us write $\mathrm{in}_l : P \overset{\mathrm{Ar}}{\twoheadrightarrow} P \oplus Q$ and $\mathrm{in}_r : Q \overset{\mathrm{Ar}}{\twoheadrightarrow} P \oplus Q$ for the two injections, and $[\sigma_1, \sigma_2] : P_1 \oplus P_2 \overset{\mathrm{Ar}}{\twoheadrightarrow} Q$ for the co-pairing of $\sigma_1 : P_1 \overset{\mathrm{Ar}}{\twoheadrightarrow} Q$ and $\sigma_2 : P_2 \overset{\mathrm{Ar}}{\twoheadrightarrow} Q$. We use similar notations for the corresponding n-ary construction. This lets us decompose any $\sigma : P \overset{\mathrm{Ar}}{\twoheadrightarrow} Q$ as

$$\sigma = [\mathrm{in}_{f_\sigma(i)} \odot \downarrow(\sigma_i) \mid i \in I] : \sum_{i \in I} \downarrow M_i \overset{\mathrm{Ar}}{\twoheadrightarrow} \sum_{j \in J} \downarrow N_j$$

where $f_\sigma : I \to J$ is a function, $\sigma_i : M_i \overset{\mathrm{Ga}}{\twoheadrightarrow} N_{f_\sigma(i)}$ is a (necessarily) negative strategy in Games; and $\downarrow(-)$ is the functorial action of $\downarrow$ in Games defined in the obvious way. This also lets us define the functorial action of $\otimes$: if $\sigma : P \overset{\mathrm{Ar}}{\twoheadrightarrow} Q$ and $\sigma' : P' \overset{\mathrm{Ar}}{\twoheadrightarrow} Q'$,

$$\sigma \otimes \sigma' : [\mathrm{in}_{(f_\sigma(i), f_{\sigma'}(i'))} \odot \downarrow(\sigma_i \parallel \sigma'_{i'}) \mid (i, i') \in I \times I'] : P \otimes P' \overset{\mathrm{Ar}}{\twoheadrightarrow} Q \otimes Q'$$

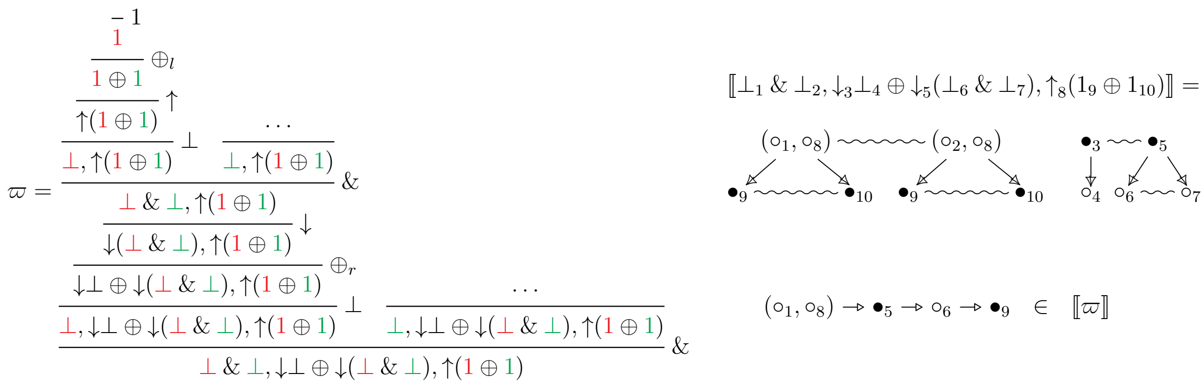


Figure 8: A proof in MALLP and its interpretation

making Arenas a symmetric monoidal category with coproducts, where $\otimes$ distributes over coproducts in the sense that the canonical morphisms $(P \otimes Q_1) \oplus (P \otimes Q_2) \overset{\mathrm{Ar}}{\twoheadrightarrow} P \otimes (Q_1 \oplus Q_2)$ and $0 \overset{\mathrm{Ar}}{\twoheadrightarrow} P \otimes 0$ are isomorphisms – Arenas may be regarded as the free coproduct completion of a category of negative games and strategies.

Finally, Arenas has a **tensorial negation**, *i.e.* a (necessarily self-adjoint) functor $\neg : \mathsf{Arenas} \to \mathsf{Arenas}^{\mathrm{op}}$ together with a family of bijections

$$\varphi_{P,Q,R} : \mathsf{Arenas}[P \otimes Q, \neg R] \cong \mathsf{Arenas}[P, \neg(Q \otimes R)]$$

natural in P, Q and R and subject to a coherence condition. On arenas, we define $\neg P = \downarrow P^{\perp}$, extended to strategies in the obvious way with the functorial action of $\downarrow$ and the compact closed structure of Games. Altogether we get a **dialogue category** with coproducts [54] hence a model of Multiplicative Additive Tensorial Logic, or equivalently MALLP. A proof ϖ of $\vdash N_1, \dots N_n, P$ with a positive formula is interpreted as a morphism

$$[\![\varpi]\!] : \bigotimes_{1 \leq i \leq n} [\![N_i]\!]^{\perp} \overset{\mathrm{Ar}}{\twoheadrightarrow} [\![P]\!]$$

while a proof ϖ of a sequent $\vdash N_1, \dots, N_n$ is interpreted as $[\![\varpi]\!] : \bigotimes_{1 \leq i \leq n} [\![N_i]\!]^{\perp} \overset{\mathrm{Ar}}{\twoheadrightarrow} \neg 1$.

The categorical structure should make it plain how the rules are interpreted; we only comment the introduction rules for shifts: the introduction of $\downarrow$ directly matches the natural isomorphism $\varphi_{\Gamma^{\perp}, M^{\perp}, 1}$; while the introduction rule for $\uparrow$ first composes $[\![\varpi]\!] : [\![\Gamma]\!]^{\perp} \overset{\mathrm{Ar}}{\twoheadrightarrow} P$ with the unit of the continuation monad $P \overset{\mathrm{Ar}}{\twoheadrightarrow} \neg\neg P$, before applying $\varphi^{-1}_{\Gamma, \neg P, 1}$.

We display in Figure 8 a branch of a proof and its corresponding interpretation. As in Figure 5, we omit the $\vdash$ symbol and color units to disambiguate the rules. On the right hand side, we first show the game interpreting the sequent, and the maximal augmentation of the strategy $[\![\varpi]\!]$ corresponding to the branch of the proof displayed on the left hand side. We observe that this branch is completely linear – in fact, we will see that this is true of all strategies obtained as the interpretation of proofs in tensorial logic, a property that in the next subsection we will call *sequential innocence*.

4.3 Full completeness for $\mathsf{MALLP}^\flat$

Now, we refine the interpretation in order to obtain full completeness. From now on and for the remainder of this paper, we will restrict to the fragments $\mathsf{MALL}^\flat$ and $\mathsf{MALLP}^\flat$, respectively of MALL and MALLP, without the additive units 0 and $\top$. While the methodology we present here does extend in their presence, they come with technical complications that are a significant obstacle to our objective of keeping this paper as simple as possible and focused on the conceptual ideas. The reader will find in Appendix B a generalization of the constructions for full MALL and MALLP.

Strategies coming from proofs satisfy constraints of two different natures. The first two conditions, *totality* and *sequential innocence*, are *causal*: they capture the causal patterns of strategies arising from MALLP proofs. The third condition, *exhaustivity*, is *positional* and expresses that complete positions of strategies should validate the linearity constraints by exhausting all resources in their complete positions.

4.3.1 Totality and Sequential Innocence

Our first two conditions, *totality* and *sequential innocence*, are intrinsic to strategies, meaning that they restrict their causal shapes without enriching the interpretation of types.

Totality. In game semantics, *proofs* (as opposed to *programs*) are traditionally interpreted as strategies that are *total*, in the sense that they always have a response to any move by the environment. Game semantics for proofs makes formal a debate between two players, arguing about the validity of a formula: Player aims to establish the truth of the formula, while Opponent attempts to falsify it. In this view, a proof should yield a strategy that never gives up, and has a valid counter-argument to any attack by Opponent.

In our games, totality may be formulated as follows.

Definition 20. *A strategy $\sigma : A$ is **total** if for any $\mathrm{q} \in \sigma$ maximal in σ (for rigid embedding), the maximal events (for $\leq_{\mathrm{q}}$) of q have positive polarity.*

Regarding strategies as descriptions of normal forms, totality is a normalization property – any exploration of the normal form by Opponent will uncover new parts of the term and will not trigger divergence. Just as terms with a normal form are not usually closed under composition (considering *e.g.* $\delta\,\delta$ in the pure λ-calculus), total strategies are not in general stable under composition as two total strategies may enter in a *livelock*, never producing an observable result. Getting total strategies to compose often requires some technology [28]; but here as our games are finite, compositionality of totality is easy.

Sequential Innocence. In Hyland-Ong games, *innocent strategies* are those whose behaviour only depends on a partial sub-history of the play called the *P-view*. Causal game semantics reveal that, in fact, P-views are simply the underlying causal structure. In traditional game semantics this causal structure is derived: strategies are defined and composed as sets of general plays, and appear a posteriori to be representable as sets of P-views. In contrast, in our strategies the causal structure is *primitive* – we get to see the elephant directly. This makes innocence appear very differently from its presentation in traditional game semantics: we must simply restrict the causal shapes to those that follow the tree-like inductive structure of proofs.

Definition 21. *A strategy $\sigma : A$ is* ***sequential innocent*** *iff any $\mathrm{q} \in \sigma$ is forest-shaped, and O-branching: if $a \rightarrowtriangle_{\mathrm{q}} a_1$ and $a \rightarrowtriangle_{\mathrm{q}} a_2$ with $a_1, a_2 \in |\mathrm{q}|$ distinct,* $\mathrm{pol}(a_1) = \mathrm{pol}(a_2) = -$.

If A is an arena, then this means that for any $\mathrm{q} \in \sigma$ and $a \in |\mathrm{q}|$, the causal history of a in q is a linearly ordered causal chain, which is alternating:

$$a_0^- \rightarrowtriangle_{\mathrm{q}} a_1^+ \rightarrowtriangle_{\mathrm{q}} a_2^- \rightarrowtriangle_{\mathrm{q}} a_3^+ \rightarrowtriangle_{\mathrm{q}} \dots \rightarrowtriangle_{\mathrm{q}} a \,.$$

Note that because arenas are forest-shaped, each move that is not minimal in A has a unique antecedent in A. From the conditions imposed on augmentations, if a_i appears in a causal chain as above, then its antecedent must also appear before. Let us call the antecedent of a_i its **justifier**. Then by courtesy, in a chain as above the justifier of a_{2n+2}^- must be a_{2n+1}^+; and the justifier of a_{2n+1}^+ must be one of the earlier negative events. So, this is exactly a *P-view*; making more concrete the intuitions suggested above. Globally, σ may then be regarded as a prefix-closed set of linearly ordered causal chains (P-views) as above branching only at Opponent moves. Two branching chains (P-views) may be either *compatible* (if they are both prefixes of a common augmentation), or *conflicting* (if not).

This link with more traditional structures of innocence in game semantics is a strength of the presentation of strategies as sets of augmentations rather than as sets of configurations, sets of plays [53] or closure operators [5]. As in Hyland-Ong games, this also means that strategies have a simple inductive structure, aiding definability.

In traditional game semantics, proving that innocence is stable under composition is tricky. In contrast here, stability of sequential innocence under composition is very easy:

Proposition 22. *If $\sigma : A \overset{\mathrm{Ar}}{\rightarrowtriangle} B$ and $\tau : B \overset{\mathrm{Ar}}{\rightarrowtriangle} C$ are sequential innocent, then so is $\tau \odot \sigma$.*

Proof. It suffices to show that if $\mathrm{q} \in \sigma$ and $\mathrm{p} \in \tau$ are causally compatible, then every event e in $\mathrm{p} \circledast \mathrm{q}$ has at most one immediate antecedent. Looking for a contradiction, assume that

$$e_1 \rightarrowtriangle_{\mathrm{p} \circledast \mathrm{q}} e \qquad\qquad e_2 \rightarrowtriangle_{\mathrm{p} \circledast \mathrm{q}} e \,.$$

If e is an external Opponent move, by courtesy $e_1, e_2 \rightarrowtriangle_{A^\perp \parallel C} e$, so $e_1 = e_2$ since arenas are forest-shaped. Otherwise e is positive for either σ or τ, say *w.l.o.g.* σ. Then, by an analysis of immediate causality in an interaction (essentially Lemma 2.10 of [19]) along with courtesy of τ, we have $e_1, e_2 \rightarrowtriangle_\sigma e$, so $e_1 = e_2$ since σ is sequential innocent. □

In other words, no *causal join* can emerge in an interaction between strategies that do not perform causal joins. Structural morphisms are sequential innocent, and all other constructions on strategies are easily shown to preserve sequential innocence. One can wonder why stability of innocence is so easy here, compared to traditional games. It seems that in traditional games, the complexity comes from the back and forth between P-views (the *causal* structure) and plays, which we completely avoid here.

Finally, an observation on the name *sequential innocence*: in concurrent games, the notion above appears as a sequential specialization of a more general notion of *parallel innocence* [20]. Parallel innocent strategies have no side-effect but may perform computations in parallel – they include strategies for *parallel-or* [21], or strategies arising from the parallel evaluation of purely functional programming languages [20].

4.3.2 Exhaustive Strategies

For the simply-typed λ-calculus, totality and innocence suffice to obtain definability; but not here: indeed, totality allows *affine behaviour*[13] whereas we want strict linearity.

Several existing mechanisms could be used here to ensure strict linearity. Our choice of name, *exhaustive strategies*, reminds one of the *exhausting strategies* of Murawski and Ong [57], in which one asserts that all moves of the game should be somehow reachable by strategies. Unlike what our name suggests, we opt instead for a simple and elegant construction due to Melliès [50] and then refined by Melliès and Tabareau [54]. The construction works by enriching arenas with a notion of *payoff*.

Definition 23. *An* ***arena with payoff*** *is an arena A with $\kappa_A : \mathscr{C}(A) \to \{-1, 0, +1\}$ such that for A non-empty, A is positive (*resp. *negative) iff $\kappa_A(\emptyset) = -1$ (*resp. *$\kappa_A(\emptyset) = +1$).*

Configurations $x \in \mathscr{C}(A)$ such that $\kappa_A(x) = 0$ are called **exhaustive** – for games coming from $\mathsf{MALLP}^\flat$, we will see that those are exactly the maximal configurations. For non-maximal configurations, κ assigns the *responsibility* of non-exhaustivity, *i.e.* points out which of the two players is stalling. Configurations $x \in \mathscr{C}(A)$ such that $\kappa_A(x) = +1$ are called **winning**: the responsibility of non-exhaustivity is assigned to Opponent. Dually, configurations $x \in \mathscr{C}(A)$ such that $\kappa_A(x) = -1$ are called **losing**.

Here, we make three observations. Firstly, the reader can observe the proximity with the winning conditions of [27]: the main difference is the existence of *neutral* positions, or *draws*, with null payoff. Secondly, unlike *e.g.* [63], the objective of strategies will be to at least ensure a draw, *i.e.* either reach an exhaustive configuration, or a state where the responsibility of non-exhaustivity may be assigned to Opponent.

Constructions. Let us show how the constructions on arenas extend in the presence of payoff functions. For units, $\kappa_1(\emptyset) = -1$ and $\kappa_1(\{\updownarrow\}) = 0$ – the payoff on $\bot$ is defined dually, with $\kappa_{A^\perp} = -\kappa_A$. For lifts, we set $\kappa_{\downarrow N}(\emptyset) = -1$, and $\kappa_{\downarrow N}(\{\bullet\} \cup x_N) = \kappa_N(x_N)$.

For positive $P = \sum_{i \in I} \downarrow N_i$ and $Q = \sum_{j \in J} \downarrow M_j$, we set $\kappa_{P \oplus Q}(x_P) = \kappa_P(x_P)$ if $x_P \in \mathscr{C}(P)$ and symmetrically for Q. For the tensor, we first set $\kappa_{P \otimes Q}(\emptyset) = -1$. Non-empty configurations of $P \otimes Q$ necessarily have the form $\{\updownarrow_{(i,j)}\} \cup (x_{N_i} \parallel x_{M_j})$, written $x_P \otimes x_Q$

[13] In fact, the proof of Theorem 27 shows that without exhaustivity, the model is fully complete for Polarized Multiplicative Additive Affine Logic (MAALP) [48].

where $x_P = \{\updownarrow_i\} \cup x_{N_i}$ and $x_Q = \{\updownarrow_j\} \cup x_{M_j}$. We then set

$$\kappa_{P\otimes Q}(x_P \otimes x_Q) = \kappa_P(x_P) \otimes \kappa_Q(x_Q)\,,$$

where, for $\alpha, \beta \in \{-1, 0, 1\}$, we set $\alpha \otimes \beta = 0$ iff $\alpha = \beta = 0$, $\alpha \otimes \beta = -1$ if $\alpha = -1$ or $\beta = -1$, and $\alpha \otimes \beta = 1$ otherwise. Finally, $\kappa_{N \parr M}$ is defined dually. A non-maximal configuration of $P \otimes Q$ is winning if it is winning or exhaustive on both sides, whereas a non-maximal configuration of $M \parr N$ must be winning on at least one side.

For each $\mathsf{MALLP}^\flat$ formula A we may build by induction on A, following the definitions above, an arena with payoff $[\![A]\!]$. We mention in passing the following straightforward lemma, where we say that $x \in \mathscr{C}(A)$ **is $+$-maximal in** $\mathscr{C}(A)$ iff for any $y \in \mathscr{C}(\sigma)$ such that $x \subseteq^+ y$ we have $x = y$; and symmetrically for $-$-maximal configurations.

Lemma 24. *For any* $\mathsf{MALLP}^\flat$ *formula* A*,* (1) *if* x *is* $+$*-maximal in* $\mathscr{C}([\![A]\!])$*, then* $\kappa_{[\![A]\!]}(x) \geq 0$*;* (2) *if* x *is* $-$*-maximal in* $\mathscr{C}([\![A]\!])$*, then* $\kappa_{[\![A]\!]}(x) \leq 0$*; and* (3) x *is maximal iff* $\kappa_{[\![A]\!]}(x) = 0$*.*

If we were to interpret all units, 0 would yield the empty arena with $\kappa_0(\emptyset) = -1$, failing the lemma above – this is the reason why the proof of definability we present here does not directly apply to additive units, which require more elaborate constructions.

Exhaustive strategies. We may now define *exhaustive strategies.* If $\sigma : A$ is a strategy and $x \in \mathscr{C}(\sigma)$, we say that it is **$+$-maximal** if σ has no further move to play, *i.e.* for any $y \in \mathscr{C}(\sigma)$ such that $x \subseteq^+ y$ we have $x = y$.

Definition 25. *Let* P, Q *be positive arenas with payoff. A strategy* $\sigma : P \overset{\mathrm{Ar}}{\nrightarrow} Q$ *is* ***exhaustive*** *if for any* $x_P \parallel x_Q \in \mathscr{C}(\sigma)$ $+$-maximal *we have* $\kappa_{P^\perp}(x_P) \parr \kappa_Q(x_Q) \geq 0$*.*

The proof that exhaustive strategies are stable under composition is exactly as in the proof of stability of winning strategies in [27]; other constructions on strategies are direct. From now on, we consider all arenas to be equipped with a payoff function, and all strategies to be total, sequential innocent and exhaustive. Altogether we get a dialogue category with coproducts, that we will keep referring to as Arenas.

The reader may wonder why we call those strategies *exhaustive* rather than *winning.* For us, the use of *winning* in game semantics usually conveys the idea that winning strategies witness logical validity. But here, exhaustivity does not ensure logical validity. It is perfectly conceivable, for instance, to have a programming language with recursion and divergence but with a strict linearity discipline that will ensure exhaustivity but where two definable exhaustive $\sigma : A$ and $\tau : A^\perp$ simultaneously exist. Then, the exhaustivity mechanism only ensures that their interaction yields an exhaustive configuration of A.

4.4 Full Completeness

To obtain full completeness for $\mathsf{MALLP}^\flat$ it remains to prove definability.

To any sequent $\vdash N_1, \ldots, N_n, [P]$ of $\mathsf{MALLP}^\flat$ and strategy

$$\sigma : \bigotimes_{1\leq i\leq n} [\![N_i]\!]^\perp \overset{\text{Ar}}{\twoheadrightarrow} [\![[P]]\!]\,,$$

where $[\![[P]]\!]$ means $\neg 1$ if there is no P, we associate a proof of $\vdash N_1, \ldots, N_n, [P]$ whose interpretation yields σ. This is done, as expected, by induction on the number of events in σ and the size (number of symbols) in the sequent.

We start by taking care of a few easy cases. If one of the N_i is $\perp$ or starts with $\parr$, then we apply directly the corresponding rule, not changing the game and strategy up to iso. If one of the N_i – say N_n is a product $N_n^1 \& N_n^2$, then the interpretation of the context is, up to isomorphism, a product, so that σ can be regarded as inhabiting:

$$\sigma : ([\![N_1]\!]^\perp \otimes \ldots \otimes [\![N_{n-1}]\!]^\perp \otimes [\![N_n^1]\!]^\perp) \oplus ([\![N_1]\!]^\perp \otimes \ldots \otimes [\![N_{n-1}]\!]^\perp \otimes [\![N_n^2]\!]^\perp) \overset{\text{Ar}}{\twoheadrightarrow} [\![[P]]\!]$$

hence σ is a co-pair $[\sigma_1, \sigma_2]$. By induction hypothesis, each σ_i is defined with a proof, and hence σ may be defined via the introduction rule for &.

Hence, we can assume that all arenas for N_i have the form $\uparrow P_i$, so that the game for

$$\sigma : \bigotimes_{1\leq i\leq n} \downarrow [\![P_i]\!]^\perp \overset{\text{Ar}}{\twoheadrightarrow} [\![[P]]\!]$$

has a unique negative minimal move corresponding to the shifts on the left hand side (this also holds in the case where the tensor is empty, as its unit 1 has exactly one event). We now distinguish several cases, depending on the shape of $[P]$. Of these, the crucial case – by far the most subtle – is that where there is one positive formula, of the form $Q_1 \otimes Q_2$.

Lemma 26. *Let $(P_k)_{1\leq k\leq n}, Q_1, Q_2$ be arenas, and consider a strategy*

$$\sigma : \bigotimes_{1\leq i\leq n} \downarrow P_k^\perp \overset{\text{Ar}}{\twoheadrightarrow} Q_1 \otimes Q_2\,.$$

Then, up to reordering of the context there are strategies

$$\sigma_1 : \bigotimes_{1\leq k\leq p} \downarrow P_k^\perp \overset{\text{Ar}}{\twoheadrightarrow} Q_1 \qquad\qquad \sigma_2 : \bigotimes_{p+1\leq k\leq n} \downarrow P_k^\perp \overset{\text{Ar}}{\twoheadrightarrow} Q_2$$

such that $\sigma = \sigma_1 \otimes \sigma_2$.

Proof. W.l.o.g. we can assume that neither Q_1 nor Q_2 is 1. Then, the game has the shape

$$\sigma : \downarrow(\|_{1\leq k\leq n} \sum_{i\in I_k} \uparrow M_{k,i}^\perp) \overset{\text{Ar}}{\twoheadrightarrow} \sum_{(l_1,l_2)\in L_1\times L_2} \downarrow(N_{l_1} \parallel N_{l_2})\,.$$

where $Q_i = \sum_{l\in L_i} \downarrow N_{l_i}$ and $P_k^\perp = M_k = \|_{1\leq k\leq n} \sum_{i\in I_k} \uparrow M_{k,i}^\perp$. After the unique minimal negative move, σ starts by playing some (l_1, l_2), and then resumes as a negative strategy

$$\sigma' \;:\; \|_{1\leq k\leq n} \sum_{i\in I_k} \uparrow M_{k,i}^\perp \overset{\text{Ga}}{\twoheadrightarrow} N_{l_1} \parallel N_{l_2}\,.$$

But then, there is a partition of the components of the parallel composition on the left hand side into those that may be accessed through N_{l_1}, through N_{l_2}, and those (in principle) that will *not* be accessed. Indeed, recall that augmentations in σ' are forest-shaped (because σ is sequential innocent). Two augmentations q_1 and q_2 visiting one component $M_k = \sum_{i\in I_k} \uparrow M_{k,i}^\perp$ cannot be compatible, so they contain respectively conflicting Opponent events. But since conflict is local in arenas, this is only possible if q_1 and q_2 either share the same minimal event, or if their minimal events are conflicting. In both cases, they start in the same component, N_{l_1} or N_{l_2}. So for each $1 \leq k \leq n$, M_k may be accessed only via N_{l_1}, or via N_{l_2}. Reordering the context we can rewrite the game for σ' as

$$\sigma' \ : \ \Gamma_1 \parallel \Gamma_2 \parallel \Gamma_3 \ \overset{\mathrm{Ga}}{\twoheadrightarrow} \ N_{l_1} \parallel N_{l_2} \,.$$

where all components of $\Gamma_1 = \|_{1\leq k\leq p} \sum_{i\in I_k} \uparrow M_{k,i}^\perp$ (*resp.* $\Gamma_2 = \|_{p\leq n} \sum_{i\in I_k} \uparrow M_{k,i}^\perp$) are accessed through N_{l_1} (*resp.* N_{l_2}) and only, and components of $\Gamma_3 = \|_{n_2\leq k\leq n} \uparrow M_{k,i}$ are not accessed. But then Γ_3 must be empty. Indeed taking $x \in \mathscr{C}(\sigma)$ is maximal, then it is $-$-maximal in the game so $\kappa(x) \leq 0$ by Lemma 24 and $\kappa(x) \geq 0$ since σ is exhaustive, so $\kappa(x) = 0$. But then it follows that x is maximal in the game, so Γ_3 must indeed be accessed if non-empty. Then, σ' decomposes as $\sigma_1' : \Gamma_1 \overset{\mathrm{Ga}}{\twoheadrightarrow} N_{l_1}$ and $\sigma_2' : \Gamma_2 \overset{\mathrm{Ga}}{\twoheadrightarrow} N_{l_2}$, yielding

$$\sigma_1 = \mathrm{in}_{l_1} \odot (\downarrow\sigma_1') : \Delta_1 \overset{\mathrm{Ar}}{\twoheadrightarrow} Q_1 \qquad \sigma_2 = \mathrm{in}_{l_2} \odot (\downarrow\sigma_2') : \Delta_2 \overset{\mathrm{Ar}}{\twoheadrightarrow} Q_2$$

(where $\Delta_1 = \otimes_{1\leq k\leq p} \downarrow M_k = \downarrow\Gamma_1$ and $\Delta_2 = \otimes_{p\leq k\leq n} \downarrow M_k = \downarrow\Gamma_2$) such that $\sigma = \sigma_1 \otimes \sigma_2$. From the fact that σ is exhaustive, along with Lemma 24 and the fact that any configuration of σ may be extended to a $-$-maximal one, it follows that σ_1 and σ_2 are exhaustive. □

With that, we can finally wrap up and conclude:

Theorem 27. Arenas *is fully complete for* $\mathsf{MALLP}^\flat$.

Proof. We resume the proof where it was before Lemma 26, *i.e.* we must define

$$\sigma : \bigotimes_{1\leq i\leq n} \downarrow [\![P_i]\!]^\perp \ \overset{\mathrm{Ar}}{\twoheadrightarrow} \ [\![[P]]\!] \,.$$

If $P = 1$ then a $+$-maximal configuration must be neutral on both sides, which is only possible if $\kappa(\{\updownarrow\}) = 0$ on the left. Because the context contains no $\perp$, this in turn is only possible if $n = 0$, but then $\vdash 1$ is provable. If $P = Q_1 \oplus Q_2$ is a coproduct; then after the initial move on the left, σ must either play on Q_1 or on Q_2 (say *e.g.* on Q_1) hence $\sigma = \mathrm{in}_l \circ \sigma'$ with $\sigma' : \bigotimes_{1\leq i\leq n} \downarrow [\![P_i]\!]^\perp \overset{\mathrm{Ar}}{\twoheadrightarrow} [\![Q_1]\!]$. By induction hypothesis σ' may be defined, and we define σ using the introduction rule for $\oplus$. If $P = Q_1 \otimes Q_2$, we apply Lemma 26.

There are two cases left, which have to do with shifts. First, we consider the case where the positive formula is a down-shift, so that we have

$$\sigma : \bigotimes_{1\leq i\leq n} \downarrow M_i \ \overset{\mathrm{Ar}}{\twoheadrightarrow} \ \downarrow N \,.$$

Then, σ is obtained via $\varphi_{\Gamma^\perp,N^\perp,1}$ from $\sigma' : (\bigotimes_{1\leq i\leq n} \downarrow M_i)\otimes N^\perp \overset{\mathrm{Ar}}{\twoheadrightarrow} \neg 1$, which is definable by induction hypothesis. Finally, the last remaining case is that where

$$\sigma : \bigotimes_{1\leq i\leq n} \downarrow M_i \overset{\mathrm{Ar}}{\twoheadrightarrow} \neg 1\,.$$

Necessarily, after the initial negative move on the left σ immediatly plays on the right, and after the subsequent move on 1, by totality it plays on the left again, say *w.l.o.g.* on M_n. Then, removing from (all augmentations in) σ the two moves in $\neg 1$, we get

$$\sigma' : \bigotimes_{1\leq i\leq n-1} \downarrow M_i \overset{\mathrm{Ar}}{\twoheadrightarrow} \downarrow M_n$$

which is definable by induction hypothesis. It follows then that σ is definable as well, obtained via the introduction rule for $\uparrow$. At each step of the definability procedure, the number of connectives in the sequent decreases, ensuring termination. □

Behind the details of this definability procedure lies a very direct geometric correspondence between derivations in $\mathsf{MALLP}^\flat$ (up to natural commutations between rules) and sequential innocent total strategies, akin to the usual correspondence between Böhm trees and innocent strategies in the traditional sense for the simply-typed λ-calculus. This full completeness result makes it appropriate to think of strategies as normal forms for proofs modulo bureaucratic commutations between proof rules. This is also related to Melliès' result that innocent strategies (for a different but related notion of innocence) in asynchronous games form the free dialogue category [52].

The reader will find in Appendix B an extension of this result to MALLP with all units.

4.5 Relational Collapse and Full Completeness for $\mathsf{MALL}^\flat$

Finally we show how to interpret unpolarized MALL in Arenas, describe the relational collapse, and deduce full completeness. Additive units cause no further difficulty here, so we formulate our constructions in their presence even though we will only be able to conclude full completeness for $\mathsf{MALL}^\flat$.

4.5.1 Interpretation of MALL and polarized translation

We first give an interpretation of MALL which, as in Section 3.1, will not be quite sound since it will fail some required equations between proofs.

Remember that the **positive connectives** of MALL are defined as $0, 1, \otimes$ and $\oplus$; while the **negative connectives** are the others. The **polarity** of an MALL formula is defined as the polarity of its outermost connective. As for MALLP, below we denote positive formulas of MALL as P, Q and negative formulas as M, N. To any formula A of MALL we associate two arenas, $(\!|A|\!)_-$ negative and $(\!|A|\!)_+$ positive, mutually inductively with

$$\begin{array}{rclcrcl}
(\!|1|\!)_+ &=& 1 & \qquad & (\!|A \otimes B|\!)_+ &=& (\!|A|\!)_+ \otimes (\!|B|\!)_+ \\
(\!|0|\!)_+ &=& 0 & & (\!|A \oplus B|\!)_+ &=& (\!|A|\!)_+ \oplus (\!|B|\!)_+ \\
(\!|\bot|\!)_- &=& \bot & & (\!|A \parr B|\!)_- &=& (\!|A|\!)_- \parr (\!|B|\!)_- \\
(\!|\top|\!)_- &=& \top & & (\!|A \,\&\, B|\!)_- &=& (\!|A|\!)_- \,\&\, (\!|B|\!)_-
\end{array}$$

along with $(\!|P|\!)_- = \uparrow(\!|P|\!)_+$ and $(\!|N|\!)_+ = \downarrow(\!|N|\!)_-$ to insert the shifts when polarities do not match. This interpretation corresponds to translations $(-)^-$ of MALL formulas as *negative* MALLP formulas, and $(-)^+$ of MALL formulas as *positive* MALLP formulas, followed by the interpretation of MALLP formulas as arenas $[\![-]\!]$ defined in the previous section.

This interpretation can easily be extended to MALL proofs: any proof ϖ of a sequent $\vdash A_1, \dots, A_n$ is interpreted as a negative, sequential innocent, exhaustive and total strategy:

$$(\!|\varpi|\!) : (\!|A_1|\!)_-^\perp \otimes \dots \otimes (\!|A_n|\!)_-^\perp \overset{\mathrm{Ar}}{\twoheadrightarrow} \neg 1\,.$$

It is straightforward to extend this interpretation to all rules of MALL. Altogether, this exactly amounts to the translation of MALL proofs into MALLP proofs described in [54] (along with other Linear Logic connectives). Overall this gives an interpretation of MALL which, however, will not validate all the expected equations between MALL proofs.

4.5.2 Relational Collapse

To restore these missing equations, the final step is to quotient out from the interpretation all the additional behaviour corresponding to the shifts. For that purpose, Melliès' idea in [50] was to quotient the strategies coinciding on certain *stopping positions*, ignoring that they might have taken different routes to reach those positions. The same idea may also be simply presented as a *functorial collapse* to the relational model.

The relational model. The category Rel has as objects *sets*, and as morphisms from A to B *relations* $R \subseteq A \times B$ from A to B. The cartesian product of sets extends to a symmetric monoidal closed structure on Rel. Furthermore Rel is compact closed, with duality being the identity. It has biproducts, given by the disjoint union of sets. Altogether this yields an interpretation of MALL into Rel, defined on formulas with

$$\begin{array}{rcrcl}
[\![0]\!]_{\mathsf{Rel}} &=& [\![\top]\!]_{\mathsf{Rel}} &=& \emptyset \\
[\![1]\!]_{\mathsf{Rel}} &=& [\![\bot]\!]_{\mathsf{Rel}} &=& \{\star\} \\
[\![A \otimes B]\!]_{\mathsf{Rel}} &=& [\![A \parr B]\!]_{\mathsf{Rel}} &=& [\![A]\!]_{\mathsf{Rel}} \times [\![B]\!]_{\mathsf{Rel}} \\
[\![A \oplus B]\!]_{\mathsf{Rel}} &=& [\![A \,\&\, B]\!]_{\mathsf{Rel}} &=& [\![A]\!]_{\mathsf{Rel}} + [\![B]\!]_{\mathsf{Rel}}
\end{array}$$

and extended to proofs following the categorical structure. See *e.g.* [33] for details.

The collapse of games. Now, we have argued earlier in the paper that games being positional meant that they have a clean connection with relational semantics. Intuitively, the relational semantics of a proof records positions reached by completed executions. In contrast, concurrent games record all positions reached by a proof, including intermediary ones matching partial executions[14]. Thus, in principle, it would seem that the correspondence between concurrent games and relational semantics should be rather straightforward: simply forget the intermediary steps, and keep only complete positions.

Following this intuition, to any arena A we associate the set

$$\textstyle\int A = \{x \in \mathscr{C}(A) \mid \kappa_A(x) = 0\}$$

of *exhaustive* configurations. Crucially, this operation is compatible with all constructions used to interpret formulas in Arenas and Rel. For instance, for two positive arenas P, Q, we have seen that configurations with null payoff are exactly those of the form $x_P \otimes x_Q$ with x_P and x_Q of null payoff in P and Q respectively; so $\int(P \otimes Q)$ is isomorphic to the cartesian product $(\int P) \times (\int Q)$ – observe also that *shifts* leave the set of exhaustive configurations invariant, up to isomorphism. Going through all formula constructors, we establish:

Lemma 28. *For any* MALL *formula A, there is an isomorphism* $\int (\!|A|\!) \cong [\![A]\!]_{\mathsf{Rel}}$.

It remains then to extend this collapse operation to strategies.

The collapse of strategies. For $\sigma : A \overset{\mathrm{Ar}}{\twoheadrightarrow} B$, the appropriate definition seems obvious:

$$\textstyle\int \sigma = \{(x_A, x_B) \in \int A \times \int B \quad | \quad x_A \parallel x_B \text{ is } +\text{-maximal in } \mathscr{C}(\sigma)\} \in \mathsf{Rel}(\int A, \int B)\,.$$

It is immediate that $\int(-)$ preserves identities, and almost all constructions on strategies used in the interpretation up to the isomorphism of Lemma 28.

One central property, however, requires some care: *functoriality*. Indeed, to show that $\int(-)$ preserves composition there is a significant obstacle, at least conceptually: composition in Arenas is not pure relational composition. As made explicit in Proposition 11, composition of strategies is relational composition augmented with an additional *reachability* assumption, eliminating synchronized states resulting in *deadlocks*. So we have $\int(\tau \odot \sigma) \subseteq \int \tau \circ \int \sigma$, but it is not clear that the converse equality also holds.

In fact, for general strategies this functoriality property fails – it is easy to construct a situation like that of Figure 4 on arenas arising from the interpretation of formulas of MALL or MALLP. But on that respect, (sequential) innocent strategies are special in that their composition causes *no deadlocks* – this phenomenon, which seems to have been noticed independently by Boudes [15] and Melliès [50], entails the following:

Lemma 29. *For $\sigma : P \overset{\mathrm{Ar}}{\twoheadrightarrow} Q$, $\tau : Q \overset{\mathrm{Ar}}{\twoheadrightarrow} R$ sequential innocent exhaustive strategies,*

$$\textstyle\int(\tau \odot \sigma) = \int \tau \circ \int \sigma$$

[14]This is in contrast with traditional game semantics, that record all *paths* rather than positions.

Using the causal presentation of games, this may be established by analysing cycles arising when computing interactions between sequential innocent strategies, as in Section 3.2.1. By iteratively simplifying such hypothetical cycles, one may prove that they do not exist. The proof does not actually depend on sequential innocence, but on the much weaker property we call *visibility* [20]. This development is too lenghty to appear here, but the interested reader may find a detailed statement and proof as Lemma 5.32 in [16].

Finally, we conclude:

Theorem 30. *There is a functor* $\int(-) : \mathsf{Arenas} \to \mathsf{Rel}$ *preserving interpretation up to iso.*

4.5.3 Full completeness for $\mathsf{MALL}^\flat$

The functor $\int(-)$ induces an equivalence relation on strategies in Arenas, defined as $\sigma \equiv \sigma'$ iff $\int \sigma = \int \sigma'$. Because $\int(-)$ preserves the structure used in the interpretation, it follows that $\equiv$ is a congruence, so we may quotient homsets in Arenas. It remains then to conclude:

Theorem 31. *The interpretation* $(\!|-|\!) : \mathsf{MALL}^\flat \to \mathsf{Arenas}/\!\equiv$ *is fully complete.*

Proof. Although the interpretation $(\!|-|\!)$ into Arenas fails soundness in general, the interpretation in $\mathsf{Arenas}/\!\equiv$ is sound. Moreover, if $\vdash A_1, \ldots, A_n$ is an $\mathsf{MALL}^\flat$ sequent and

$$\sigma : (\!|A_1|\!)^\perp_- \otimes \ldots \otimes (\!|A_n|\!)^\perp_- \overset{\mathrm{Ar}}{\twoheadrightarrow} \neg 1\,,$$

then by Theorem 27, there is a proof ϖ in $\mathsf{MALLP}^\flat$ of the sequent $\vdash A_1^-, \ldots, A_n^-$ such that $[\![\varpi]\!] = \sigma$. Removing shifts in ϖ yields a proof ϖ' of $\vdash A_1, \ldots, A_n$ in $\mathsf{MALL}^\flat$. Finally, the interpretation of $\mathsf{MALL}^\flat$ into Arenas preserves the relational interpretation, *i.e.* $\int (\!|\varpi'|\!) = [\![\varpi']\!]_{\mathsf{Rel}}$, hence $[\![\varpi']\!]_{\mathsf{Rel}} \equiv \sigma$ as required. □

It is worth noting that as the $\mathsf{MALL}^\flat$ proofs coming from definability are obtained by erasing the shifts from $\mathsf{MALLP}^\flat$ proofs, they are *focused* proofs.

5 Conclusion

We hope that this paper will help in making more accessible the work on games models of MALL, starting with Abramsky and Melliès' paper on concurrent games via closure operators. In writing this paper we have attempted to make it as pedagogical and self-contained as possible so that besides telling the story of concurrency and additives, it may also be used as an introduction to concurrent games.

Positionality and causality. As a take-home message, we emphasize once more the *causal* and *positional* nature of deterministic concurrent strategies under their various forms. The *positional* presentation reveals a clear understanding of the similarities – and differences – between game and relational semantics. The *causal* presentation endows strategies with a concrete nature that may be leveraged to capture innocence.

In this paper, we have used the word *causal* to describe the model construction in Section 3.2.1 and *positional* to describe that in Section 3.2.2. It is in our opinion a fundamental, deep property of deterministic concurrent strategies that they enjoy such sharply different presentations. But as in this paper the qualifiers *causal* and *positional* accompany the same model, the reader may wonder to what extent these two are intrinsically related. One element of answer if that beyond the deterministic case, concurrent strategies are defined causally but the purely positional presentation given here does not survive: for a non-deterministic strategy σ on A, the behaviour of σ in a configuration $x \in \mathscr{C}(A)$ may depend on the *path* used to reach x, as this path might constrain the current augmentation (*i.e.* the non-deterministic slice) more than the configuration does.

Nevertheless, one can push the causal presentation way further than the deterministic case. Some constructions of this paper survive, in particular we have recent generalizations of the relational collapse to the *probabilistic* [18] and *quantum* [25] cases.

On sequentiality. In the end, it appears that the ability to express concurrency is not *per se* what allows us to get full completeness: in fact, the interpretation of MALLP that leads to full completeness for MALL is completely sequential. We hope to have convinced the reader that beyond concurrency, the true conceptual advance offered by Abramsky and Melliès' closure-strategies was *positionality*, and that despite their names, the family of *concurrent games* have a lot to offer to the study of *sequential* languages.

Nevertheless, even to study proof systems it is compelling to explore the use of the concurrency offered by the model. For instance, in light of *multifocusing* [24] it would seem natural to seek canonical representations of MALL proofs exploiting the parallelism inherent to concurrent games. The recent work of Castellan and Yoshida [22] goes in that direction, representing dependencies between logical rules in MALL as a disjunctive deterministic strategy [21], but precise connections remain to be explored.

Acknowledgments. The author would like to thank Olivier Laurent and Simon Castellan for numerous discussions on the contents of this paper, and insightful comments on earlier versions of this paper.

This work was supported by ANR project DyVerSe (ANR-19-CE48-0010-01) and Labex MiLyon (ANR-10-LABX-0070) of Universit de Lyon, within the program Investissements dAvenir (ANR-11-IDEX-0007), operated by the French National Research Agency (ANR).

References

[1] Samson Abramsky. Sequentiality vs. concurrency in games and logic. *Mathematical Structures in Computer Science*, 13(4):531–565, 2003.

[2] Samson Abramsky and Radha Jagadeesan. Games and full completeness for multiplicative linear logic (extended abstract). In *Foundations of Software Technology and*

Theoretical Computer Science, 12th Conference, New Delhi, India, December 18-20, 1992, Proceedings, pages 291–301, 1992.

[3] Samson Abramsky, Radha Jagadeesan, and Pasquale Malacaria. Full abstraction for PCF. *Inf. Comput.*, 163(2):409–470, 2000.

[4] Samson Abramsky and Guy McCusker. Call-by-value games. In *Computer Science Logic, 11th International Workshop, CSL '97, Annual Conference of the EACSL, Aarhus, Denmark, August 23-29, 1997, Selected Papers*, pages 1–17, 1997.

[5] Samson Abramsky and Paul-André Melliès. Concurrent games and full completeness. In *14th Annual IEEE Symposium on Logic in Computer Science, Trento, Italy, July 2-5, 1999*, pages 431–442, 1999.

[6] Aurore Alcolei, Pierre Clairambault, Martin Hyland, and Glynn Winskel. The true concurrency of herbrand's theorem. In *27th EACSL Annual Conference on Computer Science Logic, CSL 2018, September 4-7, 2018, Birmingham, UK*, pages 5:1–5:22, 2018.

[7] Aurore Alcolei, Pierre Clairambault, and Olivier Laurent. Resource-tracking concurrent games. In *Foundations of Software Science and Computation Structures - 22nd International Conference, FOSSACS 2019, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2019, Prague, Czech Republic, April 6-11, 2019, Proceedings*, pages 27–44, 2019.

[8] Jean-Marc Andreoli. Logic programming with focusing proofs in linear logic. *J. Log. Comput.*, 2(3):297–347, 1992.

[9] Patrick Baillot. *Approches dynamiques de la logique linéaire: jeux et géométrie de linteraction.* PhD thesis, PhD thesis, Aix-Marseille 2 University, 1999.

[10] Patrick Baillot, Vincent Danos, Thomas Ehrhard, and Laurent Regnier. Believe it or not, ajm's games model is a model of classical linear logic. In *Proceedings, 12th Annual IEEE Symposium on Logic in Computer Science, Warsaw, Poland, June 29 - July 2, 1997*, pages 68–75, 1997.

[11] G. Berry and Pierre-Louis Curien. Sequential algorithms on concrete data structures. *Theor. Comput. Sci.*, 20:265–321, 1982.

[12] Gérard Berry. *Modèles complètement adéquats et stables des λ-calculs typés.* PhD thesis, PhD thesis, Paris 7, 1979.

[13] Andreas Blass. A game semantics for linear logic. *Ann. Pure Appl. Logic*, 56(1-3):183–220, 1992.

[14] Richard Blute, Masahiro Hamano, and Philip J. Scott. Softness of hypercoherences and MALL full completeness. *Ann. Pure Appl. Logic*, 131(1-3):1–63, 2005.

[15] Pierre Boudes. Projecting games on hypercoherences. In *Automata, Languages and Programming: 31st International Colloquium, ICALP 2004, Turku, Finland, July 12-16, 2004. Proceedings*, pages 257–268, 2004.

[16] Simon Castellan. *Concurrent structures in game semantics. (Structures concurrentes en sémantique des jeux)*. PhD thesis, University of Lyon, France, 2017.

[17] Simon Castellan and Pierre Clairambault. Causality vs. interleavings in concurrent game semantics. In *27th International Conference on Concurrency Theory, CONCUR 2016, August 23-26, 2016, Québec City, Canada*, pages 32:1–32:14, 2016.

[18] Simon Castellan, Pierre Clairambault, Hugo Paquet, and Glynn Winskel. The concurrent game semantics of probabilistic PCF. In *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2018, Oxford, UK, July 09-12, 2018*, pages 215–224, 2018.

[19] Simon Castellan, Pierre Clairambault, Silvain Rideau, and Glynn Winskel. Games and strategies as event structures. *Logical Methods in Computer Science*, 13(3), 2017.

[20] Simon Castellan, Pierre Clairambault, and Glynn Winskel. The parallel intensionally fully abstract games model of PCF. In *30th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2015, Kyoto, Japan, July 6-10, 2015*, pages 232–243, 2015.

[21] Simon Castellan, Pierre Clairambault, and Glynn Winskel. Observably deterministic concurrent strategies and intensional full abstraction for parallel-or. In *2nd International Conference on Formal Structures for Computation and Deduction, FSCD 2017, September 3-9, 2017, Oxford, UK*, pages 12:1–12:16, 2017.

[22] Simon Castellan and Nobuko Yoshida. Causality in linear logic - full completeness and injectivity (unit-free multiplicative-additive fragment). In *Foundations of Software Science and Computation Structures - 22nd International Conference, FOSSACS 2019, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2019, Prague, Czech Republic, April 6-11, 2019, Proceedings*, pages 150–168, 2019.

[23] Simon Castellan and Nobuko Yoshida. Two sides of the same coin: session types and game semantics: a synchronous side and an asynchronous side. *PACMPL*, 3(POPL):27:1–27:29, 2019.

[24] Kaustuv Chaudhuri, Dale Miller, and Alexis Saurin. Canonical sequent proofs via multi-focusing. In *Fifth IFIP International Conference On Theoretical Computer Science - TCS 2008, IFIP 20th World Computer Congress, TC 1, Foundations of Computer Science, September 7-10, 2008, Milano, Italy*, pages 383–396, 2008.

[25] Pierre Clairambault and Marc de Visme. Full abstraction for the quantum λ-calculus. *PACMPL*, 4(POPL), 2020.

[26] Pierre Clairambault, Marc de Visme, and Glynn Winskel. Game semantics for quantum programming. *PACMPL*, 3(POPL):32:1–32:29, 2019.

[27] Pierre Clairambault, Julian Gutierrez, and Glynn Winskel. The winning ways of concurrent games. In *Proceedings of the 27th Annual IEEE Symposium on Logic in Computer Science, LICS 2012, Dubrovnik, Croatia, June 25-28, 2012*, pages 235–244, 2012.

[28] Pierre Clairambault and Russ Harmer. Totality in arena games. *Ann. Pure Appl. Logic*, 161(5):673–689, 2010.

[29] J Robin B Cockett and Robert AG Seely. Weakly distributive categories. *Journal of Pure and Applied Algebra*, 114(2):133–173, 1997.

[30] John H. Conway. *On numbers and games, Second Edition*. A K Peters, 2001.

[31] Thierry Coquand. A semantics of evidence for classical arithmetic. *J. Symb. Log.*, 60(1):325–337, 1995.

[32] Thomas David Cuvillier. *Nominal Models of Linear Logic*. PhD thesis, Queen Mary University of London.

[33] Thomas Ehrhard. The Scott model of linear logic is the extensional collapse of its relational model. *Theor. Comput. Sci.*, 424:20–45, 2012.

[34] Claudia Faggian and Mauro Piccolo. Partial orders, event structures and linear strategies. In *Typed Lambda Calculi and Applications, 9th International Conference, TLCA 2009, Brasilia, Brazil, July 1-3, 2009. Proceedings*, pages 95–111, 2009.

[35] Dan R. Ghica and Andrzej S. Murawski. Angelic semantics of fine-grained concurrency. *Ann. Pure Appl. Logic*, 151(2-3):89–114, 2008.

[36] Jean-Yves Girard. Linear logic. *Theor. Comput. Sci.*, 50:1–102, 1987.

[37] Jean-Yves Girard. Geometry of interaction 1: Interpretation of system F. In *Studies in Logic and the Foundations of Mathematics*, volume 127, pages 221–260. Elsevier, 1989.

[38] Jean-Yves Girard. A new constructive logic: Classical logic. *Mathematical Structures in Computer Science*, 1(3):255–296, 1991.

[39] Jean-Yves Girard. Locus solum: From the rules of logic to the logic of rules. *Mathematical Structures in Computer Science*, 11(3):301–506, 2001.

[40] Jean-Yves Girard. Proof-nets: the parallel syntax for proof-theory. In *Logic and Algebra*, pages 97–124. Routledge, 2017.

[41] J. M. E. Hyland and C.-H. Luke Ong. On full abstraction for PCF: I, II, and III. *Inf. Comput.*, 163(2):285–408, 2000.

[42] J Martin E Hyland and C-H Luke Ong. Fair games and full completeness for multiplicative linear logic without the mix-rule. 1993.

[43] Martin Hyland. Game semantics. *Semantics and logics of computation*, 14:131, 1997.

[44] Martin Hyland and Andrea Schalk. Glueing and orthogonality for models of linear logic. *Theor. Comput. Sci.*, 294(1/2):183–231, 2003.

[45] André Joyal. Remarques sur la théorie des jeux à deux personnes. *Gazette des sciences mathématiques du Québec*, 1(4):46–52, 1977.

[46] François Lamarche. Sequentiality, games and linear logic, 1992.

[47] Olivier Laurent. A proof of the focusing property of linear logic. Available at `https://perso.ens-lyon.fr/olivier.laurent/llfoc2.pdf`.

[48] Olivier Laurent. *Étude de la polarisation en logique, Université Aix-Marseille II*. PhD thesis, Thèse de Doctorat, 2002.

[49] Guy McCusker. *Games and full abstraction for a functional metalanguage with recursive types*. CPHC/BCS distinguished dissertations. Springer, 1998.

[50] Paul-André Melliès. Asynchronous games 4: A fully complete model of propositional linear logic. In *20th IEEE Symposium on Logic in Computer Science (LICS 2005), 26-29 June 2005, Chicago, IL, USA, Proceedings*, pages 386–395, 2005.

[51] Paul-André Melliès. Sequential algorithms and strongly stable functions. *Theor. Comput. Sci.*, 343(1-2):237–281, 2005.

[52] Paul-André Melliès. Game semantics in string diagrams. In *Proceedings of the 27th Annual IEEE Symposium on Logic in Computer Science, LICS 2012, Dubrovnik, Croatia, June 25-28, 2012*, pages 481–490, 2012.

[53] Paul-André Melliès and Samuel Mimram. Asynchronous games: Innocence without alternation. In *CONCUR 2007 - Concurrency Theory, 18th International Conference, CONCUR 2007, Lisbon, Portugal, September 3-8, 2007, Proceedings*, pages 395–411, 2007.

[54] Paul-André Melliès and Nicolas Tabareau. Resource modalities in tensor logic. *Ann. Pure Appl. Logic*, 161(5):632–653, 2010.

[55] Robin Milner. Fully abstract models of typed λ-calculi. *Theor. Comput. Sci.*, 4(1):1–22, 1977.

[56] Samuel Mimram. *Sémantique des jeux asynchrones et réécriture 2-dimensionnelle. (Asynchronous Game Semantics and 2-dimensional Rewriting Systems).* PhD thesis, Paris Diderot University, France, 2008.

[57] Andrzej S. Murawski and C.-H. Luke Ong. Exhausting strategies, joker games and full completeness for IMLL with unit. *Theor. Comput. Sci.*, 294(1/2):269–305, 2003.

[58] Arend Rensink. Posets for configurations! In *CONCUR '92, Third International Conference on Concurrency Theory, Stony Brook, NY, USA, August 24-27, 1992, Proceedings*, pages 269–285, 1992.

[59] Silvain Rideau and Glynn Winskel. Concurrent strategies. In *Proceedings of the 26th Annual IEEE Symposium on Logic in Computer Science, LICS 2011, June 21-24, 2011, Toronto, Ontario, Canada*, pages 409–418, 2011.

[60] Ken Sakayori and Takeshi Tsukada. A truly concurrent game model of the asynchronous π-calculus. In *Foundations of Software Science and Computation Structures - 20th International Conference, FOSSACS 2017, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2017, Uppsala, Sweden, April 22-29, 2017, Proceedings*, pages 389–406, 2017.

[61] Glynn Winskel. Event structures. In *Petri Nets: Central Models and Their Properties, Advances in Petri Nets 1986, Part II, Proceedings of an Advanced Course, Bad Honnef, Germany, 8-19 September 1986*, pages 325–392, 1986.

[62] Glynn Winskel. Prime algebraicity. *Theor. Comput. Sci.*, 410(41):4160–4168, 2009.

[63] Glynn Winskel. Winning, losing and drawing in concurrent games with perfect or imperfect information. In *Logic and Program Semantics - Essays Dedicated to Dexter Kozen on the Occasion of His 60th Birthday*, pages 298–317, 2012.

A Other Closure Operators From Strategies

In Section 3.3, we have studied the transformation of deterministic concurrent strategies into closure-strategies. As pointed out in the introduction, this tranformation is new. In this first appendix we review some tempting alternative definitions, that have been considered in the literature [53, 59].

A.1 Intersection of +-maximal configurations

Firstly, recall from Section 2.3 that for a domain D, the identity $\text{æ}_D : D \to\!\!\!\!\!+\!\!\!\!\to D$ is defined as $\text{æ}_D(x, y) = (x \vee y, x \vee y)$. As already pointed out in Section 3.3.2, if D is the domain of configurations of a game, this has the puzzling consequence that æ_D may actually play Opponent moves and not just Player moves. For instance, if A is the game with just one Player move $\bullet$, then as observed in Section 3.3.2, we have $\text{æ}_{\mathscr{C}^\infty(A)}(\emptyset, \{\bullet\}) = (\{\circ\}, \{\bullet\})$ – in other words, applying $\text{æ}_{\mathscr{C}^\infty(A)}$ has the effect of adding the missing Opponent dependency to an already present Player move $\bullet$. This invites the following definition [53][15]:

Definition 32. *If $\sigma : A$ is a strategy and $x \in \mathscr{C}^\infty(A)$, then we set*

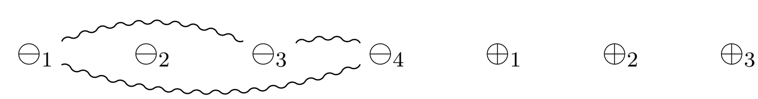
$$\mathsf{C}'(\sigma)(x) = \bigwedge\{y \in \mathscr{C}^\infty(\sigma) \mid y \textit{ is } +\textit{-maximal} \ \wedge \ x \subseteq y\} \in \mathscr{C}^\infty(A)^\top$$

Given $x \in \mathscr{C}^\infty(A)$, if there is no +-maximal $y \in \mathscr{C}^\infty(\sigma)$ such that $x \subseteq y$, then this definition yields $\mathsf{C}'(\sigma)(x) = \top$, otherwise it is their intersection. In other words, applied to a (possibly infinite) configuration $x \in \mathscr{C}^\infty(A)$, $\mathsf{C}'(\sigma)$ adds all moves that are known to appear in all +-maximal configurations containing x. This includes of course the Player events enabled by Opponent events in x, but also all Opponent events that are necessary requirements for the Player events already present in x.

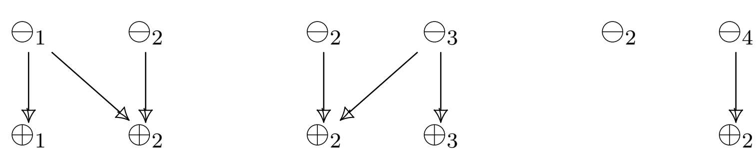
This definition is tempting, because it is analogous to the copycat closure-strategy: for any game A, it is apparent that $\mathsf{C}'(\text{æ}_A) = \text{æ}_{\mathscr{C}^\infty(A)}$. However, unfortunately it does not in general give a closure-strategy. We give below counter-examples to stability and continuity.

Example 33 (Non-stability)*.* Consider the game

A =



and the deterministic concurrent strategy $\sigma : A$ having maximal augmentations



[15] The formal setting differs superficially and a detailed proof of equivalence is out of scope of this paper, but we have checked that the problem described here also occurs in [53].

yielding, by Definition 32,

$$\begin{array}{rcl} \mathsf{C}'(\sigma)(\{\oplus_1, \oplus_2\}) & = & \{\oplus_1, \oplus_2\} \cup \{\ominus_1, \ominus_2\} \\ \mathsf{C}'(\sigma)(\{\oplus_2, \oplus_3\}) & = & \{\oplus_2, \oplus_3\} \cup \{\ominus_2, \ominus_3\} \,. \end{array}$$

But $\{\oplus_1, \oplus_2\}$ and $\{\oplus_2, \oplus_3\}$ are compatible, since $\{\oplus_1, \oplus_2, \oplus_3\} \in \mathscr{C}(A)$. Therefore, the stability condition of closure-strategies entails that we should have

$$\mathsf{C}'(\sigma)(\{\oplus_2\}) = \{\oplus_2\} \cup \{\ominus_2\} \,.$$

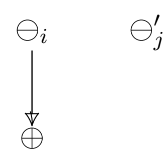
However, this is false: instead we have $\mathsf{C}'(\sigma)(\{\oplus_2\}) = \{\oplus_2\}$ since there is a $+$-maximal configuration of σ, namely $\{\ominus_4, \oplus_2\}$ which does not contain $\ominus_2$.

Note that [53] did not claim stability. We now also give a counter-example to continuity.

Example 34 (Non-continuity). Consider the game A having as events

$$\{\ominus_i \mid i \geq 0\} \cup \{\ominus'_j \geq j \geq 1\} \cup \{\oplus\} \,.$$

Causality is trivial, comprising only reflexive pairs. Minimal conflicts are described by $\ominus_i \sim\!\sim \ominus_j$ for all $i \neq j$, and $\ominus_i \sim\!\sim \ominus'_i$ for all $i \geq 1$. We then consider $\sigma : A$ the deterministic concurrent strategy defined with maximal augmentations of the form



for $i \neq j$. Then, for all $n \geq 1$, we have

$$\mathsf{C}'(\sigma)(\{\oplus, \ominus'_1, \dots, \ominus'_n\}) = \{\oplus, \ominus'_1, \dots, \ominus'_n\} :$$

no new event is added, because there are still many mutually inconsistent possible causal histories for $\oplus$. By continuity, we should therefore also have $\mathsf{C}'(\sigma)(\{\oplus\} \cup \{\ominus'_j \mid j \geq 1\}) = \{\oplus\} \cup \{\ominus'_j \mid j \geq 1\}$. However, instead we have

$$\mathsf{C}'(\sigma)(\{\oplus\} \cup \{\ominus'_j \mid j \geq 1\}) = \{\oplus\} \cup \{\ominus'_j \mid j \geq 1\} \cup \{\ominus_0\} \,.$$

Indeed, any $+$-maximal configuration of σ which includes $\{\ominus'_j \mid j \geq 1\}$ must also contain $\ominus_0$: it is the only possible cause left for $\oplus$, and is therefore included by the definition.

Both of these pathologies boil down to the fact that the configurations of a deterministic concurrent strategy $\sigma : A$ are not in general closed under intersection; unless we assume that there is no conflict between Opponent events in the game. It is noteworthy that despite these, the reachable fixpoints of $\mathsf{C}'(\sigma)$ are always the same as those of $\mathsf{C}(\sigma)$.

A.2 Least +-maximal configuration

Finally, we mention a variation of Definition 32 that also appears in the literature [59].

Definition 35. *If $\sigma : A$ is a deterministic concurrent strategy and $x \in \mathscr{C}^\infty(A)$, we set $\mathsf{C}''(\sigma)(x)$ as the* least *+-maximal $y \in \mathscr{C}^\infty(\sigma)$* s.t. *$x \subseteq y$, if such exists, and $\top$ otherwise.*

The difference with respect to Definition 32 is that we consider the least +-maximal configuration containing x rather than their intersection. This may be tempting, because it ensures that for all $x \in \mathscr{C}^\infty(\sigma)$, we always have $\mathsf{C}''(\sigma)(x) \in \mathscr{C}^\infty(\sigma)$ unless $\mathsf{C}''(\sigma)(x) = \top$ – this natural property is satisfied by neither in the definition of Proposition 13 not in Definition 32. However, this definition unfortunately fails monotonicity.

Example 36 (Non-monotonicity). Consider the game $A = \ominus_1 \sim\sim \ominus_2 \quad \oplus$ and $\sigma : A$ with maximal augmentations $\ominus_1 \rightarrowtriangle \oplus$ and $\ominus_2 \rightarrowtriangle \oplus$. Then,

$$
\begin{array}{rcl}
\mathsf{C}''(\sigma)(\{\oplus\}) & = & \top \\
\mathsf{C}''(\sigma)(\{\oplus, \ominus_1\}) & = & \{\oplus, \ominus_1\}\,,
\end{array}
$$

failing monotonicity of $\mathsf{C}''(\sigma)$. Indeed, there are two incomparable +-maximal $y_1, y_2 \in \mathscr{C}^\infty(\sigma)$ such that $\{\oplus\} \subseteq y_1, y_2$, so there is no least one, leading to $\top$.

Although it does not give a closure-strategy in general, it is noteworthy that $\mathsf{C}''(\sigma)$ also has the same *reachable* fixpoints as $\mathsf{C}(\sigma)$ and $\mathsf{C}'(\sigma)$.

B Full Completeness For MALLP

In this final section, we show how to refine the fully complete model for $\mathsf{MALLP}^\flat$ of Section 4.3 into a fully complete model for MALLP. This comes with significant technical complications in order to deal adequately with the additive units.

First, payoff is extended to additive units by setting $\kappa_0(\emptyset) = -1$, and dually, $\kappa_\top(\emptyset) = 1$. As pointed out in the text, this breaks Lemma 24 which was useful to prove definability for $\mathsf{MALLP}^\flat$, but the interpretation itself still works out, yielding for every proof a total, sequential innocent, and exhaustive strategy. We will shortly see, however, that those conditions are not enough for definability in the presence of additive units.

B.1 Locally winning strategies

Unlike multiplicative units, additive units allow a proof to leave parts of the arena unexplored. Indeed, any sequent $\vdash \Gamma, \top$ is provable by the $\top$ rule, yielding a strategy that will never visit Γ – *garbage-collects* it. This is captured by the notion of exhaustivity in the presence of additive units: an exhaustive strategy σ may garbage-collect part of the context, provided σ is able to uncover a unit $\top$ ensuring that the global payoff is 1.

However, for definability we must ensure that the uncovered $\top$ belongs to a component that will "stay with" the garbage-collected context during the inductive definability

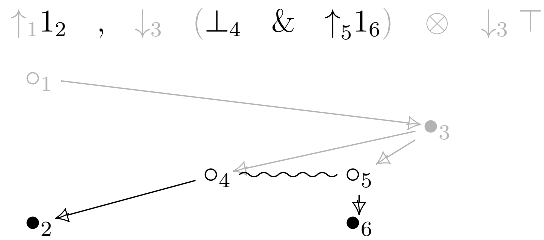


Figure 9: A winning, non-decomposable strategy

process. Unfortunately, this is not ensured by exhaustivity: we show in Figure 9 a total, sequential innocent and exhaustive strategy failing definability. The figure displays a strategy (call it σ) playing on the game for $\uparrow 1, \downarrow(\bot \& \uparrow 1) \otimes \downarrow \top$ – indices are added to emphasize the correspondence between moves and formula components. The strategy satisfies all of our conditions, even though the sequent is not provable in MALLP. Attempting to apply the definability process, one must decompose σ as a tensor of two strategies. The only way forward is defining σ' with the same moves as σ, but on sequent $\uparrow 1, \downarrow(\bot \& \uparrow 1)$. But σ' is not exhaustive anymore – the configuration $\{\circ_1, \bullet_3, \circ_5, \bullet_6\}$, which had payoff 1 in σ thanks to the presence of the $\top$ allowing us to leave part of the context unexplored, is now losing.

Definability arguments in game semantics require "good" (*i.e.* satisfying all the imposed conditions) strategies to be stable under *decomposition*, in the sense that strategies obtained by decomposing good strategies should be good. This property, which is usually for free, fails here due to the non-local behaviour of additive units. It is precisely to deal with this issue that Melliès considers in [50] a payoff for *walks* on strategies rather than simply positions. Rather than reproduce Melliès' construction, we give a variant of the mechanism, which we believe to be more explicit. Our condition, called *local exhaustivity*, expresses that "σ is exhaustive on all sub-games". To express it, we first need to enrich payoffs so that they also assign valuations to configurations on sub-games.

Definition 37. *If A is an arena and $x \in \mathscr{C}(A)$, a **sub-arena** of A is a subset $X \subseteq A$ which is up-closed for $\leq_A$, and such that there is $x \in \mathscr{C}(A)$ such that all minimal events of X are enabled in x. A **local payoff** on A consists in functions*

$$\kappa_A^X : \mathscr{C}(X) \to \{-1, 0, 1\}$$

for any sub-arena X, where X inherits from A the components of an event structure. Furthermore, those satisfy the additional properties that (1) if $X = \emptyset$, then $\kappa_A^{\emptyset}(\emptyset) = 0$; and (2) if $X \neq \emptyset$ and its minimal events are minimal in A, then $\kappa_A^X(y) = \kappa_A(y)$.

For instance, in Figure 9, the set $\{\bullet_2, \circ_4, \circ_5, \bullet_6\}$ is a sub-arena.

Besides being exhaustive globally, strategies must also be exhaustive locally in the sub-arenas they reach. If $\sigma : A$ is a strategy, $x \in \mathscr{C}(\sigma)$ is $+$-maximal, and we have $x \subseteq x_1, \ldots, x_n \in \mathscr{C}(\sigma)$ distinct configurations that are also $+$-maximal, then we write

$$\lceil x_1, \ldots, x_n \rceil_x = \{a \in A \mid \exists a' \in x_1 \cup \cdots \cup x_n,\ a' \leq_A a\} \setminus x$$

for the up-closure of $x_1, \ldots, x_n$ with x removed. By construction, it is a sub-arena of A. We then ask that each x_i is then exhaustive, localized to this sub-arena.

Definition 38. *A strategy $\sigma : A$ is* ***locally exhaustive*** *iff for all $x, x_1, \ldots, x_n \in \mathscr{C}(\sigma)$ which are all $+$-maximal and such that $x \subseteq x_1, \ldots, x_n$, for all $1 \leq i \leq n$, we have*

$$\kappa_A^{\lceil x_1, \ldots, x_n \rceil x}(x_i \setminus x) \geq 0\,.$$

We extend arena constructions with local payoff. For units, the local payoff is forced by the conditions. For $A \oplus B$, if a sub-arena X is entirely included in A and $y \in \mathscr{C}(X)$, we set $\kappa^X_{A\oplus B}(y) = \kappa^X_A(y)$ and likewise for B. If X has components in A and B, then its minimal events are necessarily minimal in A and B. We then set $\kappa^X_{A\oplus B}(y) = \kappa_{A\oplus B}(y)$. For $A \otimes B$, if X is empty then $\kappa^X_{A\otimes B}$ is forced by condition *(1)*. If its minimal events are minimal in $A \otimes B$ then $\kappa^X_{A\otimes B}$ is forced by condition *(2)*. Otherwise, X decomposes into X_A a sub-arena of A and X_B a sub-arena of B. Likewise, if $y \in \mathscr{C}(X)$, it decomposes into $y_A \in \mathscr{C}(X_A)$ and $y_B \in \mathscr{C}(X_B)$. We then set $\kappa^X_{A\otimes B}(y) = \kappa^{X_A}_A(y_A) \otimes \kappa^{X_B}_B(y_B)$. For $\downarrow N$, either $X = \downarrow N$ in which case $\kappa^X_{\downarrow N} = \kappa_{\downarrow N}$ by condition *(2)*, or X is a sub-arena of N, and we set $\kappa^X_{\downarrow N}(y) = \kappa^X_N(y)$. Other cases follow by duality, with $\kappa^X_{A^\perp}(y) = -\kappa^X_A(y)$.

Example 39. The strategy of Figure 9 is not locally exhaustive: we have $+$-maximal

$$\{\circ_1, \bullet_3\} \subseteq \{\circ_1, \bullet_3, \circ_4, \bullet_2\}, \{\circ_1, \bullet_3, \circ_5, \bullet_6\}$$

inducing the reachable sub-game $\{\circ_4, \bullet_2, \circ_5, \bullet_6\}$, which corresponds to the part of Figure 9 which is not grayed out. But then, the configuration $\{\circ_5, \bullet_6\}$ fails to be exhaustive:

$$\begin{array}{rcl}
\kappa^{\{\circ_4,\bullet_2,\circ_5,\bullet_6\}}_{(\uparrow_1 1_2)⅋(\downarrow_3(\perp_4 \& \uparrow_5 1_6)\otimes(\downarrow_3 \top))}(\{\circ_5, \bullet_6\}) & = & \kappa^{\{\bullet_2\}}_{\uparrow_1 1_2}(\emptyset)\ ⅋\ (\kappa^{\{\circ_4,\circ_5,\bullet_6\}}_{\perp_4 \& \uparrow_5 1_6}(\{\circ_5, \bullet_6\}) \otimes \kappa^{\emptyset}_{\downarrow_3 \top}(\emptyset))\\
& = & \kappa^{\{\bullet_2\}}_{1_2}(\emptyset)\ ⅋\ (\kappa_{\uparrow_5 1_6}(\{\circ_5, \bullet_6\}) \otimes \kappa^{\emptyset}_{\downarrow_3 \top}(\emptyset))\\
& = & \kappa_{1_2}(\emptyset)\ ⅋\ (\kappa_{\uparrow_5 1_6}(\{\circ_5, \bullet_6\}) \otimes \kappa^{\emptyset}_{\downarrow_3 \top}(\emptyset))\\
& = & -1\ ⅋\ (0 \otimes 0)\\
& = & -1\,.
\end{array}$$

There is a category $\mathsf{LocExAr}$ having as objects arenas with local payoff, and as morphisms total, sequential innocent strategies that are both exhaustive and locally exhaustive. Furthermore, $\mathsf{LocExAr}$ inherits from Arenas the structure of a dialogue category with coproducts, supporting the interpretation of MALLP.

B.2 Definability and Full Completeness

We now prove full completeness. From now on, *strategies* are always assumed to satisfy sequential innocence, totality, exhaustivity and locally exhaustivity.

With respect to the proof of definability of Section 4.4, the only difference is the decomposition of a tensor, which requires local exhaustivity in the presence of additive units.

Lemma 40. *Let $(P_k)_{1\leq k\leq n}, Q_1, Q_2$ be arenas, and consider*

$$\sigma : \bigotimes_{1\leq i\leq n} \downarrow P_k^\perp \stackrel{\mathrm{Ar}}{\twoheadrightarrow} Q_1 \otimes Q_2$$

a morphism in $\mathsf{LocExAr}$. *Then, up to reordering of the context there are strategies*

$$\sigma_1 : \bigotimes_{1\leq k\leq p} \downarrow P_k^\perp \stackrel{\mathrm{Ar}}{\twoheadrightarrow} Q_1 \qquad\qquad \sigma_2 : \bigotimes_{p+1\leq k\leq n} \downarrow P_k^\perp \stackrel{\mathrm{Ar}}{\twoheadrightarrow} Q_2$$

such that $\sigma = \sigma_1 \otimes \sigma_2$.

Proof. At first the proof proceeds as in Lemma 26. We first extract

$$\sigma_1 : \Delta_1 \stackrel{\mathrm{Ar}}{\twoheadrightarrow} Q_1 \qquad\qquad \sigma_2 : \Delta_2 \stackrel{\mathrm{Ar}}{\twoheadrightarrow} Q_2$$

as in the proof of Lemma 26, using the same notations. It follows easily that σ_1 and σ_2 are locally exhaustive from the fact that σ is locally exhaustive. We first consider the case where Γ_3 is empty in the construction of σ_1 and σ_2, *i.e.* $\Delta_1 \otimes \Delta_2 = \Delta = \bigotimes_{1\leq i\leq n} \downarrow P_k^\perp$.

The main novelty is that exploiting local exhaustivity, we may show that σ_1 and σ_2 are exhaustive as well. Indeed, take $x \in \mathscr{C}(\sigma_1)$ +-maximal, and consider X the set of non-empty +-maximal configurations of σ_1 – necessarily, $x \in X$. Leaving the renaming implicit, we regard X as a set of +-maximal configurations of σ. Moreover, for all $y \in X$ we have $x_0 = \{\circ, \bullet_{(l_1,l_2)}\} \subseteq y$ where $\circ$ and $\bullet_{(l_1,l_2)}$ are the initial two moves of σ. We may then use that σ is locally exhaustive, and obtain

$$\kappa^{\lceil X\rceil_{x_0}}_{(\Delta_1\otimes\Delta_2)^\perp \parr (Q_1\otimes Q_2)}(x \setminus x_0) \geq 0$$

where $\lceil X\rceil_{x_0}$ is a sub-arena of $(\Delta_1 \otimes \Delta_2)^\perp \parr (Q_1 \otimes Q_2)$. But by construction of X, this sub-arena contains no move in Δ_2 and Q_2, so it is a sub-arena $X_l \parallel X_r$ of $\Delta_1^\perp \parr Q_1$, where X_l is a sub-arena of Δ_1 and X_r is a sub-arena of Q_1. We compute:

$$\begin{aligned}
\kappa^{\lceil X\rceil_{x_0}}_{(\Delta_1\otimes\Delta_2)^\perp \parr (Q_1\otimes Q_2)}(x \setminus x_0) &= \kappa^{X_l\parallel X_r}_{(\Delta_1\otimes\Delta_2)^\perp \parr (Q_1\otimes Q_2)}(x_l \parallel x_r)\\
&= \kappa^{X_l}_{(\Delta_1\otimes\Delta_2)^\perp}(x_l) \parr \kappa^{X_r}_{(Q_1\otimes Q_2)}(x_r)\\
&= (\kappa^{X_l}_{\Delta_1^\perp}(x_l) \parr 0) \parr (\kappa^{X_r}_{Q_1}(x_r) \otimes 0)\\
&= \kappa^{X_l}_{\Delta_1^\perp}(x_l) \parr \kappa^{X_r}_{Q_1}(x_r)
\end{aligned}$$

where we have used that X_l is entirely in Δ_1 and X_r in Q_1, so the local payoffs in Δ_2 and Q_2 are null by condition *(1)* of Definition 37. But now, recall that:

$$\Delta_1 = \bigotimes_{1\leq k\leq p} \downarrow M_k \qquad\qquad Q_1 = \sum_{l_1\in L_1} \downarrow N_{l_1}\,.$$

Since σ is total, after $\circ, \bullet_{(l_1,l_2)}$ it has a response to any of the minimal events of N_{l_1}. So, each minimal event of N_{l_1} appears in at least one +-maximal configuration of σ_1, thus

$X_r = N_{l_1}$. Likewise, recall that for each $1 \leq k \leq p$, we have $M_k = \sum_{i \in I_k} \uparrow M_{k,i}^{\perp}$. Recall that Δ_1 was constructed by selecting those components M_k that were accessed by an augmentation with minimal negative event (after $\circ, \bullet_{(l_1,l_2)}$) in N_{l_1}. Therefore, X_l comprises at least one of the $\uparrow M_{k,i}^{\perp}$ for each $1 \leq k \leq p$. From these two observations, it follows directly by induction on Δ_1 and Q_1 and condition *(2)* of Definition 37 that $\kappa_{\Delta_1^{\perp}}^{X_l}(x_l) = \kappa_{\Delta_1^{\perp}}(\{\circ\} \cup x_l)$ and $\kappa_{Q_1}^{X_r}(x_r) = \kappa_{Q_1}(\{\bullet_{l_1}\} \cup x_r)$ so

$$\kappa_{(\Delta_1 \otimes \Delta_2)^{\perp} \parr (Q_1 \otimes Q_2)}^{\lceil X \rceil_{x_0}}(x \setminus x_0) = \kappa_{\Delta_1^{\perp} \parr Q_1}(x)$$

which is therefore positive, as required. Likewise, σ_2 is exhaustive as well.

In the proof of Lemma 26, we proved that Γ_3 must always be empty. In the presence of additive units, that is of course no longer true. We prove that in this case as well, σ_1 and σ_2 are still exhaustive. Consider $x \in \mathscr{C}(\sigma)$ non-empty and $+$-maximal, and write $x = \{\circ\} \cup x_l \parallel \{\bullet_{(l_1,l_2)}\} \cup (x_{l_1} \parallel x_{l_2})$. Because Γ_3 is not explored, we have $\kappa_{\Delta^{\perp}}(\{\circ\} \cup x_l) = -1$, so we must have $\kappa_{Q_1 \otimes Q_2}(\{\bullet_{(l_1,l_2)}\} \cup (x_{l_1} \parallel x_{l_2})) = 1$ to compensate, so that $\kappa_{Q_1}(\{\bullet_{l_1}\} \cup x_{l_1}) = 1$ or $\kappa_{Q_2}(\{\bullet_{l_2}\} \cup x_{l_2}) = 1$. But in fact there must be a side, Q_1 or Q_2, that always has payoff 1 independently of x. Indeed say we have $\kappa_{Q_1}(x_{l_1}) = 0$ and $\kappa_{Q_2}(x'_{l_2}) = 0$ where

$$\{\circ\} \cup x_l \parallel \{\bullet_{(l_1,l_2)}\} \cup (x_{l_1} \parallel x_{l_2}) \in \mathscr{C}(\sigma) \qquad \{\circ\} \cup x'_l \parallel \{\bullet_{(l_1,l_2)}\} \cup (x'_{l_1} \parallel x'_{l_2}) \in \mathscr{C}(\sigma)$$

These configurations are images of a unique augmentation, so each move in x_l depends either on x_{l_1} or on x_{l_2}, and likewise for x'. So the two configurations above admit as subsets $+$-maximal configurations

$$\{\circ\} \cup y_l^1 \parallel \{\bullet_{(l_1,l_2)}\} \cup (x_{l_1} \parallel \emptyset) \in \mathscr{C}(\sigma) \qquad \{\circ\} \cup y_l^2 \parallel \{\bullet_{(l_1,l_2)}\} \cup (\emptyset \parallel x_{l_2}) \in \mathscr{C}(\sigma)\,.$$

By determinism, we may now take their union

$$\{\circ\} \cup (y_l^1 \cup y_l^2) \parallel \{\bullet_{(l_1,l_2)}\} \cup (x_{l_1} \parallel x'_{l_2}) \in \mathscr{C}(\sigma)$$

which by construction has payoff -1. So, there is $i \in \{1,2\}$ so that for all $x \in \mathscr{C}(\sigma)$ non-empty $+$-maximal, we have $\kappa_{Q_i}(x_{l_i}) = 1$. Say *w.l.o.g.* that it is $i = 2$. Then, we form

$$\sigma_1 : \Delta_1^{\perp} \overset{\mathrm{Ar}}{\twoheadrightarrow} Q_1 \qquad \sigma_2 : (\Delta_2 \otimes \Delta_3)^{\perp} \overset{\mathrm{Ar}}{\twoheadrightarrow} Q_2$$

as previously, but with a larger domain for σ_2. By construction we have $\sigma = \sigma_1 \otimes \sigma_2$, and σ_2 is exhaustive by construction. The proof that σ_1 and σ_2 satisfy all the required conditions is as in the case above with Γ_3 empty. □

With that, we can now complete the proof of:

Theorem 41. LocExAr *is fully complete for* MALLP.

Proof. The rest of the proof is as in Theorem 27. □

From there, the exact same construction as in Section 4.5 can be applied in order to get a fully complete model for MALL with all units. We omit the details, which are unchanged.