On the dualization problem in graphs, hypergraphs, and lattices

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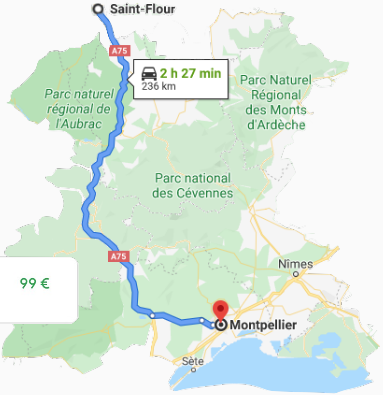
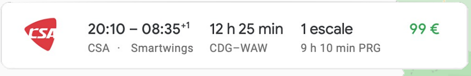
Typical question:

Given input I, find the best solution
from all feasible solutions of I.

Examples:

- shortest path to Montpellier
- cheapest flight to Warsaw
- best answer to a query
- ...

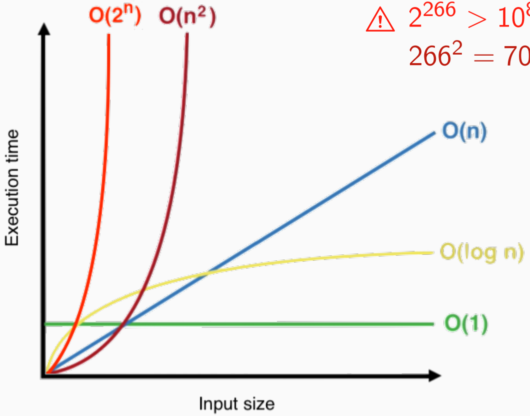




https://www.maps.google.fr

Let n be input size, e.g., number of roads in the network

Efficient algorithm : runs in poly(n)-time



⚠ $2^{266} > 10^{80}$

$266^2 = 70756$

**based on Daniel Ko's chart*

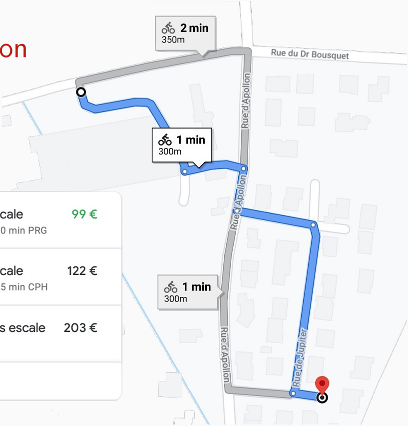
https://medium.com/@dankomong/big-o-notation-using-ruby-a357d85bb9b1

Typical question:

Given input I, list all solutions in I.

Examples:

- bike itineraries to a destination
- flights to Warsaw
- answers to a query
- ...



CSA	20:10 – 08:35+1 CSA · Smartwings	12 h 25 min CDG–WAW	1 escale 9 h 10 min PRG	99 €
SAS	14:30 – 22:30 SAS	8 h 0 min CDG–WAW	1 escale 4 h 55 min CPH	122 €
AF	13:00 – 15:20 Air France	2 h 20 min CDG–WAW	Sans escale	203 €
	114 autres vols			

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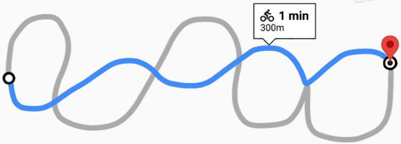
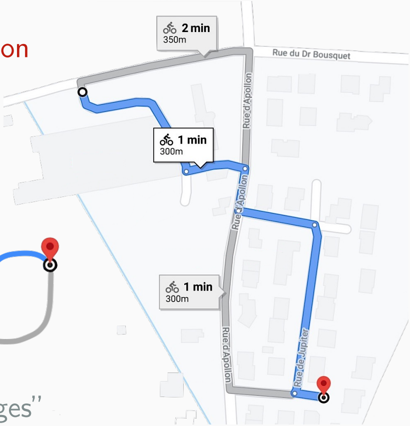


$2^{m/2}$ different paths
with m the number of "edges"



https://www.maps.google.fr

Let n be input size, e.g., number of roads in the network

Let d be output size, $\approx$number of solutions

output-polynomial

algo. stops in $\text{poly}(n+d)$*-time*

incremental-polynomial

outputs i^{th} *solution in* $\text{poly}(n+i)$*-time*

polynomial-delay

$\text{poly}(n)$*-time between two cons. outputs*

Typical question:

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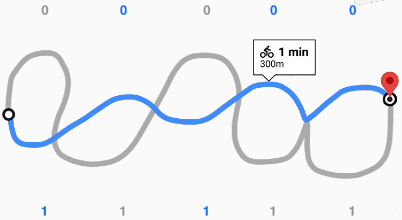

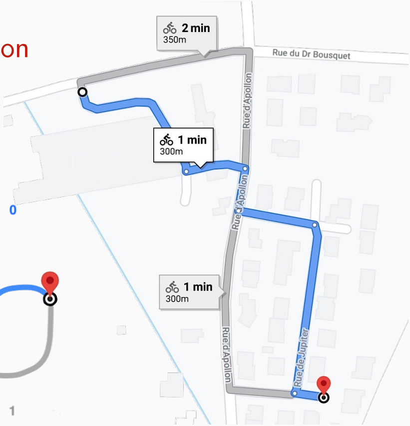

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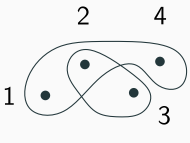
Definitions:

- hypergraph: family of subsets $\mathcal{H} \subseteq 2^X$ on **ground set** X
- transversal of $\mathcal{H}$: $T \subseteq X$ s.t. $T \cap E \neq \emptyset$ for any $E \in \mathcal{H}$
- $Tr(\mathcal{H})$: set of (inclusion-wise) minimal transervals of $\mathcal{H}$

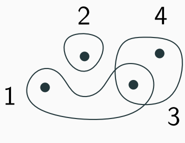
  it is a hypergraph!

$\rightarrow$ two hypergraphs $\mathcal{H}$ and $\mathcal{G}$ are called dual if $\mathcal{G} = Tr(\mathcal{H})$

and $Tr(Tr(\mathcal{H})) = \mathcal{H}$!



$\mathcal{H}$



$\mathcal{G}$

Hypergraph Dualization

input: two hypergraphs $\mathcal{H}$ and $\mathcal{G}$ on same ground set.

question: are $\mathcal{H}$ and $\mathcal{G}$ dual?

Minimal Transversals Enumeration (Trans-Enum)

input: a hypergraph $\mathcal{H}$.

output: the set $\mathcal{G} = Tr(\mathcal{H})$ of minimal transversals of $\mathcal{H}$.

Theorem (Fredman and Khachiyan, 1996)

There is a $N^{o(\log N)}$ quasi-polynomial time algorithm solving Hypergraph Dualization where $N = |\mathcal{H}| + |\mathcal{G}|$.

$\rightarrow$ *generation version is incremental*

Minimal Transversals Enumeration (Trans-Enum)

input: a hypergraph $\mathcal{H}$.

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Equivalent to:

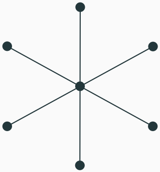
- translating from a positive CNF to a positive DNF
- ✗ enumerating the minimal dominating sets of a graph
- enumerating the minimal set coverings of a hypergraph
- enumerating database repairs

Are harder than Trans-Enum:

- lattice dualization problems
- meet-irreducibles/implicational bases translations
- characteristic models/Horn clauses translations

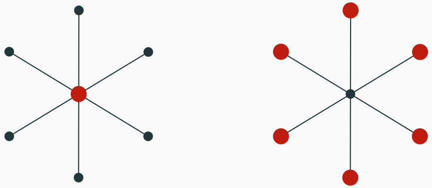
Definitions:

- graph G: a set of vertices $V(G)$, together with a set of edges $E(G) \subseteq \{\{x,y\} \mid x,y \in V(G),\ x \neq y\}$
- stable set: set of pairwise non-adjacent vertices
- clique: set of pairwise adjacent vertices
- triangle-free: does not contain a triangle ()



- $N(v)$: neighborhood of vertex v
- dominating set (DS): $D \subseteq V(G)$ s.t. $V(G) = D \cup N(D)$

  "D can see everybody else"
- minimal dominating set: inclusion-wise minimal DS



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  "D can see everybody else"
- minimal dominating set: inclusion-wise minimal DS
- private neighbor of $v \in D$:

  vertex that is $\begin{cases} \text{dominated by } v, \text{ and} \\ \text{not dominated by } D \setminus \{v\} \end{cases}$ (possibly v)
- irredundant set: $S \subseteq V(G)$ s.t. every $x \in S$ has a priv. neighbor

Observation

A DS is minimal if and only if it is irredundant.

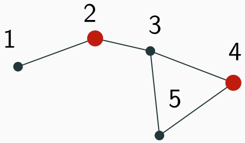
if all its vertices have a private neighbor.

Minimal DS Enumeration (Dom-Enum)

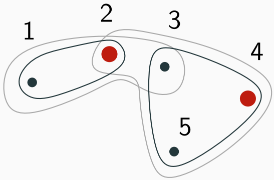
input: a n-vertex graph G.

output: the set $\mathcal{D}(G)$ of minimal DS of G.

A particular case of Trans-Enum



G



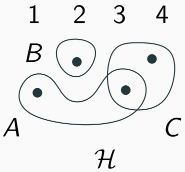
$\mathcal{H}$

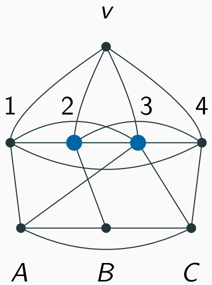
Minimal DS Enumeration (Dom-Enum)

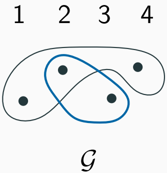
input: a n-vertex graph G.

output: the set $\mathcal{D}(G)$ of minimal DS of G.

Equivalent to Trans-Enum [Kanté et al., 2014]







Minimal DS Enumeration (Dom-Enum)

input: a n-vertex graph G.

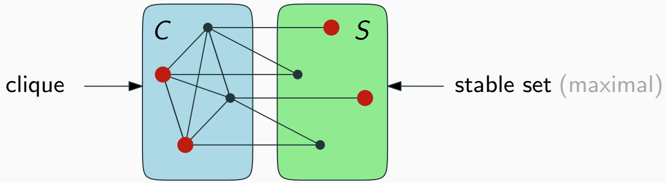
output: the set $\mathcal{D}(G)$ of minimal DS of G.

Dream goal: an output-poly. poly(N) algorithm, $N = n + |\mathcal{D}(G)|$

General case: open, best is quasi-polynomial $N^{o(\log N)}$

Known cases:

- **output poly.:** $\log(n)$-degenerate graphs
- **incr. poly.:** chordal bipartite graphs, bounded conformality graphs
- **poly. delay:** degenerate, line, and chordal graphs
- **linear delay:** permutation and interval graphs, graphs with bounded clique-width, split and P_6-free chordal graphs



Proposition (Kanté, Limouzy, Mary, and Nourine, 2014)

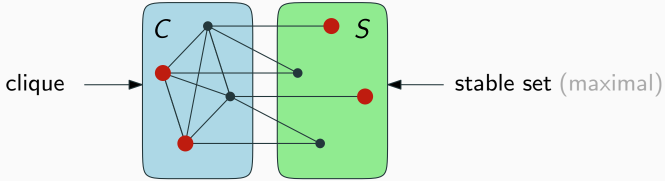
A set $D \subseteq V(G)$ is a minimal DS of G iff D dominates S and every $v \in D$ has a private neighbor in S.

Then: $D \cap S = \{$all vertices not dominated by $D \cap C\}$

Enumeration: complete every irredundant set $X \subseteq C$ in S

→ the family of such X's is an independence set system

→ can be enumerated with linear delay



Theorem (Kanté, Limouzy, Mary, and Nourine, 2014)

There is a linear-delay (and poly. space) algorithm enumerating minimal dominating sets in split graphs.

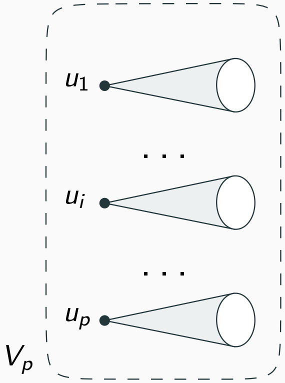
Then: $D \cap S = \{$all vertices not dominated by $D \cap C\}$

Enumeration: complete every irredundant set $X \subseteq C$ in S

$\rightarrow$ the family of such X's is an independence set system

$\rightarrow$ can be enumerated with linear delay

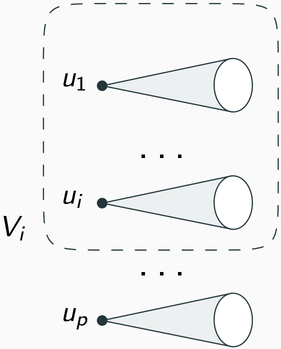
Goal: enumerating minimal DS one neighborhood at a time



Peeling: sequence $(V_0, \dots, V_p)$ s.t.

1. $V_p = V(G)$
2. for $i \in \{1, \dots, p\}$:
   $V_{i-1} = V_i \setminus \{u_i\} \setminus N(u_i)$
3. $V_0 = \emptyset$

Goal: enumerating minimal DS one neighborhood at a time

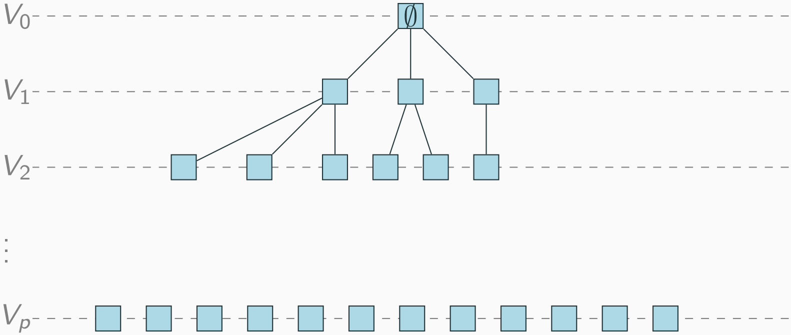


Dominating set (DS) of V_i:

- $D \subseteq V(G)$ s.t. $V_i \subseteq D \cup N(D)$
  "D can see everybody else in V_i"

Plan:

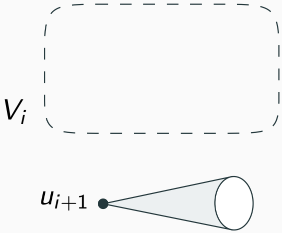
1. given minimal DS of V_i
   allowing vertices of $G - V_i$
2. enumerate those of V_{i+1}
   allowing vertices of $G - V_{i+1}$



Important wanted properties:

- no cycle (no *repetition*, using a parent relation: lex. order)
- no leaf before level p (no *exponential blowup*)

Goal: extend each minimal DS D of V_i to a minimal DS of V_{i+1}



Observation:

- possibly D **minimally** dominates V_{i+1}
- if not then $D \cup \{u_{i+1}\}$ does

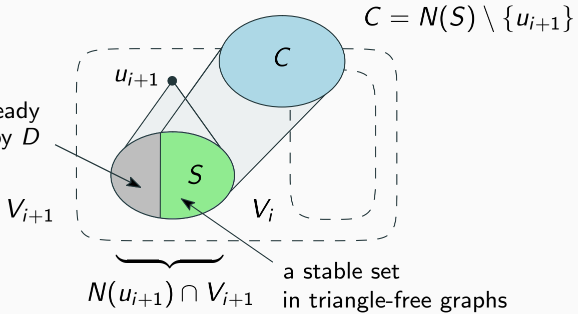
extension is always possible, hence

$$|\text{minimal DS of } V_i| \leq |\text{minimal DS of } V_{i+1}| \leq |\text{minimal DS of } G|$$

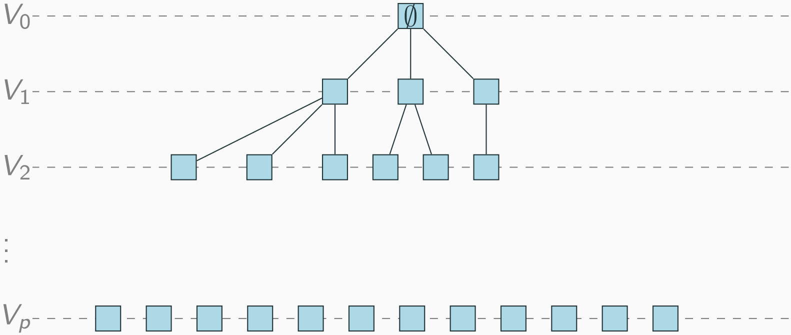
candidate extension of D: minimal set X s.t. $D \cup X$ dominates V_{i+1}

Lemma

|candidate extensions of D| ≤ |minimal DS of G|

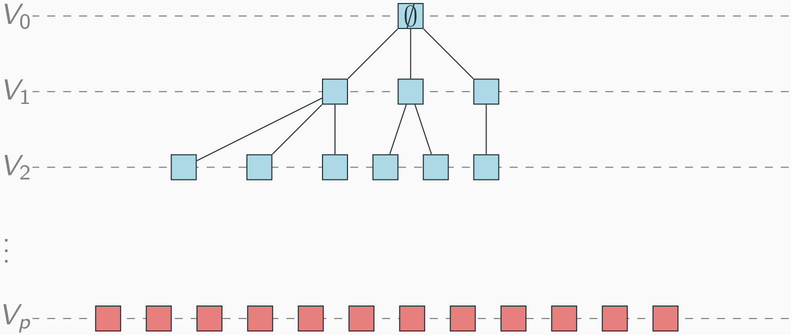


- if u_{i+1} dom. by D: they only have to dominate S
  $\rightarrow$ exactly the minimal DS of Split(C, S)
- if u_{i+1} not dom. by D: they should also dom. u_{i+1}
  irredundant $\{t\} \cup Q$ s.t. $\begin{cases} t \in N(u_{i+1}) \\ Q \text{ minimal DS of Split}(C, S) \end{cases}$



For each minimal DS of V_i:

- compute all candidate extensions; in time $O(\text{poly}(n) \cdot |\mathcal{D}(G)|)$
- only keep the $X \cup D$'s that are minimal and children of D



Theorem (Bonamy, D., Heinrich, and Raymond, 2019)

The set $\mathcal{D}(G)$ of minimal DS of any triangle-free graph G can be enumerated in time $O(\text{poly}(n) \cdot |\mathcal{D}(G)|^2)$ and polynomial space.

Theorem (Bonamy, D., Heinrich, Pilipczuk, and Raymond)

The set $\mathcal{D}(G)$ of minimal DS of any graph G can be enumerated in time $O(n^{2^{t+1}} \cdot |\mathcal{D}(G)|^{2^t})$ and poly. space where $t = \omega(G) + 1$.

Future work:

- complexity improvements? delay is still open for bipartite graphs

Theorem (Bonamy, D., Heinrich, and Raymond, 2019)

Deciding if a vertex set S can be extended into a minimal DS is NP-complete in bipartite graphs.

- extensions to other classes?

  $K_t + K_2$-free, paw-free, diamond-free ✓

  C_4-free? ✗

  (in)comparability and unit disk? ✗

- extensions to other classes?

  $K_t + K_2$-free, paw-free, diamond-free ✓

  C_4-free? ✗ chordal graphs ✓ split graphs ✓

  (in)comparability graphs? ✗

Theorem (D. and Nourine, 2019)

The set $\mathcal{D}(G)$ of minimal DS of any P_7-free chordal graph G can be enumerated with linear delay and poly. space.

Theorem (Bonamy, D., Micek, and Nourine, 2020)

The set $\mathcal{D}(G)$ of minimal DS can be enumerated with incremental and polynomial delay (and poly. space) in the comparability and incomparability graphs of posets of bounded dimension.

unit disk graphs? ✗

Minimal Transversals Enumeration (Trans-Enum)

input: a hypergraph $\mathcal{H}$.

output: the set $\mathcal{G} = Tr(\mathcal{H})$ of minimal transversals of $\mathcal{H}$.

Equivalent to:

- translating from a positive CNF to a positive DNF
- ✗ enumerating the minimal dominating sets of a graph
- enumerating the minimal set coverings of a hypergraph
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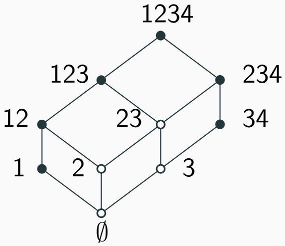
Are harder than Trans-Enum:

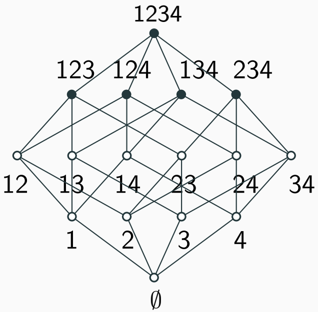
$\equiv ?$

- ✗ lattice dualization problems
- ✗ meet-irreducibles/implicational bases translations
- characteristic models/Horn clauses translations

Definitions:

- lattice $\mathcal{L} = (\mathcal{X}, \subseteq)$: order obtained by inclusions of a family $\mathcal{X} \subseteq 2^X$ over a **ground set** X that is
  - containing X: $X \in \mathcal{X}$
  - closed by intersection: $A, B \in \mathcal{X} \implies A \cap B \in \mathcal{X}$





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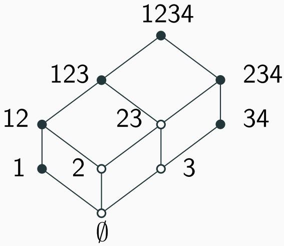
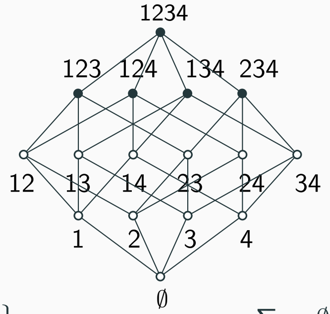


$\Sigma = \{13 \to 2,\ 4 \to 3\}$



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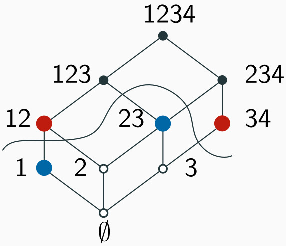

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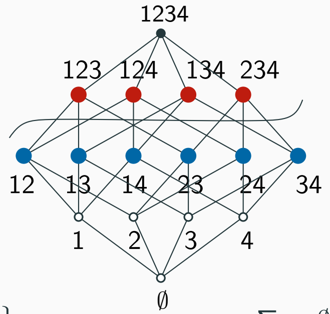

$\Sigma = \emptyset$

Theorem (D. and Nourine, 2019)

The dualization in lattices given by implicational bases of dimension two cannot be solved in output-polynomial time unless P=NP.

→ output quasi-polynomial time algorithms in subclasses of distributive lattices (lately improved by Khaled Elbassioni)
with Lhouari Nourine, and Takeaki Uno

Theorem (D., Nourine, and Vilmin, 2019)

Translating between meet-irreducible elements and implicational bases can be done in output quasi-polynomial time algorithm for ranked convex geometries.

Minimal Transversals Enumeration (Trans-Enum)

input: a hypergraph $\mathcal{H}$.

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Are harder than Trans-Enum:

- lattice dualization problems

  Dim. 2 IB lattices ✗ imp. in poly. time unless P=NP

  Acyclic IB lattices? ✗ Distributive lattices ✓

- meet-irreducibles/implicational bases translations

  Acyclic IB lattices? ✗ Distributive lattices ✓

  Ranked IB lattices ✓

Is meet enumeration possible in output quasi-polynomial time?

Are lattice dualization & meet enumeration **equivalent?**

Big question: can Hypergraph Dualization be solved in poly. time?

Equivalent to:

- translating from a positive CNF to a positive DNF
- enumerating the minimal dominating sets of a graph

    C_4-free? ✗

    (in)comparability and unit disk? ✗

Are harder than Trans-Enum:

- lattice dualization problems

≡? Acyclic IB lattices? ✗

- meet-irreducibles/implicational basestranslations

    Acyclic IB lattices? ✗

- characteristic models/Horn clauses translations

Thank you!