Dualization in lattices given by implicational bases

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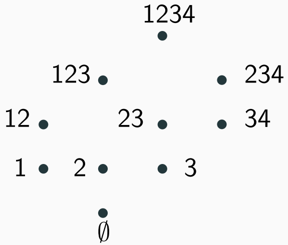
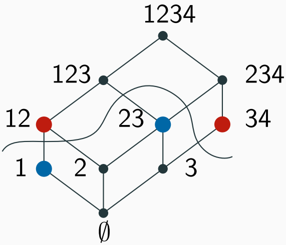
Implicational bases

- implicational base (X, Σ): set Σ of implications $A \to b$ over a ground set X, i.e., $A, \{b\} \subseteq X$
- closed set of Σ: set $C \subseteq X$ such that $A \not\subseteq C$ or $b \in C$ for every implication $A \to b \in \Sigma$
- $\phi(T)$: the smallest closed set of Σ containing T



The set of closed sets of the implicational base $\Sigma = \{13 \to 2,\ 4 \to 3\}$ on ground set $X = \{1, 2, 3, 4\}$

- lattice $\mathcal{L}(\Sigma)$: set of closed sets of Σ, ordered by inclusion.
- antichain of $\mathcal{L}(\Sigma)$: (inclusion-wise) incomparable closed sets of Σ
- two antichains $\mathcal{A}$ and $\mathcal{B}$ are dual if they **partition** the lattice into two parts: elements that are above $\mathcal{A}$, denoted by $\uparrow \mathcal{A}$, and those that are below $\mathcal{B}$, denoted $\downarrow \mathcal{B}$



The lattice $\mathcal{L}(\Sigma)$ of closed sets of the implicational base $\Sigma = \{13 \to 2,\ 4 \to 3\}$ on ground set $X = \{1, 2, 3, 4\}$

Lattice dualization (Dual):

input: an implicational base (X, Σ), two antichains $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{L}(\Sigma)$.

question: are $\mathcal{A}$ and $\mathcal{B}$ dual in $\mathcal{L}(\Sigma)$?

Lattice dualization, enumeration version (Dual-Enum):

input: an implicational base (X, Σ) and an antichain $\mathcal{B}$ of $\mathcal{L}(\Sigma)$.

output: the dual antichain $\mathcal{A}$ of $\mathcal{B}$ in $\mathcal{L}(\Sigma)$.

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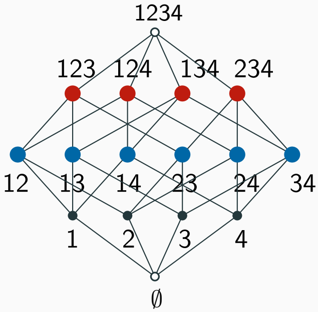
Harder than Hypergraph Dualization:

When Σ is empty (the lattice is Boolean)



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CoNP-complete in general:

In the general case when the implicational base has no restriction on the size of the premises (Babin and Kuznetsov, 2017)

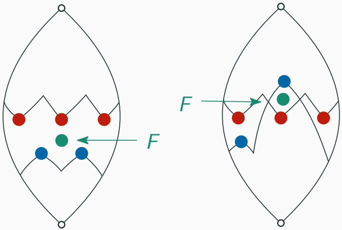
$$\{a_1, \ldots, a_m\} \to b$$

Open cases: implications bases with premises of size one (the lattice is distributive), two, or any fixed $k \in \mathbb{N}$

Theorem (D. and Nourine, 2019)

Dual is is coNP-complete for implicational bases with premises of size at most two.

- Polynomial certificate: a closed set $F \in \uparrow \mathcal{A} \cap \downarrow \mathcal{B}$, or a closed set F such that that both $F \notin \uparrow \mathcal{A}$ and $F \notin \downarrow \mathcal{B}$
- Reduction: from (positive) 1-in-3 3SAT
  In the fashion of Kavvadias et al., 2000.



- Let φ be a n-variable m-clause instance of 1-in-3 3SAT:

$$(x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (x_3 \vee x_4 \vee x_5)$$

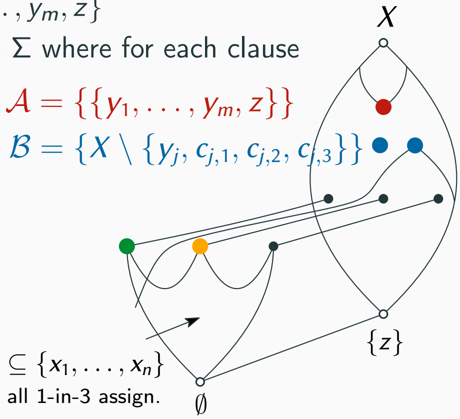
$$C_1 \qquad\qquad C_2 \qquad\qquad C_3$$

- Let $X = \{x_1, \ldots, x_n, y_1, \ldots, y_m, z\}$
- And the implicational base Σ where for each clause $C_j = (c_{j,1} \vee c_{j,2} \vee c_{j,3})$:

$$\begin{aligned} c_{j,1}c_{j,2} &\rightarrow z \\ c_{j,1}c_{j,3} &\rightarrow z \\ c_{j,2}c_{j,3} &\rightarrow z \\ zc_{j,1} &\rightarrow y_j \\ zc_{j,2} &\rightarrow y_j \\ zc_{j,3} &\rightarrow y_j \\ y_j &\rightarrow z \end{aligned}$$

$\mathcal{A} = \{\{y_1, \ldots, y_m, z\}\}$

$\mathcal{B} = \{X \setminus \{y_j, c_{j,1}, c_{j,2}, c_{j,3}\}\}$



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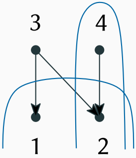
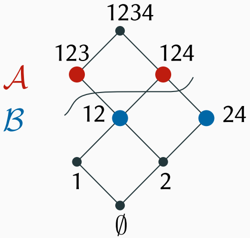


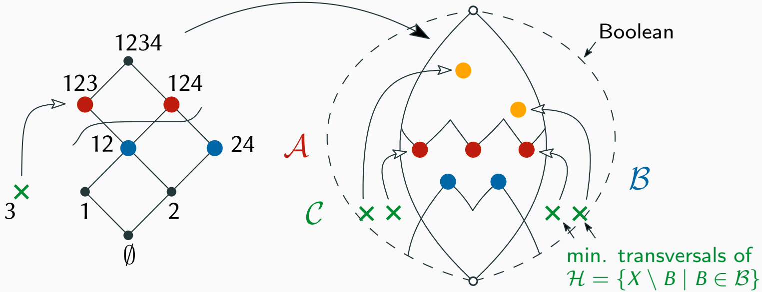


The lattice $\mathcal{L}(\Sigma)$ of closed sets of the implicational base

$\Sigma = \{3 \to 1,\ 3 \to 2,\ 4 \to 2\}$ on ground set $X = \{1, 2, 3, 4\}$

General steps:

- dualize the lattice as if it was Boolean
- close the solutions in the true lattice
- only keep actual solutions



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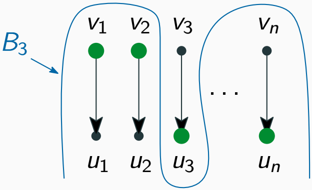
Boolean dualization on arbitrary lattices: limitations

Limitation:

There are instances with an **exponential gap** between Boolean solutions and actual solutions



$$\mathcal{A} = \{\{u_1, \dots, u_n\}\}$$

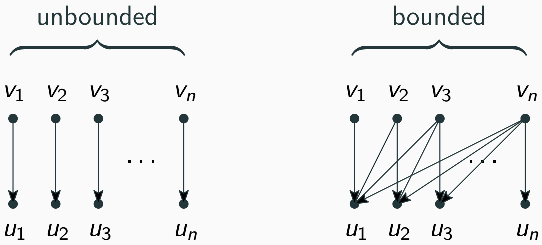
$$\mathcal{B} = \{B_i = X \setminus \{u_i, v_i\}\}$$

$$\mathcal{C} = \{u_1, v_1\} \times \dots \times \{u_n, v_n\}$$

An implicational base on ground set $X = \{u_1, v_1, \dots, u_n, v_n\}$

Independent-width of an implicational base

- independent implications: implications $A_1 \to b_1, \ldots, A_k \to b_k$ s.t. $b_i \notin \phi(A_1 \cup \cdots \cup A_{i-1} \cup A_{i+1} \cup \cdots \cup A_k)$ for any $i \in [k]$
- independent-width: the size of a maximum set of independent implications in Σ

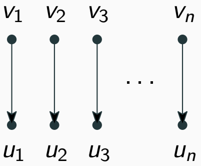


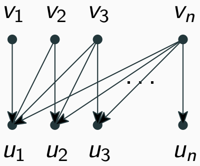
Lemma

The number of min. transversals of $\mathcal{H} = \{X \setminus B \mid B \in \mathcal{B}\}$ is bounded by $|X|^k \cdot \mathcal{A}$ where k is the independent-width of Σ

Theorem (D. and Nourine, 2019)

Dual can be solved in quasi-polynomial time in lattices coded by implicational bases of bounded independent-width.





Thank you!