# Enumerating minimal dominating sets and variants in chordal bipartite graphs

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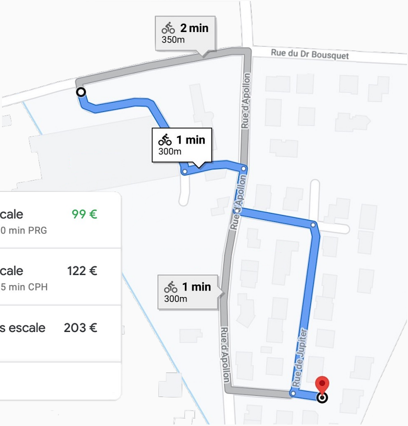
# Enumeration problems

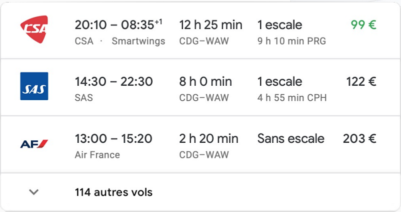
## Typical question:

*Given input $I$, list all solutions in $I$.*

## Examples:

- paths to a destination
- flights to a city
- answers to a query



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| CSA | 20:10 – 08:35+1<br>CSA · Smartwings | 12 h 25 min<br>CDG–WAW | 1 escale<br>9 h 10 min PRG | 99 € |
| SAS | 14:30 – 22:30<br>SAS | 8 h 0 min<br>CDG–WAW | 1 escale<br>4 h 55 min CPH | 122 € |
| AF | 13:00 – 15:20<br>Air France | 2 h 20 min<br>CDG–WAW | Sans escale | 203 € |
| | 114 autres vols | | | |

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# Two perspectives about complexity

**Input-sensitive:** in terms of input size

**Theorem (Moon & Moser, IJM 65)**

*There is an $O(3^{n/3})$-time algorithm enumerating all the maximal cliques of a n-vertex graph.*

$\rightarrow$ *basically upper-bounds the number of objects*

**Output-sensitive:** in terms of input + output size

**Theorem (Tsukiyama et al., SICOMP 77)**

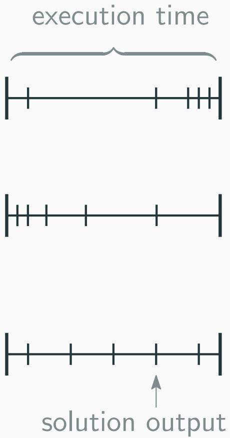
*There is a $O(n + m + d)$-time algorithm enumerating all the $d$ maximal cliques of a n-vertex m-edge graph.*

$\rightarrow$ *many techniques (reverse search, backtrack search, saturations algorithms, ordered generation, etc.)*

# Efficiency for the output-sensitive approach

Let $n$ be input size, e.g., number of vertices of a graph

Let $d$ be the output size, e.g., number[1] of max. cliques




**out**put-**poly**nomial
*stops in* poly$(n + d)$ *time*

**inc**remental-**poly**nomial
*outputs* $i^{th}$ *solution in* poly$(n + i)$ *time*

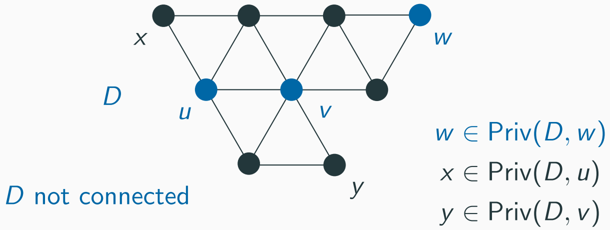
**poly**nomial **delay**
poly$(n)$ *time between consec. outputs*

[1] For simplicity as solutions are of poly size

## Neighborhoods

Let $D$ be a subset of vertices and $u \in D$:

- $N(u)$: neighborhood of $u$ i.e., vertices adjacent to $u$
- $N[u]$: closed neighborhood of $u$ i.e., vertices adjacent to $u + u$
- private neighborhood of $u$: $\mathrm{Priv}(D, u) := N[u] \setminus N[D \setminus \{u\}]$
- connected: $G[D]$ is connected



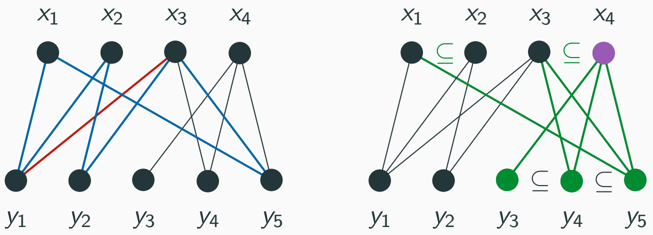
# Chordal bipartite

A graph is **chordal bipartite** if it is bipartite and its cycles of length $\geq 6$ have a chord i.e. induced cycles have length $\leq 4$

**Definition**

A vertex $v \in V(G)$ is **weak-simplicial** if

- $N(v)$ is an independent set
- for every $x, y \in N(v)$, either $N(x) \subseteq N(y)$ or $N(y) \subseteq N(x)$[2]




[2]Which implies inclusions of the distance-2 neighborhood within $N(v)$

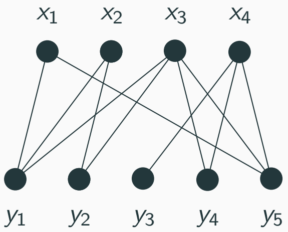
# Elimination ordering

**Definition**

An ordering $v_1, \ldots, v_n$ of $V(G)$ is a **weak-simplicial elimination ordering (wseo)** if $v_i$ is weak-simplicial in $G_i := G[v_1, \ldots, v_i]$, $\forall i$

**Theorem (Kurita et al., IWOCA 19)[3]**

*A graph is chordal bipartite iff it admits a* ***wseo***




**wseo**: $(x_1, x_3, y_1, y_5, x_2, y_2, y_3, y_4, x_4)$

[3] Also follows from previously-known observations e.g. Uehara (ICALP 02)

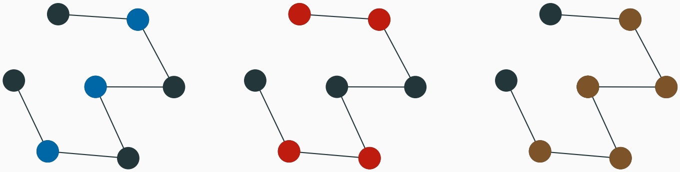
## Domination

A set $D$ of vertices is

- dominating: $\forall v \in V(G),\ N[v] \cap D \neq \varnothing$ i.e. $N[D] = V(G)$
- total dominating: $\forall v \in V(G),\ N(v) \cap D \neq \varnothing$
- connected dominating: dominating + connected
- minimal: inclusion-wise minimal

**Observation**

*A connected dom. set of cardinal $\geq 2$ is a total dom. set*[4]



[4] So the combinaison total + connected makes little sense

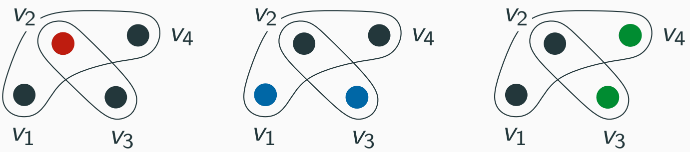
# Transversals

A hypergraph is a pair $(V, \mathcal{H})$ s.t. $\mathcal{H} \subseteq 2^V$

A set $T$ of vertices is

- a transversal: $\forall E \in \mathcal{H},\ E \cap T \neq \varnothing$
- minimal: inclusion-wise minimal

A private edge of $v \in T$ is an edge $E$ s.t. $E \cap T = \{v\}$



# Problems

**Min Dom Enumeration (Dom·Enum)**

input: a graph $G$

output: the set $\mathrm{MDS}(G)$ of all min. dom. sets of $G$

**Min Total Dom Enumeration (TDom·Enum)**

output: the set $\mathrm{MTDS}(G)$ of all min. tot. dom. sets of $G$

**Min Connected Dom Enumeration (CDom·Enum)**

output: the set $\mathrm{MCDS}(G)$ of all min. con. dom. sets of $G$

These problems are related to the following

**Min Transversals Enumeration (Trans·Enum)**

input: a hypergraph $(V, \mathcal{H})$

output: the set $\mathrm{MT}(\mathcal{H})$ of min. transversals of $\mathcal{H}$

## Link with transversals

It is easily observed that:

- Dom·Enum asks to hit closed neighborhoods
- TDom·Enum asks to hit open neighborhoods

These problems are in fact equivalent

**Theorem (Kanté et al., SIDMA 14)**

- Dom·Enum $\equiv$ Trans·Enum
- TDom·Enum $\equiv$ Trans·Enum

*Moreover, if* $\mathcal{S}(G)$ *is the set of minimal separators of* $G$*, then for any graph* $G$*, we have* $\mathrm{MCDS}(G) = \mathrm{MT}(\mathcal{S}(G))$

However in general

- $\mathcal{S}(G)$ is exponential
- $\mathcal{S}(G)$ is hard to compute [BDK$^+$24]

# Status in general

**Theorem (Kanté et al., SIDMA 14)**

- Dom·Enum $\equiv$ Trans·Enum
- TDom·Enum $\equiv$ Trans·Enum

*Moreover, if* $\mathcal{S}(G)$ *is the minimal separators of* $G$*, then for any graph* $G$*, we have* $\mathsf{MCDS}(G) = \mathsf{MT}(\mathcal{S}(G))$

Unfortunately the complexity status of Trans·Enum is open

Best-known algorithm is output-quasi-poly [FK96]

Decision version is among the few natural NP-intermediate problems

What about chordal bipartite graphs?

# Status in chordal bipartite graphs

**Theorem (Golovach et al., DAM 16)**

*In chordal bipartite graphs:*

- Dom·Enum *admits a inc-poly algorithm*
- TDom·Enum *admits an poly-delay algorithm*

Two questions arise:

- can we improve to poly-delay for Dom·Enum?
- can we obtain output-poly for CDom·Enum?

**Theorem (Castelo, D., and Gomes)**

*In chordal bipartite graphs:*

- Dom·Enum *admits a poly-delay algorithm*
- TDom·Enum *admits a poly-delay algorithm*
- CDom·Enum *admits an inc-poly algorithm*

## Techniques

**Theorem (Castelo, D., and Gomes)**

*In chordal bipartite graphs:*

- Dom·Enum *admits a poly-delay algorithm*
- TDom·Enum *admits a poly-delay algorithm*
- CDom·Enum *admits an inc-poly algorithm*

Techniques:

- sequential method + **wseo** for Dom·Enum + TDom·Enum
- conformality of min. sep. + [KBEG07] for CDom·Enum

This talk:

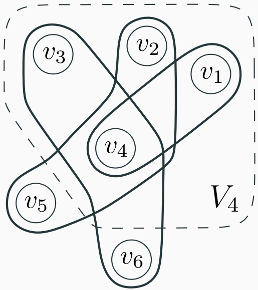
- sequential method + **wseo** for Dom·Enum
  via the enumeration of $MT(\mathcal{H})$
  for $\mathcal{H}$ the closed neighborhoods of $G$

# Sequential method

**Introduced**[5] by Eiter, Gottlob & Makino (STOC 2002)

**Goal** given $\mathcal{H} := \{N[v] : v \in V(G)\}$

- define a peeling of $\mathcal{H}$ according to a vertex ordering
- list min. transversals of increasing portions of $\mathcal{H}$




**Sub-hypergraph induced** by the $i$ first vertices:

$$V_i := \{v_1, \ldots, v_i\}$$

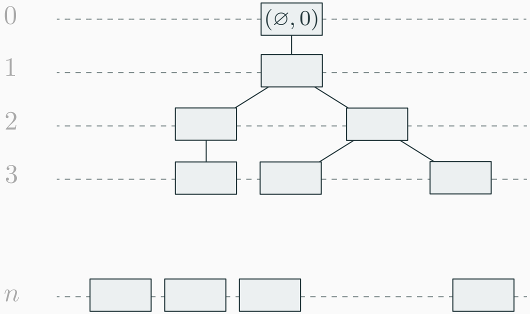
$$\mathcal{H}_i := \{E \in \mathcal{H} : E \subseteq V_i\}$$

**Put** $\mathrm{MT}(\mathcal{H}_0) := \{\emptyset\}$

**Goal**: for all $0 \leq i < n$
given $\mathrm{MT}(\mathcal{H}_i)$
enumerate $\mathrm{MT}(\mathcal{H}_{i+1})$

[5]As a generalization of an algorithm by Lawler et al. (SICOMP 80)

# Sequential method




**Wanted properties:**

(A) no cycle: to avoid repetitions

(B) no leaf before level $n$: to avoid useless computation

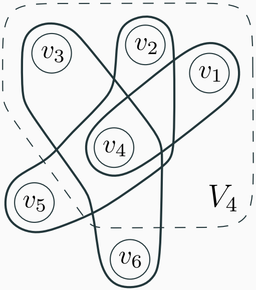
(C) efficient computation of children

## (A) No cycle: parent relation

Let $(T, i)$ with $1 < i \leq n$ and $T \in \mathrm{MT}(\mathcal{H}_i)$

**Parent** of $(T, i)$: set $T^*$ obtained by repeating

*while there exists a vertex $v \in T$ with no private edge in $\mathcal{H}_{i-1}$ remove one such vertex of smallest label*



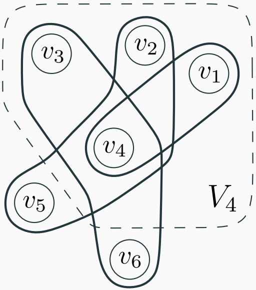

**Properties:**

- $T^*$ is uniquely defined
- $T^*$ belongs to $\mathrm{MT}(\mathcal{H}_{i-1})$

Put Children$(T^*, i)$ as the set of minimal transversals of $\mathcal{H}_{i+1}$ whose parent is $T^*$

# (B) No stopping branch: extension

Let $(T^*, i)$ with $1 \leq i < n$ and $T^* \in \mathrm{MT}(\mathcal{H}_i)$




**Properties:** either

- $T^*$ belongs to $\mathrm{MT}(\mathcal{H}_{i+1})$; or
- $T^* \cup \{v_{i+1}\}$ does

Moreover this set is actually a child of $(T^*, i)$ we may call trivial child

**Proof sketch:** edges of $\mathcal{H}_{i+1}$ not in $\mathcal{H}_i$ are those intersecting $v_{i+1}$; moreover, private edges of $T^*$ are included in $V_i$ hence may not be lost by adding $v_{i+1}$

## (C) Children generation

**Goal:** generate Children($T^*, i$) given $T^* \in Tr(\mathcal{H}_i)$ and $1 \leq i < n$

What is the shape of non-trivial children?

**Observations:**

- parts of $\mathcal{H}_{i+1}$ are already hit by $T^*$
- those that remain are precisely edges in

$$\Delta_{i+1} := \{E \in \mathcal{H}_{i+1} : E \cap T^* = \varnothing,\ v_{i+1} \in E\}$$

**Lemma**

*childrens of $T^*$ are of the form $T^* \cup X$ for $X \in \mathrm{MT}(\Delta_{i+1})$*

Not every such $X$ gives rise to a child[6]

But this approach is enough if $\mathrm{MT}(\Delta_{i+1})$ is of poly size

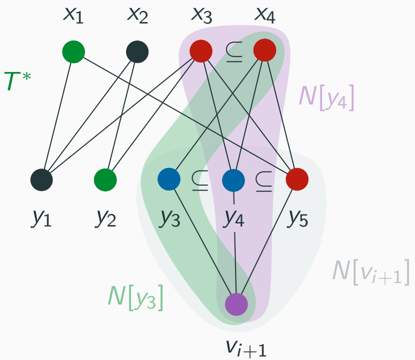
[6] As $X$ may steal private edges

# Chordal bipartite case

Consider the underlying structure of $G$ + **weso** $v_1, \ldots, v_n$

Sets in $\Delta_{i+1}$ are either $N[v_{i+1}]$, or $N[u]$ for $u \in N[v_{i+1}]$ called blue

They can be hit with $v_{i+1}$, blue $u$'s, or other $w \in N[u]$ called red



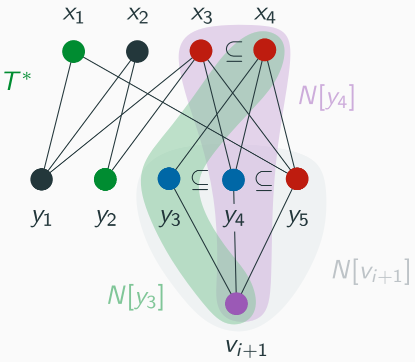
# Chordal bipartite case

Let $B$ be the blue vertices, $R$ the red ones

**Lemma**

*Let* $X \in \mathrm{MT}(\Delta_{i+1})$. *Then*

- $|X \cap R \cap N(v_{i+1})| \leq 1$
- $|X \cap R \cap N^2(v_{i+1})| \leq 1$ *with* $N^2$ *the distance-2 neighborhood*

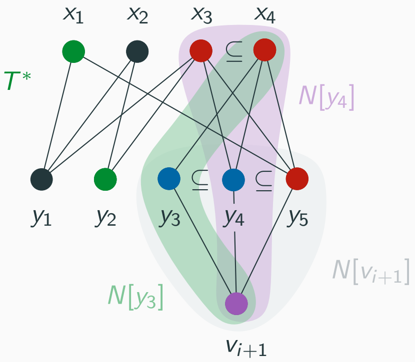


# Chordal bipartite case

**Lemma**

*Let $X \in \mathrm{MT}(\Delta_{i+1})$. Then exactly one of the following holds*

- $X = \{v_{i+1}\}$
- $X \subseteq B$, *in which case* $X = B$
- $X \subseteq R$, *in which case* $|X| \leq 2$
- $X = \{r\} \cup (B \setminus N(r))$ *for some* $r \in N^2(v_{i+1})$



## Chordal bipartite case

**Lemma**

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- $X = \{r\} \cup (B \setminus N(r))$ *for some* $r \in N^2(v_{i+1})$

The number of such $X$ is poly in $n$

They can be enumerated in poly time

**Theorem (Castelo, D., and Gomes)**

Dom·Enum *can be solved with poly delay in chordal bip. graphs*

Adapting $\mathcal{H}$ to open neighb. yields the same for TDom·Enum

# Open questions

**Theorem (Castelo, D., and Gomes)**

*In chordal bipartite graphs:*

- Dom·Enum *admits a poly-delay algorithm*
- TDom·Enum *admits a poly-delay algorithm*
- CDom·Enum *admits an inc-poly algorithm*

A natural question is whether these extend to bipartite graphs:

- poly delay for Dom·Enum?
- poly delay for TDom·Enum?
- output-poly for CDom·Enum?

Inc-poly is known for the first one [BDH+20] an can most probably be adapted for the second one ; nothing is known for CDom·Enum

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