Translating between the representations of a ranked convex geometry

Oscar Defrain, Lhouari Nourine, Simon Vilmin
LIMOS, Université Clermont Auvergne, France

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Kyoto University, Japan

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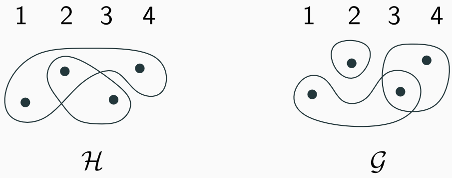
- hypergraph: family of subsets $\mathcal{H} \subseteq 2^X$ on **ground set** X
- transversal of $\mathcal{H}$: $T \subseteq X$ s.t. $T \cap E \neq \emptyset$ for any $E \in \mathcal{H}$
- $Tr(\mathcal{H})$: set of (inclusion-wise) minimal transervals of $\mathcal{H}$
  it is a hypergraph!
- two hypergraphs $\mathcal{H}$ and $\mathcal{G}$ are called dual if $\mathcal{G} = Tr(\mathcal{H})$



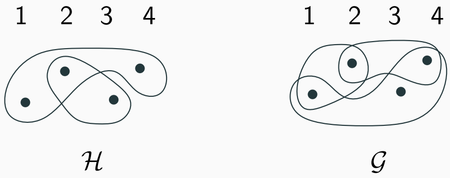
Hypergraph Dualization parenthesis

- hypergraph: family of subsets $\mathcal{H} \subseteq 2^X$ on **ground set** X
- independent set of $\mathcal{H}$: $S \subseteq X$ s.t. $E \not\subseteq S$ for any $E \in \mathcal{H}$
- $MIS(\mathcal{H})$: set of (inclusion-wise) maximal independent sets of $\mathcal{H}$

  it is a hypergraph!

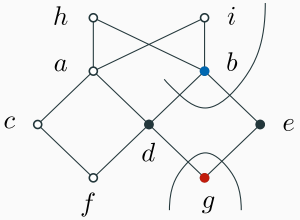
$\rightarrow$ two hypergraphs $\mathcal{H}$ and $\mathcal{G}$ are dual iff $\overline{\mathcal{G}} = MIS(\mathcal{H})$

$\overline{\mathcal{G}} = \{X \setminus E \mid E \in \mathcal{G}\}$



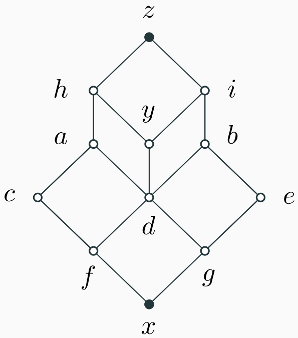
Partially ordered sets (posets)

- poset $P = (V, \leq)$: binary relation $\leq$ on V which is transitive, reflexive, and antisymmetric ($a \leq b$ and $b \leq a \implies a = b$)
- meet $a \wedge b$: x s.t. $x \geq y$ for all y below both a and b, if it exists
- join $a \vee b$: x s.t. $x \leq y$ for all y above both a and b, if it exists



Lattices: iconic definition

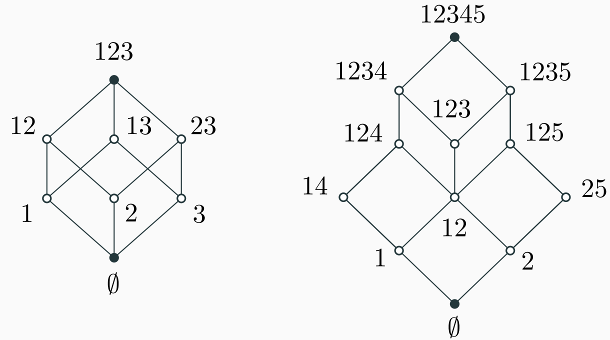
- lattice $\mathcal{L} = (V, \leq)$: poset in which $a \wedge b$ and $a \vee b$ are defined for every single pair $a, b \in V$ of elements
- meet $a \wedge b$: x s.t. $x \geq y$ for all y below both a and b, if it exists
- join $a \vee b$: x s.t. $x \leq y$ for all y above both a and b, if it exists



Folklore

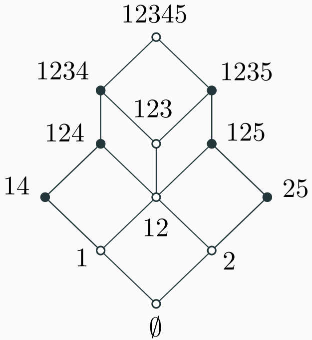
To every lattice $\mathcal{L}$ corresponds a family of sets $\mathcal{C} \subseteq 2^X$ closed by intersection such that $\mathcal{L} \cong (\mathcal{C}, \subseteq)$, with ground set X as the top.

$\rightarrow$ Compact representation?



Representations of a lattice: meet-irreducibles

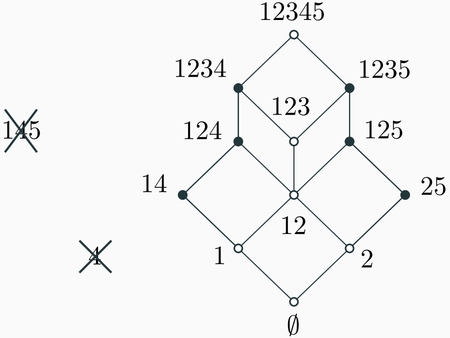
- meet-irreducible elements: elements with a unique successor
- more formally: elements $M \in \mathcal{L}$ s.t.
  $\forall A, B \in \mathcal{L},\ M = A \cap B \implies M = A$ or $M = B$
- $\mathcal{M}(\mathcal{L})$: set of meet-irreducible elements of $\mathcal{L}$



Folklore

Every lattice $\mathcal{L}$ can be reconstructed by **intersection** *of the family $\mathcal{M}(\mathcal{L})$ of its meet-irreducible elements.*

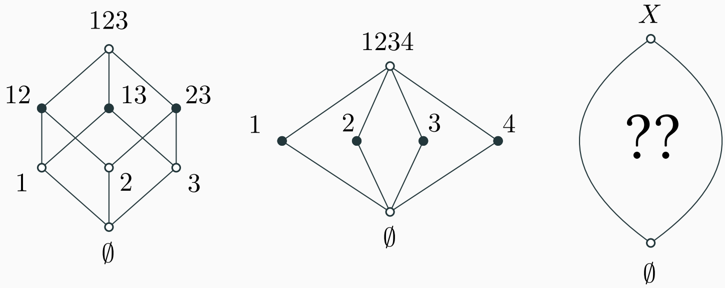
$\rightarrow$ Compact representation?



Folklore

Every lattice $\mathcal{L}$ can be reconstructed by ***intersection*** *of the family $\mathcal{M}(\mathcal{L})$ of its meet-irreducible elements.*

→ Compact representation? it depends



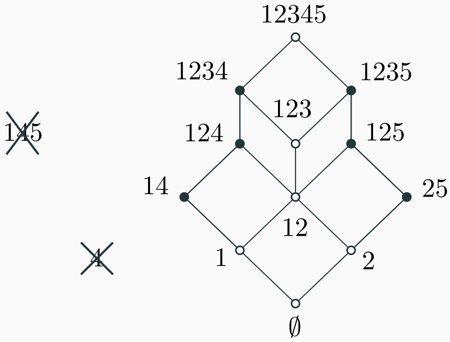
- implicational base (X, Σ): set Σ of implications $A \to b$ over a ground set X, i.e., $A, \{b\} \subseteq X$
- closed set of Σ: set $C \subseteq X$ that satisfies the implications of Σ, i.e., such that $A \not\subseteq C$ or $b \in C$ for all $A \to b \in \Sigma$
- $\mathcal{C}_\Sigma$: set of all closed sets of Σ

$$
\begin{aligned}
X &= \{1, 2, 3, 4, 5\} \\
4 &\to 1 \\
5 &\to 2 \\
3 &\to 1 \\
3 &\to 2 \\
45 &\to 3
\end{aligned}
$$

Folklore

Every lattice $\mathcal{L}$ can be reconstructed as the **closed sets** *of some implicational base (X, Σ), and many such imp. bases exist.*

→ Compact representation?



$\mathcal{L}_\Sigma = (\mathcal{C}_\Sigma, \subseteq)$

$X = \{1, 2, 3, 4, 5\}$

$4 \rightarrow 1$

$5 \rightarrow 2$

$3 \rightarrow 1$

$3 \rightarrow 2$

$45 \rightarrow 3$

Folklore

*Every lattice $\mathcal{L}$ can be reconstructed as the **closed sets** of some implicational base (X, Σ), and many such imp. bases exist.*

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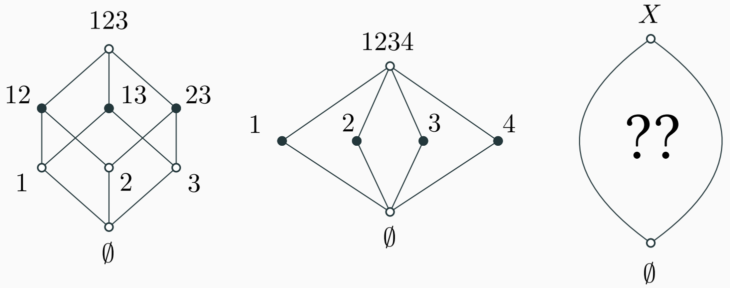


Horn formulas and Characteristic Models

Folklore

*Every lattice corresponds to the models of a Horn expression, defined as **clauses** with at **most one positive litteral**.*

Example:

$$\varphi = (\neg 4 \vee 1) \wedge (\neg 5 \vee 2) \wedge (\neg 3 \vee 1) \wedge (\neg 3 \vee 2) \wedge (\neg 4 \vee \neg 5 \vee 3)$$

In Horn logic:

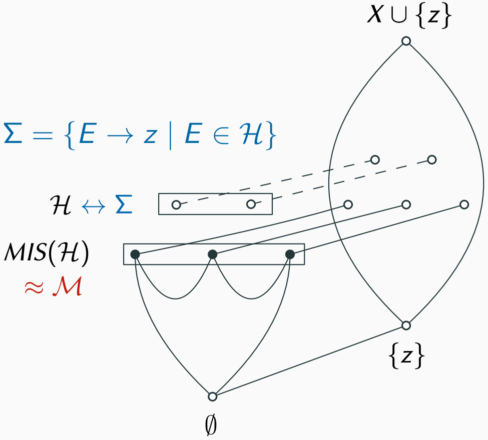
- implicational base: Horn clauses of φ
- closed sets: models of φ + ground set
- meet-irreducibles: known as the characteristic models

Question:

What is better? implicational bases/meet-irreducibles?

Folklore

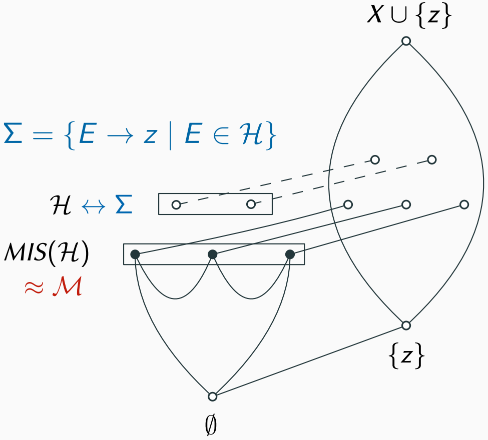
There are lattices $\mathcal{L}$ for which the size of $\mathcal{M}(\mathcal{L})$ is ***exponential*** *in that of Σ, hence exponential in $|X|$, and vice versa.*



Relation between the two representations

Theorem (Khardon, 1995)

Translating between the representations of a lattice is harder than hypergraph dualization.



Translating between the representations of a lattice

Meet-irreducible and Implicational Base identifications

input: an implicational base (X, Σ) and a family of sets $\mathcal{M} \subseteq 2^X$.

question: is $\mathcal{M} = \mathcal{M}(\mathcal{L}_\Sigma)$?

Meet-irreducible enumeration

input: an implicational base (X, Σ).

output: the set $\mathcal{M}(\mathcal{L}_\Sigma)$.

Implicational base enumeration

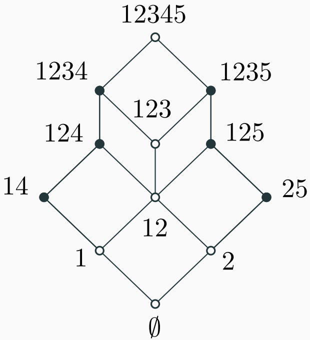
input: Two sets X and $\mathcal{M} \subseteq 2^X$.

output: An implicational base* (X, Σ) such that $\mathcal{M} = \mathcal{M}(\mathcal{L}_\Sigma)$.

Dream goal: an algorithm running in poly(N)-time,

$$N = |X| + |\mathcal{M}| + |\Sigma|$$

- implication-graph $G(\Sigma)$: directed graph on vertex set X, with an arc (a, b) if there exists $A \to b \in \Sigma$ s.t. $a \in A$.
- acyclic implicational base: s.t. $G(\Sigma)$ is acyclic.
- acyclic convex geometry (ACG): lattice that admits an acyclic implicational base



$X = \{1, 2, 3, 4, 5\}$

$4 \to 1$

$5 \to 2$

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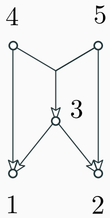
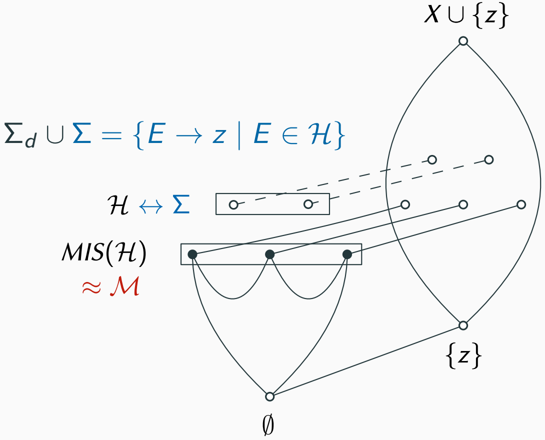
$45 \to 3$



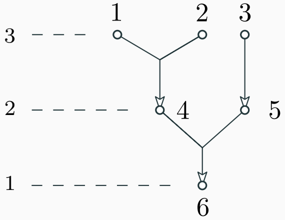
Theorem (D., Nourine and Vilmin, 2019)

Translating between the representations of an acyclic convex geometry is harder than the dualization in distributive lattices.



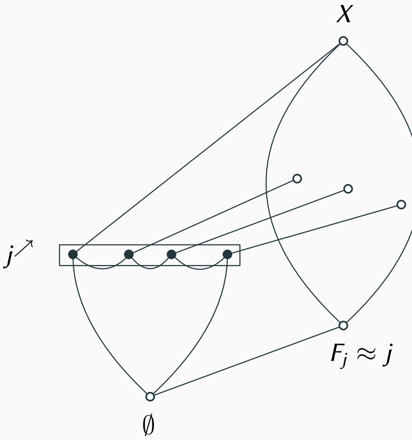
Sublass: ranked convex geometries

- ranked implicational base: Σ that admits a rank function $\rho : X \to \mathbb{N}$ s.t. $A \to b \in \Sigma \implies \rho(a) = \rho(b) + 1, \ \forall a \in A$
- ranked convex geometry (RCG): lattice that admits a ranked implicational base



Folklore

If $\mathcal{L}$ is a convex geometry then the set $\mathcal{M}(\mathcal{L})$ is ***partitioned*** *by the family $\{j^{\nearrow} = \mathrm{Max}_{\subseteq}\{C \in \mathcal{L} \mid j \notin C\},\ j \in X\}$.*



Enumerating meet-irreducible in convex geometries

Folklore

*If $\mathcal{L}$ is a convex geometry then the set $\mathcal{M}(\mathcal{L})$ is **partitioned** by the family $\{j^{\nearrow} = \mathrm{Max}_{\subseteq}\{C \in \mathcal{L} \mid j \notin C\},\ j \in X\}$.*

Theorem (D., Nourine and Vilmin, 2019)

*Let $\mathcal{L}$ be an **acyclic** convex geometry, and $j \in X$. Then enumerating $j^{\nearrow} = \mathrm{Max}_{\subseteq}\{C \in \mathcal{L} \mid j \notin C\}$ is harder than the dualization in lattices given by **acyclic** implicational bases.*

Theorem (D., Nourine and Vilmin, 2019)

*The meet-irreducible elements of a ranked convex geometry given by (X, Σ) can be enumerated in **output quasi-polynomial** time using hypergraph dualization.*

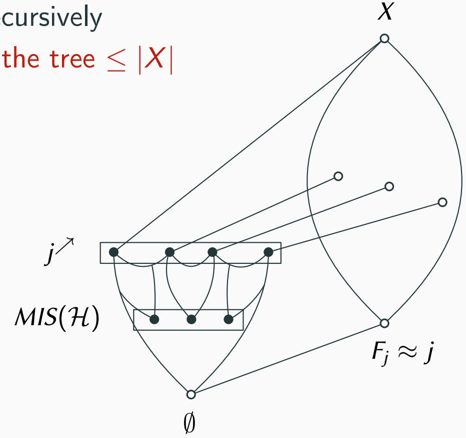
Algorithm outline:

- partition the solutions

$\rightarrow$ using hypergraph dualization, rank by rank

- explore recursively

$\rightarrow$ height of the tree $\leq |X|$

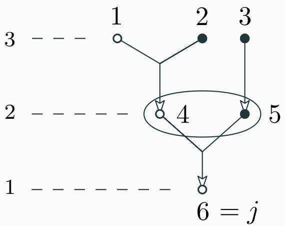


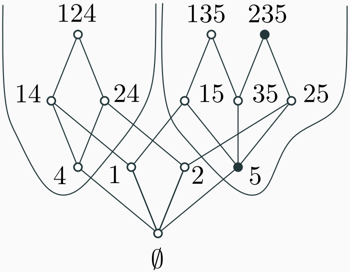
At first:

- $B = \{j\}$: set of elements that should not be implied

Then:

- $\mathcal{H}_B = \{A \mid A \rightarrow b \in \Sigma,\ b \in B\}$ on ground set rank $\rho(B) + 1$
- for each $S \in MIS(\mathcal{H}_B)$, $\hat{S} = \{x \in X \mid \rho(x) = \rho(S),\ x \notin S\}$
- recursively call on $\hat{S}$





Constructing the ranked implicational base of a RCG

- $\phi(C)$: smallest set in $\mathcal{L}$ containing C
  obtained by intersecting $\{M \in \mathcal{M}(\mathcal{L}),\ C \subseteq M\}$
- minimal generator of j: minimal set $A \subseteq X \setminus \{j\}$ s.t. $j \in \phi(A)$

Folklore

The min. transversals of the hypergraph $\mathcal{H}_j = \{X \setminus M_j \mid M_j \in j^{\nearrow}\}$ *on ground set* $V = X \setminus \bigcap_{M_j \in j^{\nearrow}} M_j$ *are the min. generators of* j.

- $\text{pred}(j) = \{a \in X \mid \exists A \to j \in \Sigma,\ a \in A\}$
  ⚠ how to compute from $\mathcal{M}(\mathcal{L})$?

Theorem (D., Nourine and Vilmin, 2019)

Let (X, Σ) *be the critical base we want to compute, and* $j \in X$. *Then* $A \to j \in \Sigma$ *iff* $A \in Tr(\mathcal{H}_j[\text{pred}(j)])$.

- $\phi(C)$: smallest set in $\mathcal{L}$ containing C
  obtained by intersecting $\{M \in \mathcal{M}(\mathcal{L}),\ C \subseteq M\}$
- minimal generator of j: minimal set $A \subseteq X \setminus \{j\}$ s.t. $j \in \phi(A)$

Lemma

Let $j \in X$, $M_j \in j^{\nearrow}$ and $a \notin M_j$.
Then $a \in pred(j)$ iff $M_j \cup \{a, j\} = \phi(M_j \cup \{a, j\})$.

$\rightarrow$ can be solved in polynomial time in $|X| + |\mathcal{M}(\mathcal{L})|$!

Theorem (D., Nourine and Vilmin, 2019)

The ranked implicational base of a ranked convex geometry can be constructed in ***output quasi-polynomial*** *time from $\mathcal{M}(\mathcal{L})$ using hypergraph dualization.*

Further work

Extending the algorithm:

- for meet-irreducible enumeration: how can implications out-of-rank be integrated?

  it appears that implications $a \rightarrow b$ are not harmful
- for implicational base construction: how to generalize the characterization of $\mathrm{pred}(j)$ to other classes?

Recognizing RCG:

- easy from (X, Σ)
- in coNP from $\mathcal{M}(\mathcal{L})$, non trivial !

$\rightarrow$ is it in P?