Hypergraph dualization with FPT-delay parameterized by the degeneracy and dimension

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Enumeration problems

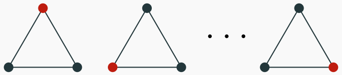
Typical question:

Given input I, list all objects of type X in I.

Examples:

- cycles, cliques, stable sets, dominating sets of a graph
- transversals/min. transversals of a hypergraph
- antichains of a partial order
- variable assignments satisfying a formula
- answers to a query
- trains to Paris leaving tomorrow before 10:00
- ...

Remark: possibly many objects!
$3^{n/3} \approx 1.4422^n$



Two perspectives about complexity

Input-sensitive: in terms of input size

Theorem (Moon & Moser, 65)

There is an $O(3^{n/3})$-time algorithm enumerating all the maximal cliques of a n-vertex graph.

$\rightarrow$ *basically upper-bounds the number of objects*

Output-sensitive: in terms of input+output size

Theorem (Tsukiyama et al., 77)

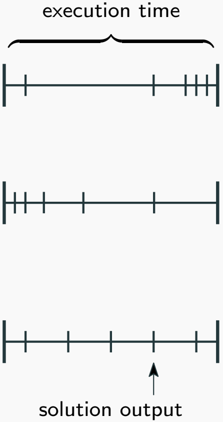
There is a $O(n + m + d)$-time algorithm enumerating all the d maximal cliques of a n-vertex m-edge graph.

$\rightarrow$ *many techniques (reverse search, backtrack search, saturations algorithms, ordered generation, etc.)*

Efficiency for the output-sensitive approach

Let n be input size, e.g., number of vertices of a graph

Let d be the # of solutions, e.g., number of max. cliques



output-polynomial

algo. stops in $\text{poly}(n + d)$*-time*

incremental-polynomial

outputs i^{th} *solution in* $\text{poly}(n + i)$*-time*

polynomial-delay

$\text{poly}(n)$*-time between two cons. outputs*

Definitions:

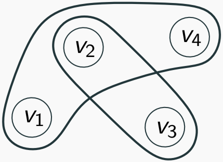
- hypergraph: family of subsets $\mathcal{H} \subseteq 2^V$ on vertex set V

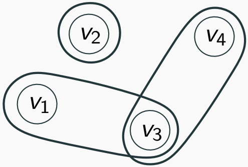
  called Sperner if $A \not\subset B$ for any two $A, B \in \mathcal{H}$
- transversal of $\mathcal{H}$: $T \subseteq V$ s.t. $T \cap E \neq \emptyset$ for every $E \in \mathcal{H}$
- $Tr(\mathcal{H})$: set of (inclusion-wise) minimal transervals of $\mathcal{H}$

  it is a Sperner hypergraph!
- two Sperner hypergraphs $\mathcal{H}$ and $\mathcal{G}$ are dual if $\mathcal{G} = Tr(\mathcal{H})$

  and we have $Tr(Tr(\mathcal{H})) = \mathcal{H}$

$\mathcal{H}$





$\mathcal{G}$

Hypergraph dualization: definitions

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Observation

*A transversal T of $\mathcal{H}$ is minimal **iff** for each vertex $v \in T$ there is a (so-called private for v) edge $E \in \mathcal{H}$ such that $E \cap T = \{v\}$.*

Hypergraph Dualization

input: two (Sperner) hypergraphs $\mathcal{H}$, $\mathcal{G}$ on a same vertex set.

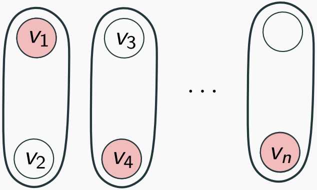
question: are $\mathcal{H}$ and $\mathcal{G}$ dual?

Minimal Transversals Enumeration (Trans-Enum)

input: a (Sperner) hypergraph $\mathcal{H}$.

output: the set $\mathcal{G} = Tr(\mathcal{H})$ of its minimal transversals.

$\# \text{ sol} = 2^{n/2}$



Hypergraph dualization: best known algorithm

Hypergraph Dualization

input: two (Sperner) hypergraphs $\mathcal{H}$, $\mathcal{G}$ on a same vertex set.

question: are $\mathcal{H}$ and $\mathcal{G}$ dual?

Theorem (Fredman & Khachiyan, 1996)

There is an $N^{o(\log N)}$ quasi-polynomial time algorithm solving Hypergraph Dualization where $N = |\mathcal{H}| + |\mathcal{G}|$.

Rough idea: pick an element x_i with high frequency in $\mathcal{H}$ or $\mathcal{G}$, and reduce the problem to the dualization of two separate instances not containing x_i

Yields a quasi-polynomial incremental delay algorithm

Hypergraph dualization: tractable parameters

Can we do better?

Yes, when some parameters are bounded[1]:

- maximum degree[2] & degeneracy[3]: $n^{O(k)}$ delay
- dimension[4]: $(n+m+i)^{O(k)}$ incremental delay
- clique number[5]: $(n+m+d)^{2^{O(k)}}$ total time

Question: can we improve to FPT times $f(k) \cdot N^{O(1)}$?

Known: FPT incremental for max. degree (delay is open)

This talk: FPT delay parameterized by degeneracy + dimension

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[1] these are polynomial for fixed value of k, called XP/slice-wise polynomial

[2] maximum number of edges a vertex intersects

[3] the minimum over all vertex left-to-right orderings of max. left degree

[4] maximum size of an edge

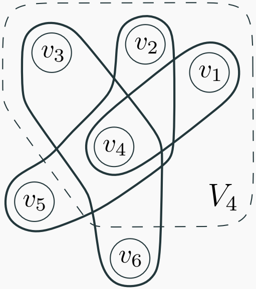
[5] of the underlying graph for hypergraphs of neighbourhoods

Ordered generation: peeling and definition

Introduced by Eiter, Gottlob & Makino (STOC 2002)

Goal: augment min. tr. one (edge) neighborhood at a time

Consider a vertex ordering $v_1, \ldots, v_n$ of a hypergraph $\mathcal{H}$



Sub-hypergraph induced by the *i* first vertices:

$$V_i := \{v_1, \ldots, v_i\}$$
$$\mathcal{H}_i := \{E \in \mathcal{H} : E \subseteq V_i\}$$

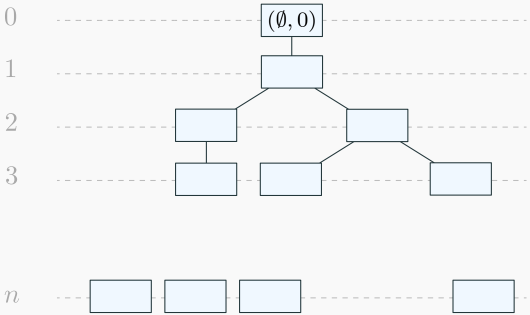
Put $Tr(\mathcal{H}_0) := \{\emptyset\}$

Goal: for all $0 \leq i \leq n$

given $Tr(\mathcal{H}_i)$

enumerate $Tr(\mathcal{H}_{i+1})$

Ordered generation: solution graph



Wanted properties:

(A) no cycle: to avoid repetitions

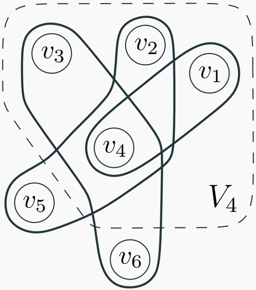
(B) no leaf before level n: to avoid useless computation

(A) No cycle: parent relation

Let (T, i) with $1 \leq i \leq n$ and $T \in Tr(\mathcal{H}_i)$

Parent of (T, i): set T^* obtained by repeating

while there exists a vertex $v \in T$ with no private edge in $\mathcal{H}_{i-1}$ remove one such vertex of smallest label



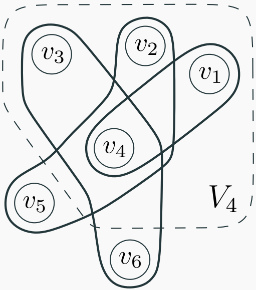
Properties:

- T^* is uniquely defined
- T^* belongs to $Tr(\mathcal{H}_{i-1})$

Put Children(T^*, i) as the set of minimal transversals of $\mathcal{H}_{i+1}$ whose parent is T^*

(B) No stopping branch: extension

Let (T^*, i) with $1 \leq i \leq n$ and $T^* \in Tr(\mathcal{H}_i)$



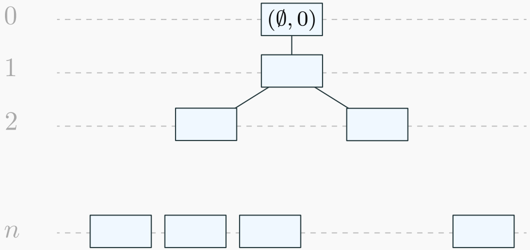
Properties: either

- T^* belongs to $Tr(\mathcal{H}_{i+1})$; or
- $T^* \cup \{v_{i+1}\}$ does

Moreover this set is actually a child of (T^*, i)

Proof sketch: edges of $\mathcal{H}_{i+1}$ not in $\mathcal{H}_i$ are those intersecting v_{i+1}; moreover, private edges of T^* are included in V_i hence may not be lost by adding v_{i+1}

Ordered generation: the theorem FPT style



Theorem

There is an FPT-delay algorithm for Trans-Enum whenever there is one for children generation given any (T^*, i).

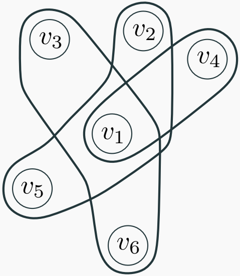
Proof sketch: worst case delay is twice the height of the solution tree times the computation of the next child

Parameters: formal definitions

Let $\mathcal{H}$ be a hypergraph

The degeneracy is the minimum, over all (left-to-right) vertex ordering $v_1, \ldots, v_n$, of the maximum (left) degree $|\{E \in \mathcal{H}_i : v_i \in E\}| \leq k$, $1 \leq i \leq n$

The dimension is the maximum size of an edge in $\mathcal{H}$



Degeneracy 1

Dimension 3

Children generation: brute force approach

Let us denote by k the degeneracy and d the dimension

Let $v_1, \dots, v_n$ be the degeneracy ordering

Goal: generate Children(T^*, i) given $T^* \in Tr(\mathcal{H}_i)$ and $1 \leq i \leq n$

Observations:

- childrens of T^* are of the form $T^* \cup X$ for X a minimal transversal of $\text{inc}_{i+1}(v_{i+1}) := \{E \in \mathcal{H}_{i+1} : v_{i+1} \in E\}$
- $|\text{inc}_{i+1}(v_{i+1})| \leq k$
- $|E| \leq d$ for any $E \in \text{inc}_{i+1}(v_{i+1})$

Brute force approach: compute $\bigcup \text{inc}_{i+1}(v_{i+1})$ in $n^{O(1)}$ time and select among the $2^{k \cdot d}$ obtained subsets those that are minimal transversal of $\mathcal{X}$ and children of (T^*, i).

This can be optimized to $k^d \cdot n^{O(1)}$ by guessing private edges

Children generation: limitations

Unfortunately, this approach fails if we relax

- the dimension: W[1]-hard p.b. degeneracy[6]
- the degeneracy: para-NP-hard p.b. the dimension

FPT for these parameters stays open

Preliminary steps of interest include:

- FPT p.b. the degeneracy for hypergraphs of neighborhoods
  $\equiv$ minimal dominating sets enumeration p.b. degeneracy
- FPT p.b. by dimension in some classes of hypergraphs?
- FPT p.b. generalizations of the above combination
  e.g. degeneracy and edge-intersection/conformality

[6] This even holds for any chosen optimal degeneracy ordering