Master internship proposal:
Copoints enumeration in graph convexities.

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1 Context

A *convexity space* (or *closure system*) on a finite set V is a family $\mathcal{F} \subseteq 2^V$ closed under intersections and containing V; see, e.g., [vdV93]. The sets in $\mathcal{F}$ are called *convex* sets, and when ordered by inclusion they are known to form a lattice $\mathcal{L} = (\mathcal{F}, \subseteq)$. Convexity spaces appear in many fields of computer science and mathematics: logic, databases, knowledge spaces, argumentation, geometry, and combinatorics, to cite a few. In graphs, several models of convexities arise. The most important is that of *geodesic convexity*, where a subset of vertices C of a graph G is *convex* if any vertex on a u–v shortest path belongs to C whenever u and v belong to C. In particular, note that every singleton vertex is convex, and that the set of all vertices is also a convex. Maximal (inclusion-wise) convex sets not containing a vertex v are called *copoints* of v; see Figure 1 for an example.

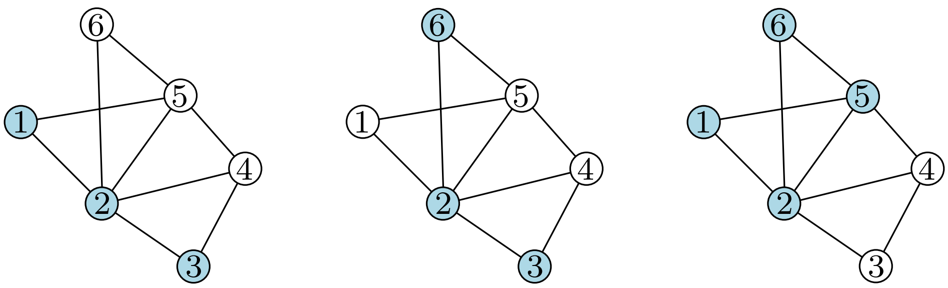

Figure 1: The copoints of vertex 4 in blue.

Enumerating all the copoints of a graph turns out to be a (very) particular case of a long-standing open problem dealing with the enumeration of characteristic models of a Horn formula [Kha95]. It is not difficult to see that the number of copoints may be exponential in n: an example can be constructed by extending the graph in Figure 1. Thus, looking for a running time polynomially bounded by the size of the input is not a reasonable—let alone meaningful—efficiency criterion. Rather, we aim for a so-called output-polynomial time algorithm whose running time is polynomially bounded by the sizes of both the input and output data. To date, however, no better than output-exponential time algorithms are known for the problem [Kha95]. Better algorithms have been exhibited in some restricted convexity spaces such as k-meet-semidistributive lattices [BMN17], modular lattices [Wil00], ranked convex geometries [DNV21], or convexity spaces of bounded Carathéodory number [NV21]. However, and despite their importance, the problem has not yet been addressed in the context of graph convexities.

2 Goal of the internship

The goal of the internship is to initiate the study of copoints enumeration for graph convexities and classes, a novel line of research. As a first direction, we would like to explore the complexity of copoints enumeration (with respect to the geodesic convexity) in graphs classes enjoying strong structural properties, like chordal graphs or bridged graphs. In a second line of research, it would be interesting to study the problem for stronger notions of convexity in graphs, such as *gated convexity*, *monophonic convexity*, or P_3-*convexity*. Finally, another direction concerns the parameterized study of this problem, aiming at improving known XP algorithms such as the one in [NV21] parameterized by the Carathéodory number to FPT for graph convexities.

3 Possibility of extension into a Ph.D.

If successful, the internship could be prolongated to a Ph.D. for a starting date between October 2025 and January 2026 as part of the PARADUAL ANR JCJC project which aims at studying enumeration problems through the lens of parameterized complexity.

4 Prerequisites

No prerequisites other than an interest for graphs, combinatorics, and algorithms.

References

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[Kha95] Roni Khardon. Translating between horn representations and their characteristic models. *Journal of Artificial Intelligence Research*, 3:349–372, 1995.

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