TACL 2011, the Fifth International Conference on

Topology, Algebra, and Categories in Logic

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Preface

TACL 2011, the 5th Conference on *Topology, Algebra, and Categories in Logic*, was held in Marseilles, France, on July 26-30 2011.

The conference was dedicated to the memory of Leo Esakia, who passed away in November 2010. Leo was the founder of the logic school in Georgia. He made numerous important contributions to the study of algebraic and topological semantics of modal and intuitionistic logics. Several landmark theorems in the field bear his name. Leo also contributed immensely to establishing a strong research community working on algebraic, topological, and categorical methods in logic. In particular, he was one of the driving forces in establishing the TACL conference series.

The editors wish to thank the contributing scientists who, with their work and ideas, primarily contributed to the success of the conference and to the wealthiness of TACL research area. They would also like to thank the anonymous reviewers and members of the Program Committee for their collaboration in the process of selecting the contributions.

A special thank goes to the French institutions that made the organization of the conference possible through their financial support:

- Laboratoire d'Informatique Fondamentale de Marseille (LIF, UMR 6166),
- Laboratoire des Sciences de l'Information et des Systèmes (UMR 6168)
- Institut de Mathématiques de Luminy (UMR 6206),
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Marseilles, July 14, 2011,
Luigi Santocanale, Nicola Olivetti, Yves Lafont

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SCIENTIFIC LEGACY OF LEO ESAKIA

GURAM BEZHANISHVILI

This talk will review Leo Esakia's main contributions to intuitionistic and modal logics. It will also have biographical sketches of Leo Esakia, and will discuss his influence on several generations of Georgian (and non-Georgian) mathematicians, as well as his main mathematical interests, most of which are central to the TACL conference series.

NEW MEXICO STATE UNIVERSITY
E-mail address: gbezhani@math.nmsu.edu
URL: http://www.math.nmsu.edu/~gbezhani/

INTUITIONISTIC MODALITIES IN TOPOLOGY AND ALGEBRA

MAMUKA JIBLADZE

In this talk I will review recent results obtained by the Esakia school on intuitionistic modalities, which are motivated by topological, algebraic and categorical considerations. The resulting modalities arise on the lattices of open sets of topological spaces as coderivative operators (duals of the topological derivative). I will discuss algebraic properties of such operators arising from various classes of topological spaces.

Razmadze Mathematical Institute, Tbilisi, Georgia.
E-mail address: mamuka.jibladze@gmail.com

TOPOLOGICAL SEMANTICS OF MODAL LOGIC

DAVID GABELAIA

One of the focal points of research at Leo Esakia's logic group in Tbilisi has been the topological interpretations of modal diamond first put forth my McKinsey and Tarski in the 1940s. In this talk I will survey this research area and report some recent results obtained by Leo Esakia, his students and colleagues. I will try to delineate the main trends, methods and constructions, as well as the remaining open problems in this line of research.

Razmadze Mathematical Institute, Tbilisi, Georgia.
E-mail address: gabelaia@gmail.com

TOPOLOGICAL SEMANTICS OF POLYMODAL PROVABILITY LOGIC

LEV BEKLEMISHEV

Topological semantics of provability logic is well-known since the work of Harold Simmons and Leo Esakia in the 1970s. The diamond modality can be interpreted as a topological derivative operator acting on a scattered topological space. Although quite natural and complete, this semantics has not been much used in the study of provability logics because of the more convenient Kripke semantics. The situation turns out to be different for the polymodal provability logic GLP that has been applied to proof-theoretic analysis of Peano arithmetic. GLP is known to be incomplete w.r.t. any class of Kripke frames.

We study natural topological models of GLP where modalities correspond to derivative operators on a polytopological space $(X, \tau_0, \tau_1, \dots)$. We call such a space *GLP-space* if, for all n, topologies τ_n are scattered, $\tau_n \subseteq \tau_{n+1}$, and $d_n(A)$ is open in τ_{n+1}, for any $A \subseteq X$. Here $d_n(A)$ denotes the set of limit points of A w.r.t. topology τ_n. GLP-spaces are exactly the spaces validating all the axioms of GLP. We show that GLP is complete w.r.t. the class of GLP-spaces (joint work with David Gabelaia).

On the other hand, completeness w.r.t. natural ordinal GLP-spaces turns out to be dependent on large cardinal axioms of set theory and various facts on reflecting stationary sets. In particular, it is consistent (relative to ZFC) that GLP is incomplete. However, under the assumption of large cardinal axioms one can establish at least some partial completeness results. Under the assumption V=L we show that the bimodal fragment of GLP is complete w.r.t. the cardinal $\aleph_\omega$ (taken with the interval topology and the club filter topology).

Steklov Mathematical Institute, Moscow
E-mail address: lbekl@yandex.ru
URL: http://www.mi.ras.ru/~bekl/

HOMOTOPY TYPE THEORY

STEVE AWODEY

Homotopy type theory refers to a new interpretation of Martin-Löf's system of intensional, constructive type theory into abstract homotopy theory. Propositional equality is interpreted as homotopy and type isomorphism as homotopy equivalence. Logical constructions in type theory then correspond to homotopy-invariant constructions on spaces, while theorems and even proofs in the logical system inherit a homotopical meaning.

In parallel, Vladimir Voevodsky (IAS) has recently proposed a comprehensive, computational foundation for mathematics based on this homotopical interpretation of type theory. The Univalent Foundations Program posits a new *univalence axiom* relating propositional equality on the universe with homotopy equivalence of small types. The program is currently being implemented with the help of the automated proof assistant Coq.

This talk will survey some of these recent developments.

Carnegie Mellon, Pittsburgh
E-mail address: `awodey@cmu.edu`
URL: `http://www.andrew.cmu.edu/user/awodey/`

RELATIVIZING THE SUBSTRUCTURAL HIERARCHY†

NIKOLAOS GALATOS

Substructural logics are generalizations of classical and intuitionistic logic and they include among others relevance, many-valued, and linear logic. Their algebraic semantics are classes of residuated lattices, examples of which include Boolean algebras, Heyting algebras, lattice-ordered groups, ideal lattices of rings, and relation algebras. Residuated frames provide relational, Kripke-style, semantics for substructural logics. They differ from Kripke frames for modal and intuitionistic logic in that they are not based on distributive logics (so two sets of worlds need to be considered) and in that the accessibility relation is ternary. The apparent complexity of the structure does not allow direct combinatorial manipulation of the frames as in modal and intuitionistic logic. However, the generality of residuated frames essentially encompasses proof theory, and proof-theoretic tools can be extended and applied to residuated frames yielding results on both logic and algebra. Moreover, they provide a way to organize and generalize proof-theoretic systems (from sequents to hypersequents and beyond) in view of a hierarchy of formulas.

We will first survey the above in a historically inaccurate but pedagogically instructive and very accessible way, showing how they can all be derived in a natural fashion from simple algebraic properties of residuated lattices. Having obtained a deep and fundamental algebraic understanding of the substructural hierarchy and of residuated frames, we will show how to relativize the hierarchy to the involutive and to the distributive cases and we will obtain two new applications of the theory.

†Partly based on joint work with A. Ciabattoni and K. Terui, and with P. Jipsen.

University of Denver
E-mail address: ngalatos@du.edu
URL: http://web.cs.du.edu/~ngalatos/

THE POSSIBLE VALUES OF CRITICAL POINTS BETWEEN VARIETIES OF ALGEBRAS

PIERRE GILLIBERT

Why do so many representation problems in algebra, enjoying positive solutions in the finite case, have counterexamples of minimal cardinality either $\aleph_0$, $\aleph_1$, $\aleph_2$ and no other cardinality? By a representation problem, we mean that we are given two functors with the same codomain, and we compare there range. We also assume that the involved categories are equipped with a notion of "cardinality".

Examples of such representation problems cover various fields of mathematics. Here are a few examples, among many:

(1) Every (at most) countable Boolean algebra is generated by a chain (cf. [6, Theorem 172]), but not every Boolean algebra is generated by a chain (cf. [6, Lemma 179]). It is an easy exercise to verify that in fact, every subchain C of the free Boolean algebra F on $\aleph_1$ generators is countable, thus F cannot be generated by C.

(2) Every dimension group with at most $\aleph_1$ elements is isomorphic to $K_0(R)$ for some (von Neumann) regular ring R (cf. [1, 5]), but there is a dimension group with $\aleph_2$ elements which is not isomorphic to $K_0(R)$ for any regular ring R (cf. [14]).

(3) Every distributive algebraic lattice with at most $\aleph_1$ compact elements is isomorphic to the congruence lattice of some lattice (cf. [7, 8, 9]), but not every distributive algebraic lattice is isomorphic to the congruence lattice of some lattice (cf. [15]); the minimal number of compact elements in a counterexample, namely $\aleph_2$, is obtained in [11].

J. Tůma and F. Wehrung introduced in [13] a particular case of the kind of representation problem considered above, concentrated in the notion of critical point between two varieties of (universal) algebras. The critical point is related to the following statement:

($*$) For each algebra in the first variety there is an algebra in the second variety with the same congruence lattice.

If ($*$) does not hold for two varieties, the critical point is the minimal number of compact congruences of a counterexample.

For example, the critical point between the variety of all lattices and the variety of all groups is $\aleph_2$ (cf. [12]). The critical point between variety of all majority algebras and the variety of all lattices is $\aleph_2$ (cf. [10]).

In this talk we shall study the case where both varieties are locally finite, and the second variety satisfies a "smallness" condition, which turns out to be satisfied if the variety is finitely generated and congruence-modular. Under these conditions, either ($*$) is satisfied or the critical point is at most $\aleph_2$ (cf. [3]). The proof relies on the *Armature Lemma* (cf. [4]) it allows to construct a counterexample to ($*$), using a diagram counterexample to the generalization of ($*$).

We shall also show that in the case of varieties of lattices, the local finiteness assumption can be dropped modulo a rather weak assumption on the simple members of the second variety, moreover, under that assumption, the containment between two congruence classes can occur only for the obvious reason, namely, the first variety is contained in either the second variety or its dual variety (cf. [2]).

References

[1] G. A. Elliott, *On the classification of inductive limits of sequences of semisimple finite-dimensional algebras*, J. Algebra **38** (1976), 29–44.

[2] P. Gillibert, *The possible values of critical points between varieties of lattices*, preprint 2010. Available online at `http://hal.archives-ouvertes.fr/hal-00468048/`.

[3] P. Gillibert, *The possible values of critical points between strongly congruence-proper varieties of algebras*, preprint 2011. Available online at `http://hal.archives-ouvertes.fr/hal-00587671/`.

[4] P. Gillibert and F. Wehrung, *From objects to diagrams for ranges of functors*, to appear in Springer Lecture Notes in Mathematics, 165 pages. Available online at `http://hal.archives-ouvertes.fr/hal-00462941/`.

[5] K. R. Goodearl and D. E. Handelman, *Tensor products of dimension groups and K_0 of unit-regular rings*, Canad. J. Math. **38**, no. 3 (1986), 633–658.

[6] G. Grätzer, "Lattice Theory: Foundation". Birkhäuser Verlag, Basel, 2011. xxix+613 p. ISBN: 978-3-0348-0017-4

[7] A. P. Huhn, *On the representation of distributive algebraic lattices, I*, Acta Sci. Math. **45** (Szeged) (1983), 239–246.

[8] A. P. Huhn, *On the representation of distributive algebraic lattices. II*, Acta Sci. Math. **53** (1989), 3–10.

[9] A. P. Huhn, *On the representation of distributive algebraic lattices. III*, Acta Sci. Math. **53** (1989), 11–18.

[10] M. Ploščica, *Non-representable distributive semilattices*, J. Pure Appl. Algebra **212** (2008), no. 11, 2503–2512.

[11] P. Růžička, *Free trees and the optimal bound in Wehrung's theorem*, Fund. Math. **198** (2008), no. 3, 217–228.

[12] P. Růžička, J. Tůma, and F. Wehrung, *Distributive congruence lattices of congruence-permutable algebras*, J. Algebra **311** (2007), no. 1, 96–116.

[13] J. Tůma and F. Wehrung, *A survey of recent results on congruence lattices of lattices*, Algebra Universalis **48** (2002), no. 4, 439–471.

[14] F. Wehrung, *Non-measurability properties of interpolation vector spaces*, Israel J. Math. **103** (1998), no. 1, 177–206.

[15] F. Wehrung, *A solution to Dilworth's congruence lattice problem*, Adv. Math. **216** (2007), no. 2, 610–625.

Charles University in Prague, Faculty of Mathematics and Physics, Department of Algebra, Sokolovská 83, 186 00 Prague, Czech Republic.

E-mail address: `gilliber@karlin.mff.cuni.cz, pgillibert@yahoo.fr`

A FEW PEARLS IN THE THEORY OF QUASI-METRIC SPACES

JEAN GOUBAULT-LARRECQ

Quasi-metrics are just like metrics, except you don't require the distance from x to y to be the same as from y to x: think of distance as measuring the effort it takes to go from x to y, and comparing climbing up and down a mountain. I'll show that several classic results in the theory of metric spaces transfer to quasi-metric spaces — with some complications. And I'd like to mention some of the results that please me most:

1. Completeness splits into two notions, a strong one (Smyth completeness) and a weak one (Yoneda completeness, due to Rutten, Bonsangue, and van Breugel), but both can be characterized elegantly through the notion of formal balls, as shown by Kostanek-Waszkiewicz and by Romaguera-Vallero, both in 2010. My contribution here will be in a simpler presentation of the Romaguera-Vallero Theorem, using Erné's notion of c-space and sobriety (equivalently, the domain-theoretic notion of abstract bases, and rounded ideal completion).
2. The space of probability distributions on a quasi-metric space is itself quasi-metric, and I'll mention a quasi-metric form of the Kantorovitch-Rubinstein theorem, itself an infinite version of linear programming duality.
3. Generalizing the Kantorovitch-Hutchinson metric, one can compare states from infinite-state 2 player turn-based stochastic games, in such a way that close states yield close values of the (min-, max-) expected payoff. This is called a simulation distance.
4. There is a simple modal logic, interpreted on the latter games, which characterizes similarity, and simulation distances.

I only have a modest contribution in Item 1. Items 2 and 3 are from one of my papers at FOSSACS'08. Item 4 is new.

ENS Cachan, CNRS, INRIA
E-mail address: goubault@lsv.ens-cachan.fr
URL: http://www.lsv.ens-cachan.fr/~goubault/

THROUGH THE LOOKING-GLASS: UNIFICATION, PROJECTIVITY, AND DUALITY

VINCENZO MARRA

Solving equations has long featured as a main item in the mathematician's agenda, at the very least since Diophantus' *Arithmetica* appeared. Unification is the modern version of the quest for general solutions to systems of equations within a class of domains. If the class in question is equationally definable, one speaks of unification modulo an equational theory. A straightforward universal-algebraic reformulation shows that, in this case, one is really concerned with solving equations in a given variety of algebras. A tight connection between unification problems in a variety and finitely presented projective objects in the variety was established by Ghilardi in 1997. When coupled with Gabriel's and Ulmer's 1971 categorial abstraction of finitely presentable object, Ghilardi's work extends the scope of the theory of unification to all locally small categories.

What tools do we have to understand the structure of the set of solutions to a unification problem in such general contexts, or even just in varieties of algebras? I argue that key insights can usually be gained by dualising the problem — provided, of course, that an efficient description of the dual category is available. I support this claim with selected examples where such a description is known, including Boolean algebras, distributive lattices, rings, and MV-algebras. (The featured talk by Luca Spada discusses unification for MV-algebras in greater detail.) Through these case studies, an interesting pattern begins to emerge. Stepping from syntax to semantics through the looking-glass provided by a duality theory allows to develop an analogy between the structure of the set of solutions to a unification problem, and such homotopy-theoretic notions as the universal covering space of a sufficiently connected topological space. I discuss some preliminary results indicating that this analogy may indeed be fruitful.

Dipartimento di Informatica e Comunicazione, Università degli Studi di Milano, Italy
E-mail address: marra@dico.unimi.it
URL: http://marra.dico.unimi.it

A CATEGORICAL ACCOUNT OF KRIVINE'S CLASSICAL REALIZABILITY

THOMAS STREICHER

After revisiting the basic ideas of Krivine's classical realizability we give an algebraic reformulation of the basic ingredients by identifying the notion of an Abstract Krivine Structure (aks). Such structures may be understood as an extension of combinatory logic by stacks (i.e continuations) and control operators. We prove that in a precise sense Cohen forcing is the commutative case of classical realizability. Further, we show how classical realizability over an arbitrary aks gives rise to a "classical realizability tripos" this way subsuming Krivine's highly original work in more traditional terms, namely the categorical account of realizability as originated by Martin Hyland's work in the early 1980s. Finally, we sketch how to build forcing models on top of classical realizability and how such a 2 step construction can be understood as a single classical realizability construction. This is a necessary requirement for building classical models of ZF validating stronger and stronger choice principles.

Technical University Darmstadt
E-mail address: `streicher@mathematik.tu-darmstadt.de`
URL: `http://www.mathematik.tu-darmstadt.de/~streicher/`

Generalised type setups for dependently sorted logic

Peter Aczel
Schools of Mathematics and Computer Science
The University of Manchester
petera@cs.man.ac.uk

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1 Introduction

Dependently sorted logic is intended to be a generalisation of many sorted logic that allows sorts to depend on variables in the kind of way that types can depend on variables in dependent type theories. The notion of a *generalised Type Setup (gTS)* is a generalisation of the notion of a *type setup*. The latter notion was previously introduced by me, in about 2004, as a suitable abstract notion of type theory for specifying the sorts and terms dependent on contexts of variable declarations, needed to formulate a dependently sorted logic. The aim of my paper will be to review the notion of a type setup and introduce the more general notion.

2 Some earlier examples of notions of dependently sorted logics

A seminal example of a logic having dependent sorts is the logic for equational reasoning in Cartmell's *Generalised Algebraic (GA) theories*, [Cartmell, 1978, Cartmell, 1986]. A typical example of a GA theory is the axiom system for the notion of a category which has a sort Obj of objects and a sort constructor Hom for forming a sort $Hom(x, y)$ of maps $x \to y$, for $x, y : Obj$ and has function symbols, i,e, term constructors, id and $comp$ for forming identity terms $id(x)$, for $x : Obj$ and composition maps $comp(x, y, z, u, v) : Hom(x, z)$ for $x, y, z : Obj$, $u : Hom(x, y)$, $v : Hom(y, z)$. The notion of a GA theory, being purely equational, does not capture the full generality of the idea of a dependently sorted logic, which allows first order formulae which are not equations. Nevertheless the formulation of the general notion of a signature for the language of a GA theory and the definition of the syntactic categories of contexts of variable declarations, sorts and terms is already rather complicated.

Another significant example of a logic having dependent sorts is Makkai's *first order logic with dependent sorts (FOLDS)*,[Makkai, 1995]. While $FOLDS$ has the usual logical constants for first order logic it avoids having individual constants or function symbols and so can get away with a simple notion of sort because the only terms are the variables so that each sort must either be a constant sort or else must have the form $s(x_1, \ldots, x_n)$ where s is an n-place sort constructor, for $n > 0$, and $x_1, \ldots, x_n$ is a list of variables.

The notion of a *Logic-enriched type theory*, [Aczel and Gambino, 2002], [Gambino and Aczel, 2006], provides another kind of example of a notion of logic with dependent sorts, the sorts being the dependent types of a type theory. Here the sorts and terms are expressions of the type theory, the type theory itself having a rather complicated structure. In order to give a precise definition of the idea of a logic-enrichment of a type theory it is necessary to have a precise definition of the notion of a type theory. In [Gambino and Aczel, 2006] a concrete syntactic notion of *standard pure type theory* was presented that had certain forms of judgement and allowed an arbitrary system of rules for deriving judgements.

3 Type Setups

It has seemed desirable to formulate a more general notion of dependently sorted logic which would include the above examples. What seems to be needed is a more general, more abstract, notion of *dependent type theory*. Already, in [Cartmell, 1978, Cartmell, 1986], Cartmell introduced the notions of *category with attributes and contextual category*, claiming that, for his GA theories, contextual categories are the algebraic structures

> *'that structurally correspond exactly to the syntactically defined theories'*

After Cartmell's work category theorists have come up with many variations on the notions of category with attributes and contextual category; e.g. category with display maps, [Taylor, 1986], comprehension category, [Jacobs, 1991], category with families, [Dybjer, 1996]. These are intended to capture at various levels of abstraction the syntax and/or semantics of dependent type theories.

But none of these notions have seemed to me to be quite right as an abstract notion of 'type theory' suitable for a sufficiently general formulation of the notion of a dependently sorted logic over a 'type theory' that would give a smooth generalisation of the usual way first order logic is presented. I have introduced the notion of a *type setup* for that purpose, see [Aczel, 2009], and the notion has been investigated by Joao Belo, [Belo, 2007, Belo, 2009].

In contrast to the above earlier notions, type setups make use of an explicit notion of variable with contexts as finite sequences of variable declarations, $x : A$, and substitutions as finite sequences of variable assignments, $x := a$. From a purely category theoretic point of view the explicit use of variables is not essential. Nevertheless, even in our abstract setting we prefer to keep to the logically familiar use of variables so as to have a smooth generalisation of the traditional presentation of many-sorted logic.

The contexts, Γ, of a type setup are the objects of a category $\mathcal{C}$ whose maps $\gamma : \Gamma' \to \Gamma$, play the role of substitutions. Associated with each context Γ is a set of Γ-types and, for each Γ-type A there is a set of Γ-terms of type A. The substitutions $\gamma : \Gamma' \to \Gamma$ act, functorially on types and terms so that, for each Γ-type A, $A[\gamma]$ is a Γ'-type and $a[\gamma]$ is a Γ'-term of type $A[\gamma]$ for each Γ-term a of type A. As with the earlier category theoretic notions for dependent types, a fundamental ingredient of the notion of a type setup are axioms for the 'comprehension extension' of a context. In a type setup a given a Γ-type A can be extended by adding a new variable declaration $x : A$ to obtain the context $(\Gamma, x : A)$, provided that the variable x is Γ-free; i.e. has not been already declared in Γ. In addition, given a substitution $\gamma : \Delta \to \Gamma$ and a Γ-term a of type A, the substitution γ can be extended to a substitution $(\gamma, x := a) : \Delta \to (\Gamma, x : A)$ such that $x[(\gamma, x := a)] = a$.

Given a type setup and a signature of sorted predicate symbols we have all the ingredients needed to formulate the syntax of formulae of a dependently sorted logic, with a notion of Γ-formula inductively generated from the atomic Γ-formulae using the usual connectives and quantifiers, the quantifiers having the forms $(\forall x : A)$, $(\exists x : A)$ where A is a Γ-type and the variable x is Γ-free. The action of the substitutions of the type setup on formulae can be defined by structural recursion on formulae in the usal way and a natural deduction style axiomatisation of intuitionistic logic using sequents can be formulated.

4 The notion of a Generalised Type Setup (gTS)

Concrete dependent type theories generally use untyped variables, but use contexts which are finite sequences of variable declarations, that associate a type with each declared variable. So the contexts form a tree structure of finite sequences, with the empty context at the root and each context having as its children its extensions by adding a new variable declaration. The notion of a contextual category also has such a tree structure. Other notions of category for type dependency, such as the notions of category with attributes and category with families, are more aimed at the semantics of type dependency and do not impose the tree structure.

As with concrete dependent type theories the notion of a type setup also uses a tree structure of finite sequences of variable declarations. I now think that this extra structure, which only complicates the presentation, is unnecessary for the purpose of the formulation of dependently sorted logic and the new notion of generalised type setup avoids the extra structure. Of course the notion of a gTS keeps the fundamental notion of context extension, as do all the various competing notions.

In my talk I hope to explain the notion of a gTS, describe intuitionistic predicate logic with equality over a gTS and, if there is time, outline an application of a logic-enriched type theory.

References

[Aczel, 2009] P. Aczel, *Predicate Logic over a Type Setup*, slides in `http://www.cs.man.ac.uk/ petera/papers.html`, of a talk given at the meeting "Philosophy and Foundations of Mathematics: Epistemological and Ontological Aspects", a conference dedicated to Per Martin-Lof on the occasion of his retirement, Uppsala, May 2009.

[Aczel and Gambino, 2002] P. Aczel and N. Gambino, *Collection Principles in Dependent Type Theory*, **Types for Proofs and Programs** (P. Callaghan, Z. Luo,, J. McKinna, and R. Pollack, editors), Lecture Notes in Computer Science, vol 2277, Springer, pp. 1-23, 2002.

[Belo, 2007] J. Belo, *Dependently Sorted Logic*, **TYPES'07**, Proceedings of the 2007 international conference on Types for proofs and programs, Springer-Verlag Berlin, Heidelberg, 2008

[Belo, 2009] J. Belo, Ph.D. thesis, Manchester University, 2009.

[Cartmell, 1978] J. Cartmell, *Generalised Algebraic theories and Contextual Categories*, Ph.D. thesis, Univ. Oxford, 1978.

[Cartmell, 1986] J. Cartmell, *Generalised Algebraic theories and Contextual Categories*, Annals of Pure and Applied Logic, 32:209-243, 1986.

[Dybjer, 1996] P. Dybjer, *Internal Type Theory*, **Types for Proofs and Programs**, (S. Berardi and M. Coppo, editors), Lecture Notes in Computer Science, vol 1158, Springer, pp. 120-134, 1996.

[Gambino and Aczel, 2006] N. Gambino and P. Aczel, *The Generalised Type-Theoretic Interpretation of Constructive Set Theory*, Journal of Symbolic Logic, vol.71, pp. 67-103, 2006.

[Jacobs, 1991] B. Jacobs, *Categorical Type Theory*, Ph.D. thesis, Nijmegen, 1991.

[Makkai, 1995] M. Makkai, *First Order Logic with Dependent Sorts, with Applications to Category Theory*, preprint, McGill University, 1995.

[Taylor, 1986] P. Taylor, *Recursive domains, indexed category theory and polymorphism*, Ph.D. thesis, Cambridge University, 1986

THE EQUATIONAL THEORY OF KLEENE LATTICES

H. ANDRÉKA, SZ. MIKULÁS AND I. NÉMETI

Abstract. Languages and families of binary relations are standard interpretations of Kleene algebras. It is known that the equational theories of these interpretations coincide. We investigate the identities valid in these interpretations when we expand the signature of Kleene algebras with the meet operation. In both cases meet is interpreted as intersection. We show that there are more identities valid in language algebras than in relation algebras (exactly three more in some sense). We also look at the picture when we exclude the identity from the signature. We prove that in this case the equational theories of the two kinds of interpretations coincide again.

1. Introduction

Kleene algebras (KA) are extensively investigated in language theory and in programming logics, see, e.g. [1, 2, 4, 7]. Two prominent types of Kleene algebras are language algebras and relation algebras. They are defined as follows. Let Σ be a set (alphabet) and Σ^* denote the free monoid of all finite words over Σ including the empty word λ. The class of *language Kleene algebras* is defined as the class of subalgebras of algebras of the form

$$(\wp(\Sigma^*), +, ;, ^*, 0, 1)$$

where $+$ is set union, ; is complex concatenation (of words)

$$X\,;Y = \{wv : w \in X, v \in Y\}\,, \tag{1}$$

* is the Kleene star operation

$$X^* = \{w_1 \dots w_n : w_1, \dots, w_n \in X \text{ for some natural number } n\}\,, \tag{2}$$

0 is the empty language and 1 is the singleton language consisting of the empty word λ. We will denote the class of language Kleene algebras by LKA.

The class of *relational Kleene algebras* is defined as the class of subalgebras of algebras of the form

$$(\wp(W), +, ;, ^*, 0, 1)$$

where $W = U \times U$ for some set U, $+$ is set union, ; is relation composition

$$x\,;y = \{(u,v) \in W : (u,w) \in x \text{ and } (w,v) \in y \text{ for some } w\}\,, \tag{3}$$

* is reflexive-transitive closure, 0 is the emptyset and 1 is the identity relation restricted to W

$$1 = \{(u,v) \in W : u = v\}\,. \tag{4}$$

We will denote the class of relational Kleene algebras by RKA.

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It is well known that the same equations are true in language Kleene algebras and in relational Kleene algebras:

$$\mathsf{Eq}(\mathsf{LKA}) = \mathsf{Eq}(\mathsf{RKA}) \tag{5}$$

where $\mathsf{Eq}(\mathsf{K})$ denotes the set of equations valid in the class K of algebras. Equation (5) can be established by showing that LKA and RKA have the same free algebra, viz. the algebra of regular languages, see, e.g. [6] for the argument in the context of dynamic algebras.

The meet $\cdot$ operation is not included into the similarity type of Kleene algebras, although it has a natural interpretation in both language and relation algebras, viz. intersection. Thus we can expand the signature of Kleene algebras by meet, and define Kleene algebras with meet, or *Kleene lattices* [5, 3]. The class of *language Kleene lattices*, LKL, is defined similarly to LKA with the additional requirement that $\cdot$ is interpreted as intersection. The class of *relational Kleene lattices*, RKL is defined as the analogous expansion of RKA.

The question that we address in this paper is whether the language and relational interpretations of Kleene lattices have the same equational theory. One of the main results is that there are more valid equations in LKL than in RKL, but $\mathsf{Eq}(\mathsf{LKL})$ is finitely axiomatisable over $\mathsf{Eq}(\mathsf{RKL})$: Theorem 3.1. The reason for the different equational theories is that the free algebra is no longer atomic in the lattice case, e.g. the term $x \cdot y$ is smaller than the term x, and similarly, the term $1 \cdot y$ is smaller than the term 1. However, 1 is still an atom in every language algebra, while not necessary an atom in a relation algebra. So the culprit is the identity operation 1 interacting with meet $\cdot$. This motivates the following definition of *identity-free Kleene lattices.*

Recall the standard abbreviation x^+ for $x;x^*$. Let KA^- denote the class of generalised subreducts of elements of KA to the signature $(+, ;, ^+, 0)$. We will use similar notation for other classes of algebras: LKA^- and LKL^- denote language algebras, and RKA^- and RKL^- denote relation algebras of the similarity types where 1 and * are replaced by $^+$. The other main result is that if we omit occurrences of 1 (even implicitly as in x^*), then the equational theories of language and relation algebras coincide: $\mathsf{Eq}(\mathsf{LKL}^-) = \mathsf{Eq}(\mathsf{RKL}^-)$, see Theorem 2.1.

2. Equations valid in algebras with meet but no identity

The following theorem says that if the identity operation is not present, even implicitly in the *-operation, then the same equations hold in language and in relation algebras.

Theorem 2.1. *The equational theories of* LKL^- *and* RKL^- *coincide:*

$$\mathsf{Eq}(\mathsf{LKL}^-) = \mathsf{Eq}(\mathsf{RKL}^-)\,.$$

The key observation is the following technical lemma which establishes a connection between language and relation algebras.

Lemma 2.2. *For every term τ in which none of $1, 0, +, ^*, ^+$ occurs, there are a "characteristic" language algebra $\mathfrak{A}_\tau$, word w_τ and valuation k_τ of the variables such that*

(1) $w_\tau \in \tau^{\mathfrak{A}_\tau}[k_\tau]$

(2) $\mathsf{RKL}^- \models \tau \leq \sigma$ *whenever* $w_\tau \in \sigma^{\mathfrak{A}_\tau}[k_\tau]$ *for any term* σ.

Proof of Theorem 2.1. Assume $\mathsf{LKL}^- \models \tau \leq \sigma$. It is easy to see that both language and relation algebras have the following "continuity" property: there is a set $\{\tau_i : i \in I\}$ such that none of $0, +, ^+$ occurs in τ_i for any $i \in I$, and also

$$\tau^{\mathfrak{A}}[k] = \bigcup\{\tau_i^{\mathfrak{A}}[k] : i \in I\} \tag{6}$$

for any $\mathfrak{A} \in \mathsf{LKL}^- \cup \mathsf{RKL}^-$ and for any appropriate valuation k of the variables. In particular, (6) holds when $\mathfrak{A}$ is the free algebra of RKL^-, since the free algebra is in $\mathbf{SP}\mathsf{RKL}^-$.

Let $i \in I$ be arbitrary. Then $\mathsf{LKL}^- \models \tau_i \leq \sigma$ by equation (6). Let $\mathfrak{A}_{\tau_i}$, w_{τ_i} and k_{τ_i} be such that they satisfy the conditions of Lemma 2.2. Then $w_{\tau_i} \in \tau_i^{\mathfrak{A}_{\tau_i}}[k_{\tau_i}]$ by these conditions, and so $w_{\tau_i} \in \sigma^{\mathfrak{A}_{\tau_i}}[k_{\tau_i}]$ by $\mathsf{LKL}^- \models \tau_i \leq \sigma$. Hence $\mathsf{RKL}^- \models \tau_i \leq \sigma$ by Lemma 2.2. Thus $\mathsf{RKL}^- \models \tau \leq \sigma$ by equation (6) (when $\mathfrak{A}$ is the free algebra of RKL^-). By this, Theorem 2.1 has been proved. □

3. Equations valid in Kleene lattices

Now we turn to equations valid in language algebras LKL. All the equations valid in relation algebras are valid in language algebras, too. Hint: use the Cayley-representation

$$f(X) = \{(w, wx) : w \in \Sigma^* \text{ and } x \in X\}.$$

However, more equations are valid in language algebras than in relation algebras. In fact, it is easy to check that the following equations are not valid in relation algebras, while they are valid in language algebras, since $1 = \{\lambda\}$ and λ cannot be written as a concatenation of non-empty words.

$$(x \,;\, y) \cdot 1 = (x \cdot 1) \,;\, (y \cdot 1) \tag{7}$$

$$(x \cdot 1) \,;\, y = y \,;\, (x \cdot 1) \tag{8}$$

$$(z + (x \cdot 1) \,;\, y)^* = z^* + (x \cdot 1) \,;\, (z + y)^* \tag{9}$$

Our next result states that the above three additional equations axiomatise $\mathsf{Eq}(\mathsf{LKL})$ over $\mathsf{Eq}(\mathsf{RKL})$.

Theorem 3.1. $\mathsf{Eq}(\mathsf{LKL})$ *is finitely axiomatisable over* $\mathsf{Eq}(\mathsf{RKL})$*:*

$$\mathsf{Eq}(\mathsf{RKL}) \cup \{(7), (8), (9)\} \vdash \mathsf{Eq}(\mathsf{LKL}).$$

To prove Theorem 3.1, we state some lemmas first. We call a term τ to be in *normal form* if τ is of form

$$\eta \,;\, \tau'$$

with either η or τ' possibly missing, where η is of form $(x_1 \,;\, \ldots \,;\, x_n) \cdot 1$ with n a natural number and $x_1, \ldots, x_n$ distinct variables, and τ' is a term in which 1 does not occur and * occurs only in the form of $^+$, i.e. τ' is a term in the language of LKL^-. Let $\mathcal{E} = \mathsf{Eq}(\mathsf{RKL}) \cup \{(7), (8), (9)\}$.

Lemma 3.2. *Assume* $\mathcal{E}$. *Each term* τ *is provably equivalent to a finite sum of terms in normal form.*

The proof of the following corollary uses Lemma 2.2.

Corollary 3.3. *If* $\mathsf{LKL} \models \tau \leq \sigma$ *such that* 1 *and* * *do not occur in* τ *(but* $^+$ *may occur in* τ*, and* 1 *and* $^+$ *may occur in* σ*), then* $\mathsf{RKL} \models \tau \leq \sigma$*.*

Lemma 3.4. *Assume* $\mathcal{E}$*. Let* $x_1, \ldots, x_n$ *be variables,* τ *be an arbitrary term, and let* τ' *denote the term we obtain from* τ *by replacing each occurrence of* x_i *with* $x_i + 1$ *for* $i \leq n$*. Then* $\mathcal{E}$ *proves*

$$((x_1 \,;\, \ldots \,;\, x_n) \cdot 1) \,;\, \tau = ((x_1 \,;\, \ldots \,;\, x_n) \cdot 1) \,;\, \tau' .$$

Proof of Theorem 3.1. Assume that $\mathsf{LKL} \models \tau \leq \sigma$. By Lemma 3.2, $\mathcal{E}$ proves that τ is equivalent to a sum of terms in normal form, say $\tau = \sum \eta_i \,;\, \tau_i$. By the equations for join + in $\mathcal{E}$ expressing that + is supremum then it is enough to prove for each i that $\eta_i \,;\tau_i \leq \sigma$. We will omit the indices i, so it is enough to prove that $\mathcal{E} \vdash \eta \,;\, \tau \leq \sigma$, where $\eta \leq 1$ and 1 does not occur in τ. We know that $\mathsf{LKL} \models \eta \,;\, \tau \leq \sigma$. Let η' and σ' be the terms we obtain from η and σ by replacing all the variables x_j occurring in η with $x_j + 1$. Then $\mathsf{LKL} \models \eta' \,;\, \tau \leq \sigma'$, because we get this if we choose any evaluation for the variables occurring in η such that they contain the identity. Since $1 \leq \eta'$ (all operations are monotone), we have $\mathsf{LKL} \models \tau \leq \sigma'$. By Corollary 3.3 then $\mathsf{RKL} \models \tau \leq \sigma'$, so $\mathcal{E}$ proves $\tau \leq \sigma'$. Also, $\mathcal{E}$ proves $\eta \,;\, \sigma = \eta \,;\, \sigma'$ by Lemma 3.4. Now, we get $\eta \,;\, \tau \leq \eta \,;\, \sigma' = \eta \,;\, \sigma \leq \sigma$, and we are done. $\square$

References

[1] J.H. Conway. *Regular Algebra and Finite Machines*. Chapman and Hall, 1971.

[2] Z. Ésik and L. Bernátsky. Equational properties of Kleene algebras of relations with conversion. *Theoretical Computer Science*, 137:237–251, 1995.

[3] P. Jipsen. From semirings to residuated Kleene lattices. *Studia Logica*, 76(2):291–303, 2004.

[4] D. Kozen. A completeness theorem for Kleene algebras and the algebra of regular events. *Information and Computation*, 110:366–390, 1994.

[5] D. Kozen. On action algebras. In J. van Eijck and A. Visser, editors, *Logic and Information Flow*, pages 78–88. MIT Press, 1994.

[6] I. Németi. Every free algebra in the variety generated by the representable dynamic algebras is separable and representable. *Theoretical Computer Science*, 17:343–347, 1982.

[7] V. Pratt, Action logic and pure induction. In J. van Eijck, editor, *Logics in AI: European Workshop JELIA '90* pages 97–120. Springer, 1990.

Alfréd Rényi Institute of Mathematics
of the Hungarian Academy of Sciences
Budapest P.O.Box 127, H-1364 Hungary
andreka@renyi.hu, nemeti@renyi.hu,
http://www.renyi.hu/˜andreka/, http://www.renyi.hu/˜nemeti/

Department of Computer Science and Information Systems
Birkbeck, University of London
Malet Street, London WC1E 7HX, UK
szabolcs@dcs.bbk.ac.uk.
http://www.dcs.bbk.ac.uk/˜szabolcs/

A topos-theoretic approach to Stone-type dualities

Extended Abstract for a Presentation at TACL 2011

Olivia Caramello

DPMMS, University of Cambridge,

Wilberforce Road, Cambridge CB3 0WB, U.K.

O.Caramello@dpmms.cam.ac.uk

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We present a general topos-theoretic interpretation of 'Stone-type dualities'; by this term we refer, following the standard terminology, to a class of dualities or equivalences between categories of preordered structures and categories of posets, locales or topological spaces, a class which notably includes the classical Stone duality for Boolean algebras (or, more generally, for distributive lattices), the duality between spatial frames and sober spaces, the equivalence between preorders and Alexandrov spaces, the Lindenbaum-Tarski duality between sets and complete atomic Boolean algebras, and the Birkhoff's duality between finite distributive lattices and finite posets.

We introduce an abstract framework in which all of these dualities are interpreted as instances of just one topos-theoretic phenomenon, and in which several new dualities are introduced. In fact, the known dualities, as well as the new ones, all arise from the application of one 'general machinery for generating dualities' to specific 'sets of inputs' which vary from case to case.

Conceptually, our 'machinery' arise from the process of functorially transferring topos-theoretic invariants across two different sites of definition of the same topos, according to the method 'toposes as bridges' introduced in [2]. Specifically, the dualities between a given category $\mathcal{K}$ of preorders and a category of locales are generated by assigning to each structure $\mathcal{C}$ of $\mathcal{K}$, equipped with a subcanonical Grothendieck topology $J_{\mathcal{C}}$ in such a way that the morphisms in the category $\mathcal{K}$ induce morphisms of the associated sites, the locale $Id_{J_{\mathcal{C}}}(\mathcal{C})$ of $J_{\mathcal{C}}$-ideals on $\mathcal{C}$, and from the inverse process of functorially recovering $\mathcal{C}$ from the locale $Id_{J_\mathcal{C}}(\mathcal{C})$ (equivalently, from the topos $\mathbf{Sh}(\mathcal{C}, J_\mathcal{C}) \simeq \mathbf{Sh}(Id_{J_\mathcal{C}}(\mathcal{C}))$) through a topos-theoretic invariant (equivalently, a frame-theoretic invariant). The frame-theoretic invariants which enable us to identify the principal ideals on $\mathcal{C}$ among all the ideals in $Id_{J_\mathcal{C}}(\mathcal{C})$, and hence to recover a poset $\mathcal{C}$ from the topos $\mathbf{Sh}(\mathcal{C}, J_\mathcal{C})$ through a topos-theoretic invariant, are generalized notions of compactness; specifically, if the Grothendieck topologies $J_\mathcal{C}$ are 'uniformly defined' by a frame-theoretic invariant C of families of elements of a frame (technically, C-induced in the sense of Definition 3.22 [2]) then the principal $J_\mathcal{C}$-ideals on $\mathcal{C}$ are precisely the ideals in $Id_{J_\mathcal{C}}(\mathcal{C})$ which are C-compact (in the sense that every covering of them in $Id_{J_\mathcal{C}}(\mathcal{C})$ admits a refinement satisfying C).

The target categories of locales in the dualities can also be naturally characterized in terms of the invariant C, as categories of locales which possess a basis of C-compact elements satisfying some specific invariant properties (cf. Theorem 3.28 [3]).

To obtain covariant equivalences with categories of locales, one relies on the well-known possibility of assigning a geometric morphism $[\mathcal{C}, \mathbf{Set}] \to [\mathcal{D}, \mathbf{Set}]$ to a given functor $\mathcal{C} \to \mathcal{D}$ in a canonical way.

Given a duality (resp. equivalence) between a category $\mathcal{K}$ of preorders and a category of locales, we have a general methodology for 'lifting' it to a duality (resp. equivalence) between $\mathcal{K}$ and a category of topological spaces. This methodology relies on the possibility of defining, for any set of points of a Grothendieck toposes $\mathcal{E}$ indexed by a set I, a natural topology on the set I, which we call the *subterminal topology* (cf. Definition 2.2 [3]), and of making this construction functorial. While the dualities (resp. equivalences) with locales (or more generally with posets) generated through our method have an essentially constructive nature, this process of 'lifting' to dualities (resp. equivalences) with categories of topological spaces might require, depending on the case, some form of the axiom of choice.

The interest of the notion of subterminal topology lies in its level of generality, which encompasses that of classical topology (every topological space arises from this construction in a canonical way), as well as in its formulation as a topos-theoretic invariant admitting a 'natural behaviour' with respect to sites. Indeed, this notion allows us to recover, with natural choices of sites of definition and of sets of points of toposes, many interesting topological spaces considered in the literature, leaving at the same time enough freedom to construct new ones with particular properties. Our method for building dualities or equivalences with categories of locales starting from Morita-equivalences of the form

$$\mathbf{Sh}(\mathcal{C}, J_\mathcal{C}) \simeq \mathbf{Sh}(Id_{J_\mathcal{C}}(\mathcal{C}))$$

can be further generalized to Morita-equivalences of the form

$$\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Sh}(\mathcal{D}, K),$$

where $\mathcal{C}$ and $\mathcal{D}$ are preordered categories. This generalization allows an abstract symmetric definition of the functors yielding the dualities, and provides us with an additional degree of freedom in building dualities or equivalences between categories of preordered structures. Grothendieck Comparison Lemma turns out be an extremely fruitful source of Morita-equivalences to which we can apply our methods; we illustrate this point by generating several new dualities or equivalences. In particular, we establish a duality which naturally generalizes Birkhoff's duality for finite distributive lattices, a duality which extends the well-known duality between algebraic lattices and sup-semilattices, and a 'finitary version' of Lindenbaum-Tarski duality.

Our theory (whose details are all given in [3]) provides a unified perspective on the subject of Stone-type dualities, in that several well-known dualities are easily recovered as applications of it. Anyway, what we consider to be the main interest of our topos-theoretic machinery is, apart from the conceptual enlightenment that it brings into the world of classical dualities, its inherent technical flexibility. In fact, one can generate infinitely many new dualities by applying it. We illustrate this by discussing various examples of dualities generated by using our method. We recover the classical Stone duality for distributive lattices (and Boolean algebras), the Alexandrov duality between preorders and Alexandrov spaces, the Lindenbaum-Tarski duality, the duality between spatial frames and sober spaces, (a simplified version of) Moshier and Jipsen's topological duality for meet-semilattices (cf. [5]), and we establish several new dualities, including a localic duality for meet-semilattices, an equivalence between the category of posets and a category of spatial locales (equivalently, a category of sober topological spaces), and a localic duality for k-frames (for a regular cardinal k, as defined in [8]).

The different 'ingredients' that our 'machinery' for generating dualities with categories of locales or topological spaces takes as 'inputs' are: the initial category $\mathcal{K}$ of preordered structures, the subcanonical Grothendieck topologies $J_{\mathcal{C}}$ on the structures $\mathcal{C}$ in $\mathcal{K}$, the topos-theoretic invariant enabling one to recover a structure $\mathcal{C}$ from the topos $\mathbf{Sh}(\mathcal{C}, J_{\mathcal{C}})$ and, if a duality with topological spaces is to be generated, appropriate sets of points of the toposes $\mathbf{Sh}(\mathcal{C}, J_{\mathcal{C}})$ (and functions between them). In fact, the more general approach mentioned above provides us with an additional degree of freedom in the choice of ingredients. Given such ingredients, dualities are generated in an automatic and 'uniform' way by the 'machine', as different concrete instances of a unique abstract pattern; in this way, the problem of building dualities gets reduced in many important cases to the much easier one of choosing appropriate sets of ingredients for this 'machine'.

Finally, we discuss how our topos-theoretic interpretation, combined with the method 'toposes as bridges' of [2], can be fruitfully exploited for obtaining results connecting properties of preorders and properties of the corresponding locales or topological spaces, as well as for establishing adjunctions between various kinds of categories (including reflections from various categories of preordered structures to the category of frames and reflections between categories of posets satisfying some generalized 'distributive law' and full subcategories of them consisting of posets satisfying certain 'topological conditions').

We conclude by suggesting some future research directions opened up by our theory.

References

[1] M. Artin, A. Grothendieck and J. L. Verdier, *Théorie des topos et cohomologie étale des schémas*, Lecture Notes in Math., vols 269, 270 and 305 (Springer-Verlag, 1972).

[2] O. Caramello, The unification of Mathematics via Topos Theory (2010), *arXiv:math.CT/1006.3930v1*.

[3] O. Caramello, A topos-theoretic approach to Stone-type dualities (2011), *arXiv:math.CT/1103.3493v1*.

[4] M. Mislove, When are order scattered and topologically scattered the same?, in *Order: Description and Roles*, Annals of Discrete Mathematics (23) (North-Holland, 1984).

[5] M. A. Moshier and P. Jipsen, Topological Duality and Lattice Expansions Part I: a topological construction of canonical extensions, preprint available at
`http://www1.chapman.edu/~jipsen/preprints/topologicalDual2.pdf`.

[6] P. T. Johnstone, *Sketches of an Elephant: a topos theory compendium. Vols. 1-2*, vols. 43-44 of *Oxford Logic Guides* (Oxford University Press, 2002).

[7] P. T. Johnstone, *Stone spaces*, Cambridge Studies in Advanced Math. No. 3 (Cambridge Univ. Press, 1982).

[8] J. J. Madden, k-frames, *J. Pure Appl. Alg.* 70(1,2) (1991), 107-127.

Algorithmic correspondence and canonicity for non-distributive logics (Extended abstract)

Willem Conradie, Alessandra Palmigiano

Introduction

Sahqvist theory has a long and distinguished history within modal logic, going back to [9] and [10]. The Sahlqvist theorem in [9] gives a syntactic definition of a class of modal formulas, the *Sahlqvist class*, each member of which defines an elementary class of frames and is canonical — two highly desirable properties. As it turns out, both these properties (singularly and in combination) are algorithmically undecidable [1], so decidable approximations, such as the Sahlqvist class, are of great interest.

Over the years, many extensions, variations, and analogues of this result have appeared. These includes, among others, (a) generalizations to non-Boolean modal logics, e.g., the Sahlqvit theorem for distributive modal logic of [6], and (b) both syntactic characterizations and algorithmic approaches properly extending the Sahlqvist class, such as the inductive formulas in [8] and the SQEMA algorithm of [2]. In [3] we combined these two trends, by devising an analogue of SQEMA, called ALBA, for distributive modal logic.

The present work aims at extending this theory to logics based on *non-distributive* propositional logic, in line with what is done e.g. in [6], in [7], and in [3], by proving an algorithmic correspondence result for modal logics whose algebraic semantics is based on arbitrary lattices with operators (LO's). We identify the appropriate extension of the Sahlqvist and inductive inequalities to this more general setting. The proof of the canonicity and elementarity of these inequalities takes the shape of an algorithm (called ND-ALBA) which tries to eliminate variables from the inequality, by replacing them, if possible, with special variables, ranging only over the completely join- and meet-irreducible elements of *perfect lattices* (defined below). In this generalized context, perfect lattices play the same role of complete and atomic Boolean algebras in the classical setting. In particular, their completely join- and meet-irreducible elements are the states of their dual relational structures, of which perfect lattices are the "complex algebras". It can be shown that ND-ALBA is correct and that all inequalities on which it terminates successfully are elementary and canonical.

1 (Perfect) lattices with operators, canonical extensions, and languages

For definiteness' sake we fix a signature containing $\wedge$, $\vee$, $\circ$, $\star$, $\Diamond$, $\Box$, $\lhd$ and $\rhd$, i.e., the signature of [6], augmented with the binary connectives $\circ$ and $\star$, called *fusion* and *fission*, respectively. The algebras under consideration will be of the form $\mathbb{A} = (A, \wedge, \vee, \circ, \star, \Diamond, \Box, \lhd, \rhd, 0, 1)$, where $(A, \wedge, \vee, 0, 1)$ is a perfect lattice, $\circ$ (binary) and $\Diamond$ (unary) are join-preserving operations, $\star$ (binary) and $\Box$ (unary) are meet-preserving operations, while $\lhd(a \wedge b) = \lhd a \vee \lhd b$ and $\rhd(a \vee b) = \rhd a \wedge \rhd b$.

The terms of the language associated with these LO's are given by

$$\varphi ::= \bot \mid \top \mid p \in \mathsf{PROP} \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \Diamond\varphi \mid \Box\varphi \mid \lhd\varphi \mid \rhd\varphi \mid \varphi \star \psi \mid \varphi \circ \psi,$$

where PROP is some denumerably infinite set of variables.

An LO $\mathbb{A} = (A, \wedge, \vee, \circ, \star, \Diamond, \Box, \lhd, \rhd, 0, 1)$ is *perfect* if it is (a) complete, (b) join-generated by its completely join-irreducible elements (the set of which is denoted by J^∞), (c) meet-generated by its completely meet-irreducible elements (the set of which is denoted by M^∞), and moreover if (d) $\circ$ and $\Diamond$ are completely join-preserving, (e) $\star$ and $\Box$ are completely meet-preserving, while (f) $\lhd \bigwedge S = \bigvee \lhd S$ and $\rhd \bigvee S = \bigwedge \rhd S$ for all $S \subseteq A$. It follows that in perfect LO's the operations $\circ$, $\star$, $\Diamond$, $\Box$, $\lhd$ and $\rhd$ are residuated. We denote their residuals by $\backslash_\circ$, $/_\circ$, $\backslash_\star$, $/_\star$, $\blacksquare$, $\blacklozenge$, $\blacktriangleleft$ and $\blacktriangleright$, respectively.

To talk about perfect LO's, we expand our language with these residuals, as well as with two sets of special variables, called, respectively, *nominals*, ranging over J^∞, denoted $\mathbf{i}, \mathbf{j}$, etc., and *co-nominals*, ranging over M^∞, denoted $\mathbf{m}, \mathbf{n}$, etc. A term or inequality in this language which contains no variables from PROP but only (possibly) nominals and co-nominals will be called *pure*.[1]

[1] The terminology is inspired by hybrid logic.

Every LO $\mathbb{A}$ can be embedded into its *canonical extension* (see, e.g., [5]), i.e., into a perfect LO $\mathbb{A}^\sigma$, in such a way that (a) every element of $\mathbb{A}^\sigma$ is both a meet of joins and a join of meets of (images of) elements $\mathbb{A}$ (if (a) holds, we say that $\mathbb{A}$ is *dense* in $\mathbb{A}^\sigma$) and (b) for every $S, T \subseteq \mathbb{A}$, if $\bigwedge S \leq \bigvee T$, then $\bigwedge S' \leq \bigvee T'$ for some finite $S' \subseteq S$ and $T' \subseteq T$, where the meet is taken in $\mathbb{A}^\sigma$ (if (b) holds, we say that $\mathbb{A}$ is *compact* in $\mathbb{A}^\sigma$). The canonical extension of $\mathbb{A}$ always exists and is unique up to an isomorphism fixing $\mathbb{A}$. An inequality $s \leq t$ is called *canonical* if $\mathbb{A} \models s \leq t$ implies $\mathbb{A}^\sigma \models s \leq t$.

The (two-sorted) relational duals of perfect lattices are called RS-frames (see [4]). The relational duals of perfect LO's are relational structures based on RS-frames, which, for the sake of this abstract, will also be referred to as RS-frames. As is the case with Kripke frames, every formula (i.e., term in the language above) defines a second-order condition on RS-frames. Formulas whose frame conditions are equivalent to first-order formulas are called *elementary*. We will exploit the observation that, since the completely join- and meet-irreducible elements of a perfect LO correspond to the individuals in its dual RS frame, pure inequalities correspond to elementary formulas.

2 Sahlqvist and inductive inequalities

As in [6], we will use the notion of a signed *generation tree* of a term s, which is here denoted by $*s$, where $* \in \{+,-\}$; the positive and negative trees are denoted by $+s$ and $-s$ respectively. Recall that every node in a signed generation tree passes its sign to its children, except those labelled by $\lhd$ and $\rhd$, whose children have the opposite sign.

Syntactically Join-Friendly (SJF)	Syntactically Right Adjoint (SRA)
$+ \quad \vee \quad \Diamond \quad \lhd \quad \circ$	$+ \quad \wedge \quad \Box \quad \rhd \quad \star$
$- \quad \wedge \quad \Box \quad \rhd \quad \star$	$- \quad \vee \quad \Diamond \quad \lhd \quad \circ$

Table 1: SJF and SRA nodes

An *order type* over $n \in \mathbb{N}$ is an n-tuple $\epsilon \in \{1, \partial\}^n$. For every order type ϵ, let ϵ^∂ be its *opposite* order type, i.e., for every $1 \leq i \leq n$, we have $\epsilon^\partial_i = 1$ iff $\epsilon_i = \partial$. For any term $s(p_1, \ldots p_n)$, any order type ϵ over n, and any $1 \leq i \leq n$, an *ϵ-critical node* in a signed generation tree of s is a (leaf) node $+p_i$ if $\epsilon_i = 1$ or $-p_i$ if $\epsilon_i = \partial$. An *ϵ-critical branch* in the tree is a branch from an ϵ-critical node. The intuition, which will be built upon later, is that variable occurrences corresponding to ϵ-critical nodes are *to be solved for, according to ϵ*. For every term $s(p_1, \ldots p_n)$ and every order type ϵ, we say that $+s$ (resp. $-s$) *agrees with* ϵ, and write $\epsilon(+s)$ (resp. $\epsilon(-s)$), if every leaf in the signed generation tree of $+s$ (resp. $-s$) is ϵ-critical.

Definition 2.1. Nodes in signed generation trees will be called *syntactically join-friendly (SJF)* and *syntactically right adjoint (SRA)* according to the specification given in table 1. A branch in a signed generation tree $*s$ is called a *good branch* if it is the concatenation of two paths P_1 and P_2, one of which may possibly be of length 0, such that P_1 is a path from the leaf consisting (apart from variable nodes) only of SRA-nodes, and P_2 consists (apart from variable nodes) only of SJF-nodes. A branch is *excellent* if it is good and in P_1 there are no binary nodes other than $+\wedge$ or $-\vee$. A good branch is called *join-friendly* if P_1 has length 0.

Definition 2.2. Given an order type ϵ and a strict partial order $<_\Omega$ on the variables $p_1, \ldots p_n$, the signed generation tree $*s$ of a term $s(p_1, \ldots p_n)$ is *(Ω, ϵ)-inductive* if for all $1 \leq i \leq n$, every ϵ-critical branch with leaf labelled p_i is good, and moreover, for every binary SRA node $*(\alpha \circledast \beta)$ on the branch, if $*\circledast \notin \{+\wedge, -\vee\}$, then for some $\gamma \in \{\alpha, \beta\}$,

1. $\epsilon^\partial(*\gamma)$, and
2. $p_j <_\Omega p_i$ for every p_j occurring in γ.

We will refer to $<_\Omega$ as the *dependency order* on the variables. An inequality $s \leq t$ is *(Ω, ϵ)-inductive* if the trees $+s$ and $-t$ are both (Ω, ϵ)-inductive. An inequality $s \leq t$ is *inductive* if it is (Ω, ϵ)-inductive for some Ω and ϵ.

Definition 2.3. (cf. [6]) Given an order type ϵ, the signed generation tree $*s$ of a term $s(p_1, \ldots p_n)$ is *ϵ-Sahlqvist* if every ϵ-critical branch is excellent. An inequality $s \leq t$ is *ϵ-Sahlqvist* if the trees $+s$ and $-t$ are both ϵ-Sahlqvist. An inequality $s \leq t$ is *Sahlqvist* if it is ϵ-Sahlqvist for some ϵ. (Note that this definition is a special case of Definition 2.2.)

Example 2.4. The inequality $\varphi_1 \leq \psi_1 := (\Box p_1 \circ \Box\lhd p_1) \circ \Diamond p_2 \leq \lhd p_1 \star \Diamond\Box p_2$ is ϵ-Sahlqvist for $\epsilon = (1, 1)$. The inequality $\varphi_2 \leq \psi_2 := (\Box p \star \Diamond\top) \circ \Box\lhd q \leq \Box(\lhd q \circ \Box p)$ is (Ω, ϵ)-inductive with $p <_\Omega q$ and $\epsilon = (1, 1)$. The inequality $\varphi_3 \leq \psi_3 := \Diamond(\Box\lhd(q \circ r) \wedge \Box(p \star \Box q)) \leq \lhd\Box(p \wedge r) \vee \Diamond p$ is (Ω, ϵ)-inductive with $p <_\Omega q <_\Omega r$ and $\epsilon_p = \partial$, $\epsilon_q = \epsilon_r = 1$. The inequality $\varphi_4 = s \leq t := \Box(\rhd q \star p) \leq \Diamond(\lhd q \circ \Box p)$ is *not* inductive. Indeed, for every ϵ-critical branch in $+s$ and $-t$ to be good, the only possible ϵ is $(1, 1)$. Given this ϵ, if $+s$ is to be (Ω, ϵ)-inductive, it will have to be the case that $q <_\Omega p$, and similarly, for $-t$ to be (Ω, ϵ)-inductive, it will have to be the case that $p <_\Omega q$, which is impossible if $<_\Omega$ is to be a strict partial order.

The next Theorem is a corollary of Theorems 3.1 and 3.2, below.

Theorem 2.5. *Every inductive inequality (and hence every Sahlqvist inequality) is elementary and canonical.*

3 The algorithm

ND-ALBA takes an inequality $\varphi \leq \psi$ as input and then proceeds in three stages. The *first stage* eliminates all propositional variables occurring only positively or only negatively, by suitable substitution with $\bot$ and $\top$, obtaining an inequality $\varphi' \leq \psi'$. The *initial system* (S, Ineq), consisting of a set S (initialized to $\varnothing$) and an inequality Ineq (initialized to $\varphi' \leq \psi'$) is formed.

The *second stage* (called the reduction stage) transforms S and Ineq through the application of its transformation rules. The aim is to eliminate all "wild" propositional variables from S and Ineq in favour of "tame" nominals and co-nominals. A system for which this has been done will be called *pure* or *purified*. The actual eliminations are effected through the Ackermann-rules (we give only the right Ackerman-rule, the left one is dual):

Right Ackermann-rule: If $S = \{\alpha_i \leq p \mid 1 \leq i \leq n\} \cup \{\beta_j(p) \leq \gamma_j(p) \mid 1 \leq j \leq m\}$ where (a) p does not occur in $\alpha_1, \ldots, \alpha_n$, (b) $\beta_1(p), \ldots, \beta_m(p)$ are positive in p, (c) $\gamma_1(p), \ldots, \gamma_m(p)$ are negative in p, and (e) p does not occur in Ineq, then

- $S := \{\beta_j(\bigvee_{i=1}^n \alpha_i) \leq \gamma_j(\bigvee_{i=1}^n \alpha_i) \mid 1 \leq j \leq m\}$, and
- $\mathsf{Ineq} := \mathsf{Ineq}$.

The other rules are used to bring S and Ineq into the appropriate shape which make the applications of the Ackermann-rules possible. These rules include **four approximation rules**. For example, the

Left-positive approximation rule. We write $\varphi(!x)$ to indicate that the variable x occurs exactly once in φ, and we let $\varphi(\gamma/!x)$ be the formula obtained by substituting γ for x in $\varphi(!x)$. If $\mathsf{Ineq} = \varphi \leq \psi$, and $\varphi = \varphi'(\gamma/!x)$, and $+x \prec +\varphi'(!x)$, and the branch of $+\varphi'(!x)$ starting at $+x$ is join-friendly, then:

- $S := S \cup \{\mathbf{j} \leq \gamma\}$, and
- $\mathsf{Ineq} := \varphi'(\mathbf{j}) \leq \psi$

where $\mathbf{j}$ is the first nominal not previously occurring in S or Ineq.

The rule above is sound by the distributive behaviour of join-friendly operation occurrences and the join generation of perfect LO's by J^∞. The other three approximation rules provide for the other combinations of φ occurring on the left/right of the inequality and being positive/negative in x.

The residuation rules operate on the inequalities in S, and are based on the residuation of the operations:

$$\frac{\Diamond\alpha \leq \beta}{\alpha \leq \blacksquare\beta} \qquad \frac{\alpha \leq \Box\beta}{\blacklozenge\alpha \leq \beta} \qquad \frac{\lhd\alpha \leq \beta}{\blacktriangleleft\beta \leq \alpha} \qquad \frac{\alpha \leq \rhd\beta}{\beta \leq \blacktriangleright\alpha} \qquad \frac{\alpha \circ \beta \leq \gamma}{\alpha \leq \gamma /_\circ \beta} \qquad \frac{\alpha \circ \beta \leq \gamma}{\beta \leq \alpha \backslash_\circ \gamma} \qquad \frac{\alpha \circ \beta \leq \gamma}{\beta \leq \alpha \backslash_\circ \gamma} \qquad \frac{\alpha \leq \beta \star \gamma}{\beta \backslash_\star \alpha \leq \gamma}$$

Once all propositional variables have been eliminated, this phase terminates and returns the pure quasi-inequality $\&S \Rightarrow \mathsf{Ineq}$. If this is not possible the algorithm reports failure (this may be because the original inequality is not canonical, or not elementary, but not necessarily).

Theorem 3.1. *If an inequality is successfully purified by ND-ALBA, then it is elementary and canonical.*

Theorem 3.2. *ND-ALBA successfully purifies all inductive and Sahlqvist inequalities.*

Example 3.3. Consider the inductive inequality $\varphi_2 = (\Box p \star \Diamond\top) \circ \Box\lhd q \leq \Box(\lhd q \circ \Box p)$ from example 2.4. The initial system is

$$S_0 = \varnothing \text{ and } \mathsf{Ineq}_0 = (\Box p \star \Diamond\top) \circ \Box\lhd q \leq \Box(\lhd q \circ \Box p).$$

Applying the left-positive approximation rule twice and also the right-negative approximation rule transforms this into

$$S_1 = \{\mathbf{j} \leq (\Box p \star \Diamond\top),\ \mathbf{k} \leq \Box\lhd q,\ (\lhd q \circ \Box p) \leq \mathbf{m}\} \text{ and } \mathsf{Ineq}_1 = \mathbf{j} \circ \mathbf{k} \leq \Box\mathbf{m}.$$

Now the residuation rules for $\star$ and for $\Box$ can be applied to the first inequality in S_1. The residuation rule for $\circ$ can be applied to the third inequality in S_1. This results in the system

$$S_2 = \{\blacklozenge(\mathbf{j}/_{\star}\Diamond\top) \leq p,\ \mathbf{k} \leq \Box\lhd q,\ \lhd q \leq \mathbf{m}/_{\circ}\Box p\} \text{ and } \mathsf{Ineq}_2 = \mathbf{j} \circ \mathbf{k} \leq \Box\mathbf{m}.$$

Now the right Ackermann-rule can be applied to eliminate p:

$$S_3 = \{\mathbf{k} \leq \Box\lhd q,\ \lhd q \leq \mathbf{m}/_{\circ}\Box\blacklozenge(\mathbf{j}/_{\star}\Diamond\top)\} \text{ and } \mathsf{Ineq}_3 = \mathbf{j} \circ \mathbf{k} \leq \Box\mathbf{m}.$$

Next we apply the residuation rule for $\lhd$ to the second inequality in S_3, yielding the system

$$S_4 = \{\mathbf{k} \leq \Box\lhd q,\ \blacktriangleleft(\mathbf{m}/_{\circ}\Box\blacklozenge(\mathbf{j}/_{\star}\Diamond\top)) \leq q\} \text{ and } \mathsf{Ineq}_4 = \mathbf{j} \circ \mathbf{k} \leq \Box\mathbf{m},$$

to which the right Ackermann-rule can be applied to eliminate q, giving the pure system

$$S_5 = \{\mathbf{k} \leq \Box\lhd\blacktriangleleft(\mathbf{m}/_{\circ}\Box\blacklozenge(\mathbf{j}/_{\star}\Diamond\top))\} \text{ and } \mathsf{Ineq}_5 = \mathbf{j} \circ \mathbf{k} \leq \Box\mathbf{m}.$$

It follows that φ_2 is elementary and canonical. In fact, on perfect LO's it is equivalent to the quasi-inequality $\mathbf{k} \leq \Box\lhd\blacktriangleleft(\mathbf{m}/_{\circ}\Box\blacklozenge(\mathbf{j}/_{\star}\Diamond\top)) \Rightarrow \mathbf{j} \circ \mathbf{k} \leq \Box\mathbf{m}$.

References

[1] A. Chagrov and L. A. Chagrova, *The truth about algorithmic problems in correspondence theory*, in Advances in Modal Logic, G. Governatori, I. Hodkinson, and Y. Venema, eds., vol. 6, College Publications, 2006, pp. 121–138.

[2] W. Conradie, V. Goranko, and D. Vakarelov, *Algorithmic correspondence and completeness in modal logic I: The core algorithm SQEMA*, Logical Methods in Computer Science, 2(1:5) (2006).

[3] W. Conradie and A. Palmigiano, *Algorithmic correspondence and canonicity for distributive modal logic*, Submitted.

[4] M. Gehrke, *Generalized Kripke frames*, Studia Logica, 84 (2006), pp. 241–275.

[5] M. Gehrke and J. Harding, *Bounded lattice expansions*, J. Algebra, 238 (2001), pp. 345–371.

[6] M. Gehrke and Y. Nagahashi, H. Venema, *A Sahlqvist theorem for distributive modal logic*, Annals of Pure and Applied Logic, 131 (2005), pp. 65–102.

[7] M. Gehrke, A. Palmigiano, and H. Priestley, *Order-theoretic correspondence*, in preparation.

[8] V. Goranko and D. Vakarelov, *Elementary canonical formulae: Extending Sahlqvist theorem*, Annals of Pure and Applied Logic, 141(1-2) (2006), pp. 180–217.

[9] H. Sahlqvist, *Correspondence and completeness in the first and second-order semantics for modal logic*, in Proceedings of the 3rd Scandinavian Logic Symposium, Uppsala 1973, S. Kanger, ed., 1975, pp. 110–143.

[10] J. F. A. K. van Benthem, *Modal Logic and Classical Logic*, Bibliopolis, 1983.

Canonical extension of coherent categories

Dion Coumans

In 1951, Jónsson and Tarski introduced the notion of canonical extension of a Boolean algebra with operators [4]. Since then, their ideas have been developed further, which has led to a smooth theory of canonical extensions applicable in a broad setting such as distributive lattices and even partially ordered sets [2, 3]. This theory has proved to be a powerful tool in the algebraic study of propositional logics. Generalizing the notion of canonical extension to the categorical setting opens the door to the application of those techniques in the study of predicate logics.
In his thesis [9], Magnan claims that the topos of types construction, introduced by Makkai in [10], is a natural generalization of canonical extension to the categoricial setting. Furthermore, at a talk at PSSL in 1999, his then PhD advisor Reyes announced (but did not prove) that this construction may be used to prove interpolation for different first order logics [12]. In [1], Butz gives a logical description of Makkai's topos of types, also drawing attention to the connection with canonical extension.
We will give a short overview of this earlier work and then describe an alternative construction of the topos of types which is a natural extension of the algebraic construction of canonical extensions. This alternative point of view will hopefully enable applications of the topos of types in the study of predicate logics, for example with respect to interpolation problems.
We will focus on distributive lattices and their categorical counterparts. For a distributive lattice $\mathbf{L}$, its canonical extension $\mathbf{L}^\delta$ may be concretely described as the downset lattice of the poset $(\mathcal{P}_\mathbf{L}, \supseteq)$ of prime filters of $\mathbf{L}$ ordered by reverse inclusion. The assignment $\mathbf{L} \mapsto \mathbf{L}^\delta$ extends to a functor $(\_)^\delta : \mathbf{DL} \to \mathbf{DL}^+$, from the category of distributive lattices to the category of completely distributive algebraic lattices, which is left adjoint to the forgetful functor $\mathbf{DL}^+ \to \mathbf{DL}$.
The categorical analogue of a distributive lattice is a coherent category, *i.e.*, a category $\mathbf{C}$ which has finite limits, stable images and the property that, for all $A \in \mathbf{C}$, $Sub_\mathbf{C}(A)$ has stable finite joins. We write **Coh** for the category of all (small) coherent categories, with structure preserving functors between them. Coherent categories provide semantics for *coherent logic*, the fragment of first order logic with only the connectives $\wedge$, $\vee$, $\top$, $\bot$ and $\exists$. Note that in a coherent category $\mathbf{C}$, for each $A \in \mathbf{C}$, $Sub_\mathbf{C}(A)$ is a distributive lattice. This enables the interpretation of the propositional connectives. As $\mathbf{C}$ has images, for each $f : A \to B$, the pullback functor $f^* : Sub_\mathbf{C}(B) \to Sub_\mathbf{C}(A)$ has a left adjoint $\exists_f$, which enables the interpretation of the existential quantifier (see *e.g.* D1 in [5]).
For a coherent category $\mathbf{C}$, we write $T(\mathbf{C})$ for its topos of types (see [10]). There is a natural embedding $[\_] : \mathbf{C} \to T(\mathbf{C})$. In [9], it is shown that this construction has categorical properties corresponding to the algebraic properties of the canonical extension of a distributive lattice. For example, for all A in $\mathbf{C}$, the distributive lattice $Sub_{T(\mathbf{C})}([A])$ is the canonical extension of the distributive lattice $Sub_\mathbf{C}(A)$, and, in case $\mathbf{C}$ is a Heyting category (a coherent category in which all pullback morphisms between subobject lattices have a right adjoint), the mapping $[\_]$ preserves this additional structure.
In [1], Butz gives a logical description of the topos of types. He first considers the category **cdCoh** of coherent categories with the additional property that all subobject lattices are complete completely distributive, pullbacks preserve arbitrary joins of subobjects, and existential quantification (left adjoints to the pullback morphisms between subobject lattices) distributes over filtered meets. By using the general

construction of a syntactic category he shows that the forgetful functor **cdCoh** $\to$ **Coh** has a left adjoint $\mathcal{I}\colon \mathbf{Coh} \to \mathbf{cdCoh}$. To get to the topos of types, he remarks that, for a category $\mathbf{D}$ in **cdCoh**, there is a natural Grothendieck topology on $\mathbf{D}$ given by defining a sieve $\{A_i \xrightarrow{\alpha_i} A\}$ to be a cover iff $A = \bigvee_I \exists_{\alpha_i} A_i$, where the join is taking in $Sub_{\mathbf{D}}(A)$. For a coherent category $\mathbf{C}$, the topos $Sh(\mathcal{I}(\mathbf{C}), J)$ is equivalent to the topos of types $T(\mathbf{C})$.
We will describe an alternative construction of the topos of types $T(\mathbf{C})$, for a coherent category $\mathbf{C}$. Our construction is inspired by the work of Pitts. In [13, 14], he defines, for a coherent category $\mathbf{C}$, its *topos of filters* $\Phi(\mathbf{C})$, which is a categorical generalisation of the functor which sends a lattice $\mathbf{L}$ to the lattice $Idl(Flt(\mathbf{L}))$ of ideals of the lattice of filters of $\mathbf{L}$. In his description of $\Phi(\mathbf{C})$ the correspondence between coherent categories and so-called *polyadic distributive lattices* plays a crucial role. We will exploit this connection to give an alternative description of the topos of types construction, which is closely related to the canonical extension construction for distributive lattices.
We will now outline our construction and its connections to the earlier work. We start with the definition of polyadic distributive lattices (pDLs) and we will see that, for a coherent category $\mathbf{C}$, the functor $Sub_{\mathbf{C}}\colon \mathbf{C}^{op} \to \mathbf{DL}$, which sends an object of $\mathbf{C}$ to the distributive lattice of its subobjects, is a pDL from which we may recover $\mathbf{C}$ (up to equivalence).

Definition 1 *Let* $\mathbf{B}$ *be a category with finite limits. A* polyadic distributive lattice *(pDL) over* $\mathbf{B}$ *is a functor* $P\colon \mathbf{B}^{op} \to \mathbf{DL}$ *such that, for every morphism* $I \xrightarrow{\alpha} J$ *in* $\mathbf{B}$, $P(\alpha)\colon P(J) \to P(I)$ *has a left adjoint* $\exists^P_\alpha$ *satisfying*

1. *Frobenius reciprocity, i.e., for all* $a \in P(I)$, $b \in P(J)$,

$$\exists^P_\alpha(a \wedge P(\alpha)(b)) = \exists^P_\alpha(a) \wedge b$$

2. *Beck-Chevalley condition, i.e., for every pullback square*

$$\begin{array}{ccc} Q & \xrightarrow{\alpha'} & J \\ {\scriptstyle \beta'}\downarrow & & \downarrow{\scriptstyle \beta} \\ I & \xrightarrow[\alpha]{} & K \end{array}$$

in $\mathbf{B}$, $P(\beta)\exists^P_\alpha = \exists^P_{\alpha'}(\beta')$.

We will often omit the superscript P *in* $\exists^P_\alpha$. *A* pDL morphism *from* $P\colon \mathbf{B}^{op} \to \mathbf{DL}$ *to* $P'\colon \mathbf{B}'^{op} \to \mathbf{DL}$ *is a pair* (K, τ), *where* $K\colon \mathbf{B} \to \mathbf{B}'$ *is a finite limit preserving functor and* $\tau\colon P \to P' \circ K$ *is a natural transformation satisfying, for all* $I \xrightarrow{\alpha} J$, $\exists^{P'}_\alpha \circ \tau_I = \tau_J \circ \exists^P_\alpha$. *We write* **pDL** *for the category of polyadic distributive lattices.*

A coherent logic naturally gives rise to a pDL $\mathcal{F}m\colon \mathbf{B}^{op} \to \mathbf{DL}$. The contexts and terms form the objects and morphisms of the category $\mathbf{B}$. For each context $\vec{x} = [x_0, \ldots, x_{n-1}]$, $\mathcal{F}m(\vec{x})$ consists of the formulae in the variables $\vec{x}$ (modulo provable equivalence). The morphisms $\mathcal{F}m(\alpha)$ are given by substitution of terms in formulae and their adjoints are given by existential quantification (for more background see [8]).
As stated above, for a coherent category $\mathbf{C}$, the functor $Sub_{\mathbf{C}}\colon \mathbf{C}^{op} \to \mathbf{DL}$ is a pDL. This assignment naturally extends to a functor $\mathcal{S}\colon \mathbf{Coh} \to \mathbf{pDL}$. Conversely, for a polyadic distributive lattice P over

$\mathbf{B}$, we define a coherent category $\mathcal{A}(P)$ whose objects are pairs (I, a), where $I \in \mathbf{B}$ and $a \in P(I)$. A morphism $(I, a) \to (J, b)$ is an element $f \in P(I \times J)$ which is, in the internal language of P, a functional relation $\{x \mid a(x)\} \to \{y \mid b(y)\}$. This yields a functor $\mathcal{A}\colon \mathbf{pDL} \to \mathbf{Coh}$, which is left adjoint to $\mathcal{S}$ [13].

Proposition 2 *The functors $\mathcal{A}\colon \mathbf{pDL} \leftrightarrows \mathbf{Coh}\colon \mathcal{S}$ form an adjunction, $\mathcal{A} \dashv \mathcal{S}$, and, for each $\mathbf{C} \in \mathbf{Coh}$, the counit at $\mathbf{C}$, $\epsilon_{\mathbf{C}}\colon \mathcal{A}(\mathcal{S}(\mathbf{C})) \to \mathbf{C}$, is an equivalence of categories.*

One may show that, for a polyadic distributive lattice P over $\mathbf{B}$, the functor $P^\delta = (\_)^\delta \circ P\colon \mathbf{B}^{op} \to \mathbf{DL}$ is again a pDL over $\mathbf{B}$. In particular, for a coherent category $\mathbf{C}$, $\mathcal{S}(\mathbf{C})^\delta$ is a pDL. The following proposition relates this construction to canonical extension of distributive lattices.

Proposition 3 *Let $\mathbf{L}$ be a distributive lattice. Viewing $\mathbf{L}$ as a coherent category, $\mathcal{A}(\mathcal{S}(\mathbf{L})^\delta) \simeq \mathbf{L}^\delta$.*

We now get to our main theorem. Let $\mathbf{C}$ be a coherent category. For all $A \in \mathbf{C}$, $\mathcal{S}(\mathbf{C})^\delta(A)$ is a complete completely distributive lattice and therefore it is in particular a frame. Using that $\mathcal{S}(\mathbf{C})^\delta$ is a pDL, it follows from the description of internal locales in $\mathbf{Set}^{\mathbf{C}^{op}}$ given in [6], that $\mathcal{S}(\mathbf{C})^\delta$ is a locale in $Set^{\mathbf{C}^{op}}$.

Theorem 4 *For a coherent category $\mathbf{C}$, the topos of types $T(\mathbf{C})$ is equivalent to the topos of sheaves over the internal locale $\mathcal{S}(\mathbf{C})^\delta$ in $Set^{\mathbf{C}^{op}}$.*

We will give a sketch of the proof. To ease the notation, we write $X_{\mathbf{C}}$ for the internal locale $\mathcal{S}(\mathbf{C})^\delta$. Using a general construction in [11], we may describe an (external) site $(\mathbf{C} \ltimes X_{\mathbf{C}}, J)$ such that the topos $Sh(\mathbf{C} \ltimes X_{\mathbf{C}}, J)$ is equivalent to the topos of sheaves over the internal locale $X_{\mathbf{C}}$. The objects of $\mathbf{C} \ltimes X_{\mathbf{C}}$ are pairs (A, u), where $A \in \mathbf{C}$ and $u \in X_{\mathbf{C}}(A)$. A morphism $(A, u) \to (B, v)$ is a morphism $A \xrightarrow{\alpha} B$ in $\mathbf{C}$ such that $u \leq X_{\mathbf{C}}(\alpha)(v)$. The Grothendieck topology J on $\mathbf{C} \ltimes X_{\mathbf{C}}$ is given by: a sieve $\{(A_i, u_i) \xrightarrow{\alpha_i} (A, u)\}_{i \in I}$ is a cover iff

$$\bigvee\{\exists_{\alpha_i} u_i \mid i \in I\} = u.$$

Makkai defines the topos of types $T(\mathbf{C})$ as $T(\mathbf{C}) = Sh(\tau\mathbf{C}, J_p)$. Here, $\tau\mathbf{C}$ is the category of types of $\mathbf{C}$, whose objects are pairs (A, ρ), where $A \in \mathbf{C}$ and ρ is a prime filter in $Sub_{\mathbf{C}}(A)$. A morphism $(A, \rho) \to (A', \rho')$ is an equivalence class $[\alpha]$ of so-called 'local continuous maps'. This is the categorical version of the poset of prime filters of a distributive lattice. For a detailed description of the category of types the reader is refered to [10]. The Grothendieck topology J_p on $\tau\mathbf{C}$ is the topology generated by the singleton covers, that is, a sieve $\{(A_i, \rho_i) \xrightarrow{[\alpha_i]} \}(A, \rho)\}_{i \in I}$ is a cover iff there exists $i \in I$ s.t. $\exists_{[\alpha_i]}\rho_i = \rho$, where $\exists_{[\alpha_i]}\rho_i = \{V \in Sub_{\mathbf{C}}(A) \mid \alpha_i^*(V) \in \rho_i\}$.

For a distributive lattice $\mathbf{L}$, the prime filters of $\mathbf{L}$ correspond to the completely join irreducible elements $\mathcal{J}^\infty(\mathbf{L}^\delta)$ of its canonical extension $\mathbf{L}^\delta$. Furthermore, $\mathbf{L}^\delta$ is join generated by $\mathcal{J}^\infty(\mathbf{L}^\delta)$. Recall that, for $A \in \mathbf{C}$, $X_{\mathbf{C}}(A) = Sub_{\mathbf{C}}(A)^\delta$. Hence, the completely join irreducible elements of $X_{\mathbf{C}}(A)$ correspond to the prime filters in $Sub_{\mathbf{C}}(A)$ and they join-generate $X_{\mathbf{C}}(A)$. Using this, we prove that the topoi $Sh(\mathbf{C} \ltimes X_{\mathbf{C}}, J)$ and $Sh(\tau\mathbf{C}, J_p)$ are equivalent by considering the full subcategory $\mathbf{D}$ of $\mathbf{C} \ltimes X_{\mathbf{C}}$ consisting of the objects of the form (A, x), where $A \in \mathbf{C}$ and $x \in \mathcal{J}^\infty(X_{\mathbf{C}}(A))$. The induced Grothendieck topology J' on $\mathbf{D}$ is the topology generated by the singleton covers. Two applications of the Comparison Lemma (see [7]) now yield

$$Sh(\mathbf{C} \ltimes X_{\mathbf{C}}, J) \simeq Sh(\mathbf{D}, J') \simeq Sh(\tau\mathbf{C}, J_p) = T(\mathbf{C}),$$

thereby completing the proof that taking sheaves over the internal locale $X_{\mathbf{C}} = \mathcal{S}(\mathbf{C})^\delta$ in $Set^{\mathbf{C}^{op}}$ yields the topos of types $T(\mathbf{C})$.

We conclude by mentioning some current research topics. A first question is how the mapping $\mathbf{C} \mapsto \mathcal{A}(\mathcal{S}(\mathbf{C})^{\delta})$ relates to the functor $\mathcal{I}$: $\mathbf{Coh} \to \mathbf{cdCoh}$ described by Butz. One would expect, especially in view of Proposition 3, that, for a coherent category $\mathbf{C}$, $\mathcal{A}(\mathcal{S}(\mathbf{C})^{\delta})$ is equivalent to $\mathcal{I}(C)$. However, for example for the coherent category $\mathbf{Set}$, the category $\mathcal{A}(\mathcal{S}(\mathbf{Set})^{\delta})$ does not have the property that existential quantification distributes over filtered meets, whence $\mathcal{A}(\mathcal{S}(\mathbf{Set})^{\delta})$ is not in $\mathbf{cdCoh}$.
Furthermore, our construction of the topos of types, as described in Theorem 4, is very close to the algebraic construction of canonical extensions. Therefore, we hope that it will enable us to translate canonical extension methods used in the study of propositional logics to the predicate setting. In particular, we intend to apply our construction in the study of interpolation problems for various first order logics.

References

[1] C. Butz, Saturated models of intuitionistic theories, Annals of Pure and Applied Logic 129 (2004), p. 245-275.

[2] M. Gehrke and B. Jónsson, Bounded distributive lattices with operators, Mathematica Japonica 40 (1994), p. 207-215.

[3] M. Gehrke and H.A. Priestley, Canonical extensions and completions of posets and lattices, Rep. on Math. Logic 43 (2008), p.133-152.

[4] B. Jónsson and A. Tarski, Boolean algebras with operators I, Amer. J. Math. 73 (1951), p. 891-939.

[5] P.T. Johnstone, Sketches of an elephant, Oxford Logic Guides 43 and 44, Clarendon Press, Oxford (2002).

[6] A. Joyal and M. Tierney, An extension of the Galois theory of Grothendieck, Memoires of the Amer. Math. Soc. 51, no. 309 (1984).

[7] A. Kock and I. Moerdijk, Presentations of étendues, Cahiers de topologie et géométrie differentielle catégoriques 32-2 (1991), p. 145-164.

[8] F.W. Lawvere, Equality in hyperdoctrines and comprehension schema as an ajoint functor, in: A. Heller (Ed.), Proc. New York Symp. on Applications of Categorical Algebra, Amer. Math. Soc. (1970), p. 1-14.

[9] F. Magnan, Le topos des types et le topos des filltres en logique catégorique, Ph.D. Thesis, Université de Montréal, 1999.

[10] M. Makkai, The topos of types, in: M. Lerman, J.H. Schmerl, R.I. Soare (Eds.), Logic Year 1979-80, Lecture Notes in Mathematics, vol. 859, Springer, Berlin (1981), p. 157-201.

[11] I. Moerdijk, Continuous fibrations and inverse limits of toposes, Compositio Mathematica 58 (1986), p. 45-72.

[12] G. Reyes, Topos of Types, talk at the 70th Peripathetic Seminar on Sheaves in Logic, Cambridge 1999.

[13] A.M. Pitts, An application of open maps to categorical logic, Journal of Pure and Applied Algebra 29 (1983), p. 313-326.

[14] A.M. Pitts, Conceptual completeness for first-order intuitionistic logic: an application of categorical logic, Annals of Pure and Applied Logic 41 (1989), p. 33-81.

Topological duality for arbitrary lattices via the canonical extension

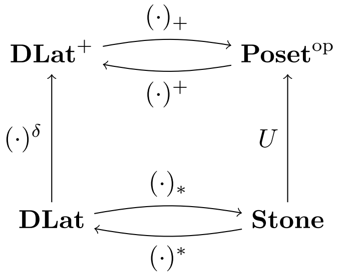
Andrew Craig, Mai Gehrke, Sam van Gool

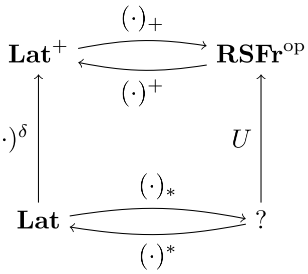
Topological dualities for Boolean algebras and distributive lattices [8] form the basis for semantics for logics based on propositional languages. The theory of canonical extensions for Boolean algebras and distributive lattices [6] forms an algebraic framework for topological duality theory.

Canonical extensions have been generalized to the context of lattices[1] which are not necessarily distributive [5], and even to partially ordered sets [1]. The ensuing theory has been successfully applied to the study of substructural logics. Using canonical extensions, relational semantics for additional operators can be obtained, even in settings where no topological duality theory is available [1].

Regarding topological duality, there have also been several attempts to generalize the theory to lattices which are not necessarily distributive, e.g., [9, 7]. As we will argue, Hartung's approach [7] is the most natural of these from the point of view of canonical extensions. Moreover, Hartung was the first to describe the topological morphisms dual to lattice homomorphisms in general. It was shown in [1], using methods from Sahlqvist correspondence theory, that these morphisms arise naturally from the theory of canonical extensions. In this paper we give an equally natural account of the *topology* on the spaces in the dual of a lattice. In addition, we will show that the spaces considered in [7] are not sober and that another choice is possible where the dual is based on two spectral spaces. Finally we will give a first result relating these two choices for the dual space of a lattice.

The starting point of our approach to topological duality for arbitrary lattices is a discrete duality between the category $\mathbf{Lat}^+$ of *perfect lattices* and the category **RSFr** of *RS frames*, described in [3]. We aim to 'topologize' this duality, drawing inspiration from the distributive case, where Stone introduced topology to generalize Birkhoff's discrete duality between finite distributive lattices and posets to a duality between arbitrary distributive lattices and Stone (spectral) spaces, as indicated in the diagram on the left. By analogy, the diagram on the right provides our road map for the case of arbitrary lattices.





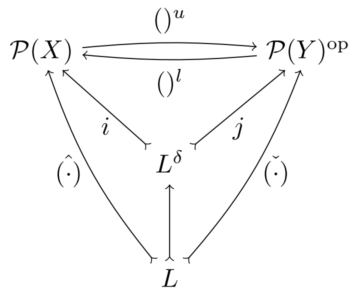
In the diagram on the right, the functor $(\cdot)^\delta$ sends a lattice L to its canonical extension L^δ, which is in $\mathbf{Lat}^+$, and the pair $((\cdot)_+, (\cdot)^+)$ is the discrete duality described in [3], of which we will now briefly recall the relevant details.

The objects of the category **RSFr** are tuples (X, Y, R), with $R \subseteq X \times Y$ a relation satisfying certain first-order properties, called *RS frames* in [3]. Given an arbitrary lattice L, walking through this diagram yields the RS frame $(X, Y, R) = (L^\delta)_+$, which minimally represents L^δ: X is the set of completely join-irreducibles of L^δ, Y is the set of completely meet-irreducibles of L^δ, and $R \subseteq X \times Y$ is the order in L^δ from X to Y. The lattice L^δ is now represented in both partial orders $\mathcal{P}(X)$ and $\mathcal{P}(Y)^{\mathrm{op}}$ as the lattice of stable sets under the Galois connection $()^u : \mathcal{P}(X) \leftrightarrows \mathcal{P}(Y)^{\mathrm{op}} : ()^l$ induced by R.

The composition $L \hookrightarrow L^\delta \to \mathcal{P}(X)$ and similarly for $\mathcal{P}(Y)^{\mathrm{op}}$ yield representations of the original lattice L inside $\mathcal{P}(X)$ and $\mathcal{P}(Y)^{\mathrm{op}}$, given by sending a lattice element $a \in L$ to $\hat{a} := \{x \in X \mid x \leq_{L^\delta} a\}$, and

[1] Throughout this abstract, we use the word 'lattice' to mean 'bounded lattice', *i.e.*, lattice with 0 and 1.

$\check{a} := \{y \in Y \mid a \leq_{L^\delta} y\}$. The map $\hat{(\cdot)}$ is a $\wedge$-embedding of L into $\mathcal{P}(X)$, and $\check{(\cdot)}$ is a $\vee$-embedding of L into $\mathcal{P}(Y)^{\mathrm{op}}$.
Summarizing, we have the following diagram.

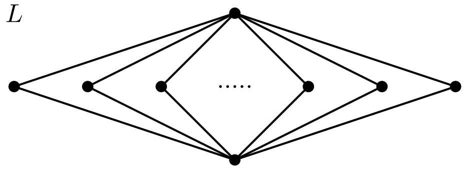


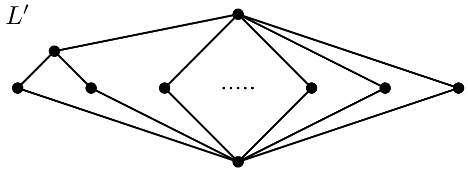
Denote by $D^\wedge(L)$ the sublattice of $\mathcal{P}(X)$ generated by the image of $\hat{(\cdot)}$, and by $D^\vee(L)$ the sublattice of $\mathcal{P}(Y)$ generated by the image of $\check{(\cdot)}$. We observe that the Galois connection $(()^u, ()^l)$ restricts to the distributive lattices $D^\wedge(L)$ and $D^\vee(L)$, and that L can also be represented as the lattice of stable pairs under this restricted Galois connection.
The lattices $D^\wedge(L)$ and $D^\vee(L)$ yield 'topological structure', in the following two ways. Firstly, the lattice $D^\wedge(L)$, being a collection of subsets, can be used to generate a topology τ_X on X, in a manner that we will describe below, and similarly $D^\vee(L)$ can be used to generate a topology τ_Y on Y. From the topologized RS frame $(X, \tau_X; Y, \tau_Y; R)$, the original lattice L can be retrieved, and we will see below that this yields a very natural description of Hartung's topological duality in terms of the canonical extension.
Secondly, if we denote by (X_S, σ_X) and (Y_S, σ_Y) the Stone dual spaces of the distributive lattices $D^\wedge(L)$ and $D^\vee(L)$, respectively, and by $S \subseteq X_S \times Y_S$ the relation dual to the Galois connection $(()^u, ()^l)$, then the tuple $(X_S, \sigma_X; Y_S, \sigma_Y; S)$ will also suffice to retrieve the original lattice L.
Our goal is to study both of these topological structures in more detail as well as relations between them.
In order to do so, our first result is a construction of the distributive lattices $D^\wedge(L)$ and $D^\vee(L)$ completely in terms of L, so that referring to its canonical extension is no longer necessary. We do so by defining a finitary version of a classical construction by Bruns and Lakser of the injective hull of a meet-semilattice [2]. Call a finite subset $M \subseteq L$ *join-admissible* if, for all $a \in L$,

$$\bigvee M \wedge a = \bigvee_{m \in M} (m \wedge a).$$

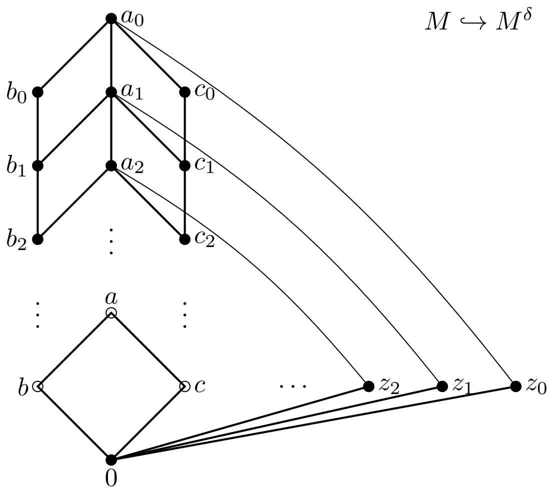
We now say a subset $A \subseteq L$ is an *admissible downset* if A is a downset such that for all join-admissible $M \subseteq A$, $\bigvee M \in A$. We then prove the following characterisation theorem.
Theorem. *The poset of finitely generated admissible downsets, ordered by inclusion, is a distributive lattice which is isomorphic to* $D^\wedge(L)$*. Moreover,* $\hat{(\cdot)} : L \to D^\wedge(L)$ *is the unique distributive* $\wedge$*-extension of* L *which preserves admissible joins and in which* L *is join-dense.*
Of course we have an order-dual characterisation for $D^\vee(L)$.
If the lattice L is distributive, then any finite join is admissible, so that the admissible downsets coincide with the lattice ideals. Moreover, the finitely generated lattice ideals are simply the principal ideals, so that in this case, L is isomorphic to $D^\wedge(L)$, and similarly, L is isomorphic to $D^\vee(L)$.
Now recall, from how we first defined $D^\wedge(L)$, that $D^\wedge(L)$ comes with a set representation on $X = J^\infty(L^\delta)$, which we want to use to generate a topology τ_X on X. If L is distributive, then we can simply take the sets in $D^\wedge(L)$ as a basis for the open sets of a topology on X to obtain the Stone dual space of L. However, if L is not distributive, then taking $D^\wedge(L)$ to be a basis for the open sets may destroy all the information about

the lattice, as the following example shows.
Let $L = M_\infty$ be a countable antichain with top and bottom, as in the diagram on the left, and let L' be the poset obtained from L by adding a join of two elements in the antichain, as in the diagram on the right.





Both lattices L and L' are equal to their canonical extension, and the sets X_L and $X_{L'}$ of completely join-irreducible elements of L and L' are both equal to the infinite antichain. For each a in the antichain, the set $\hat{a}$ is the singleton $\{a\}$, so that the topology generated by taking the sets $\hat{a}$ to be a basis of open sets is the discrete topology on an infinite set. This topology is not compact, and it is impossible to recover any information about L and L' from it.
Alternatively, one could take the complements $(\hat{a})^c$ as a subbasis for the open sets of a topology on $X_L := J^\infty(L^\delta)$. The topology thus generated on X_L is the cofinite topology, which is compact. However, this topology is not sober, and, moreover, X_L is homeomorphic to $X_{L'}$ with this topology, even though the lattices L and L' are not isomorphic.
We are now ready to give an account of the general set-up of Hartung's duality, relative to canonical extensions. Let τ_X be the topology on X, generated by taking the complements of the sets in $D^\wedge(L) \subseteq \mathcal{P}(X)$ as a basis, and similarly define the topology τ_Y on Y. Then the topological RS frame $(X, \tau_X; Y, \tau_Y; R)$, with $R \subseteq X \times Y$ the restriction of the order $\leq_{L^\delta}$, is exactly the structure dual to the lattice L defined by Hartung, now put in the context of canonical extensions. The topologies τ_X and τ_Y have some peculiar properties: they are not sober in general, as we saw above, and although they always have a subbasis of compact-open sets, there are examples in which the intersection of two compact-open sets is not compact, so in general, the compact-open sets do not form a basis for the topology.
As mentioned above, the lattice L may also be understood topologically by considering the tuple $(X_S, \sigma_X; Y_S, \sigma_Y; S)$, arising from the Galois connection $()^u : D^\wedge(L) \leftrightarrows D^\vee(L) : ()^l$ by Stone duality for distributive lattices. This topological representation has the advantage of involving two sober spaces. We also observe that the spaces (X, τ_X) and (Y, τ_Y) can be embedded in (X_S, σ_X) and (Y_S, σ_Y), respectively.
It is natural to ask how these two choices of duals are related. We can provide the following preliminary results. While the complements of the sets in $D^\wedge(L)$ form a basis for the topology of (X, τ_X), these sets are not necessarily all compact in τ_X as shown by the following example (see explanation on the next page).



In the above diagram, the elements of the original lattice M are drawn as filled dots, and the three additional elements a, b and c of the canonical extension M^δ are drawn as transparent dots. The set $X = J^\infty(M^\delta)$ is $\{b_i, c_i, z_i \mid i \geq 0\} \cup \{b, c\}$. In the space (X, τ_X) associated to M, we have that $\{(\hat{a}_n)^c\}_{n=0}^\infty$ is an open cover of $(\hat{b}_0 \cup \hat{c}_0)^c = \{z_i : i \geq 0\}$ with no finite subcover.
As a consequence, even the sobrification of (X, τ_X) will not equal (X_S, σ_X) in general. However, if we use the lattice representation $D^\wedge(L) \hookrightarrow \mathcal{P}(X)$ to generate a quasi-uniform (or ordered uniform) space on X instead of a topological space, then, using results from [4], we may obtain (X_S, σ_X) (or the Priestley space corresponding to (X_S, σ_X)) by quasi-uniform (or ordered uniform) space completion. To be more specific, the version dealing with ordered uniform spaces is already available in [4]: The ordered Hausdorff completion of the ordered uniform Pervin space corresponding to *any* representation of a bounded distributive lattice, and thus in particular $D^\wedge(L) \hookrightarrow \mathcal{P}(X)$, is equal to the Priestley space of that lattice, see [4, Theorem 1.6, p. 4] and the comments in Section 6 of that paper. We note that a quasi-uniform space version of the results in [4] will soon be available in the journal version of that paper, thus allowing us to obtain (X_S, σ_X) directly as a quasi-uniform completion of the quasi-uniform Pervin space corresponding to the lattice representation $D^\wedge(L) \hookrightarrow \mathcal{P}(X)$. This provides a relation between the points of the two spaces (X, τ_X) and (X_S, σ_X) which we want to explore further. Of course order-dual comments hold for the spaces (Y, τ_Y) and (Y_S, σ_Y).

References

[1] J. Michael Dunn, Mai Gehrke, and Alessandra Palmigiano, *Canonical extensions of ordered algebraic structures and relational completeness of some substructural logics*, Journal of Symbolic Logic **70** (2005), no. 3, 713–740.

[2] G. Bruns and H. Lakser, *Injective hulls of semilattices*, Canadian Mathematical Bulletin **13** (1970), no. 1, 115–118.

[3] Mai Gehrke, *Generalized Kripke Frames*, Studia Logica **84** (2006), no. 2, 241–275.

[4] Mai Gehrke, Serge Grigorieff, and Jean-Eric Pin, *A Topological Approach to Recognition*, Automata, Languages and Programming (Samson Abramsky et al., ed.), vol. 6199, Lecture Notes in Computer Science, no. 2, Springer, 2010, 37th International Colloquium (ICALP 2010), pp. 151–162.

[5] Mai Gehrke and John Harding, *Bounded lattice expansions*, Journal of Algebra **238** (2001), no. 1, 345–371.

[6] Mai Gehrke and Bjarni Jónsson, *Bounded distributive lattices with operators*, Mathematica Japonica **40** (1994), no. 2, 207–215.

[7] Gerd Hartung, *An extended duality for lattices*, General algebra and applications (K. Denecke and H.-J. Vogel, eds.), Heldermann-Verlag, Berlin, 1993, pp. 126–142.

[8] M. H. Stone, *Topological representations of distributive lattices and Brouwerian logics*, Časopis pro Peštování Matematiky a Fysiky **67** (1937), 1–25.

[9] Alasdair Urquhart, *A topological representation theory for lattices*, Algebra Universalis **8** (1978), no. 1, 45–58.

Residuated Park Theories: Extended Abstract

Z. Ésik
Dept. of Computer Science, University of Szeged
6720 Szeged, Árpád tér 2, Hungary

1 Introduction

The semantics of recursion is usually described by fixed points of functions, functors, or other constructors, cf. [2, 14]. Least fixed points of monotone or continuous functions on cpo's or complete lattices have been widely used to give semantics to functional programs and various programming logics such as the μ-calculus. The parameterized least fixed point operation †, in conjunction with function composition and the cartesian operations (or Lawvere theory operations) and the binary supremum operation satisfies several nontrivial equations, such as the well-known De Bakker-Scott-Bekić equation, [1, 3]. One would naturally like to have a complete description of *all* valid (in)equations in the form of a system of axioms. Our aim in this paper is to provide such complete descriptions.

The program carried out in this paper stems on one side from [11] and [5, 7, 8], and from [12, 13] on the other side. All theorems formulated in this abstract are new unpublished results. Complete proofs can be found in [9].

2 Theories and residuated theories

In any category we write composition $f \cdot g$ of morphisms $f : a \to b$ and $g : b \to c$ in diagrammatic order and we let $\mathbf{1}_a$ denote the identity morphism $a \to a$.

A *Lawvere theory* is a small category T whose objects are the nonnegative integers such that each object n is the n-fold coproduct of object 1 with itself. We assume that every Lawvere theory comes with distinguished coproduct injections $i_n : 1 \to n$, $i \in [n] = \{1, \ldots, n\}$. Thus, for any sequence of morphisms $f_1, \ldots, f_n : 1 \to p$ there is a unique $f : n \to p$ with $i_n \cdot f = f_i$, for all $i \in [n]$. We denote this unique morphism f by $\langle f_1, \ldots, f_n \rangle$ and call it the *tupling* of the f_i. When $n = 0$, we also write 0_p. It is clear that $\mathbf{1}_n = \langle 1_n, \ldots, n_n \rangle$ for all n. We require that $\mathbf{1}_1 = 1_1$, so that $\langle f \rangle = f$ for all $f : 1 \to p$. Since an object $n+m$ is the coproduct of objects n and m with respect to coproduct injections $\kappa_{n,n+m} : n \to n+m$ and $\lambda_{m,n+m} : m \to n+m$ that are tuplings of the injections i_{n+m}, $i \in [n+m]$, any theory is equipped with a *pairing* operation mapping a pair of morphisms (f, g) with $f : n \to p$ and $g : m \to p$ to $\langle f, g \rangle : n+m \to p$. Also, we can define for $f : n \to p$ and $g : m \to q$ the morphism $f \oplus g : n+m \to p+q$ as $\langle f \cdot \kappa_{p,p+q},\, g \cdot \lambda_{q,p+q} \rangle$.

When A is any set, the *Lawvere theory of functions* $\mathbf{F}_A$ has as morphisms $n \to p$ all functions $A^p \to A^n$ (note the reversal of the arrow). The composition $f \cdot g$ of $f : n \to p$ and $g : p \to q$ is their function composition which is a function $A^q \to A^n$. The distinguished coproduct injections are the projections. It is well-known that each Lawvere theory can be embedded by an object and coproduct preserving functor in a theory $\mathbf{F}_A$, for some set A. Thus, each theory can be faithfully represented as a theory of functions. Also, each theory T may be seen as a many-sorted algebra satisfying certain equational axioms whose set of sorts is the set $\mathbb{N} \times \mathbb{N}$ of all ordered pairs of nonnegative integers, see e.g. [2] for details.

An *ordered theory* is a theory T equipped with a partial order $\leq$ defined on each hom-set $T(n, p)$ such that composition is monotone and for all $f, g : n \to p$, $f \leq g$ iff $i_n \cdot f \leq i_n \cdot g$ for all $i \in [n]$. A *semilattice ordered theory* is an ordered theory T such that each home-set is semilattice ordered. Alternatively, a semilattice ordered theory may be viewed as a theory T equipped with a binary operation $\vee$, defined on each hom-set $T(n, p)$ such that $(T(n, p), \vee)$ is an upper semilattice, moreover

$$\begin{aligned} f \cdot g &\leq (f \vee f') \cdot (g \vee g'), \quad f, f' : n \to p,\ g, g' : p \to q \\ i_n \cdot (f \vee g) &\leq (i_n \cdot f) \vee (i_n \cdot g), \quad f, g : n \to p,\ i \in [n], \end{aligned}$$

where for any $f, g : n \to p$, $f \leq g$ is an abbreviation for $f \vee g = g$.

Suppose that T is an ordered theory. Then any morphism $g : p \to q$ in T induces a monotone function $T(n, p) \to T(n, q)$ by right composition: $f \mapsto f \cdot g$, for all $f : n \to p$. When this function has a right adjoint, we have a Galois connection that defines a (left) residuation operation.

Definition 2.1 *Suppose that T is an ordered theory. We call T a* ***residuated ordered theory*** *if T is* *equipped with a binary operation*

$$\begin{array}{rcl} T(n,q) \times T(p,q) & \to & T(n,p), \quad n,p,q \geq 0 \\ (h,g) & \mapsto & h \Leftarrow g \end{array}$$

such that $f \cdot g \leq h$ *iff* $f \leq (h \Leftarrow g)$ *for all* $f : n \to p$, $g : p \to q$ *and* $h : n \to q$. *We call* $h \Leftarrow g$ *the* (left) residual of h by g.

A ***residuated semilattice ordered theory*** *is a semilattice ordered theory which is a residuated ordered theory.*

Note that in a residuated ordered theory, for any $h : n \to q$ and $g : p \to q$, $h \Leftarrow g$ is the greatest morphism $f : n \to p$ with $f \cdot g \leq h$. Thus, an ordered theory can be turned into a residuated ordered theory in at most one way.

Suppose that L is a complete lattice. Let $\mathbf{Mon}_L$ denote the theory of all monotone functions on L which is a subtheory of $\mathbf{F}_L$. Equipped with the pointwise partial order, $\mathbf{Mon}_L$ is a residuated semilattice ordered theory.

3 Residuated Park theories

Definition 3.1 *A* ***residuated Park theory*** T *is a residuated ordered theory equipped with a dagger operation* $^\dagger : T(n, n+p) \to T(n,p)$ *for* $n, p \geq 0$ *satisfying the following conditions:*

$$f \leq g \;\; \Rightarrow \;\; f^\dagger \leq g^\dagger, \quad f,g : n \to n+p \tag{1}$$

$$f \cdot \langle f^\dagger, \mathbf{1}_p \rangle \;\; \leq \;\; f^\dagger, \quad f : n \to n+p \tag{2}$$

$$f^\dagger \cdot g \;\; \leq \;\; (f \cdot (\mathbf{1}_n \oplus g))^\dagger, \quad f : n \to n+p, \; g : p \to q \tag{3}$$

$$(g \Leftarrow \langle g, \mathbf{1}_p \rangle)^\dagger \;\; \leq \;\; g, \quad g : n \to p. \tag{4}$$

A ***residuated semilattice ordered Park theory*** *is a residuated Park theory which is semilattice ordered.*

Note that when T is semilattice ordered, (1) can be expressed by the inequation

$$f^\dagger \;\; \leq \;\; (f \vee g)^\dagger, \quad f, g : n \to n+p.$$

It is not difficult to show that in any residuated Park theory it holds that

$$f \cdot \langle f^\dagger, \mathbf{1}_p \rangle = f^\dagger, \quad f : n \to n+p \quad \text{and} \quad f^\dagger \cdot g = (f \cdot (\mathbf{1}_n \oplus g))^\dagger, \quad f : n \to n+p, \; g : p \to q.$$

Moreover, the *least fixed point rule* (or *Park induction*) holds:

$$f \cdot \langle g, \mathbf{1}_p \rangle \leq g \;\; \Rightarrow \;\; f^\dagger \leq g, \quad f : n \to n+p, \; g : n \to p.$$

Thus, any residuated Park theory is a *Park theory* as defined in [5].

When L is a complete lattice, $\mathbf{Mon}_L$ is a residuated semilattice ordered Park theory. The dagger operation maps a morphism $f : n \to n+p$, i.e., a monotone function $f : L^{n+p} \to L^n$ to the monotone function $f^\dagger : L^p \to L^n$ such that for each $y \in L^p$, $f^\dagger(y)$ is the least fixed point of the monotone endofuntction $L^n \to L^n$, $x \mapsto f(x,y)$.

A *term* in the language of theories equipped with a dagger operation is a well-formed expression composed from sorted morphism variables (or letters) $f : n \to p$ and the symbols $i_n : 1 \to n$, $i \in [n]$, $n \geq 0$, by the theory operations and dagger. A term in the language of theories equipped with some additional operations such as $\vee$ or the residuation and star operations introduced later in the sequel may involve those additional operations. Each term t has a source n and a target p, noted $t : n \to p$. When the variables f are interpreted as morphisms of appropriate source and target in a theory T equipped with dagger or the other additional operations, each term $t : n \to p$ denotes a morphism $n \to p$ of T. When T is ordered, we say that an inequation $t \leq t'$ between terms $t, t' : n \to p$ holds in T if under each interpretation of the variables by morphisms in T, the morphism denoted by t is less than or equal to the morphism denoted by t' in the ordering of T. We say that the equation $t = t'$ holds in T if both $t \leq t'$ and $t' \leq t$ hold. Clearly, when T is semilattice ordered, $t \leq t'$ holds iff $t \vee t' = t'$ does. Examples of inequations that hold in all theories $\mathbf{Mon}_L$, where L is any complete lattice are given in Definition 3.1.

The equational properties of the dagger operation in the theories $\mathbf{Mon}_L$ in conjunction with the theory operations are captured by the axioms of iteration theories axiomatized by the "Conway equations" and an equation associated with each finite (simple) group [6]. As the next result shows, this infinite collection of equational axioms may be replaced by a much simpler one if we enlarge the set of operations with residuation.

Theorem 3.2 *An inequation between terms in the language of theories equipped with a* † *operation holds in all theories* $\mathbf{Mon}_L$*, where* L *is a complete lattice iff it holds in all residuated Park theories.*

An equation between terms in the language of theories equipped with a dagger operation and an operation $\vee$ *holds in all theories* $\mathbf{Mon}_L$*, where* L *is a complete lattice iff it holds in all residuated semilattice ordered Park theories.*

4 Dagger vs. star

Call an ordered theory T *strict* if for each $n, p \geq 0$ there exists a least morphism $n \to p$, denoted $\perp_{n,p} : n \to p$, such that $\perp_{n,p} \cdot g = \perp_{n,q}$. Every Park theory is strict with $\perp_{n,p} = (\mathbf{1}_n \oplus 0_p)^\dagger$. For any morphism $f : n \to n + p$ in a *semilattice ordered theory*, let us define

$$f^\tau = f \cdot (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) \vee (0_n \oplus \mathbf{1}_n \oplus 0_p) : n \to n + n + p.$$

(We assume that $\cdot$ has higher precedence than $\vee$.)

Definition 4.1 *A* ***residuated star Park theory*** *is a strict residuated semilattice ordered theory equipped with a star operation* $^* : T(n, n+p) \to T(n, n+p)$, $n, p \geq 0$ *satisfying*

$$f^* = (f^\tau)^* \cdot \langle \perp_{n,n+p}, \mathbf{1}_{n+p} \rangle, \quad f : n \to n + p \tag{5}$$

$$f^* \leq (f \vee g)^*, \quad f, g : n \to n + p \tag{6}$$

$$f^* \cdot (\mathbf{1}_n \oplus g) \leq (f \cdot (\mathbf{1}_n \oplus g))^*, \quad f : n \to n + p,\ g : p \to q \tag{7}$$

$$f \cdot \langle f^*, 0_n \oplus \mathbf{1}_p \rangle \vee (\mathbf{1}_n \oplus 0_p) \leq f^*, \quad f : n \to n + p \tag{8}$$

$$(g \Leftarrow \langle g, 0_n \oplus \mathbf{1}_p \rangle)^* \leq (g \Leftarrow \langle g, 0_n \oplus \mathbf{1}_p \rangle), \quad g : n \to n + p \tag{9}$$

It is clear that the star operation is monotone in any star Park theory.

Let T be a strict (residuated) semilattice ordered theory. If T is equipped with a dagger operation, define a star operation by $f^* = (f^\tau)^\dagger$, for all $f : n \to n + p$. Let T_* denote the resulting theory. If T is equipped with a star operation, define $f^\dagger = f^* \cdot \langle \perp_{n,p}, \mathbf{1}_p \rangle$, for all $f : n \to n + p$. The resulting theory is denoted $T_\dagger$.

Theorem 4.2 *The assignments* $T \mapsto T_*$ *and* $T \mapsto T_\dagger$ *are inverse bijections between residuated semilattice ordered Park theories and residuated star Park theories.*

Actually by introducing appropriate morphisms, it follows that the category of residuated semilattice ordered Park theories is isomorphic to the category of residuated star Park theories.

Theorem 4.3 *An (in)equation between terms in the language of theories equipped with operations* $\vee$ *and* * *and constants* $\perp_{n,p}$ *holds in all theories* $\mathbf{Mon}_L$*, where* L *is any complete lattice iff it holds in all residuated star Park theories.*

We end this section by presenting an interesting property of the star operation in residuated star Park theories.

Theorem 4.4 *Suppose that* T *is a strict residuated semilattice ordered theory equipped with a star operation. Then* T *is a residuated star Park theory iff (5), (7) hold and for each* $f : n \to n + p$, f^* *is the least morphism* $g : n \to n + p$ *such that*

$$(\mathbf{1}_n \oplus 0_p) \vee f \vee g \cdot \langle g, 0_n \oplus \mathbf{1}_p \rangle \leq g. \tag{10}$$

5 Regular tree languages

In this section, we present a computer science application. When Σ is ranked alphabet (or signature) and $n \geq 0$, we denote by $T_\Sigma(X_n)$ the set of all Σ-terms (or Σ-trees) in the variables $x_1, \ldots, x_n$, cf. [10]. We recall that a (Σ-)tree language is a subset of $T_\Sigma(X_n)$, for some $n \geq 0$. The theory $\mathbf{TreeLang}_\Sigma$ has as morphisms $n \to p$ all n-tuples of tree languages in $T_\Sigma(X_p)$. Composition is defined by OI-substitution [4] and for each $i \in [n]$, $n \geq 0$, the ith distinguished morphism $1 \to n$ is the language $\{x_i\}$ containing only the variable x_i. Equipped with the pointwise inclusion, $\mathbf{TreeLang}_\Sigma$ is uniquely a residuated semilattice ordered Park theory and a residuated star Park theory. It contains the theory $\mathbf{Reg}_\Sigma$ of regular (finite tree automaton recognizable) tree languages as a subtheory. (For unexplained notions we refer to [10] or [9].)

Definition 5.1 *Suppose that T is a strict semilattice ordered theory. We call a morphism $f : 1 \to p$* strict *if*

$$f \cdot \langle 1_p, \ldots, (i-1)_p, \bot_{1,p}, (i+1)_p, \ldots, p_p \rangle \quad = \quad \bot_{1,p}$$

for all $i \in [p]$. We call f distributive *if $f \cdot \langle 1_{p+1}, \ldots, (i-1)_{p+1}, i_{p+1} \vee (i+1)_{p+1}, (i+2)_{p+1}, \ldots, (p+1)_{p+1} \rangle =$*

$$f \cdot \langle 1_{p+1}, \ldots, (i-1)_{p+1}, i_{p+1}, (i+2)_{p+1}, \ldots, (p+1)_{p+1} \rangle \vee$$
$$f \cdot \langle 1_{p+1}, \ldots, (i-1)_{p+1}, (i+1)_{p+1}, (i+2)_{p+1}, \ldots, (p+1)_{p+1} \rangle$$

for all $i \in [p]$.

For example, for each letter $\sigma \in \Sigma_p$, the morphism $\{\sigma(x_1, \ldots, x_p)\} : 1 \to p$ in $\mathbf{TreeLang}_\Sigma$ is strict and distributive. More generally, a tree language $L : 1 \to p$ in $\mathbf{TreeLang}_\Sigma$ is strict and distributive iff each tree $t \in L$ contains exactly one occurrence of each variable $x_i \in X_p$.
For any term t over Σ in the language of theories equipped with operations $\vee$ and †, we let $|t|$ denote the morphism denoted by t in $\mathbf{Reg}_\Sigma$ when each letter $\sigma \in \Sigma_p$ is interpreted as the language $\{\sigma(x_1, \ldots, x_p)\} : 1 \to p$.

Theorem 5.2 *Suppose that t and t' are terms $n \to p$ over Σ in the language of theories equipped with operations $\vee$ and † (or *). Then $|t| = |t'|$ iff $t = t'$ holds in all residuated semilattice ordered Park theories under all interpretations of the letters in Σ by strict distributive morphisms.*

This theorem extends a result from [12] from regular word languages to regular tree languages. Actually for regular word languages it is stronger than the corresponding result in [12] since it involves only one-sided residuation.

Remark 5.3 *The dagger operation and the star operation may be replaced by their "scalar versions", i.e., operations mapping a morphism $f : 1 \to 1 + p$ to a morphism $f^\dagger : 1 \to p$ or $f^* : 1 \to 1 + p$. See [9].*

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References

[1] H. Bekić, Definable operation in general algebras, and the theory of automata and flowcharts, Technical report, IBM Vienna, 1969. Reprinted in: *Programming Languages and Their Definition - Hans Bekic (1936-1982)*, LNCS 177, Springer 1984, 30–55.

[2] S.L. Bloom and Z. Ésik, *Iteration Theories*, Springer, 1993.

[3] J.W. DeBakker and D. Scott, A theory of programs, Technical Report, IBM Vienna, 1969.

[4] J. Engelfriet and E. M. Schmidt, IO and OI, Parts I and II, *J. Comput. System Sci.*, 15(1977), 328–353 and 16(1978), 67–99.

[5] Z. Ésik, Completeness of Park induction.*Theoret. Comput. Sci.*, 177(1997), no. 1, 217–283.

[6] Z. Ésik, Group axioms for iteration. *Inform. and Comput.*, 148(1999), no. 2, 131–180.

[7] Z. Ésik, Axiomatizing the least fixed point operation and binary supremum. In: *Computer Science Logic (Fischbachau, 2000)*, LNCS 1862, 2000, 302–316.

[8] Z. Ésik, Axiomatizing the equational theory of regular tree languages, *J. Logic and Algebraic Programming*, 79(2010), 189–213.

[9] Z. Ésik, Residuated Park theories, arXiv:1102.1139v1, 2011.

[10] F. Gécseg and M. Steinby, *Tree Automata*, Akadémiai Kiadó, Budapest, 1986.

[11] D. Kozen, A completeness theorem for Kleene algebras and the algebra of regular events. *1991 IEEE Symposium on Logic in Computer Science (Amsterdam, 1991)*, and *Inform. and Comput.*, 110(1994), no. 2, 366–390.

[12] V. Pratt, Action logic and pure induction. *Logics in AI (Amsterdam, 1990)*, LNCS, 478, Springer, Berlin, 1991, 97–120,

[13] L. Santocanale, On the equational definition of the least prefixed point. *Theoretical Computer Science*, 295(2003), 341–370.

[14] A.K. Simpson and G.D. Plotkin, Complete axioms for categorical fixed-point operators. *LICS 2000*, IEEE Press, 2000, 30–41.

Substructural Logic for Orientable and Non-Orientable Surfaces

Christophe Fouqueré[1] Virgile Mogbil[1] Gabriele Pulcini[2]

[1]LIPN – UMR7030, CNRS – Université Paris 13 (France),
{Christophe.Fouquere, Virgile.Mogbil}@lipn.univ-paris13.fr
[2]Department of Philosophy, University of Chieti-Pescara (Italy),
pulcini@di.ens.fr

Abstract

We present a generalization of Permutative logic (PL) [1] which is a non-commutative variant of Linear logic suggested by some topological investigations on the geometry of linear proofs. The original logical status based on a variety-presentation framework is simplified by extending the notion of q-permutation to the one of pq-permutation [7]. Whereas PL is limited to orientable structures, we characterize the whole range of topological surfaces, orientable as well as non-orientable. The system we obtain is a surface calculus that enjoys both cut elimination and focussing properties and comes with a natural phase semantics whenever explicit context is considered.

Among the different viewpoints considered for studying proofs of Linear Logic, let us recall that its graph-theoretical representations may be seen as topological objects and considered as surfaces on which usual proofs are drawn without crossing edges [2, 6, 5]. Following that interpretation, Gaubert [3] provided a way to *compute* surfaces. Moreover, based on the fact that the exchange rule may model topological operations, non-commutative variants of Multiplicative Linear logic (MLL) were developed: planar logic [5], the calculus of surfaces [3] and permutative logic (PL) [1]. In all these cases the conclusions of the proofs are drawn on disjoint oriented circles, more precisely orientable surfaces with boundary. E.g. in PL the underlying structure is that of a permutation, which is a product of disjoint cycles, together with a natural number to express the number of tori, actually a topological invariant of the surface. The shape of such sequents is called a q-permutation and geometrically studied in [7] by one of the present authors. One of the main topological results given in [7] is related to Massey classification theorem [4]: any orientable surface, possibly with boundary, is homeomorphic either to a sphere or to a finite connected sum of tori, possibly with boundary.

We consider in this paper a generalization to surfaces orientable or not. Massey theorem states in that case that it may be homeomorphic also to a finite connected sum of projective planes, possibly with boundary. For that purpose, we consider in our work pq-permutations which are simply obtained from q-permutations by replacing the single index q with an ordered couple (p, q) of positive integers for counting tori handles and projective planes. The shape of our sequents still integrates the topology of a surface and non-trivial exchange rules correspond to surface transformations, following what is done in PL but also in Melliès planar logic [5]. After presenting the logical system, we prove a few logical properties: it enjoys both cut elimination and the focussing property, as PL does. We give a phase semantics that is sound and complete with respect to the calculus. Though a phase semantics may seem a too elementary result, it allows us to tackle the problem of contextual structures. The aim of our work is to shed new light on the relationship between topology and logic.

1 sPL: A Sequent Calculus for Surfaces

Formulas of sPL are inductively built from a countable infinite set of atoms $\mathcal{A} = \{a, b, c, \ldots, a^\perp, b^\perp, c^\perp, \ldots\}$ and the two usual multiplicative connectives $⅋$ and $\otimes$, together with a unary bar operation $(\bar{\ })$ that models the inversion of the orientation:

$$F ::= F \in \mathcal{A} \mid \bar{F} \mid F_1 ⅋ F_2 \mid F_1 \otimes F_2$$

IDENTITY GROUP

$$\frac{}{\vdash_0^0 (A, A^\perp)}\text{ax.} \qquad \frac{\vdash_q^p \Sigma, (\Gamma, A) \qquad \vdash_{q'}^{p'} \Xi, (\Delta, A^\perp)}{\vdash_{q+q'}^{p+p'} \Sigma, \Xi, (\Gamma, \Delta)}\text{cut}$$

ORIENTABLE STRUCTURAL RULES

$$\frac{\vdash_q^p \Sigma, (\Gamma, \Delta)}{\vdash_q^p \Sigma, (\Gamma), (\Delta)}\text{cylinder} \qquad \frac{\vdash_q^p \Sigma, (\Gamma), (\Delta)}{\vdash_{q+1}^p \Sigma, (\Gamma, \Delta)}\text{torus} \qquad \frac{\vdash_q^p \Sigma}{\vdash_q^p \overline{\Sigma}}\text{invert}$$

NON-ORIENTABLE STRUCTURAL RULES

$$\frac{\vdash_q^p \Sigma, (\Gamma, \Delta)}{\vdash_q^{p+1} \Sigma, (\Gamma, \overline{\Delta})}\text{Möbius} \qquad \frac{\vdash_q^p \Sigma, (\Gamma), (\Delta)}{\vdash_q^{p+2} \Sigma, (\Gamma, \overline{\Delta})}\text{Klein}$$

LOGICAL RULES

$$\frac{\vdash_q^p \Sigma, (\Gamma, A, B)}{\vdash_q^p \Sigma, (\Gamma, A \parr B)}\parr \qquad \frac{\vdash_q^p \Sigma, (\Gamma, A) \qquad \vdash_{q'}^{p'} \Xi, (\Delta, B)}{\vdash_{q+q'}^{p+p'} \Sigma, \Xi, (\Gamma, A \otimes B, \Delta)}\otimes$$

Table 1: Sequent calculus for sPL

The negation is defined as usual by de Morgan duality and preserves the bar operation. A *sequent* is denoted $\vdash_q^p \Gamma$ where Γ is a multiset of cyclic sequences which are formulas separated by ',' within parenthesis, and p and q are integers with the intuition they denote a pq-permutation (p for projective planes). We write () for an empty cycle. Type derivations are built from the rules of table 1.

Remark that the key rules of divide and merge in PL are also in sPL as respectively cylinder and torus acting as orientable rules.

2 Cut Elimination and Focussing Property

Theorem 1 (cut-elimination)**.** *Any proof of a sPL sequent can be rewritten into a cut-free proof of the same sequent.*

A standard proof by case analysis need a particular attention for commutative conversions involving the Möbius or the Klein rule. This result can also be obtained as a consequence of *focalization.*

A focalized sequent calculus, called foc-sPL, may be defined with sequents of the form $\vdash_q^p \Gamma|\Sigma$ where Γ is the focus – a distinguished cyclic sequence – and Σ is a multiset of cyclic sequences of formulas separated by ';'. The cut rule acts only on focusses. The main ingredients are the following ones: a focus rule is added and Klein and torus rules are changed in the following way:

$$\frac{\vdash_q^p |(\Gamma); \Sigma}{\vdash_q^p \Gamma|\Sigma}\text{focus} \qquad \frac{\vdash_q^p \Gamma, \Lambda, \Delta|\Sigma}{\vdash_{q+1}^p \Gamma, \Delta, \Lambda|\Sigma}\text{torus'} \qquad \frac{\vdash_q^p \Gamma, \Lambda, \Delta|\Sigma}{\vdash_q^{p+2} \Gamma, \Delta, \overline{\Lambda}|\Sigma}\text{Klein'}$$

In such a presentation the defocus rule is simply a special case of the cylinder rule (with () neutral w.r.t. ';'). As it follows from topological considerations, structures of proofs in foc-sPL may be normalized in such a way that cylinder applications arrive only at the end of a proof construction. Such proofs are called *maximally focalized* and it is then possible to prove a cut-elimination property on them.

Proposition 2 (Maximal Focalization)**.** *A sequent is provable in foc-sPL if and only if there exists a proof such that cylinder rules are applied only at the end. Moreover cuts in a maximally focalized proof in foc-sPL may be eliminated.*

Sketch. As the cut rule is applied to focalized formulas and that cut and cylinder rules commute, one may consider that cuts are applied to sequents of the form $\vdash_q^p \Gamma|$. The rest is done by case analysis. □

We are finally able to prove that provability in sPL and foc-sPL are equivalent, hence a cut-elimination theorem for sPL follows.

Proposition 3 (Focussing property). *A sequent is provable in sPL if and only if it is provable in foc-sPL.*

3 Phase Semantic

A *phase space* is provided that is proved to be *complete and valid* with respect to the calculus. This should be considered as a first step towards a better understanding of the calculus and its relation to geometry. In fact, this is not at all obvious if we notice that there is not yet satisfying proof semantics for non-commutative logic (NL) even though its phase space has been given together with its sequent calculus (by Ruet in his 1997'thesis). What is the main difficulty when turning to a calculus of surfaces? Or equivalently what makes NL an easier situation? The orientation has to be taken into account, more than that the context cannot be neglected. In NL, the non-commutative structure is an order variety. Hence a formula on which an operation is applied may be 'extracted' from its context: the structure of the semantics is close to what is required with Linear Logic. This is no more true in the calculus of surfaces as *see-saw* structural rules are not valid: one is required to deal explicitly with the context.

For that purpose, a *support* phase space $\texttt{Supp}(M)$ interpreting formulas is embedded into a *context* phase space $\texttt{Con}(M)$ interpreting sequents. The two phase spaces are defined from an associative monoid and give rise to two closure operations $\perp$ and $\dagger$ in such a way that the fundamental proposition is provable:

Proposition 4. *Let $\mathcal{M} = (M, \star, 1)$ be a (not necessarily commutative) associative monoid with neutral element 1, let $F, G \subset \mathcal{M}$,*

$$(F \star G^{\perp\perp})^{\perp\perp} = (F \star G)^{\perp\perp}$$
$$(F \star G^{\dagger\dagger})^{\dagger\dagger} = (F \star G)^{\dagger\dagger}$$

We consider as usual that a *fact* is a subset A of the support phase space such that $A^{\perp\perp} = A$. Operations are defined on facts:

$$A \otimes B \stackrel{\text{def}}{=} (A \star B)^{\perp\perp} \qquad A \parr B \stackrel{\text{def}}{=} (B^{\perp} \star A^{\perp})^{\perp} \qquad 1 \stackrel{\text{def}}{=} \perp^{\perp} \qquad \perp \text{ is given}$$
$$A \oplus B \stackrel{\text{def}}{=} (A \cup B)^{\perp\perp} \qquad A \,\&\, B \stackrel{\text{def}}{=} A \cap B \top \stackrel{\text{def}}{=} \mathcal{M} \qquad \mathbf{0} \stackrel{\text{def}}{=} \top^{\perp}$$

Although the general lines for proving soundness and validity of the model are standard, their proofs are more complex as they require to consider explicitly the context.

4 Principal line of current work: Relaxation

Relaxation is the binary relation induced by structural transformations – *divide*, *merge*, *Möbius* and *Klein* – on the set of *pq*-permutations. We write $\alpha \prec \beta$, β *relaxes* α, for meaning that the *pq*-permutation α can be rewritten into β through a suitable series of applications of structural rules. Since each structural rule increases the topological genus of the transformed surface, relaxation turns out to impose a partial order on the set of *pq*-permutations. We pose the problem of providing an algorithm for the decision of relaxation, namely an effective procedure able to answer to the question '$\alpha \prec \beta$?' being given two *pq*-permutations α and β.

Two parallel solutions have been already afforded in case of *q*-permutations, namely in case of combinatorial structures encoding orientable surfaces. The first one has been considered in [1] and consists in interpreting orientable transformations – *divide* and *merge* – as the effect of composing *q*-permutations with a suitable transposition. Being established such an algebraic correspondence, the solution comes straightforwardly by stressing very standard achievements in theory of permutations. The other solution provides a geometrical and interactive approach. For answering to our question '$\alpha \prec \beta$?' we compute the surface $\mathscr{S}_\alpha * \mathscr{S}_\beta$ obtained by composing, through identification of paired edges occurring on the boundaries, the two surfaces $\mathscr{S}_\alpha$ and $\mathscr{S}_\beta$ respectively corresponding to α and β. In [7], it is proved that the topological genus of $\mathscr{S}_\alpha * \mathscr{S}_\beta$ provides information enough to decide relaxation.

The passage from q to *pq*-permutations turns out to be critical from the point of view of the decision of relaxation. Whereas orientable transformations exclusively act at the level of the combinatorial structure of *pq*-permutations, *Möbius* and *Klein* also affect their supports so as to make impossible any resort to theory of permutations. On the contrary, we guess that the just mentioned geometrical procedure might admit a very natural extension in order to include non-orientable transformations and surfaces. As a matter of fact, it is an often remarked logical phenomenon that genuinely interactive approaches allow to avoid technical problems due to syntactical bureaucracy.

5 At the end...

Our logical system sPL generalizes Permutative logic, and in this way it is an embedding of the multiplicative Cyclic Linear logic (CyLL) and MLL. A completely unexplored field of research is that one of proof-nets for PL and sPL. The starting point should be the criterion for proof-nets of Planar Logic which just consists in requiring, together with the logical correctness, the planarity of the graph [5]. In this direction the main difficulty is that structural rules are usually 'transparent' with respect to the syntax of proof-nets so as we need to recover this kind of information by stressing the geometrical structure of the net. As far as PL is concerned, some useful results could be borrowed from [6], whereas the non-orientable side of the question misses at all of contributions.

Finally we developed a framework allowing to characterize the relationship between logic and orientable as well as not orientable surfaces. Semantical issues have not yet been explored and we are peculiarly interested in denotational semantic to give a topological interpretation of formulas and proofs. The existence of a phase semantics for sPL may provide some alternative algebraic tools for studying the geometry of 2-manifolds. A standard application of phase semantics consists in singling out redundant rules, namely rules which are superfluous with respect to the deductive power of the system, typically the cut rule. Now, since the basic topological transformations are embodied into our system, the classical classification theorem might find an interesting alternative proof when addressed in terms of semantics.

References

[1] Jean-Marc Andreoli, Gabriele Pulcini, and Paul Ruet. Permutative logic. In C.-H. Luke Ong, editor, *CSL*, volume 3634 of *Lecture Notes in Computer Science*, pages 184–199. Springer, 2005.

[2] G. Bellin and A. Fleury. Planar and braided proof-nets for multiplicative linear logic with mix. *Arch. Math. Log.*, 37(5-6):309–325, 1998.

[3] Christophe Gaubert. Two-dimensional proof-structures and the exchange rule. *Mathematical Structures in Computer Science*, 14(1):73–96, 2004.

[4] W. S. Massey. *A basic course in algebraic topology.* Springer, 1991.

[5] Paul-André Melliès. A topological correctness criterion for non-commutative logic. In P. Ruet T. Ehrhard, J-Y. Girard and P. Scott, editors, *Linear Logic in Computer Science*, volume 316 of *London Mathematical Society Lecture Notes Series.* Cambridge University Press, 2004.

[6] François Métayer. Implicit exchange in multiplicative proofnets. *Mathematical Structures in Computer Science*, 11(2):261–272, 2001.

[7] Gabriele Pulcini. A geometrical procedure for computing relaxation. *Ann. Pure Appl. Logic*, 158(1-2):80–89, 2009.

On varieties generated by standard BL-algebras

Zuzana Haniková*
Institute of Computer Science, ASCR
182 07 Prague, Czech Republic
Email address: zuzana@cs.cas.cz

This paper is a contribution to the theory of standard BL-algebras, where 'standard' means that the domain is the real interval $[0, 1]$ and BL refers to Hájek's Basic Fuzzy Logic. The study of BL-algebras was initiated in [5]. In [2] it has been shown that BL is complete w. r. t. standard BL-algebras. The paper [1] gives a characterization of BL-generic BL-chains, and [3] shows that each logic given by a particular single standard BL-algebra can be finitely axiomatized. [4] shows how to extend this result to finite sets of standard BL-algebras.

In this work, we address the problem of arbitrary sets of standard BL-algebras, and we want to argue that passing from finite sets of BL-algebras to infinite ones brings nothing new. We prove the following.

Theorem 1. *If $\mathbb{V}$ is a subvariety of $\mathbb{BL}$ generated by a set of standard BL-algebras, then $\mathbb{V}$ is generated by a* finite *set of standard BL-algebras.*

This result offers answers to some important questions, such as, what the cardinality is of the class of subvarieties of **BL** generated by classes of its standard algebras, or how to axiomatize them.

We build on, and assume familiarity with, [3] and the notions therein, in particular, the notion of $\mathrm{Fin}(\boldsymbol{A})$ and a crucial result of that paper stating that, for two standard BL-algebras $\boldsymbol{A}$ and $\boldsymbol{B}$, we have $\mathrm{Var}(\boldsymbol{A}) \subseteq \mathrm{Var}(\boldsymbol{B})$ iff $\mathrm{Fin}(\boldsymbol{A}) \subseteq \mathrm{Fin}(\boldsymbol{B})$. Also, for each standard BL-algebra, there is a canonical standard BL-algebra that generates the same variety and can be expressed in alphabet ∞Ł, $\infty\Pi$, Ł, G, Π (see [3]). W. l. o. g., we may assume that we work with canonical, pairwise non-isomorphic standard BL-algebras. We proceed by a case study involving finitely many cases, relying on the following statement.

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Lemma 2. *Let* $\mathbb{K} = \bigcup_{i \in I} \mathbb{K}_i$, $\mathbb{L} = \bigcup_{i \in I} \mathbb{L}_i$ *be classes of algebras of the same signature. Assume* $\mathrm{Var}(\mathbb{K}_i) = \mathrm{Var}(\mathbb{L}_i)$ *for* $i \in I$. *Then* $\mathrm{Var}(\mathbb{K}) = \mathrm{Var}(\mathbb{L})$.

Let $k \in \mathbb{N}$. A standard BL-algebra is said to be of type Ł $\oplus$ k.Ł if it starts with an Ł-component, which is followed by an ordinal sum containing exactly k other Ł-components (as well as possibly other components, the order is immaterial). Analogously, the type $\overline{\text{Ł}} \oplus k$.Ł either starts with a component other than Ł, or has no first component, and has exactly k Ł-components altogether. The type $\overline{\text{Ł}}$ has no Ł-components. Further, a standard BL-algebra is said to be of type Ł $\oplus \infty$Ł if it starts with an Ł-component, which is followed by an ordinal sum containing infinitely many other Ł-components; analogously for $\overline{\text{Ł}} \oplus \infty$Ł.

Given a class $\mathbb{K}$ of standard BL-algebras, we decompose it into $\mathbb{K}_0$, consisting of all algebras in $\mathbb{K}$ starting with an Ł-component, and $\mathbb{K}_1$, consisting of all algebras in $\mathbb{K}$ not starting with an Ł-component. Relying on Lemma 2, we tackle $\mathbb{K}_0$ and $\mathbb{K}_1$ separately, trying to replace each class by a suitable finite counterpart.

Lemma 3. *If the set* $\{k \in \mathbb{N} | \exists \boldsymbol{A} \in \mathbb{K}_0(\boldsymbol{A}$ *of type* Ł $\oplus$ k.Ł$)\}$ *is infinite, or if* Ł $\oplus \infty$Ł $\in \mathbb{K}_0$, *then* $\mathrm{Var}(\mathbb{K}_0) = \mathbb{BL}$.

If a class of standard algebras $\mathbb{K}_0$ generates the variety $\mathbb{BL}$, then we have $\mathrm{Var}(\mathbb{K}_0) = \mathrm{Var}(\text{Ł} \oplus \infty\text{Ł}) = \mathrm{Var}(\infty\text{Ł})$; thus the variety generated by $\mathbb{K}_0$ is also generated by a single standard canonical BL-algebra ∞Ł.

On the other hand, if $\mathbb{K}_0$ does not satisfy the conditions of Lemma 3, then there is a $k_0 \in \mathbb{N}$ such that each algebra $\boldsymbol{A} \in \mathbb{K}_0$ has at most k_0 Ł-components. In such a case, $\mathbb{K}_0$ generates a strict subvariety of $\mathbb{BL}$.

In exactly analogous way, one shows that the class $\mathbb{K}_1$ either generates the variety $\mathbb{SBL}$, or the number of Ł-components in each $\mathbb{A} \in \mathbb{K}_1$ is bounded by some $k_1 \in \mathbb{N}$, and in the latter case $\mathbb{K}_1$ generates a strict subvariety of $\mathbb{SBL}$. In the former case, $\mathbb{K}_1$ can be replaced by the single algebra $\Pi \oplus \infty$Ł, which also generates $\mathbb{SBL}$.

In the rest, we address the problem of finding, for a given a class $\mathbb{K}$ of canonical standard (S)BL-algebras with at most k Ł-components each, a *finite* class $\mathbb{K}'$ of (canonical) standard BL-algebras such that $\mathrm{Var}(\mathbb{K}) = \mathrm{Var}(\mathbb{K}')$. Building on Lemma 2, we partition $\mathbb{K}$ into finitely many classes $\mathbb{K}^i, i \leq k$, depending on the number i of Ł-components, and solve the problem separately for each i.

For $k \in \mathbb{N}$, we denote $\mathbb{L}^k$ the class of all (isomorhism types of) canonical standard BL-algebras with exactly k Ł-components. Clearly for $i \leq k_0$ we have $\mathbb{K}^i \subseteq \mathbb{L}^i$. Moreover, in a way analogous to the previous case, we proceed separately for algebras starting with an Ł-component and for algebras not starting with an Ł-component. We address the latter case; the former case can be viewed as a special case.

Fix $k \in \mathbb{N}$. For $\boldsymbol{A} \in \mathbb{L}^k$, we may write

$$\boldsymbol{A} = \boldsymbol{A}_0 \oplus \text{Ł} \oplus \boldsymbol{A}_1 \oplus \cdots \oplus \boldsymbol{A}_{k-1} \oplus \text{Ł} \oplus \boldsymbol{A}_k$$

where $\boldsymbol{A}_0$ is non-empty (by assumption) and each $\boldsymbol{A}_j$, $1 \le j \le k$ is either an empty sum $\emptyset$, or a finite ordinal sum of G's and Π's, or it is $\infty\Pi$.

Definition 4. *For canonical standard BL-algebras $\boldsymbol{A}, \boldsymbol{B}$, let $\boldsymbol{A} \preceq \boldsymbol{B}$ iff* $\mathrm{Var}(\boldsymbol{A}) \subseteq \mathrm{Var}(\boldsymbol{B})$. *Moreover, let $\emptyset \preceq \boldsymbol{A}$ for any standard BL-algebra $\boldsymbol{A}$.*

Lemma 5. *$\preceq$ on $\mathbb{L}^k$ is the product order of its restrictions to each of $\{0, \dots, k\}$; i. e., for $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{L}^k$, where $\boldsymbol{A} = \boldsymbol{A}_0 \oplus Ł \oplus \boldsymbol{A}_1 \oplus \cdots \oplus \boldsymbol{A}_{k-1} \oplus Ł \oplus \boldsymbol{A}_k$ and $\boldsymbol{B} = \boldsymbol{B}_0 \oplus Ł \oplus \boldsymbol{B}_1 \oplus \cdots \oplus \boldsymbol{B}_{k-1} \oplus Ł \oplus \boldsymbol{B}_k$, we have $\boldsymbol{A} \preceq \boldsymbol{B}$ iff for each $j \le k$, $\boldsymbol{A}_j \preceq \boldsymbol{B}_j$.*

It is illuminating to study the behaviour of $\preceq$ on $\mathbb{L}^0$ first. It is easy to show that $\infty\Pi$ is the top element of $\mathbb{L}^0$ w. r. t. $\preceq$; by definition, $\emptyset$ is the bottom element. Moreover, if $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{L}^0$ are finite ordinal sums (of G and Π-components), then $\boldsymbol{A} \preceq \boldsymbol{B}$ iff $\boldsymbol{A}$ is a subsum of $\boldsymbol{B}$. Employing Higman's theorem, $\preceq$ on $\mathbb{L}^0$ is a w.q.o. and hence, $\mathbb{L}^0$ *has no infinite $\preceq$-antichains.*

Further, we investigate $\preceq$-chains in $\mathbb{L}^0$. Let $\{\boldsymbol{A}_i\}_{i \in I}$ be a non-empty $\preceq$-chain in $\mathbb{L}^0$. Then $\{\boldsymbol{A}_i\}_{i \in I}$ has a supremum (in $\mathbb{L}^0$), and $\mathrm{Var}(\{\boldsymbol{A}_i\}_{i \in I}) = \mathrm{Var}(\sup(\{\boldsymbol{A}_i\}_{i \in I}))$.

Using the above results on $\mathbb{L}^0$, we investigate the case of $\mathbb{L}^k$ for a fixed $k \in \mathbb{N}$. By a standard argument, if $\preceq$ on $\mathbb{L}^0$ is a w.q.o., then so is $\preceq$ on $\mathbb{L}^k$ using Lemma 5. Hence, in particular, $\mathbb{L}^k$ has no infinite $\preceq$-antichains.

Moreover, again using Lemma 5, if $\{\boldsymbol{A}_i\}_{i \in I}$ is a $\preceq$-chain in $\mathbb{L}^k$, then $\mathrm{Var}(\{\boldsymbol{A}_i\}_{i \in I}) = \mathrm{Var}(\sup(\{\boldsymbol{A}_i\}_{i \in I}))$.

Let $\mathbb{K} \subseteq \mathbb{L}^k$. Let $\{\boldsymbol{A}_i\}_{i \in I}$ be a $\preceq$-chain in $\mathbb{K}$. We say that $\{\boldsymbol{A}_i\}_{i \in I}$ is *maximal* in $\mathbb{K}$ iff there is no $\boldsymbol{B} \in \mathbb{K}$ such that $\boldsymbol{A}_i \prec \boldsymbol{B}$ for each $i \in I$.

Lemma 6. *Let $\mathbb{K} \subseteq \mathbb{L}^k$. Let $\{\boldsymbol{A}_i\}_{i \in I}$, $\{\boldsymbol{B}_{i'}\}_{i' \in I'}$ be two maximal $\preceq$-chains in $\mathbb{K}$. If $\{\boldsymbol{B}_{i'}\}_{i' \in I'}$ has a top element in $\mathbb{K}$, then* $\sup(\{\boldsymbol{A}_i\}_{i \in I}) \not\prec \sup(\{\boldsymbol{B}_{i'}\}_{i' \in I'})$.

Now we are ready to tackle the main statement of this paper. A class $\mathbb{K} \subseteq \mathbb{L}^k$ of standard (canonical, pairwise non-isomorphic) BL-algebras is given. We need to find a finite $\mathbb{K}' \subseteq \mathbb{L}^k$ such that $\mathrm{Var}(\mathbb{K}) = \mathrm{Var}(\mathbb{K}')$. Let us denote $\mathbb{D}_0 = \mathbb{K}$.

Definition 7. *Let* $\mathbb{D}_0 \subseteq \mathbb{L}^k$*. For* $n \in \mathbb{N}$*, define*

$$\mathbb{D}_{n+1} = \{\boldsymbol{A} | \boldsymbol{A} = \sup(\{\boldsymbol{A}_i\}_{i \in I}) \text{ for some maximal chain } \{\boldsymbol{A}_i\}_{i \in I} \text{ in } \mathbb{D}_n\}$$

Theorem 8. *(i)* $\mathrm{Var}(\mathbb{D}_n) = \mathrm{Var}(\mathbb{D}_{n+1})$ *for each* $n \in \mathbb{N}$

(ii) There is an $n \leq k+2$ *such that*

(a) $\mathbb{D}_n = \mathbb{D}_{n+1}$

(b) $\mathbb{D}_n$ *is finite*

The key point in the proof is observing that, for $n \geq 1$, that if $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{D}_n$, then $\boldsymbol{A} \prec \boldsymbol{B}$ implies $\boldsymbol{B} \notin \mathbb{D}_{n-1}$; moreover, if, for $n \in \mathbb{N}$, we have $\boldsymbol{A} \in \mathbb{D}_{n+1} \setminus \mathbb{D}_n$, then, for at least n distinct elements $j \in J = \{0, \ldots, k\}$ we have $(\boldsymbol{A})_j = \infty\Pi$ (by induction on n).

Corollary 9. *Let* $\mathbb{K} \subseteq \mathbb{L}^k$ *be a class of BL-algebras. Then* $\mathbb{K}' = \mathbb{D}_{k+2}$ *is a finite set of standard algebras in* $\mathbb{L}^k$ *generating the same variety as* $\mathbb{K}$*.*

To conclude, we have provided, for a given class $\mathbb{K}$ of standard BL-algebras, a finite class of BL-algebras generating the same subvariety of $\mathbb{BL}$, by first dealing with the case that $\mathbb{K}$ generates $\mathbb{BL}$ or $\mathbb{SBL}$, and then addressing the case that the number of Ł-components in elements of $\mathbb{K}$ is bounded. In the latter case, we proceeded again by cases, distinguishing by exact number i of Ł-components. For each such case, we have replaced the algebras in $\mathbb{K}^i$ with a suitable finite counterpart.

References

[1] Paolo Aglianò and Franco Montagna. Varieties of BL-algebras I: General properties. *Journal of Pure and Applied Algebra*, 181(2–3):105–129, 2003.

[2] Roberto Cignoli, Francesc Esteva, Lluís Godo, and Antoni Torrens. Basic fuzzy logic is the logic of continuous t-norms and their residua. *Soft Computing*, 4(2):106–112, 2000.

[3] Francesc Esteva, Lluís Godo, and Franco Montagna. Equational characterization of subvarieties of BL-algebras generated by t-norm algebras. *Studia Logica*, 76(2):161–200, 2004.

[4] Nikolaos Galatos. Equational bases for joins of residuated-lattice varieties. *Studia Logica*, 76(2):227–240, 2004.

[5] Petr Hájek. *Metamathematics of Fuzzy Logic*, volume 4 of *Trends in Logic*. Kluwer, Dordrecht, 1998.

Conservativity of Boolean algebras with operators over semilattices with operators

A. Kurucz,[1] Y. Tanaka,[2] F. Wolter[3] and M. Zakharyaschev[4]

[1] Department of Informatics
King's College London, UK
agi.kurucz@kcl.ac.uk

[2] Faculty of Economics
Kyushu Sangyo University, Japan
ytanaka@ip.kyusan-u.ac.jp

[3] Department of Computer Science
University of Liverpool, UK
wolter@liverpool.ac.uk

[4] Department of Computer Science
Birkbeck College London, UK
michael@dcs.bbk.ac.uk

The problems considered in this paper originate in recent applications of large scale ontologies in medicine and other life sciences. The profile *OWL 2 EL* of the *OWL 2* Web Ontology Language,[1] used for this purpose, is based on the description logic $\mathcal{EL}$ [7]. The syntactic terms of $\mathcal{EL}$, called *concepts*, are interpreted as sets in first-order relational models. Concepts are constructed from atomic concepts and constants for the whole domain and empty set using intersection and existential restrictions of the form $\exists R.C$, R a binary relation and C a concept, which are understood as $\exists y\,(R(x,y) \wedge C(y))$. From a modal logic point of view, concepts are modal formulas constructed from propositional variables and the constants $\top$, $\bot$ using conjunction and diamonds. An *$\mathcal{EL}$-theory* is a set of inclusions (or implications) between such concepts, and the main reasoning problem in applications of $\mathcal{EL}$ in life sciences is to decide whether an $\mathcal{EL}$-theory entails a concept inclusion when interpreted over a class of relational structures satisfying certain constraints on its binary relations. Standard constraints in *OWL 2 EL* are transitivity and reflexivity, for which reasoning in $\mathcal{EL}$ is PTime-complete, as well as symmetry and functionality, for which reasoning is ExpTime-complete [1, 2].

As in modal logic, apart from reasoning over relational models, one can try to develop a purely syntactical reasoning machinery using a calculus. In other words, we can define a more general algebraic semantics for $\mathcal{EL}$: the underlying algebras are *bounded meet-semilattices with monotone operators* (SLOs, for short), constraints are given by *equational theories* of SLOs, and the reasoning problem is *validity of quasi-equations* in such equational theories. The resulting more general entailment problem is not necessarily complete with respect to the 'intended' relational semantics. This paper presents our initial results in an attempt to clarify which equational theories of SLOs are complete in this sense and which are not. We also prove that the *completeness problem*—given a finitely axiomatised equational theory of SLOs, decide whether it is complete with respect to the relational semantics—is algorithmically undecidable, which establishes a principle limitation regarding possible answers to our research question.

An *$\mathcal{EL}$-equation* is an expression of the form $\varphi \leq \psi$, where φ and ψ are *terms* that are built from variables x_j, $j \geq 1$, using meet $\wedge$, unary operators f_i, for $i \in I$, and constants 1 and 0. An *$\mathcal{EL}$-theory*, $\mathcal{T}$, is a set of $\mathcal{EL}$-equations; and an *$\mathcal{EL}$-quasi-equation* is an expression of the form $(\varphi_1 \leq \psi_1)\ \&\ \cdots\ \&\ (\varphi_n \leq \psi_n) \to (\varphi \leq \psi)$, where the $\varphi_i \leq \psi_i$ and $\varphi \leq \psi$ are $\mathcal{EL}$-equations. The class of SLOs $\mathfrak{A} = (A, \wedge, 0, 1, f_i)_{i \in I}$ validating all equations in $\mathcal{T}$ is the variety $\mathsf{V}(\mathcal{T})$. The 'intended'

[1] http://www.w3.org/TR/owl2-overview/

relational semantics of $\mathcal{EL}$ is given by *$\mathcal{EL}$-structures* $\mathfrak{F} = (\Delta, R_i)_{i \in I}$, which consist of a set $\Delta \neq \emptyset$ and binary relations R_i on it. Every such $\mathfrak{F}$ gives rise to the *complex algebra* $\mathfrak{F}^+ = (2^\Delta, F_i)_{i \in I}$ of $\mathfrak{F}$, where 2^Δ is the full Boolean set algebra over Δ and $F_i(X) = \{x \in \Delta \mid \exists y \in X\, xR_iy\}$, for $X \subseteq \Delta$. Complex algebras (CAs) are special cases of *Boolean algebras with normal and* $\vee$*-additive operators* (BAOs, for short). The class of *bounded distributive lattices with normal and* $\vee$*-additive operators* is denoted by DLO. Slightly abusing notation (and remembering the signatures of DLOs and BAOs), we may assume that CA $\subseteq$ BAO $\subseteq$ DLO $\subseteq$ SLO.

Given a class $\mathcal{C}$ of SLOs, an $\mathcal{EL}$-theory $\mathcal{T}$ and a quasi-equation $\boldsymbol{q}$, we say that $\boldsymbol{q}$ *follows from* $\mathcal{T}$ *over* $\mathcal{C}$ and write $\mathcal{T} \models_{\mathcal{C}} \boldsymbol{q}$ if $\mathfrak{A} \models \boldsymbol{q}$, for every $\mathfrak{A} \in \mathcal{C}$ with $\mathfrak{A} \models \mathcal{T}$. An $\mathcal{EL}$-theory $\mathcal{T}$ is said to be *$\mathcal{C}$-conservative* if $\mathcal{T} \models_{\mathcal{C}} \boldsymbol{q}$ implies $\mathcal{T} \models_{\mathrm{SLO}} \boldsymbol{q}$, for every quasi-equation $\boldsymbol{q}$. We call $\mathcal{T}$ *complete* if it is CA-conservative.

A standard way of establishing completeness of a modal logic is by showing that its axioms generate what Goldblatt [3] calls a 'complex variety.' This notion works equally well in the $\mathcal{EL}$ setting: We say that an $\mathcal{EL}$-theory $\mathcal{T}$ is *complex* if every $\mathfrak{A} \in \mathsf{V}(\mathcal{T})$ is embeddable in some $\mathfrak{F}^+ \in \mathsf{V}(\mathcal{T})$. The following theorem provides our main tool for investigating completeness of $\mathcal{EL}$-theories:

Theorem 1. *For every $\mathcal{EL}$-theory $\mathcal{T}$,*

$$\mathcal{T} \text{ is complex} \quad \text{iff} \quad \mathcal{T} \text{ is complete} \quad \text{iff} \quad \mathcal{T} \text{ is BAO-conservative.}$$

The proof of this theorem uses the fact that all $\mathcal{EL}$-equations correspond to Sahlqvist formulas in modal logic. Therefore, every $\mathfrak{A} \in$ BAO validating an $\mathcal{EL}$-theory $\mathcal{T}$ is embeddable into some $\mathfrak{F}^+ \in$ CA validating $\mathcal{T}$. It also follows from the 'correspondence' part of Sahlqvist's theorem that the class of $\mathcal{EL}$-structures validating any $\mathcal{EL}$-theory is first-order definable. For example,

– $x \leq f(x)$ defines reflexivity;

– $f(f(x)) \leq f(x)$ defines transitivity;

– $x \wedge f(y) \leq f(f(x) \wedge y)$ defines symmetry;

– $f(x) \wedge f(y) \leq f(x \wedge y)$ defines functionality;

– $f(x \wedge y) \wedge f(x \wedge z) \leq f(x \wedge f(y) \wedge f(z))$ defines linearity over quasi-orders.

(We refer the reader to [6] for first steps towards a correspondence theory for $\mathcal{EL}$.) In contrast to modal logic, however, the 'completeness' part of Sahlqvist's theorem does not hold. The possibly simplest example of an *incomplete* $\mathcal{EL}$-theory is $\{f(x) \leq x\}$ (to see that this theory is not complex, it is enough to consider the SLO with three elements $0 < a < 1$ and the operation f such that $f(a) = f(0) = 0$ and $f(1) = 1$).

SLOs validating the reflexivity and transitivity equations above (but without 0 and 1 in the signature) have been studied by Jackson [4] under the name '*closure semilattices*' (*CSLs*). He proves that every CSL is embeddable into a BAO validating reflexivity and transitivity. With a slight modification of his technique, we can obtain:

Theorem 2. *The $\mathcal{EL}$-theory $\{x \leq f(x),\ f(f(x)) \leq f(x)\}$ is complete.*

A more general completeness result has been proved by Sofronie-Stokkermans [9]:

Theorem 3 ([9]). *Every $\mathcal{EL}$-theory consisting of equations of the form $f_1 \dots f_n(x) \leq f(x)$, $n \geq 0$, is complete.*

This result implies that reflexivity or transitivity alone is also complete. Using modifications of Sofronie-Stokkermans' techniques, we can also cover symmetry, functionality, and some combinations thereof.

Theorem 4. *The following $\mathcal{EL}$-theories are complete:*

- $\{x \wedge f(y) \leq f(f(x) \wedge y)\}$ (*symmetry*);
- $\{f(x) \wedge f(y) \leq f(x \wedge y)\}$ (*functionality*);
- $\{x \leq f(x),\, f(f(x)) \leq f(x),\, x \wedge f(y) \leq f(f(x) \wedge y)\}$ (*reflexivity, transitivity and symmetry*).

In general, completeness is not preserved under unions of $\mathcal{EL}$-theories. For example:

Theorem 5. *Neither the union $\mathcal{T}_1$ of reflexivity and functionality, nor the union $\mathcal{T}_2$ of symmetry and functionality is complete.*

Interestingly, in both cases one can easily restore completeness by adding the equation $f(x) \leq x$ to $\mathcal{T}_1$, and by adding $f(f(x)) \leq x$ to $\mathcal{T}_2$. (Observe that these equations are consequences of $\mathcal{T}_1$ and $\mathcal{T}_2$ in modal logic.)

We also have a full picture of extensions of

$$\mathcal{T}_{\mathrm{S5}} = \{x \leq f(x),\, f(f(x)) \leq f(x),\, x \wedge f(y) \leq f(f(x) \wedge y)\},$$

using that these equations axiomatise the well-known modal logic $\mathcal{S}5$, and *normal* CSLs in [4]:

Theorem 6. *The $\mathcal{EL}$-theory $\mathcal{T}_{\mathrm{S5}} \cup \{f(x) \wedge f(y) \leq f(x \wedge y)\}$ is incomplete. All other (countably infinitely many) extensions of $\mathcal{T}_{\mathrm{S5}}$ are complete.*

As a first step towards general completeness results, we note the following analogue of completeness preservation under fusions of modal logics [5]. We call $\mathcal{T}_1 \cup \mathcal{T}_2$ a *fusion* of $\mathcal{EL}$-theories $\mathcal{T}_1$ and $\mathcal{T}_2$ if the sets of the f_i-operators occurring in $\mathcal{T}_1$ and $\mathcal{T}_2$ are disjoint.

Theorem 7. *The fusion of complete $\mathcal{EL}$-theories is also complete.*

The proofs of Theorems 3 and 4 go via two steps: (1) by embedding any SLO validating $\mathcal{T}$ into a DLO validating $\mathcal{T}$, and then (2) by embedding this DLO into a BAO validating $\mathcal{T}$, using various extensions of Priestley's [8] DL-to-BA embedding to the operators f_i. As concerns step (1), we have the following result:

Theorem 8. *Every $\mathcal{EL}$-theory containing only equations where each variable occurs at most once in the left-hand side is DLO-conservative.*

An interesting example, showing that the condition on the number of occurrences of variables in the left-hand side of equations in Theorem 8 cannot be dropped, is given by the $\mathcal{EL}$-theory

$$\mathcal{T}_{\mathrm{S4.3}} = \{x \leq f(x),\, f(f(x)) \leq f(x),\, f(x \wedge y) \wedge f(x \wedge z) \leq f(x \wedge f(y) \wedge f(z))\}.$$

Observe first that $\mathcal{T}_{\mathrm{S4.3}}$ defines a relation which is reflexive, transitive and right-linear, that is, $\forall x, y, z (R(x,y) \wedge R(x,z) \rightarrow R(y,z) \vee R(z,y))$. The modal logic determined by this frame condition is known as $\mathcal{S}4.3$, and the $\mathcal{EL}$-equations above axiomatise, if added to the equations for BAOs, the corresponding variety. However, one can show the following:

Theorem 9. *$\mathcal{T}_{\mathrm{S4.3}}$ is not DLO-conservative.*

Proof. Consider the quasi-equation

$$\boldsymbol{q} = \big(f(x) \wedge y = x \wedge f(y)\big) \;\rightarrow\; \big(f(x) \wedge f(y) = f(x \wedge y)\big)$$

and the SLO $\mathfrak{A} = (A, \wedge, 0, 1, f)$, where

$$\begin{aligned} &A = \{0, a, b, c, d, e, 1\},\\ &a \wedge b = a \wedge c = b \wedge c = 0,\\ &d = a \vee b,\ e = b \vee c,\ 1 = d \vee e,\\ &f(a) = d,\ f(c) = e,\ \text{and } f(x) = x \text{ for the remaining } x \in A. \end{aligned}$$

One can check that $\mathcal{T}_{S4.3} \models_{DLO} \boldsymbol{q}$; on the other hand, $\mathfrak{A} \models \mathcal{T}_{S4.3}$, $\mathfrak{A} \not\models \boldsymbol{q}$, and so $\mathcal{T}_{S4.3} \not\models_{SLO} \boldsymbol{q}$. □

Finally, we analyse the completeness problem for $\mathcal{EL}$-theories from the algorithmic point of view and show that it is impossible to give an effective syntactic criterion for completeness:

Theorem 10. *It is undecidable whether a finite set $\mathcal{T}$ of $\mathcal{EL}$-equations is complete.*

The proof of this result proceeds in two steps. First, we show the following by reduction of the undecidable halting problem for Turing machines:

Theorem 11. *Triviality of finite sets of $\mathcal{EL}$-equations is undecidable; more precisely, no algorithm can decide, given a finite set $\mathcal{T}$ of $\mathcal{EL}$-equations, whether $\mathcal{T} \models_{SLO} 0 = 1$.*

In the second step, we prove that, for every $\mathcal{EL}$-theory $\mathcal{T}$, the following two conditions are equivalent:

– the fusion of $\mathcal{T}$ and $\{f(x) \leq x\}$ is complete;

– $\mathcal{T} \models_{SLO} 0 = 1$.

Theorem 10 is then an immediate consequence of Theorem 11 and this equivalence.

References

[1] F. Baader, S. Brandt, and C. Lutz. Pushing the $\mathcal{EL}$ envelope. In *Proc. of IJCAI*, 2005.

[2] F. Baader, S. Brandt, and C. Lutz. Pushing the $\mathcal{EL}$ envelope further. In *Proc. of OWLED Workshop on OWL: Experiences and Directions*, 2008.

[3] R. Goldblatt. Varieties of complex algebras. *Annals of Pure and Applied Logic*, 38:173–241, 1989.

[4] M. Jackson. Semilattices with closure. *Algebra Universalis*, 52:1–37, 2004.

[5] M. Kracht and F. Wolter. Properties of independently axiomatizable bimodal logics. *Journal of Symbolic Logic*, 56(4):1469–1485, 1991.

[6] A. Kurucz, F. Wolter, and M. Zakharyaschev. Islands of tractability for relational constraints: towards dichotomy results for the description logic $\mathcal{EL}$. In L. Beklemishev, V. Goranko, and V. Shehtman, editors, *Advances in Modal Logic, Volume 8*, pages 271–291. College Publications, 2010.

[7] C. Lutz and F. Wolter. Mathematical logic for life science ontologies. In *WoLLIC*, pages 37–47, 2009.

[8] H. Priestley. Representations of distributive lattices by means of ordered Stone spaces. *Bull. London Math. Soc.*, 2:186–190, 1970.

[9] V. Sofronie-Stokkermans. Locality and subsumption testing in $\mathcal{EL}$ and some of its extensions. In C. Areces and R. Goldblatt, editors, *Advances in Modal Logic, Volume 7*, pages 315–339. College Publications, 2008.

The unification type of Łukasiewicz logic and MV-algebras is nullary[☆]

Luca Spada[a]

[a]*Dipartimento di Matematica, Università degli Studi di Salerno, Via Ponte don Melillo, 84084 Fisciano (SA), Italy*

The classical, syntactic unification problem is: given two terms s, t (built from function symbols and variables), find a *unifier* for them, that is, a uniform replacement of the variables occurring in s and t by other terms that makes s and t identical. When the latter syntactical identity is replaced by equality modulo a given equational theory E, one speaks of E*-unification.* Unsurprisingly, E-unification can be far harder than syntactic unification even when the theory E comes from the least exotic corners of the mathematical world. For instance, it may well be impossible to uniformly decide whether two terms admit at least one unifier, i.e. whether they are *unifiable* at all; and even when the two terms indeed are unifiable, there may well be no most general unifier for them, contrary to the situation in the syntactic case. In light of these considerations, perhaps the most basic piece of information one would like to have about E in connection with unification issues is its *unification type.*

In order to present the result, a quick glance to unification is given below, however we refer the reader to [1] and references therein for a far more complete introduction to the subject. Given a first order language constituted by a set of function symbols $\mathscr{F}$ and an infinite set of variables $\mathscr{V} = \{X_1, X_2, \ldots\}$, we let $\mathsf{Term}_{\mathscr{V}}(\mathscr{F})$ be the *term algebra* built from $\mathscr{F}$ and $\mathscr{V}$ in the usual manner. A *substitution* is a mapping $\sigma\colon \mathscr{V} \to \mathsf{Term}_{\mathscr{V}}(\mathscr{F})$ that acts identically to within a finite number of exceptions; substitutions compose in the obvious manner. An *equational theory* over the signature $\mathscr{F}$ is a set $E = \{(l_i, r_i) \mid i \in I\}$ of pairs of terms $l_i, r_i \in \mathsf{Term}_{\mathscr{V}}(\mathscr{F})$, where I is some index set. The set of equations E axiomatises the *variety of algebras* consisting of the models of the theory E, written $\mathbb{V}_E$.

Now a (*symbolic*) *unification problem modulo* E is a finite set of pairs

$$\mathscr{E} = \{(s_j, t_j) \mid s_j, t_j \in \mathsf{Term}_{\mathscr{V}}(\mathscr{F})\ ,\ j \in J\}\,,$$

for some finite index set J. A *unifier* for $\mathscr{E}$ is a substitution σ such that

$$E \models \sigma(s_j) \approx \sigma(t_j)\,,$$

for each $j \in J$, i.e. such that the equality $\sigma(s_j) = \sigma(t_j)$ holds in every algebra of the variety $\mathbb{V}_E$ in the usual universal-algebraic sense. The problem $\mathscr{E}$ is *unifiable* if it admits at least one unifier. The set $U(\mathscr{E})$ of unifiers for $\mathscr{E}$ can be partially ordered as follows. If

[☆]This paper is based on a joint work with **Vincenzo Marra (University of Milan)**, for a final version of this work please visit the *Home page* below where it will shortly be available.

Email address: `lspada@unisa.it`

Home page: `http://logica.dmi.unisa.it/lucaspada/`

σ and τ are substitutions and $V \subseteq \mathscr{V}$ is a set of variables, we say that σ is *more general* than τ (with respect to E and V), written $\tau \preceq_E^V \sigma$, if there exists a substitution ρ such that

$$E \models \tau(X) \approx (\rho \circ \sigma)(X)$$

holds for every $X \in V$. This amounts to saying that τ is an instantiation of σ, but only to within E-equivalence, and only as far as the set of variables V is concerned. We endow $U(\mathscr{E})$ with the relation $\preceq_E^V$, where V is the set of variables occurring in the terms s_j, t_j with $(s_j, t_j) \in \mathscr{E}$, as j ranges in J. The relation $\preceq_E^V$ is a pre-order, hence it can be canonically made into a partial order $\leqslant_E^V$, by taking the quotient set $U(\mathscr{E})/{\sim}$, where $\sim$ identifies σ and τ if and only if $\tau \preceq_E^V \sigma$ and $\sigma \preceq_E^V \tau$ both hold. Since we are interested in unifiers modulo equivalence, we call *unifiers* also the members of the quotient set, even though its elements actually are equivalence classes of unifiers.

The *unification type* of the unification problem $\mathscr{E}$ is:

1. *unitary*, if $\leqslant_E^V$ admits a maximum;
2. *finitary*, if $\leqslant_E^V$ admits no maximum, but admits finitely many maximal elements $[\mu_1], \ldots, [\mu_u]$ such that every $[\sigma] \in U(\mathscr{E})/{\sim}$ lies below some $[\mu_i]$;
3. *infinitary*, if $\leqslant_E^V$ admits infinitely many maximal elements $\{[\mu_i] \in U(\mathscr{E})/{\sim} \mid i \in I\}$, for I an infinite index set, such that every $[\sigma] \in U(\mathscr{E})/{\sim}$ lies below some $[\mu_i]$;
4. *nullary*, if none of the preceding cases applies.

It is understood that the list above is arranged in decreasing order of desirability. In the best, unitary case, any element of the maximum equivalence class $[\mu]$ is called a *most general unifier* for $\mathscr{E}$, or *mgu* for short. An mgu is then unique up to the relation $\sim$, whence one speaks of *the* mgu for $\mathscr{E}$. If $[\mu]$, on the other hand, is maximal but not a maximum, then any element of $[\mu]$ is called a *maximally general unifier*. The *unification type* of the equational theory E is now defined to be the worst unification type occurring among the unification problems $\mathscr{E}$ modulo E.

Establishing the unification type of a theory seems to be a problem of an essentially syntactical flavour, however, in [4] Ghilardi presents a particularly useful categorical characterisation of the unification type of a class of structures. In Ghilardi's approach a unification problem $\mathscr{E}$, as in the above, is modelled by the algebra finitely presented by the relations $s_j = t_j$, and a unifier is modelled by a morphism $u\colon A \to P$, with P a finitely presented projective algebra P. Unifiers are pre-ordered via comparison arrows and a hierarchy of unification types is given as above, *mutatis mutandi*. Ghilardi's main general result is that the algebraic unification type defined along these lines coincides with the traditional, symbolic unification type.

This result has two main advantages: on the one hand it allows a syntax-free treatment of unification, on the other hand it establishes the categorical invariance of the unification type, which is therefore preserved under categorical equivalence. In this work we use the aforementioned approach through projectivity to establish the unification type of MV-algebras.

MV-algebras arose in mathematics as the Lindenbaum-Tarski algebras of *Łukasiewicz* (*infinite-valued propositional*) *logic*: a logical system with a many-valued semantics going back to the 1920's. The standard reference is [3]. In modern terms one says that

MV-algebras are the *equivalent algebraic semantics* of Łukasiewicz logic. Since, in the appropriate setting, the correspondence by Blok and Pigozzi [2] between a logical system and its equivalent algebraic semantics can stated as a categorical equivalence, MV-algebras and Łukasiewicz logic have the same unification type. The interval (of truth values) $[0,1] \subseteq \mathbb{R}$ can be made into an MV-algebra with neutral element 0 by defining $x \oplus y = \min\{x+y, 1\}$ and $\neg x = 1 - x$; MV-algebras can be characterised as that class of structures which satisfies exactly the equational properties of the above algebra $\langle [0,1], \oplus, \neg, 0 \rangle$.

Our main result is the following

Theorem. *The unification type of the variety of MV-algebras is nullary. Specifically, consider the unification problem in the language of MV-algebras*

$$\mathscr{E} = \{ (X_1 \vee \neg X_1 \vee X_2 \vee \neg X_2 \, , \, 1) \} \; . \tag{$\star$}$$

Then the partially ordered set of unifiers for $\mathscr{E}$ contains a co-final chain of order-type ω.

Coupling algebraic unification with Stone-type dualities often leads to decisive topological insight. In [4, Theorem 5.7], for instance, Ghilardi used the basic duality between finitely presented distributive lattices and finite partially ordered sets to show that the unification type of distributive lattices is nullary. In the same way, to establish our result, we perform a further step across categories. Indeed, specialists know that the full subcategory of finitely presented MV-algebras is dually equivalent to a category of rational polyhedra whose morphisms, called $\mathbb{Z}$*-maps*, are continuous, piece-wise linear functions with integer coefficients.

Under this duality the finitely generated algebra presented by ($\star$) is precisely the boundary $\mathfrak{B}$ of $[0,1]^2$. If $[0,1]^2$ is endowed with its Euclidean metric topology, then $\mathfrak{B}$ inherits a subspace topology that makes it homeomorphic to $\mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, the standard unit circle in the plane. In particular, $\mathfrak{B}$ is not simply-connected: it is connected, but its fundamental (Poincaré) group is not trivial. It is also an easy matter to obtain a characterisation of the duals of projective finitely presented MV-algebras. These rational polyhedra are precisely those obtainable as retracts of unit cubes $[0,1]^n$ by $\mathbb{Z}$-maps, for some positive integer n; it follows that they are simply-connected. It transpires form the proof of the Theorem that these homotopical properties of $\mathfrak{B}$ and its (co-)unifiers are the deeper impediments that force ($\star$) to have nullary type.

To prove the Theorem, we first construct what can be considered (in a very strong sense) the *polyhedral* universal cover of $\mathfrak{B}$; the map representing this universal cover is called ζ, while its domain is called $\mathfrak{t}_\infty$. Such a cover can be thought of as the polyhedral correspondent of the *universal cover of the circle* (we refer to [5] for all unexplained notions in algebraic topology). The universal cover $(\mathfrak{t}_\infty, \zeta)$ is depicted in the left part of Figure 1 below.

The map ζ (projection) is a $\mathbb{Z}$-map, while the object $\mathfrak{t}_\infty$ is very close to be a rational polyhedron, but is not, as it is not compact. We therefore extract from it an infinite sequence of compact subspaces $\mathfrak{t}_i$, by symmetrically cutting $\mathfrak{t}_\infty$ above and below at a certain integer i. The restrictions ζ_i of the map ζ obviously remain $\mathbb{Z}$-maps and the $\mathfrak{t}_i$'s are seen to be projective by some recent result by Cabrer and Mundici; hence the pairs $(\zeta_i, \mathfrak{t}_i)$ are indeed dual algebraic unifiers for ($\star$). By the inclusion maps, these unifiers form an increasing sequence.

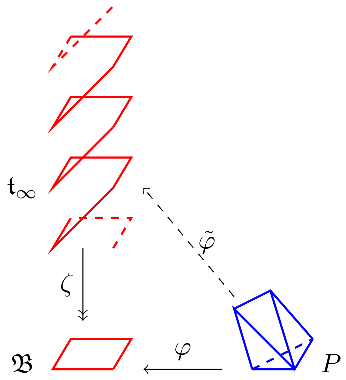


Figure 1: The lifting property of the spirals $\mathfrak{t}_\infty$.

To shows that the constructed sequence is in fact a strictly increasing, co-final sequence of unifiers for $\mathscr{E}$, the argument hinges on the lifting properties of a polyhedral universal covering space of $\mathfrak{B}$. In particular we are able to show that for any map φ from a simply-connected rational polyhedron P into $\mathfrak{B}$ there exists a *lift*, hence commuting, $\tilde{\varphi}$ from P into $\mathfrak{t}_\infty$ (see Figure 1), with the crucial property of being a $\mathbb{Z}$-map. The image of φ being compact, this amounts to say that any unifier of $\mathfrak{B}$ must be less general or equivalent to some $\mathfrak{t}_i$ in the sequence. The strictness of the order among the $\mathfrak{t}_i$ is an easy consequence of the fact that if $i < j$ then $\mathfrak{t}_i$ has strictly fewer lattice points than $\mathfrak{t}_j$ and, as $\mathbb{Z}$-maps preserve lattice points, the existence of a commuting $\mathbb{Z}$-map from $\mathfrak{t}_i$ into $\mathfrak{t}_j$ would contradict the uniqueness of the lifting above.

This result is quite in contrast with what was previously known about MV-algebras. Indeed every subvariety of MV-algebras generated by a finite chain has unitary unification type. Furthermore MV-algebras are categorical equivalent to Abelian lattice-ordered groups with strong unit (hence our result also extends to this latter class of structures), however the unification type of Abelian lattice-ordered groups (without necessarily a strong unit) is again unitary.

References

[1] F. Baader and W. Snyder. Unification theory. In J. A. Robinson and A. Voronkov, editors, *Handbook of Automated Reasoning*, pages 445–532. Elsevier and MIT Press, 2001.

[2] W. J. Blok and D. Pigozzi. *Algebraizable logics.* Mem. Amer. Math. Soc., Vol. 77. The American Mathematical Society, Providence, 1989.

[3] R. L. O. Cignoli, I. M. L. D'Ottaviano, and D. Mundici. *Algebraic foundations of many-valued reasoning*, volume 7 of *Trends in Logic—Studia Logica Library.* Kluwer Academic Publishers, Dordrecht, 2000.

[4] S. Ghilardi. Unification through projectivity. *J. Logic Comput.*, 7(6):733–752, 1997.

[5] A. Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

SUBLATTICES OF ASSOCIAHEDRA AND PERMUTOHEDRA

LUIGI SANTOCANALE AND FRIEDRICH WEHRUNG

Abstract. Grätzer asked in 1971 for a characterization of sublattices of Tamari lattices (*associahedra*). A natural candidate was coined by McKenzie in 1972 with the notion of a *bounded homomorphic image of a free lattice*—in short, *bounded lattice*. Urquhart proved in 1978 that every associahedron is bounded (thus so are its sublattices). Geyer conjectured in 1994 that every finite bounded lattice embeds into some associahedron.

We disprove Geyer's conjecture, by introducing an infinite collection of lattice-theoretical identities that hold in every associahedron, but not in every finite bounded lattice. Among those finite counterexamples, there are the *permutohedron* on four letters $\mathsf{P}(4)$, and in fact two of its subdirectly irreducible retracts, which are *Cambrian lattices of type A*.

For natural numbers m and n, we denote by $\mathsf{B}(m,n)$ the (bounded) lattice obtained by doubling a join of m atoms in an $(m+n)$-atom Boolean lattice. We prove that $\mathsf{B}(m,n)$ embeds into an associahedron iff $\min\{m,n\} \leq 1$, and that $\mathsf{B}(m,n)$ embeds into a permutohedron iff $\min\{m,n\} \leq 2$. In particular, $\mathsf{B}(3,3)$ cannot be embedded into any permutohedron. Nevertheless we prove that $\mathsf{B}(3,3)$ is a homomorphic image of a sublattice of the permutohedron on 12 letters.

This is a summary of the main results from the preprint [5].

Permutohedra and Associahedra

For each natural number n set

$$[n] := \{1, \dots, n\}, \qquad \mathcal{I}_n := \{(i,j) \in [n] \times [n] \mid i < j\}.$$

A subset $\boldsymbol{x}$ of $\mathcal{I}_n$ is *closed* if $(i,j) \in \boldsymbol{x}$ and $(j,k) \in \boldsymbol{x}$ implies that $(i,k) \in \boldsymbol{x}$, for all $i,j,k \in [n]$. A subset $\boldsymbol{x}$ of $\mathcal{I}_n$ is *open* if $\mathcal{I}_n \setminus \boldsymbol{x}$ is closed; it is *clopen* if both $\boldsymbol{x}$ and $\mathcal{I}_n \setminus \boldsymbol{x}$ are closed. A subset $\boldsymbol{x}$ of $\mathcal{I}_n$ is a *left subset* if $(i,k) \in \boldsymbol{x}$ implies $(i,j) \in \boldsymbol{x}$, for all $i,j,k \in [n]$ such that $i < j < k$. Notice that a left subset is open.

Definition. The *permutohedron of index* n, denoted by $\mathsf{P}(n)$, is the set of all clopen subsets of $\mathcal{I}_n$, partially ordered by inclusion. The *associahedron of index* n, denoted by $\mathsf{A}(n)$, is the set of all closed left subsets of $\mathcal{I}_n$, partially ordered by inclusion.

It is a well known fact [6] that $\mathsf{P}(n)$ is isomorphic (as a poset) to the set of permutations on n elements, endowed with the weak Bruhat order. Similarly, the poset $\mathsf{A}(n)$ is isomorphic to the set of all binary trees (or parenthesized words) with $n+1$ leaves, where the order is obtained by taking the reflexive and transitive closure of the relation that re-associates a tree from left to right.

The representations of $\mathsf{P}(n)$ and $\mathsf{A}(n)$ by families of subsets of $\mathcal{I}_n$ easily allow to establish that these posets are lattices. As every intersection of closed sets is closed, every union of open sets is open. For a subset $\boldsymbol{x}$ of $\mathcal{I}_n$, denote by $\operatorname{int}(\boldsymbol{x})$ (resp., $\operatorname{cl}(\boldsymbol{x})$) the largest open subset of $\boldsymbol{x}$ (resp., the least closed set containing $\boldsymbol{x}$).

Lemma. *The set* $\mathrm{cl}(\boldsymbol{x})$ *is open, for each open* $\boldsymbol{x} \subseteq \mathcal{I}_n$. *Dually, the set* $\mathrm{int}(\boldsymbol{x})$ *is closed, for each closed* $\boldsymbol{x} \subseteq \mathcal{I}_n$. *Consequently, the poset* $\mathsf{P}(n)$ *is a lattice where the meet and the join are computed by the formulas*

$$\boldsymbol{x} \wedge \boldsymbol{y} = \mathrm{int}(\boldsymbol{x} \cap \boldsymbol{y}), \quad \boldsymbol{x} \vee \boldsymbol{y} = \mathrm{cl}(\boldsymbol{x} \cup \boldsymbol{y}).$$

It is easily verified that closed left subsets are stable under intersections, so that $\mathsf{A}(n)$ is a lattice. As the closure of a left subset is a left subset and $\mathrm{int}(\boldsymbol{x}) = \boldsymbol{x}$ if $\boldsymbol{x}$ is a left subset, the associahedron $\mathsf{A}(n)$ is actually a sublattice of $\mathsf{P}(n)$ (and, as we will see later, a *lattice-theoretical retract*).

Generalized Associahedra are Cambrian Lattices

Let us fix a subset U of $[n]$ and denote by $\mathsf{D}_U(n)$ the collection of all subsets $\boldsymbol{a}$ of $\mathcal{I}_n$ such that $1 \leq i < j < k \leq n$ and $(i,k) \in \boldsymbol{a}$ implies that $(i,j) \in \boldsymbol{a}$ in case $j \in U$ and $(j,k) \in \boldsymbol{a}$ in case $j \notin U$.

Definition. We define $\mathsf{P}_U(n)$ as the collection of all closed members of $\mathsf{D}_U(n)$, and we order $\mathsf{P}_U(n)$ by set-theoretical inclusion.

The posets $\mathsf{P}_U(n)$ generalize associahedra, in the sense that $\mathsf{P}_{[n]}(n) = \mathsf{A}(n)$.

Proposition. *Each poset* $\mathsf{P}_U(n)$ *is a* $0,1$*-sublattice of* $\mathsf{P}(n)$, *where the meet and the join of elements* $\boldsymbol{x}, \boldsymbol{y} \in \mathsf{P}_U(n)$ *are given by* $\boldsymbol{x} \wedge \boldsymbol{y} = \boldsymbol{x} \cap \boldsymbol{y}$ *and* $\boldsymbol{x} \vee \boldsymbol{y} = \mathrm{cl}(\boldsymbol{x} \cup \boldsymbol{y})$, *respectively. Moreover,* $\mathsf{P}_U(n)$ *is a quotient of* $\mathsf{P}(n)$, *and in fact a lattice-theoretical retract of* $\mathsf{P}(n)$.

In the following we shall denote by $\pi_U : \mathsf{P}(n) \to \mathsf{P}_U(n)$ the projection map, that associates to each $\boldsymbol{x} \in \mathsf{P}(n)$ the largest element of $\mathsf{P}_U(n)$ below $\boldsymbol{x}$. The subset $U \subseteq [n]$ gives rise to an orientation of the Dynkin diagram of the group of permutations and, in turn, to a Cambrian congruence [4] on the lattice $\mathsf{P}(n)$.

Proposition. *The Cambrian congruence associated to* U *is the kernel of* π_U. *Consequently, the associated Cambrian lattice is* $\mathsf{P}_U(n)$.

The following Proposition exhibits the role of Cambrian lattices from an algebraic perspective.

Proposition. *Every lattice* $\mathsf{P}_U(n)$ *is subdirectly irreducible, and the diagonal map* $\pi \colon \mathsf{P}(n) \to \prod(\mathsf{P}_U(n) \mid U \subseteq [n])$, $\boldsymbol{x} \mapsto (\pi_U(\boldsymbol{x}) \mid U \subseteq [n])$ *is a subdirect product decomposition of the permutohedron* $\mathsf{P}(n)$.

The Gazpacho Identities

For $\vec{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$ with $d \geq 2$, set $\mathfrak{F}(\vec{m}) := \prod([m_i] \mid 1 \leq i \leq d)$. Define terms a_i, $\tilde{\mathsf{b}}_i$, $\mathsf{e}_{\vec{m}}$, $\mathsf{e}^*_{\vec{m}}$ in the variables $\mathsf{a}_{i,j}$ and b_i by

$$\mathsf{a}_i := \bigvee_{j=1}^{m_i} \mathsf{a}_{i,j}, \qquad \tilde{\mathsf{b}}_i := \Big(\bigvee_{i'=1}^{d} \mathsf{b}_{i'}\Big) \wedge (\mathsf{a}_i \vee \mathsf{b}_i) \qquad (\text{for } 1 \leq i \leq d),$$

$$\mathsf{e}_{\vec{m}} := \bigwedge_{i=1}^{d} (\mathsf{a}_i \vee \mathsf{b}_i), \qquad \mathsf{e}^*_{\vec{m}} := \Big(\bigvee_{i'=1}^{d} \mathsf{b}_{i'}\Big) \wedge \mathsf{e}_{\vec{m}} = \bigwedge_{i=1}^{d} \tilde{\mathsf{b}}_i.$$

Further, we define lattice terms $\mathsf{f}_i^{\sigma,\tau}$, for $2 \leq i \leq d$ and $(\sigma,\tau) \in \mathfrak{S}_d \times \mathfrak{F}(\vec{m})$, by downward induction on i (for $2 \leq i < d$), by

$$\mathsf{f}_d^{\sigma,\tau} := (\mathsf{a}_{\sigma(d),\tau\sigma(d)} \vee \tilde{\mathsf{b}}_{\sigma(1)}) \wedge (\mathsf{a}_{\sigma(d)} \vee \mathsf{b}_{\sigma(d)}),$$

$$\mathsf{f}_i^{\sigma,\tau} := (\mathsf{a}_{\sigma(i),\tau\sigma(i)} \vee \tilde{\mathsf{b}}_{\sigma(1)}) \wedge (\mathsf{a}_{\sigma(i)} \vee \mathsf{b}_{\sigma(i)}) \wedge \bigwedge_{i<j\leq d} \big(\mathsf{a}_{\sigma(i),\tau\sigma(i)} \vee \mathsf{f}_j^{\sigma,\tau}\big).$$

Let $\mathrm{Gzp}(\vec{m})$ (the *Gazpacho identity with index* $\vec{m}$) be the following lattice-theoretical identity, in the variables $\mathsf{a}_{i,j}$ and b_i, for $1 \leq i \leq d$ and $1 \leq j \leq m_i$:

$$\mathsf{e}_{\vec{m}} \leq \mathsf{e}^*_{\vec{m}} \vee \bigvee \left(\mathsf{f}_2^{\sigma,\tau} \mid (\sigma, \tau) \in \mathfrak{S}_d \times \mathfrak{F}(\vec{m})\right) . \qquad (\mathrm{Gzp}(\vec{m}))$$

Theorem. *Every associahedron satisfies* $\mathrm{Gzp}(\vec{m})$ *for each* $\vec{m} \in \mathbb{N}^d$.

The identity $\mathrm{Gzp}(1,1)$ is equivalent to the identity below:

$$(\mathsf{a}_1 \vee \mathsf{b}_1) \wedge (\mathsf{a}_2 \vee \mathsf{b}_2) \leq \left((\mathsf{a}_1 \vee \mathsf{b}_1) \wedge (\mathsf{a}_1 \vee \tilde{\mathsf{b}}_2)\right) \vee \left((\mathsf{a}_2 \vee \tilde{\mathsf{b}}_1) \wedge (\mathsf{a}_2 \vee \mathsf{b}_2)\right) , \qquad (\mathrm{Veg}_1)$$

where $\tilde{\mathsf{b}}_i := (\mathsf{b}_1 \vee \mathsf{b}_2) \wedge (\mathsf{a}_i \vee \mathsf{b}_i)$. Hence, as a consequence of the previous Theorem, we obtain that *every associahedron satisfies* (Veg_1). Yet we have:

Proposition. *The permutohedron* $\mathsf{P}(4)$ *does not satisfy the identity* (Veg_1). *In particular, it has no lattice embedding into any associahedron.*

Thus we obtain a first counterexample to Geyer's conjecture [2] that every finite bounded lattice embeds into some associahedron. A second counterexample arises from the following indentity:

$$(\mathsf{a}_1 \vee \mathsf{a}_2 \vee \mathsf{b}_1) \wedge (\mathsf{a}_1 \vee \mathsf{a}_2 \vee \mathsf{b}_2) = \bigvee_{i,j \in \{1,2\}} \left((\mathsf{a}_i \vee \tilde{\mathsf{b}}_j) \wedge (\mathsf{a}_1 \vee \mathsf{a}_2 \vee \mathsf{b}_{3-j})\right) , \qquad (\mathrm{Veg}_2)$$

with the lattice terms $\tilde{\mathsf{b}}_j := (\mathsf{b}_1 \vee \mathsf{b}_2) \wedge (\mathsf{a}_1 \vee \mathsf{a}_2 \vee \mathsf{b}_j)$, for $j \in \{1,2\}$. It is a weakening of $\mathrm{Gzp}(2,2)$, and therefore every associahedron satisfies (Veg_2). For natural numbers m and n, we denote by $\mathsf{B}(m,n)$ the lattice obtained by doubling the join of m atoms in the $(m+n)$-atom Boolean lattice.

Proposition. *The lattice* $\mathsf{B}(2,2)$ *does not satisfy the identity* (Veg_2). *In particular, it cannot be embedded into any associahedron.*

Permutohedra are not Universal

Polarized measures are the tools that we use when searching for an embedding of a lattice into some $\mathsf{P}_U(n)$. For L a join-semilattice and $U \subseteq [n]$, an *L-valued U-polarized measure* is a map $\mu \colon \mathfrak{I}_n \to L$ such that

(i) $\mu(x,z) \leq \mu(x,y) \vee \mu(y,z)$,

(ii) $y \in U$ implies that $\mu(x,y) \leq \mu(x,z)$,

(iii) $y \notin U$ implies that $\mu(y,z) \leq \mu(x,z)$,

for all $x < y < z$ in $[n]$. We say that μ satisfies the *V-condition* if for all $(x,y) \in \mathfrak{I}_n$ and all $\boldsymbol{a}, \boldsymbol{b} \in L$,

if $\mu(x,y) \leq \boldsymbol{a} \vee \boldsymbol{b}$, *then* (V)

there are $m \geq 1$ *and a subdivision* $x = z_0 < z_1 < \cdots < z_m = y$ *of* $[n]$ *such that either* $\mu(z_i, z_{i+1}) \leq \boldsymbol{a}$ *or* $\mu(z_i, z_{i+1}) \leq \boldsymbol{b}$ *for each* $i < m$.

We say that maps $\mu \colon \mathfrak{I}_n \to L$ and $\varphi \colon L \to \mathsf{P}_U(n)$ are *dual* if, for all $(x,y) \in \mathfrak{I}_n$ and all $\boldsymbol{a} \in L$, $(x,y) \in \varphi(\boldsymbol{a})$ iff $\mu(x,y) \leq \boldsymbol{a}$.

Lemma. *Let* $U \subseteq [n]$, *let* L *be a finite lattice, and let* $\mu \colon \mathfrak{I}_n \to L$ *and* $\varphi \colon L \to \mathsf{P}_U(n)$ *be dual. The following statements hold:*

(i) *μ is an L-valued U-polarized measure satisfying the V-condition iff φ is a 1-preserving lattice homomorphism,*

(ii) *$\varphi(0) = \varnothing$ iff 0 does not belong to the range of μ,*

(iii) *the range of μ generates L as a $(\vee, 0)$-subsemilattice iff φ is one-to-one.*

The following Theorem collects our results concerning embeddability of the lattices $\mathsf{B}(m,n)$ into associahedra and permutohedra.

Theorem. *Let m and n be natural numbers. The following statements hold:*

(i) *The lattice $\mathsf{B}(m,n)$ embeds into some associahedron iff* $\min\{m,n\} \leq 1$.

(ii) *The lattice $\mathsf{B}(m,2)$ has a $0,1$-lattice embedding into the Cambrian lattice $\mathsf{P}_{[m+2,2m+1]}(2m+2)$, for every positive integer m.*

(iii) *The lattice $\mathsf{B}(3,3)$ cannot be embedded into any permutohedron.*

After several unsuccessful attempts to turn the last statement into an identity holding in all permutohedra while failing in $\mathsf{B}(3,3)$, the authors wondered whether it could actually be the case that $\mathsf{B}(3,3)$ satisfies every lattice-theoretical identity satisfied by all permutohedra. This turned out to be the correct guess. In order to prove this, we needed the notion of *splitting identity* of a finite, bounded, subdirectly irreducible lattice—or, using the terminology introduced in [3], a *splitting lattice*. It is a classical result of lattice theory (cf. Freese, Ježek, and Nation [1, Corollary 2.76]) that for every splitting lattice K, there exists a largest lattice variety $\mathcal{C}_K$ which is maximal with respect to not containing K as a member. Furthermore, $\mathcal{C}_K$ can be defined by a single lattice identity, called a *splitting identity* for K, and there is an effective way to compute such an identity.

The lattices $\mathsf{B}(m,n)$ are splitting lattices. We obtained the example underlying the next Theorem with the assistance of the `Mace4` part of the `Prover9 - Mace4` software, available online on William McCune's Web page at http://www.cs.unm.edu/~mccune/prover9/.

Theorem. *Set $U := \{5,6,9,10,11\}$. Then the Cambrian lattice $\mathsf{P}_U(12)$ does not satisfy the splitting identity for $\mathsf{B}(3,3)$. Consequently, $\mathsf{B}(3,3)$ is the homomorphic image of a sublattice of $\mathsf{P}_U(12)$. In particular, it satisfies all the identities satisfied by $\mathsf{P}_U(12)$, thus all the identities satisfied by all permutohedra.*

References

[1] R. Freese, J. Ježek, and J. B. Nation, "Free Lattices", Mathematical Surveys and Monographs **42**, Amer. Math. Soc., Providence, 1995. viii+293 p. ISBN: 0-8218-0389-1

[2] W. Geyer, *On Tamari lattices*, Discrete Math. **133** (1994), 99–122.

[3] R. N. McKenzie, *Equational bases and non-modular lattice varieties*, Trans. Amer. Math. Soc. **174** (1972), 1–43.

[4] N. Reading, *Cambrian lattices*, Adv. Math. **205**, no. 2 (2006), 313–353.

[5] L. Santocanale and F. Wehrung, *Sublattices of associahedra and permutohedra*, preprint 2011. Available online at http://fr.arxiv.org/abs/1103.3488, http://hal.archives-ouvertes.fr/hal-00577258.

[6] T. Yanagimoto and M. Okamoto, *Partial orderings of permutations and monotonicity of a rank correlation statistic*, Ann. Inst. Statist. Math. **21** (1969), 489–506.

LIF, CNRS UMR 6166, Université Aix-Marseille I, 13453 Marseille Cedex 13, France
E-mail address: luigi.santocanale@lif.univ-mrs.fr
URL: http://www.lif.univ-mrs.fr/~lsantoca/

LMNO, CNRS UMR 6139, Département de Mathématiques, Université de Caen, 14032 Caen Cedex, France
E-mail address: wehrung@math.unicaen.fr
URL: http://www.math.unicaen.fr/~wehrung

ORDERED DIRECT IMPLICATIONAL BASIS OF A FINITE CLOSURE SYSTEM

K. ADARICHEVA, J. B. NATION, AND R. RAND

1. Introduction

In K. Bertet and B. Monjardet [2], it is shown that five implicational bases for a closure operator on a finite set, found in various contexts in the literature, are actually the same. The goal of this paper is to demonstrate that standard lattice-theoretic results about the "most economical way" to describe the structure of a finite lattice may be transformed into a basis for a closure system naturally associated with that lattice. We may refer to [7], where the coding of a finite lattice in the form of so-called OD-graph was first suggested.

We will call the basis directly following from this OD-graph a *D-basis*, since it is closely associated with a *D-relation* on the set of join-irreducibles of a lattice that was crucial in the studies of free and lower bounded lattices, see [5]. We show that the D-basis is a subset of *a canonical direct unit basis* that unifies the five bases discussed in [2]. The reverse inclusion does not hold, thus this newly introduced D-basis is generally shorter than the existing ones.

Recall that the main desirable feature of bases from [2] is that they be *direct*, which means that the computation of the closure of any subset can be done in a single application of each implication from the basis. While the D-basis is not direct in this meaning of this term, the closures can still be computed in a single application of each implication from the basis, provided the basis was put in a specific order prior to computation. We call the bases with this property *ordered direct*. There exists a simple and effective linear time algorithm for ordering a D-basis appropriately. The application of the D-basis can be compared to the iteration known in artificial intelligence as the *forward chaining algorithm*, see [4].

We also discuss the E-relation, introduced in [5], which leads to the definition of the E-basis in closure systems *without cycles*. In such systems the E-basis is ordered direct and it is contained in the D-basis.

We explore the connections between D-basis, E-basis and the *canonical basis* introduced in [3]. While the canonical basis has the minimal number of implications among all the bases of a closure system, it is not ordered direct. In the full version of the paper (available at http://www.math.hawaii.edu/∼jb/papers.html), we present examples of closure systems on 6-element set, for which the canonical basis cannot be ordered.

2. Lattices, closure operators and Horn formulas

The collection of closed subsets of a closure operator forms a lattice. Conversely, we can associate with every finite lattice L a particular closure system $\langle S, \phi \rangle$ in such a way that L is isomorphic to a closure lattice of that closure system. Consider $J(L) \subseteq L$, the subset of *join-irreducible elements*. We define a closure system with $S = J(L)$ and the following closure operator: $\phi(X) = [0, \bigvee X] \cap J(L)$. It is straightforward to check that the closure lattice of ϕ is isomorphic to L. This representation is called *standard*.

If $y \in \phi(X)$, then this relation between an element $y \in S$ and a subset $X \subseteq S$ in a closure system can be written in the form of implication: $X \to y$. Thus, the closure system $\langle S, \phi \rangle$ can be replaced by the set of implications:

$$\Sigma_\phi = \{X \to y : y \in S, X \subseteq S \text{ and } y \in \phi(X)\}$$

Conversely, any set of implications Σ defines a closure system.

Two sets of implications Σ and Σ' on the same set S are called *equivalent*, if they define the same closure system on S. The term *basis* is used for the set of implications Σ' satisfying some minimality condition.

Closure operator arise in logic programming because implications $X \to y$, $X \subseteq S, x \in S$, can be treated as the *Horn formulas* of propositional logic over the set of literals S.

3. D-basis

Let us translate to the language of closure systems the defining relations of a finite lattice developed in the lattice theory framework.

Given a *standard* closure system $\langle S, \phi \rangle$, let us define two auxiliary relations. The first relation is between the subsets of S: we write $X \ll Y$, if for every $x \in X$ there is $y \in Y$ satisfying $x \in \phi(y)$. We define $X \sim_{\ll} Y$, if $X \ll Y$ and $Y \ll X$.

Lemma 1. *The relation $\sim_{\ll}$ is an equivalence relation on $P(S)$. Moreover, each equivalence class has a unique minimal element with respect to the containment order.*

We will call a subset $X \subseteq S$ *a cover* of x if $x \in \phi(X)$ and $x \notin \phi(x')$, for any $x' \in X$. A subset $Y \subseteq S$ is called *a minimal cover* of an element $x \in S$, if Y is a cover of x, and for every other cover Z of x, $Z \ll Y$ implies $Y \subseteq Z$.

Definition 2. Given a standard closure system $\langle S, \phi \rangle$, we define a D-basis Σ_D as a union of two subsets of implications:

(1) $\{y \to x : x \in \phi(y) \setminus y, y \in S\}$;
(2) $\{X \to x : X$ is a minimal cover for $x\}$.

Lemma 3. *Σ_D generates $\langle S, \phi \rangle$, for any closure system $\langle S, \phi \rangle$.*

4. Direct basis versus ordered direct basis

If Σ is some set of implications, then, in notation of [2], $\pi_\Sigma(X) = X \cup \{b : A \subseteq X \text{ and } (A \to b) \in \Sigma\}$. Then $\phi(X) = \pi(X) \cup \pi^2(X) \cup \pi^3(X) \dots$. The bases for which $\phi(X) = \pi(X)$ are called *direct*. The goal of this section to implement a different approach to the concept of iteration.

Definition 4. Suppose the set of implications Σ is equipped with some linear order $<$, or, equivalently, is indexed as $\Sigma = \{s_1, s_2, \dots, s_n\}$. Define a mapping $\rho_\Sigma : P(S) \to P(S)$ associated with this ordering as follows. For any $X \subseteq S$, let $X_0 = X$. If X_k is computed and implication s_{k+1} is $A \to b$, then

$$X_{k+1} = \begin{cases} X_k \cup \{b\}, & \text{if } A \subseteq X_k, \\ X_k, & \text{otherwise.} \end{cases}$$

Finally, $\rho_\Sigma(X) = X_n$. We will call ρ_Σ an *ordered iteration* of Σ.

Definition 5. The set of implications with some linear ordering on it, $\langle \Sigma, < \rangle$, is called *an ordered direct basis*, if, with respect to this ordering $\phi_\Sigma(X) = \rho_\Sigma(X)$, for all $X \subseteq S$.

The D-basis Σ_D is, in fact, an ordered direct basis. Moreover, it does not take much computational effort to impose a proper ordering on Σ_D.

Lemma 6. *Let $<$ be any linear ordering on Σ_D such that all implications of the form $y \to x$ precede implications $X \to x$, where X is a minimal cover of x. Then, with respect to this ordering, Σ_D is an ordered direct basis.*

Corollary 7. *If $\Sigma_D = \{s_1, \dots, s_n\}$ is the D-basis of an implicational system Σ, then it requires time $O(n)$ to turn it into a direct ordered basis of Σ.*

Let Σ_c denote the canonical direct unit basis that unifies all direct bases discussed in [2].

Lemma 8. *$\Sigma_D \subseteq \Sigma_c$, in particular, D-basis is contained in every direct unit basis.*

Proposition 9. *Given any direct unit basis Σ with n implications, on set S with m elements, it requires time $O((mn)^2)$ to build the D-basis Σ_D equivalent to Σ.*

5. Closure systems without cycles and the E-basis

It turns out that the D-basis can be further reduced, when an additional property holds in a closure system $\langle S, \phi \rangle$. The results of this section use section 2.4 in [5].

We will write xDy, for $x, y \in S$, if $y \in Y$, for some minimal cover Y of x. We note that D-relation is a subset of so-called *dependence* relation δ from [6]. A sequence $x_1, x_2, \dots, x_n$, $n > 1$, is called a *D-cycle*, if $x_1 D x_2 D \dots x_n D x_1$.

Definition 10. A finite closure system $\langle S, \phi \rangle$ is said to be *without cycles*, if it does not have D-cycles.

The lattices of closed sets of such systems are known as *lower bounded*.

For every $x \in S$, consider $M(x) = \{Y \subseteq S$:Y is a minimal cover of $x\}$. Let $M^*(x) = \{Y \in M(x) : \phi(Y)$ is minimal by containment in $\phi(M(x))\}$.

We will write xEy, for $x, y \in S$, if $y \in Y$ for some $Y \in M^*(x)$. According to the definition, if xEy then xDy. On the other hand, converse is not always true.

Lemma 11. *Let $\langle S, \phi \rangle$ be a standard closure system without cycles. Consider a subset Σ_E of the D-basis that is the union of two sets of implications:*

(1) $\{y \to x : x \in \phi(y)\}$;
(2) $\{X \to x : X \in M^*(x)\}$.

Then Σ_E is a basis for $\langle S, \phi \rangle$. Moreover, this basis is ordered direct.

Proposition 12. *Suppose $\Sigma_D = \{s_1, s_2, \dots, s_n\}$ is a D-basis of some closure system $\langle S, \phi \rangle$ and $|S| = m$. It requires time $O(mn^2)$ to determine whether the closure system is without cycles, and if it is, to build its ordered direct basis Σ_E.*

6. D-basis versus canonical basis

The canonical basis was introduced by V. Duquenne and J.L. Guigues in [3], see also [1]. It is defined by means of *critical* (also, *pseudo-closed*) sets that are explicitly defined for each closure system (S, ϕ). The basis then consists of the implications $C \to \phi(C)$, where C ranges over all critical sets. Note that here the implications not necessarily have one-element sets as the conclusion. Among all the bases for the closure system, the Duquenne-Guigues canonical basis has the minimum number of implications.

It turns out that the canonical basis is not ordered direct. In our paper we present two examples that were discovered by running a computer program and checking about a million of closure systems on $5-7$ element sets.

The performance of the D-basis in comparison with Duquenne-Guigues canonical basis (in its unit form) and canonical (or optimal) unit direct basis was tested on randomly generated closure systems. The following table shows that D-basis performs consistently faster than two others on the average closure system.

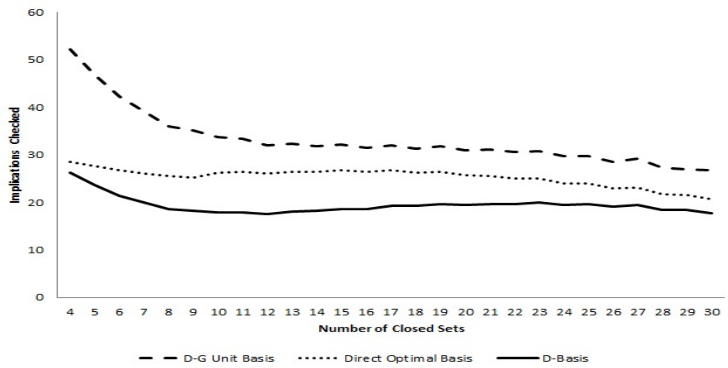


FIGURE 1. Bases comparison on domain set 7

REFERENCES

[1] N. Caspard and B. Monjardet, *The lattices of closure systems, closure operators, and implicational systems on a finite set: a survey*, Disc. Appl. Math. 127(2003), 241–269.

[2] K. Bertet and B. Monjardet, *The multiple facets of the canonical direct unit implicational basis*, Theoretical Computer Science **411**(2010), 2155–2166.

[3] J. L. Guigues and V. Duquenne, *Familles minimales d'implications informatives résultant d'une tables de données binares*, Math. Sci. Hum. **95** (1986), 5–18.

[4] W. Dowling and J.H. Gallier, *Linear-time algorithms for testing the satisfiability of propositional Horn formulae*, J. Logic Programming **3** (1984), 267–284.

[5] R. Freese, J. Ježek and J. B. Nation, *Free Lattices*, Mathematical Surveys and Monographs **42**, Amer. Math. Soc., Providence, 1995.

[6] B. Monjardet, *Arrowian characterizations of latticial federation consensus functions*, Mathematical Social Sciences **20** (1990), 51–71.

[7] J.B. Nation, *An approach to lattice varieties of finite height*, Algebra Universalis **27** (1990), no. 4, 521–543.

DEPARTMENT OF MATH SCIENCES, YESHIVA UNIVERSITY, NEW YORK, NY 10016, USA
E-mail address: adariche@yu.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAWAII, HONOLULU, HI 96822, USA
E-mail address: jb@math.hawaii.edu

YESHIVA COLLEGE, NEW YORK, NY 10033, USA
E-mail address: rrand@yu.edu

ON SCATTERED CONVEX GEOMETRIES

KIRA ADARICHEVA AND MAURICE POUZET

1. Introduction

We call a pair (X, ϕ) of a non-empty set X and a closure operator $\phi : 2^X \to 2^X$ on X *a convex geometry* [5], if it satisfies the anti-exchange axiom: for all $x \neq y$ and closed $X \subseteq A$, if $x \in \overline{X \cup \{y\}}$ and $x \notin X$, then $y \notin \overline{X \cup \{x\}}$.

The study of convex geometries in finite case was inspired by their frequent appearance in modeling various discrete structures, as well as by their juxtaposition to matroids, see [11, 12]. More recently, there was a number of publications, see, for example, [6, 4, 16, 17, 18] brought up by studies in infinite convex geometries.

Convex geometry is called *algebraic*, if the closure operator ϕ is *finitary.* Most of interesting infinite convex geometries are algebraic, such as convex geometries of relatively convex sets, subsemilattices of a semilattice, suborders of a partial order or convex subsets of a partially ordered set. In particular, the closed sets of algebraic convex geometry form an algebraic lattice, i.e. a complete lattice, whose each element is a join of compact elements. Compact elements are exactly the closures of finite subsets of X, and they form a semilattice with the respect to the join operation of the lattice.

There is a serious restriction on the structure of an algebraic lattice and its semilattice of compact elements, when the lattice is *order-scattered*, i.e. it does not contain a chain of type η, the order type of the chain of rational numbers $\mathbb{Q}$, as a sub-order. While the description of order-scattered algebraic lattices remains to be an open problem, it was recently obtained in the case of modular lattices. The description is done in the form of *obstructions*, i.e. prohibiting special types of subsemilattices in the semilattice of compact elements.

Theorem 1.1. [9] *The algebraic modular lattice is order-scattered iff the semilattice of compact elements is order-scattered and does not contain as a subsemilattice the semilattice $\mathfrak{P}^{<\omega}(\mathbb{N})$ of finite subsets of a countable set.*

This theorem motivated the current investigation, due to the fact that convex geometries almost never satisfy the modular law, see [5]. It is known that outside the modular case the list of obstructions contains the semilattice $\Omega(\eta)$ described in [10]. We show in section 5 that $\Omega(\eta)$ appears naturally as a subsemilattice of compact of elements in the convex geometries known as *multichains.* For this, one of the chains must be of type ω (the order type of natural numbers), and another of type η.

More generally, we prove in Theorem 6.3 that any algebraic convex geometry whose semilattice of compact elements K has the finite semilattice dimension will be order scattered iff K is order-scattered and it does not have a sub-semilattice isomorphic to either $\mathfrak{P}^{<\omega}(\mathbb{N})$ or $\Omega(\eta)$.

As for the other types of convex geometries, we prove the result analogous to modular case. It holds true trivially in case of convex geometries of subsemilattices and suborders of a partial order, since order scattered geometries of these types are always finite, see section 4. For the

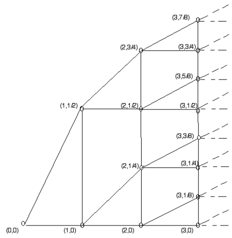


FIGURE 1. $\Omega(\eta)$

convex geometries of relatively convex sets, we analyze independent sets and reduce the problem to relatively convex sets on a line. As stated in Theorem 3.1, the only obstruction in the semilattice of compact elements in this case is $\mathfrak{P}^{<\omega}(\mathbb{N})$.

We also discuss the topological issues of the algebraic convex geometries and establish that the convex geometry of relatively convex sets is order scattered iff it is topologically scattered in product topology. This is the result analogous to Mislove's theorem for algebraic distributive lattices [15].

2. Weakly atomic convex geometries

In this section we prove a general statement about the structure of convex geometries that are *weakly atomic*, i.e., whose every interval has a cover. The result hints how to produce algebraic distributive lattice which is not a convex geometry.

Theorem 2.1. *Suppose convex geometry $C = (X, \phi)$ satisfies the property that every interval $[X, Y]$ of closed sets has a cover: $X \subseteq X' \prec Y' \subseteq Y$. Then $C = (X, \phi)$ is* spatial*, i.e. every element is a join of completely join irreducible elements. In particular, one can choose $Y \subseteq X$, define an anti-exchange operator ψ on Y, and present convex geometry as $C = (Y, \psi)$ so that $\psi(y)$ is completely join-irreducible, for every $y \in Y$.*

Corollary 2.2. *In any of the following cases, the convex geometry $C = (X, \phi)$ is spatial:*

1. *ϕ is algebraic closure operator. Equivalently, the lattice of closed sets of $C = (X, \phi)$ is algebraic.*
2. *$C = (X, \phi)$ is order scattered.*

Another example of weakly atomic convex geometry was presented recently in [6]. Since it was given in the form of *antimatroid*, i.e. the structure of open sets of convex geometry, we will provide the corresponding definition of *super solvable* convex geometry here.

Definition 2.3. Convex geometry $C = (X, \phi)$ is called *super solvable*, if there exists well-ordering $\leq_X$ on X such that, for all $A = \phi(A), B = \phi(B)$, if $A \not\subseteq B$, then $A \setminus a$ is ϕ-closed, where $a = \min(A \setminus B)$

We note that the corresponding definition of super solvable antimatroid in [6] is more restrictive in the sense that X is countable and $(X, \leq_X)$ is a chain of order type ω. Super solvable antimatroids with such definition appear as the structure associated with special ordering of elements in Coxeter groups.

Corollary 2.4. *If convex geometry $C = (X, \phi)$ is super solvable then it is spatial.*

Example of a finite super solvable convex geometry is given also by lattice of subsemilattices $Sub_\wedge(P)$ of a finite lattice P. This follows from result in [14], where it was established for more general (and dual) lattices of closure operators on finite partially ordered sets. We deal with infinite lattices $Sub_{\wedge}(P)$ in section 4.

3. Relatively convex sets

Let V be a real vector space, X a subset of V. Let $Co(V, X)$ be the collection of sets $C \cap X$, where C is a convex subset of V. Ordered by inclusion $Co(V, X)$ is an algebraic convex geometry. Several recent publications are devoted to this convex geometry [2, 3, 7].

The main goal of this section is to prove the following analogue of Theorem 1.1.

Theorem 3.1. *$Co(V, X)$ is order scattered iff the semilattice S of compact elements of $Co(V, X)$ is order scattered and does not have $\mathfrak{P}^{<\omega}(\mathbb{N})$ as a subsemilattice.*

Since the argument will involve some topological considerations, we will be able to establish also the analogue of Mislove's result [15].

Theorem 3.2. *$Co(V, X)$ is topologically scattered iff it is order-scattered.*

4. The lattice of subsemilattices and the lattice of suborders

Convex geometries of subsemilattices of a semilattice and suborders of a partially ordered set play important role in the studies of convex geometries in general due to their close connection to lattices of quasi-equational theories, see [4, 5, 17, 18].

Theorem 4.1. *If S is an infinite $\wedge$-semilattice, then the lattice $Sub_{\wedge}(S)$ of subsemilattices of S always has a copy of $\mathbb{Q}$. Thus, $Sub_{\wedge}(S)$ is order-scattered iff S is finite.*

Similar result holds for the lattice of suborders. For a partially ordered set $\langle P, \leq \rangle$, denote by $S(P)$ the strict order associated with P, i.e. $S(P) = \{(p, q) : p \leq q \text{ and } p \neq q, p, q \in P\}$. The lattice of suborders $O(P)$ is the lattice of transitively closed subsets of $S(P)$.

Theorem 4.2. *The lattice of suborders $O(P)$ of a partially ordered set $\langle P, \leq \rangle$ is order-scattered iff $S(P)$ is finite.*

5. Semilattice $\Omega(\eta)$ as an obstruction in algebraic convex geometry

As mentioned in the introduction, the semilattice $\Omega(\eta)$ does not appear in the semilattice of compact elements of an algebraic modular lattice, see [9]. We show that $\Omega(\eta)$ is a typical subsemilattice of compact elements in some special convex geometries called *multi-chains*.

The main result of this section is about particular *bi-chains*: these are structures $(E, <_1, <_2)$, where E is a countable set, $<_1$ is the order of type ω, and $<_2$ is a total order that has a suborder of type η. For each $i \in \{1, 2\}$, define a convex geometry $C_i = I(E, <_i)$ of initial segments of $(E, <_i)$. A convex geometry $C = C_1 \vee C_2$ is called *a duplex.*

Lemma 5.1. *$\Omega(\eta)$ is a subsemilattice of the semilattice of compact elements of some duplex.*

6. Order scattered algebraic convex geometries with finite $\vee$-dimension of semilattice of compact elements

In this section we characterize by obstruction the order scattered algebraic convex geometries whose semilattice of compact elements has a finite dimension.

The following conjecture is stated in [8].

Conjecture 6.1. *$L = J(P)$, the lattice of ideals of a semilattice P is order scattered iff P is order scattered, and neither $\mathfrak{P}^{<\omega}(\mathbb{N})$, nor $\Omega(\eta)$ is embeddable into P as a semilattice.*

If L is in addition modular, then $\Omega(\eta)$ cannot appear as a subsemilattice of P. In this case, the conjecture was proved in [8]. When L is a convex geometry, as was shown in section 5, $\Omega(\eta)$ may be a subsemilattice of P.

In current paper we prove the conjecture for algebraic convex geometries $L = J(P)$, for which P has a finite $\vee$-dimension.

Definition 6.2. We say that a semilattice P with 0 has $\vee$-dimension $dim_{\vee}(P) = \kappa$, if κ is the smallest cardinal for which there exist κ chains C_i, $i < \kappa$, with minimal element 0_i, and injective map $f : P \to \Pi C_i$ such that $f(a \vee b) = f(a) \vee f(b)$ and $f(0) = (0_i, i < \kappa)$.

Our central result is the next Theorem.

Theorem 6.3. *Let* $\mathbf{C}$ *be an algebraic convex geometry, let* P *be the semilattice of compact elements of* $\mathbf{C} = J(P)$. *If* $dim_{\vee}P = n < \omega$, *then* $\mathbf{C}$ *is order scattered iff* P *is order scattered and* $\Omega(\eta)$ *is not a subsemilattice of* P.

One of technical tools is the iterative application of the bracket relation $\eta \to [\eta]^2_3$, a famous unpublished theorem by F. Galvin whose proof can be found in [13].

References

[1] K. Adaricheva, *Semidistributive and co-algebraic lattices of subsemilattices*, Algebra and Logic **27** (1988), 385–395; translated from Algebra i Logika **27**, no. 6 (1988), 625–640.

[2] ———, *Join-semidistributive lattices of relatively convex sets*, Contributions to General Algebra 14, Proceedings of the Olomouc Conference 2002 (AAA 64) and the Potsdam conference 2003 (AAA 65), Verlag Johannes Heyn, Klagenfurt, 2004, 1–14. Also see on http://arxiv.org/abs/math/0403104

[3] ———, *Representing finite convex geometries by relatively convex sets*, to appear in EJofC, available at http://arxiv.org/abs/1101.1539

[4] ———, *On prevariety of perfect lattices*, Alg. Universalis **65**, 21–39.

[5] K.V. Adaricheva, V.A. Gorbunov, and V.I. Tumanov, *Join-semidistributive lattices and convex geometries*, Adv. Math., **173** (2003), 1–49.

[6] D. Armstrong, *Sorting order on a Coxeter group*, http://arxiv.org/abs/0712.1047

[7] G. M. Bergman, *On lattices of convex sets in* R^n, Algebra Universalis **53** (2005), 357–395.

[8] I. Chakir, and M. Pouzet, *The length of chains in modular algebraic lattices*, Order, 24(2007), 227–247.

[9] ———, *The length of chains in algebraic lattices*, Proceedings ISO-08, Algiers, Algeria, Nov 2-6, 2008, H. Ait Haddadene, I. Bouchemakh, M. Boudar, S. Bouroubi (Eds) LAID3, p. 379–390. arXiv:0812.2193v1 [math.CO]

[10] ———, *A characterization of well-founded algebraic lattices*, preprint.

[11] B. Dietrich, *A circuit characterization of antimatroids*, J. Comb. Theory B **43**(1987), 314–321.

[12] P.H. Edelman and R. Jamison, *The theory of convex geometries*, Geom.Dedicata **19**(1985), 247-274.

[13] R. Fraïssé, *The theory of relations*, North-Holland Pub.Co., Amsterdam, 2000.

[14] M. Hawrylicz, and V. Reiner, *The lattice of closure relations on a poset*, Algebra Universalis **30**(1993), 301–310.

[15] M. Mislove, *When are topologically scattered and order scattered are the same?*, Annals of Disc. Math. 23(1984), 61–80.

[16] M. Semenova, and A. Zamojska-Dzienio, *On lattices embeddable into lattices of order-convex sets. Case of trees*, Internat. J.Algebra Comput. **17** (2007), 1667-1712.

[17] M.V. Semenova *On lattices embeddable into lattices of suborders*, (Russian) Algebra i Logika **44** (2005),483–511; translation in Algebra and Logic **44** (2005), 270–285.

[18] F. Wehrung, *Sublattices of complete lattices with continuity conditions*, Algebra Universalis **53** (2005), 149–173.

Department of Mathematical Sciences, Yeshiva University, 245 Lexington ave., New York, NY 10016, USA

E-mail address: adariche@yu.edu

UFR Mathématiques, Université Claude-Bernard, Lyon1, 43 Bd du 11 November 1918, 69622 Villeurbanne cedex, France

E-mail address: pouzet@univ-lyon1.fr

Preferential Semantics for the Logic of Comparative Concepts Similarity

Régis Alenda[1], Nicola Olivetti[1], Camilla Schwind[2], and Dmitry Tishkovsky[3]

[1] LSIS - UMR CNRS 6168, Université de Provence – St-Jérôme, Marseille (France)
regis.alenda@lsis.org and nicola.olivetti@univ-cezanne.fr
[2] LIF - UMR CNRS 6166, Université de la Méditerranée – Luminy, Marseille (France)
camilla.schwind@lif.univ-mrs.fr
[3] School of Computer Science, The University of Manchester, Manchester (UK)
dmitry@cs.man.ac.uk

1 The Logic of Comparative Similarity

As a part of their investigations on several logics for distance and topological reasoning, Sheremet, Tishkovsky, Wolter and Zakharyaschev have presented in [4,5] the logic $\mathcal{CSL}$ of comparative similarity, whose main operator, $\leftleftarrows$, represents qualitative assertions of the form "being more similar/closer to ... than to ...".

The language $\mathcal{L}_{\mathcal{CSL}}$ of $\mathcal{CSL}$ is generated from a (countable) set of propositional variables $p_1, p_2, \ldots \in \mathcal{V}_p$ by ordinary propositional connectives together with the operator $\leftleftarrows$: $A, B ::= \bot \mid p_i \mid \neg A \mid A \sqcap B \mid A \leftleftarrows B$.

The original semantics of $\mathcal{CSL}$ is based on distance spaces. A distance space is a pair (Δ, d) where Δ is a non-empty set, and $d : \Delta \times \Delta \to \mathbb{R}^{\geq 0}$ is a *distance function* satisfying the following condition: $\forall x, y \in \Delta$, $d(x, y) = 0$ iff $x = y$. The distance between an object w and a non-empty subset X of Δ is defined by $d(w, X) = \inf\{d(w, x) \mid x \in X\}$. If $X = \varnothing$, then $d(w, X) = \infty$. $\mathcal{CSL}$-distance models are defined as a kind of Kripke models based on distance spaces:

Definition 1 ($\mathcal{CSL}$-distance model [5,4]). *A* $\mathcal{CSL}$-distance model $\mathcal{I}$ *is a triple* $\mathcal{I} = \langle \Delta, d, .^{\mathcal{I}} \rangle$ *where:*
- (Δ, d) *is a distance space.*
- $.^{\mathcal{I}}$ *is the* evaluation function *defined as usual on propositional variables and boolean connectives, and as follows for* $\leftleftarrows$*:*

$$(A \leftleftarrows B)^{\mathcal{I}} \stackrel{\text{def}}{=} \{x \in \Delta \mid d(x, A^{\mathcal{I}}) < d(x, B^{\mathcal{I}})\}.$$

Additional properties of the distance function may be assumed, namely the symmetry, the triangular inequality, and the minspace property:
(SYM) $d(x, y) = d(y, x)$,
(TRI) $d(x, y) \leq d(x, w) + d(w, x)$,
(MIN) $\forall x \in \Delta, \forall Y \subseteq \Delta, Y \neq \varnothing$ implies $\exists y \in Y$ such that $d(x, Y) = d(x, y)$.

Despite its apparent simplicity, $\mathcal{CSL}$ can be seen as a fragment, indeed a powerful one (including for instance the logic $\mathbf{S4}_u$ of topological spaces), of a general logic for spatial reasoning comprising different modal operators defined by (bounded) quantified distance expressions (namely the logic $\mathcal{QML}$ [5]). The satisfiability problem for the $\mathcal{CSL}$ logic is ExpTime-complete. Sadly, when interpreted over subspaces of $\mathbb{R}^n$, it turns out that this logic is undecidable [5].

2 Axiomatization

An axiomatization $\mathbf{CS_g}$ of $\mathcal{CSL}$ over arbitrary distance spaces is presented in figure 1. This axiomatization can be easily derived from the axiomatization of $\mathcal{CSL}$ over minspaces in [2], or from the one in [5].

(1)	$\neg(A \leftleftarrows B) \sqcup \neg(B \leftleftarrows A)$	(2)	$\neg(A \leftleftarrows C) \sqcap \neg(C \leftleftarrows B) \to \neg(A \leftleftarrows B)$
(3)	$B \to \neg(\top \leftleftarrows B)$	(4)	$(A \leftleftarrows B) \sqcap (A \leftleftarrows C) \to (A \leftleftarrows (B \sqcup C))$
(5)	$(A \leftleftarrows \bot) \to \neg(\neg(A \leftleftarrows \bot) \leftleftarrows \bot)$	$(Taut)$	All tautologies and classical rules.

$$(Mon)\quad \frac{\vdash (A \to B)}{\vdash (A \leftleftarrows C) \to (B \leftleftarrows C)}$$

Figure 1. The $\mathbf{CS_g}$ system.

Interestingly, it is shown in [5] that, when the minspace property is not assumed, $\mathcal{CSL}$ is not sensible to the symmetry alone. But it can distinguish between arbitrary distance models, distance models with (TRI), and distance models with (TRI)+(SYM) (ie. metric models). On the contrary, if we assume the minspace property, the situation changes drastically: $\mathcal{CSL}$ becomes sensible to (SYM), whereas it is blind to (TRI) (with or without (SYM)). To take into account additional properties of the distance function, additional axioms can be considered:

(MS) $\neg(\top \leftleftarrows B) \to B$

(TR) $\neg(\neg(\top \leftleftarrows A) \leftleftarrows A)$

(MT) $(A \leftleftarrows B) \to (\top \leftleftarrows \neg(A \leftleftarrows B))$

The following theorem shows the correspondence between each axiomatization and the class of models they define.

Theorem 2 (Soundness and Completeness of $\mathbf{CS_g}$).
- $\mathbf{CS_g}$ *is sound and complete for the class of arbitrary distance models.*
- $\mathbf{CS_g}$ + (TR) *is sound and complete for distance models satisfying the triangular inequality [5].*
- $\mathbf{CS_g}$ + (TR) + (MT) *is sound and complete for metric models [5].*
- $\mathbf{CS_g}$ + (MS) *is sound and complete for minspace models [2].*

3 A Preferential Semantics

$\mathcal{CSL}$ is a logic of pure qualitative comparisons. This motivates an alternative semantics where the distance function is replaced by a family of comparisons relations, one for each object. Our preferential structures are similar to the ones used for some conditional logics, that is to say a set equipped by a family of strict pre-orders encoding distance comparisons between objects (or regions) of the distance spaces: $x \leq_w y$ means "the objects of the region w are closer to the region x than to the region y". By this mean, we intend to abstract away from the numerical values of the distance function, and give a purely relational, modal-like, semantics.

Definition 3 ($\mathcal{CSL}$ preferential model [1]). *A $\mathcal{CSL}$-preferential $\mathcal{I}$ model is a triple $\langle \Delta, (\leq_w)_{w\in\Delta}, .^{\mathcal{I}}\rangle$ where:*

- *Δ is a non-empty countable set.*
- *$(\leq_w)_{w\in\Delta}$ is a family of total pre-orders indexed by the objects of Δ, each $\leq_w$ satisfying:*
  (Limit Assumption) *For all non-empty subset X of Δ:*
  $\exists x \in X$ *such that* $\forall x' \in X$, $x \leq_w x'$.
  (Weak Centering) *For all $x \in \Delta$, $w \leq_w x$.*
- *$.^{\mathcal{I}}$ is the evaluation function defined as usual for propositional variables and boolean operators, and as follows for $\leftleftarrows$:*
  $$(A \leftleftarrows B)^{\mathcal{I}} \overset{\text{def}}{=} \{w \mid \exists x \in A^{\mathcal{I}} \text{ such that } \forall y \in B^{\mathcal{I}}, x <_w y\}$$
  where $<_w$ is the strict weak order induced by $\leq_w$.

To take into account additional properties of the distance function, we can assume further properties on the pre-orders. One of the most interesting is the strong centering, $y \leq_x x$ iff $x = y$, which captures the minspace property. Preferential models satisfying the strong centering will be called *preferential min-model*. Note that the minspace property in distance spaces is not related to the limit assumption: preferential models always satisfy the limit assumption [1].

It seems that symmetric minspaces models cannot be represented in a natural way by preferential structures as defined above. In particular, it is not yet known whether symmetric minspaces can be captured by a finite set of additional properties on the pre-orders. This motivated the study of an alternative preferential semantics, closer to the distance one, where the preference relation compares pairs of objects [3].

Given a non-empty set Δ, we denote by $\mathcal{MS}_2(\Delta)$ the set of two-element multisets over Δ; its elements are denoted by $\{a, b\}$ (abusing set-notation). This set will be equipped with a total pre-order $\leq$. Intuitively, each $\{a, b\}$ represents the distance from a to b, while distance comparison will be represented by the total pre-order $\leq$. We can restate the definition of a $\mathcal{CSL}$-model in terms of preferential structures of this kind:

Definition 4 (Pair Model [3]). *A $\mathcal{CSL}$-pair-model $\mathcal{I}$ is a triple $\langle \Delta, \leq, .^{\mathcal{I}}\rangle$ where:*

- *Δ is a non empty countable set.*
- *$\leq$ is a total pre-order over $\mathcal{MS}_2(\Delta)$ satisfying the following properties:*
  (Strong Limit Assumption) *For every non empty-subsets X and Y of Δ,:*
  $\exists x \in X, \exists y \in Y,$ *such that* $\forall x' \in X, \forall y' \in Y, \{x, y\} \leq \{x', y'\}$.
  (Pair-Centering) *For every $x, y \in \Delta$, $\{x, x\} < \{x, y\}$.*
- *$.^{\mathcal{I}}$ is the* evaluation function *defined as usual on propositional variables and boolean connectives, and as follow for the concept similarity operator:*
  $$(A \leftleftarrows B)^{\mathcal{I}} = \{x \mid \exists y \in A^{\mathcal{I}}, \forall z \in B^{\mathcal{I}}, \{x, y\} < \{x, z\}\},$$
  where $<$ is the strict weak order induced by $\leq$.

Theorem 5. – *A formula is satisfiable in a general distance model iff it is satisfiable in a preferential model [1].*
– *For each minspace distance model, there exists an equivalent (ie. satisfying exactly the same formulas) preferential min-model, and vice-versa [2].*
– *For every symmetric distance minspace model, there exists an equivalent (ie. satisfying exactly the same formulas) pair-model [3].*

It is possible to show that $\mathcal{CSL}$ has the finite model property over minspaces, but this fails for $\mathcal{CSL}$ over arbitrary distance spaces with the distance semantics. On the contrary, the next theorem shows that the preferential semantics always enjoy the small model property. This fact was exploited to give several tableau calculi, grounded on the preferential semantics, for $\mathcal{CSL}$ over various classes of distance spaces [2,3,1].

Theorem 6 (Small model property [3,1]). *A formula A is satisfiable in a preferential model (resp. preferential min-model, pair-model), iff it is satisfiable in a preferential model (resp. preferential min-model, pair-model) such that the number of objects in Δ is bounded by $2^{|\mathsf{sub}(A)|}$, where $\mathsf{sub}(A)$ is the set of subformulas of A.*

4 Conclusion

There are several issues and open problems that are of interest. First is finding the additional conditions on the preferential relations needed to capture distance models with (TRI), and metric models. We expect these conditions to be similar to the ones used for the decidability and axiomatization's completeness proof in [5]. Such a preferential semantics could be used as a base to extend the tableau calculi in [2,3,1], giving practical decision procedure for these cases. Another open problem is to find a direct axiomatization of $\mathcal{CSL}$ over symmetric minspaces. As for the corresponding property in the original preferential semantics, we conjecture that such an axiomatization is likely to be infinite.

References

1. R. Alenda and N. Olivetti. Tableau calculus for the logic of comparative similarity over arbitrary distance spaces. In *Proc. LPAR 2010*, volume 6397:52–66 of *LNCS*, 2010.
2. R. Alenda, N. Olivetti, and C. Schwind. Comparative concept similarity over minspaces: Axiomatisation and tableaux calculus. In *Proc. TABLEAUX 2009*, volume 5607:17–31 of *LNCS*, 2009.
3. R. Alenda, N. Olivetti, C. Schwind, and D. Tishkovsky. Tableau calculi for csl over minspaces. In *Proc. CSL 2010*, volume 6247:52–66 of *LNCS*, 2010.
4. M. Sheremet, D. Tishkovsky, F. Wolter, and M. Zakharyaschev. Comparative similarity, tree automata, and diophantine equations. In *Proc. LPAR 05*, volume 3835:651–665 of *LNAI*, 2005.
5. M. Sheremet, F. Wolter, and M. Zakharyaschev. A modal logic framework for reasoning about comparative distances and topology. *Annals of Pure and Applied Logic*, 161:534–559(4), 2010.

Homotopical Fibring

Peter Arndt *

Over the last fifteen years the problem of fibring, i.e. of combining several logics into a single common extension, has received a lot of attention, both for its significance in applications and its interest for theoretical logic, see e.g. [3], [5], [6]. The main issues about fibring that have been studied in the literature are fibring techniques for different settings of logic — like abstract consequence relations, institutions or logics with frame semantics or topos semantics — and the preservation of metaproperties of logics under fibring — e.g. the existence of implicit connectives [3], completeness [2], or the position in the Leibniz hierarchy (which, roughly, measures the degree of algebraizability/applicability of the Lindenbaum-Tarski algebra technique) [7].

A major conceptual advance, in [1], see also [10], was the recognition of fibring as a colimit construction in an appropriate category of logics and translations between them. In particular the combination of two logics sharing a common sublogic, called their constrained fibring, was shown to be the pushout of the two logics along the inclusions of the sublogic into both of them.

However, the categories in which these considerations take place have a very restricted notion of morphism: Morphisms are translations mapping the primitive connectives, which generate the domain language, to primitive connectives of the target language. A more natural notion of morphism would be to allow primitive connectives to be mapped to derived connectives, as it happens e.g. in the $\neg\neg$-translations from intuitionistic to classical logic. Unfortunately the categories of logics with this broader notion of morphism are badly behaved, in particular colimits other than the ones from the old setting do either not exist at all or are degenerate and do not describe a combination of logics as desired. It has thus been an open problem how to combine logics along more general translations.

In this talk we present a solution: Any category of logics known to the speaker comes with a natural definition of when a translation is an equivalence of logics. It is thus open to the methods of abstract homotopy theory, e.g. those exposed in [12] or [13] — in particular the notion of homotopy colimit is defined, and this is what we call the *homotopical fibring*, or *hofibring*, of logics, and what we propose to replace the colimit construction of fibring with. The main conceptual advantages of hofibring over fibring in this setting are:

- homotopy colimits tend to exist in settings of interest where colimits do not exist
- one can always see the constituent logics as linguistic fragments of their hofibring, unlike for fibring
- invariance under equivalence: replacing the logics to be combined by equivalent ones will result in an equivalent hofibred logic

As an example we present the concrete meaning of this in a simple setting of propositional Hilbert Calculi. In this setting we consider signatures S of generating connectives given with arities (formally: a map $S \to \mathbb{N}$) and the absolutely free S-algebra $L(S)$ generated by a fixed set of propositional variables.

*Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany
peter.arndt@mathematik.uni-regensburg.de

We now work in the category whose objects are logics, presented by pairs $(S,\vdash)$, where S is a signature and $\vdash\subseteq \mathscr{P}(L(S))\times L(S)$ is a consequence relation between subsets of $L(S)$ and elements of $L(S)$. As usual we write the consequence relation in infix notation $\Gamma\vdash\varphi$ and say that Γ entails φ. A morphism $f:(S,\vdash)\to(S',\vdash')$ is an arity preserving map $f:S\to L(S')$ such that its extension $\hat{f}:L(S)\to L(S')$ to the absolutely free algebra $L(S)$ satisfies $\Gamma\vdash\varphi\Rightarrow\hat{f}(\Gamma)\vdash'\hat{f}(\varphi)$ (i.e. it is a "translation").

There is an obvious forgetful functor from logics to signatures and it is easy to see that it preserves and creates colimits. Thus the underlying signature of a colimit of logics will always be the colimit of their underlying signatures. If, as in our case, signatures are given by free algebras, this explains the problem of defective or non-existing colimits: Colimits of free algebras are not "by nature" free again, and forcing them to be so results in degeneracy.

One can call a translation $f:(S,\vdash)\to(S',\vdash')$ a *weak equivalence*, if $\Gamma\vdash\varphi\Leftrightarrow\hat{f}(\Gamma)\vdash\hat{f}(\varphi)$ (i.e. it is a "conservative translation") and if for every formula φ in the target there exists a formula in the image of $\hat{f}$ which is logically equivalent to φ (it has "dense image").

One can further call *cofibration* a morphism which is given by mapping generating connectives injectively to generating connectives (i.e. which is given by an injective, arity preserving map between the signatures).

This category of logics and translations now has the convenient structure of a so–called ABC cofibration category (see [11]), that is, we have a factorization of any morphism into a cofibration followed by a weak equivalence, satisfying some axioms. The proof of this proceeds in close analogy to the original topological setting, e.g. by constructing mapping cylinders. The results of [11] then yield a concrete construction recipe for the homotopy colimit of a given diagram as the actual colimit of a different diagram, by which we can

- express hofibring through fibring
- see that fibring is a special case of hofibring (which yields a new universal property of fibring)
- see that all homotopy colimits exist and
- transfer preservation results known from fibring to hofibring, for metaproperties which are invariant under equivalence

While existence of homotopy colimits could be inferred through any structure of ABC cofibration category, for the last group of results we make crucial use of our concrete choice of cofibrations and weak equivalences. Among the preservation results obtained by this technique, those on the existence of implicit connectives are straightforward. Preservation of completeness and position in the Leibniz hierarchy require a homotopical view on semantics first, which we will provide before applying the transfer result.

In the talk, after having explained the above central ideas for the example of Hilbert calculi, we will show that they are in fact largely independent of the chosen notion of "logic" and extend e.g. to first order logics by admitting many-sorted signatures and to the fibring of institutions via the c-parchments from [14].

To conclude, we sketch a picture of homotopical versions of other variants of fibring, like modulated fibring ([4]), metafibring ([8]) and fibring of non-truth functional logics [9], as well as of the homotopical categories given by other settings like institutions and type theory. If time remains, we hint at a variety of approaches to abstract logic, other than those aimed at fibring questions, suggested by the homotopical view point.

References

[1] A. Sernadas, C. Sernadas, and C. Caleiro. Fibring of logics as a categorial construction. Journal of Logic and Computation, 9(2):149–179, 1999.

[2] A. Zanardo, A. Sernadas, and C. Sernadas. Fibring: Completeness preservation. Journal of Symbolic Logic, 66(1):414–439, 2001.

[3] W. A. Carnielli, M. E. Coniglio, D. Gabbay, P. Gouveia, and C. Sernadas. Analysis and Synthesis of Logics - How To Cut And Paste Reasoning Systems, volume 35 of Applied Logic. Springer, 2008

[4] C. Sernadas, J. Rasga, and W. A. Carnielli. Modulated fibring and the collapsing problem. Journal of Symbolic Logic, 67(4):1541–1569, 2002.

[5] Dov M. Gabbay. Fibring Logics. Clarendon Press, 1999

[6] W.A. Carnielli; M.E. Coniglio. Combining Logics. In: The Stanford Encyclopedia of Philosophy, 2007. Ed.: E.N. Zalta. URL: http://plato.stanford.edu/entries/logic-combining/

[7] V.L. Fernández; M.E. Coniglio. Fibring in the Leibniz Hierarchy. Logic Journal of the IGPL 15, no. 5-6: 475-501, 2007.

[8] M.E. Coniglio. Recovering a logic from its fragments by meta-fibring. Logica Universalis 1, no. 2: 377-416, 2007.

[9] C. Caleiro; W. Carnielli; M.E. Coniglio; A. Sernadas; C. Sernadas. Fibring Non-Truth-Functional Logics: Completeness Preservation. Journal of Logic, Language and Information 12, no. 2: 183-211, 2003.

[10] C. Caleiro, W. A. Carnielli, J. Rasga, and C. Sernadas. Fibring of logics as a universal construction. In D. Gabbay and F. Guenthner, editors, Handbook of Philosophical Logic, 2nd Edition, volume 13, pages 123–187. Springer, 2005.

[11] A. Radulescu-Banu, Cofibrations in Homotopy Theory, URL: http://arxiv.org/abs/math/0610009

[12] J. Lurie, Higher Topos Theory, Annals of Mathematics Studies 170, Princeton University Press, 2009

[13] C. Barwick, D. M. Kan, Relative categories: Another model for the homotopy theory of homotopy theories, URL: http://arxiv.org/abs/1011.1691

[14] C. Caleiro, P. Mateus, J. Ramos, and A. Sernadas. Combining logics: Parchments revisited. In M. Cerioli and G. Reggio, editors, Recent Trends in Algebraic Development Techniques - Selected Papers, volume 2267 of Lecture Notes in Computer Science, pages 48–70. Springer, 2001.

The word problem in semiconcept algebras

Philippe Balbiani*

Institut de recherche en informatique de Toulouse

1 Introduction

Extending concept lattices to protoconcept algebras and semiconcept algebras, Herrmann *et al.* [5] and Wille [10] introduced negations in conceptual structures based on formal contexts such as double Boolean algebras. These algebras have attracted interest for their theoretical merits — basic representations have been obtained [5, 10] — and their practical relevance — applications in the field of knowledge representation have been developed [7, 8]. This interest motivated Vormbrock [9] to attack the word problem (WP) in protoconcept algebras and to demonstrate that given terms s, t, if $s = t$ is not valid in all protoconcept algebras then there exists a finite protoconcept algebra in which $s = t$ is not valid. Nevertheless, the upper bound on the size of the finite protoconcept algebra given in [9, Page 258] is not elementary. Therefore, it does not allow us to conclude — as wrongly stated in [9, Page 240] — that the WP in protoconcept algebras is **NP**-complete. Switching over to semiconcept algebras, our aim is to prove that the WP in semiconcept algebras is **PSPACE**-complete.

2 Pure double Boolean algebras

See [3, Chapter 3] or [4] for a short introduction to formal concept analysis. Formal contexts are structures of the form $\mathbb{K} = (G, M, \Delta)$ where G is a nonempty set, M is a nonempty set and Δ is a binary relation between G and M. For all $X \subseteq G$ and for all $Y \subseteq M$, let $X^{\triangleright} = \{m \in M\colon$ for all $g \in G$, if $g \in X$ then $g \; \Delta \; m\}$ and $Y^{\triangleleft} = \{g \in G\colon$ for all $m \in M$, if $m \in Y$ then $g \; \Delta \; m\}$. Let $\mathbb{K} = (G, M, \Delta)$ be a formal context. Given $X \subseteq G$ and $Y \subseteq M$, the pair (X, Y) is called "semiconcept of $\mathbb{K}$" iff $Y = X^{\triangleright}$ or $X = Y^{\triangleleft}$. Let $\underline{\mathcal{H}}(\mathbb{K}) = (\mathcal{H}(\mathbb{K}), \bot_l, \bot_r, \top_l, \top_r, \neg_l, \neg_r, \vee_l, \vee_r, \wedge_l, \wedge_r)$ be the algebraic structure of type $(0, 0, 0, 0, 1, 1, 2, 2, 2, 2)$ where $\mathcal{H}(\mathbb{K})$ is the set of all semiconcepts of $\mathbb{K}$, $\bot_l = (\emptyset, M)$, $\bot_r = (M^{\triangleleft}, M)$, $\top_l = (G, G^{\triangleright})$, $\top_r = (G, \emptyset)$, $\neg_l(X, Y) = (G \setminus X, (G \setminus X)^{\triangleright})$, $\neg_r(X, Y) = ((M \setminus Y)^{\triangleleft}, M \setminus Y)$, $(X_1, Y_1) \vee_l (X_2, Y_2) = (X_1 \cup X_2, (X_1 \cup X_2)^{\triangleright})$,

*Address: Institut de recherche en informatique de Toulouse, CNRS — Université de Toulouse, 118 route de Narbonne, 31062 TOULOUSE CEDEX 9, France.

$(X_1, Y_1) \vee_r (X_2, Y_2) = ((Y_1 \cap Y_2)^{\triangleleft}, Y_1 \cap Y_2)$, $(X_1, Y_1) \wedge_l (X_2, Y_2) = (X_1 \cap X_2, (X_1 \cap X_2)^{\triangleright})$ and $(X_1, Y_1) \wedge_r (X_2, Y_2) = ((Y_1 \cup Y_2)^{\triangleleft}, Y_1 \cup Y_2)$. It is a simple matter to check that $\underline{\mathcal{H}}(\mathbb{K})$ satisfies the following conditions: $x \wedge_l (y \wedge_l z) = (x \wedge_l y) \wedge_l z$, $x \vee_r (y \vee_r z) = (x \vee_r y) \vee_r z$, $x \wedge_l y = y \wedge_l x$, $x \vee_r y = y \vee_r x$, $\neg_l(x \wedge_l x) = \neg_l x$, $\neg_r(x \vee_r x) = \neg_r x$, $x \wedge_l (y \wedge_l y) = x \wedge_l y$, $x \vee_r (y \vee_r y) = x \vee_r y$, $x \wedge_l (y \vee_l z) = (x \wedge_l y) \vee_l (x \wedge_l z)$, $x \vee_r (y \wedge_r z) = (x \vee_r y) \wedge_r (x \vee_r z)$, $x \wedge_l (x \vee_l y) = x \wedge_l x$, $x \vee_r (x \wedge_r y) = x \vee_r x$, $x \wedge_l (x \vee_r y) = x \wedge_l x$, $x \vee_r (x \wedge_l y) = x \vee_r x$, $\neg_l(\neg_l x \wedge_l \neg_l y) = x \vee_l y$, $\neg_r(\neg_r x \vee_r \neg_r y) = x \wedge_r y$, $\neg_l \bot_l = \top_l$, $\neg_r \top_r = \bot_r$, $\neg_l \top_r = \bot_l$, $\neg_r \bot_l = \top_r$, $\top_r \wedge_l \top_r = \top_l$, $\bot_l \vee_r \bot_l = \bot_r$, $x \wedge_l \neg_l x = \bot_l$, $x \vee_r \neg_r x = \top_r$, $\neg_l \neg_l(x \wedge_l y) = x \wedge_l y$, $\neg_r \neg_r(x \vee_r y) = x \vee_r y$, $(x \wedge_l x) \vee_r (x \wedge_l x) = (x \vee_r x) \wedge_l (x \vee_r x)$ and $x \wedge_l x = x$ or $x \vee_r x = x$. We shall say that an algebraic structure $\mathcal{D} = (D, \bot_l, \bot_r, \top_l, \top_r, \neg_l, \neg_r, \vee_l, \vee_r, \wedge_l, \wedge_r)$ of type $(0, 0, 0, 0, 1, 1, 2, 2, 2, 2)$ is a pure double Boolean algebra iff the operations $\bot_l$, $\bot_r$, $\top_l$, $\top_r$, $\neg_l$, $\neg_r$, $\vee_l$, $\vee_r$, $\wedge_l$ and $\wedge_r$ satisfy the above conditions.

3 The word problem

Let Var denote a countable set of individual variables (x, y, etc). The set $t(Var)$ of all terms (s, t, etc) is given by the rule $s ::= x \mid 0_l \mid 0_r \mid 1_l \mid 1_r \mid -_l s \mid -_r s \mid (s \sqcup_l t) \mid (s \sqcup_r t) \mid (s \sqcap_l t) \mid (s \sqcap_r t)$. Let $\mathcal{D} = (D, \bot_l, \bot_r, \top_l, \top_r, \neg_l, \neg_r, \vee_l, \vee_r, \wedge_l, \wedge_r)$ be a pure double Boolean algebra. A valuation based on $\mathcal{D}$ is a function m assigning to each individual variable x an element $m(x)$ of D. m induces a function $(\cdot)^m$ assigning to each term s an element $(s)^m$ of D such that $(x)^m = m(x)$, $(0_l)^m = \bot_l$, $(0_r)^m = \bot_r$, $(1_l)^m = \top_l$, $(1_r)^m = \top_r$, $(-_l s)^m = \neg_l (s)^m$, $(-_r s)^m = \neg_r (s)^m$, $(s \sqcup_l t)^m = (s)^m \vee_l (t)^m$, $(s \sqcup_r t)^m = (s)^m \vee_r (t)^m$, $(s \sqcap_l t)^m = (s)^m \wedge_l (t)^m$ and $(s \sqcap_r t)^m = (s)^m \wedge_r (t)^m$. Now, for the WP in pure double Boolean algebras: given terms s, t, decide whether there exists a pure double Boolean algebra $\mathcal{D}$ and a valuation m based on $\mathcal{D}$ such that $(s)^m \neq (t)^m$. Our aim is to prove that the WP in pure double Boolean algebras is **PSPACE**-complete. In this respect, we need to introduce the computational complexity of the 2-sorted modal logic K_2.

4 Computational complexity of K_2

See [1, Chapter 6] or [6] for a short introduction to the computational complexity of modal logic. The language of K_2 is based on a countable set $OVar$ of object variables (P, Q, etc) and a countable set $AVar$ of attribute variables (p, q, etc). The set of all object formulas (A, B, etc) and the set of all attribute formulas (a, b, etc) are given by the rules $A ::= P \mid \bot \mid \neg A \mid (A \vee B) \mid \Box a$ and $a ::= p \mid \bot \mid \neg a \mid (a \vee b) \mid \Box A$. A formula ($\alpha$, β, etc) is either an object formula or an attribute formula. Let $\vec{OVar} = P_1, P_2, \ldots$ be an enumeration of $OVar$ and $\vec{AVar} = p_1, p_2, \ldots$ be an enumeration of $AVar$. We shall say that a substitution (Θ, θ) is normal with respect to $\vec{OVar}$ and $\vec{AVar}$ iff for all positive integers i, $\Theta(P_i) = P_i$ and $\theta(p_i) = \Box P_i$ or $\Theta(P_i) = \Box p_i$ and $\theta(p_i) = p_i$. Given a formula

α, $Var(\alpha)$ will denote the set of all variables occurring in α. A formula α is said to be nice iff $Var(\alpha) \subseteq OVar$ or $Var(\alpha) \subseteq AVar$. Let $\mathbb{K} = (G, M, \Delta)$ be a formal context. A $\mathbb{K}$-valuation is a pair (V, v) of functions where V assigns to each object variable P a subset $V(P)$ of G and v assigns to each attribute variable p a subset $v(p)$ of M. (V, v) induces a function $(\cdot)^{(V,v)}$ assigning to each formula α a subset $(\alpha)^{(V,v)}$ of $G \cup M$ such that $(P)^{(V,v)} = V(P)$, $(\bot)^{(V,v)} = \emptyset$, $(\neg A)^{(V,v)} = G \setminus (A)^{(V,v)}$, $(A \vee B)^{(V,v)} = (A)^{(V,v)} \cup (B)^{(V,v)}$, $(\Box a)^{(V,v)} = \{g \in G$: for all $m \in M$, if $m \in (a)^{(V,v)}$ then $g \ \Delta \ m\}$, $(p)^{(V,v)} = v(p)$, $(\bot)^{(V,v)} = \emptyset$, $(\neg a)^{(V,v)} = M \setminus (a)^{(V,v)}$, $(a \vee b)^{(V,v)} = (a)^{(V,v)} \cup (b)^{(V,v)}$ and $(\Box A)^{(V,v)} = \{m \in M$: for all $g \in G$, if $g \in (A)^{(V,v)}$ then $g \ \Delta \ m\}$. A formula α is said to be satisfiable iff there exists a formal context $\mathbb{K} = (G, M, \Delta)$ and a $\mathbb{K}$-valuation (V, v) such that $(\alpha)^{(V,v)}$ is nonempty. Following the line of reasoning suggested in [1, Chapter 6] or [6], one can prove that

Proposition 1 *The following decision problem is* **PSPACE**-*hard: given a nice formula α, determine whether α is satisfiable.*

Proposition 2 *The following decision problem is in* **PSPACE**: *given a formula α, determine whether α is satisfiable.*

5 Lower and upper bound for the WP

Let $\vec{OVar} = P_1, P_2, \ldots$ be an enumeration of $OVar$, $\vec{AVar} = p_1, p_2, \ldots$ be an enumeration of $AVar$ and $\vec{Var} = x_1, y_1, x_2, y_2, \ldots$ be an enumeration of Var. The function $T(\cdot)$ assigning to each nice object formula A a term $T(A)$ and the function $t(\cdot)$ assigning to each nice attribute formula a a term $t(a)$ are such that $T(P_i) = x_i$, $T(\bot) = 0_l$, $T(\neg A) = -_l T(A)$, $T(A \vee B) = T(A) \sqcup_l T(B)$, $T(\Box a) = -_l -_l -_r -_r t(a)$, $t(p_i) = y_i$, $t(\bot) = 1_r$, $t(\neg a) = -_r t(a)$, $t(a \vee b) = t(a) \sqcap_r t(b)$ and $t(\Box A) = -_r -_r -_l -_l T(A)$. Let $(s_1(\cdot), s_2(\cdot))$ be the function assigning to each nice formula α a pair $(s_1(\alpha), s_2(\alpha))$ of terms such that if α is a nice object formula then $s_1(\alpha) = T(\alpha)$ and $s_2(\alpha) = 0_l$ and if α is a nice attribute formula then $s_1(\alpha) = t(\alpha)$ and $s_2(\alpha) = 1_r$. One can prove that if α is nice then α is satisfiable iff there exists a pure double Boolean algebra $\mathcal{D}$ and a valuation m based on $\mathcal{D}$ such that $(s_1(\alpha))^m \neq (s_2(\alpha))^m$. Thus, by proposition 1,

Proposition 3 *The WP in pure double Boolean algebras is* **PSPACE**-*hard.*

Let $\vec{Var} = x_1, x_2, \ldots$ be an enumeration of Var, $\vec{OVar} = P_1, P_2, \ldots$ be an enumeration of $OVar$ and $\vec{AVar} = p_1, p_2, \ldots$ be an enumeration of $AVar$. The function $F(\cdot)$ assigning to each term s an object formula $F(s)$ and the function $f(\cdot)$ assigning to each term s an attribute formula $f(s)$ are such that $F(x_i) = P_i$, $f(x_i) = p_i$, $F(0_l) = \bot$, $f(0_l) = \Box\bot$, $F(0_r) = \Box\top$, $f(0_r) = \top$, $F(1_l) = \top$, $f(1_l) = \Box\top$, $F(1_r) = \Box\bot$, $f(1_r) = \bot$, $F(-_l s) = \neg F(s)$, $f(-_l s) = \Box\neg F(s)$, $F(-_r s) = \Box\neg f(s)$, $f(-_r s) = \neg f(s)$, $F(s \sqcup_l t) = F(s) \vee F(t)$, $f(s \sqcup_l t) = \Box(F(s) \vee F(t))$, $F(s \sqcup_r t) = \Box(f(s) \wedge f(t))$, $f(s \sqcup_r t) = f(s) \wedge f(t)$, $F(s \sqcap_l t) = F(s) \wedge F(t)$, $f(s \sqcap_l t) = \Box(F(s) \wedge F(t))$, $F(s \sqcap_r t) = \Box(f(s) \vee f(t))$ and $f(s \sqcap_r t) = f(s) \vee f(t)$. Let

$O(\cdot,\cdot)$ be the function assigning to each pair (s,t) of terms the object formula $O(s,t)$ such that $O(s,t) = \neg(F(s) \leftrightarrow F(t))$. Let $A(\cdot,\cdot)$ be the function assigning to each pair (s,t) of terms the attribute formula $A(s,t)$ such that $A(s,t) = \neg(f(s) \leftrightarrow f(t))$. One can prove that there exists a pure double Boolean algebra $\mathcal{D}$ and a valuation m based on $\mathcal{D}$ such that $(s)^m \neq (t)^m$ iff there exists a substitution (Θ,θ) such that (Θ,θ) is normal with respect to $O\vec{V}ar$ and $A\vec{V}ar$ and $O(s,t)^{(\Theta,\theta)}$ is satisfiable or $A(s,t)^{(\Theta,\theta)}$ is satisfiable. Thus, by proposition 2,

Proposition 4 *The WP in pure double Boolean algebras is in* **PSPACE**.

References

[1] Blackburn, P., de Rijke, M., Venema, Y.: *Modal Logic.* Cambridge University Press (2001).

[2] Burgmann, C., Wille, R.: *The basic theorem on preconcept lattices.* In Missaoui, R., Schmid, J. (editors): *Formal Concept Analysis.* Springer (2006) 80–88.

[3] Davey, B, Priestley, H.: *Introduction to Lattices and Order.* Cambridge University Press (2002).

[4] Ganter, B, Wille, R.: *Formal Concept Analysis: Mathematical Foundations.* Springer (1999).

[5] Herrmann, C., Luksch, P., Skorsky, M., Wille, R.: *Algebras of semiconcepts and double Boolean algebras.* Technische Universität Darmstadt (2000).

[6] Marx, M., Venema, Y.: *Local variations on a loose theme: modal logic and decidability.* In Grädel, E., Kolaitis, P., Libkin, L., Marx, M., Spencer, J., Vardi, M., Venema, Y., Weinstein, S. (editors): *Finite Model Theory and its Applications.* Springer (2007) 371–429.

[7] Priss, U.: *Formal concept analysis in information science.* Annual Review of Information Science and Technology **40** (2006) 521–543.

[8] Stumme, G.: *Formal concept analysis on its way from mathematics to computer science.* In Priss, U., Corbett, D., Angelova, G. (editors): *Conceptual Structures: Integration and Interfaces.* Springer (2002) 2–19.

[9] Vormbrock, B.: *A solution of the word problem for free double Boolean algebras.* In Kuznetsov, S., Schmidt, S. (editors): *Formal Concept Analysis.* Springer (2007) 240–270.

[10] Wille, R.: *Boolean concept logic.* In Ganter, B., Mineau, G. (editors): *Conceptual Structures: Logical, Linguistic, and Computational Issues.* Springer (2000) 317–331.

Knowledge Spaces and Interactive Realizers

Stefano Berardi, Ugo de'Liguoro

Department of Computer Science, Unversità di Torino
{stefano,deliguoro}@di.unito.it

1 Introduction

A possible interpretation of Coquand's winning strategies in backtracking games in [6] is as learning strategies. This generalizes Gold's idea of "learning in the limit" in [7], making any arithmetical set learnable (see [3]).

By learning in the limit, in its simplest form called 1-backtracking in [2], we mean the activity of making falsifiable hypothesis, or "guesses", about a law describing a given sequence of individuals, with the possibility of changing a guess as soon as it is contradicted by new evidences in the sequence. This kind of learning is successful if the limit of the sequences of guesses about the law describing the sequence is a correct guess about the sequence.

In this paper we give a topology over a set of knowledge states, providing a description of Learning in the Limit in both monotonic and non-monotonic cases. We investigate a quite general description of the original Gold's approach, by giving a topology over a set of individual we call "answers", a topology which is directly motivated by the concept of learning. The topology we have in mind is an extension of Scott's topology, akin to Lawson topology but more general than it.

We represent a knowledge state as an equivalence relation over a set of answers: two answers are equivalent if they are two alternative answers to the same question. Answers are divided into true and false answers. A knowledge state is a choice of at most one true answer in each equivalent class (that is at most one true answer for each question), and it represents the current knowledge we have about the true answers to some (usually, not all) questions considered up to a certain point. The topology has all basic opens of a Scott-like topology: the family of all knowledge states including a fixed finite knowledge state is a basic open. These basic opens represent positive information, or a finite list of questions for which we selected one true answer. We consider a second kind of basic open, representing negative information, or a set of questions for which we assume there is no true answer. A basic open of this second kind consists of all knowledge states having empty intersection with a finite set of equivalence classes. These equivalence classes represent finitely many questions for which we assume there is no true answer in the knowledge state.

We use this topology as a frame in which to state and solve several problems about learning in the limit. We define a correct learning strategy as any *continuous* map taking a state of learning and a set of possible counterexamples, and returning finitely many new answers, which we may add to the current state in order to produce a new and more accurate set of guesses. In [4] we call such strategies *interactive realizers*, of which we study the monotonic case. Interactive realizers have also been related to Kleene's realizability in [1]. A general notion of interactive reliazer also exists, and it is sketched in [5].

2 The Space of States

In this section we define a set of "possible answers", equipped with an equivalence relation relating two alternative answers to the same question, then a notion of knowledge states consisting in a consistent selection of answers, and eventually a topology over the knowledge states. Basic opens of this topology represent finite positive or negative information about a knowledge state.

Let $\mathbb{A}$ be some countable set of *atoms*, and $\sim\, \subseteq \mathbb{A}\times\mathbb{A}$ an equivalence relation. Two atoms $x, y \in \mathbb{A}$ are *compatible*, written $x\#y$, according to:

$$x\#y \;\Leftrightarrow\; x = y \vee x \not\sim y.$$

We think of an atom as a possible answer, right or wrong, to a given question, and two equivalent atoms as two alternative answers to the same question.

A set of atoms $X \subseteq \mathbb{A}$ is *consistent*, namely if and only if:

$$\forall x, y \in X.\; x\#y.$$

Equivalently X is consistent if for all $x \in \mathbb{A}$ the set $X \cap [x]$ is either empty or a singleton, where $[x]$ is the equivalence class of x w.r.t. $\sim$. We think of a consistent set of atoms as a "knowledge state", or equivalently as a selection of at most one answer for each question.

We call $\mathbb{S}_\infty = \{X \subseteq \mathbb{A} \mid X \text{ is consistent}\}$ the set of consistent sets of atoms, and $\mathbb{S} = \{X \in \mathbb{S}_\infty \mid X \in \mathcal{P}_{\mathit{fin}}(\mathbb{A})\}$ the set of *states*, using $s, t, \ldots$ to range over $\mathbb{S}$.

The set $\mathbb{S}_\infty$ is a poset by subset inclusion, and it is downward closed. It follows that $(\mathbb{S}_\infty, \cap, \subseteq)$ is an inf-semilattice with bottom $\emptyset$. $\mathbb{S}$ is closed under arbitrary but non-empty inf, as the empty inf, namely the whole $\mathbb{A}$, is not consistent in general. $\mathbb{S}_\infty$ it is not closed under union, unless the compatibility relation is trivial. We write $X \uparrow Y$ and say, by overloading terminology, that X and Y are *compatible*, if $X \subseteq Z \supseteq Y$ for some $Z \in \mathbb{S}_\infty$. Clearly the union of a family $\mathcal{U} \subseteq \mathbb{S}_\infty$ belongs to $\mathbb{S}_\infty$ if and only if all elements of $\mathcal{U}$ are pairwise compatible. Thus $\mathbb{S}_\infty$ is closed under directed sups, so it is a cpo, which has compacts $K(\mathbb{S}_\infty) = \mathbb{S}$ and it is algebraic. We now introduce a state topology, whose basic opens are all possible positive and negative information about a finite set of questions.

Definition 2.1 (State Topology) *The* state topology $(\mathbb{S}_\infty, \Omega(\mathbb{S}_\infty))$ *is generated by the subbasics A_x, B_x, with $x \in \mathbb{A}$:*

$$\begin{aligned} A_x &= \{X \in \mathbb{S} \mid x \in X\} = \{X \in \mathbb{S} \mid X \cap [x] \textit{ a singleton}\}, \\ B_x &= \{X \in \mathbb{S} \mid X \cap [x] = \emptyset\}. \end{aligned}$$

$X \in A_x$ means that the knowledge state X has selected the answer $y \in [x]$ to the question $[x]$, while $X \in B_x$ means that X has no answer to the question $[x]$.

By definition, a basic open of $\Omega(\mathbb{S}_\infty)$ has the shape:

$$\mathcal{O}_{U,V} = \bigcap_{x \in U} A_x \cap \bigcap_{y \in V} B_y,$$

for some finite $U, V \subseteq \mathbb{A}$. If $\neg\, x \# y$, that is $x \sim y$ and $x \neq y$, then $A_x \cap A_y = \emptyset$, so that $\emptyset$ is a basic open. On the other hand if $x \sim y$ then $B_x = B_y$. Therefore without loss of generality we assume U, V to be consistent, so that we shall refer to basic opens $\mathcal{O}_{s,t}$ with $s, t \in \mathbb{S}$ only.

The state topology resembles Lawson topology, and in fact it is finer than that. The *lower topology* over a poset is generated by the complements of principal filters; the *Lawson topology* is the smallest refinement of both the lower and the Scott topology. In case of the cpo $(\mathbb{S}_\infty, \subseteq)$ the Lawson topology is generated by the subbasics:

$$\overline{X\uparrow} = \{Y \in \mathbb{S}_\infty \mid X \not\subseteq Y\} \quad \text{and} \quad s\uparrow = \{Y \in \mathbb{S}_\infty \mid s \subseteq Y\},$$

for $X \in \mathbb{S}_\infty$ and $s \in \mathbb{S}$, representing the negative and positive information respectively.

Proposition 2.2 (State versus Lawson Topology) *The state topology $\Omega(\mathbb{S}_\infty)$ refines the Lawson topology over the cpo $(\mathbb{S}_\infty, \subseteq)$, and they coincide if and only if $[x]$ is finite for all $x \in \mathbb{A}$.*

3 Layered states and relative truth

In this section we define a stratification into level for the set of answers, translating the notion of non-monotonic learning. When there is a single level of answers, we may check directly whenever an answer is true or false. When there are two or more levels of answers, the truth or falsity of an answer depends, through a fixed continuous function, over the answers of smaller levels belonging the same knowledge state. Whenever we change a knowledge state the truth value of all its answers may change: this situation makes learning in the limit much more difficult to study. Let us assume the existence of a map $\mathsf{lev} : \mathbb{A} \to \mathit{Ord}$, associating to each atom x the ordinal $\mathsf{lev}(x)$, and such that any two answers to the same question are of the same level:

$$\forall x, y \in \mathbb{A}.\ x \sim y \Rightarrow \mathsf{lev}(x) = \mathsf{lev}(y).$$

If $X \in \mathbb{S}_\infty$ and $\alpha \in \mathit{Ord}$ we write $X \upharpoonright \alpha = \{x \in X \mid \mathsf{lev}(x) < \alpha\}$. We may now precise a notion of truth of an answer w.r.t. a knowledge state.

Definition 3.1 (Layered Valuations) *A map $\mathsf{tr} : \mathbb{A} \times \mathbb{S}_\infty \to \mathbb{B}$, where $\mathbb{B} = \{\mathsf{true}, \mathsf{false}\}$, is a* layered valuation*, or a* valuation *for short, if:*

1. tr *is continous, by taking $\mathbb{A}$ and $\mathbb{B}$ with the discrete toplogy, $\mathbb{S}_\infty$ with the state topology $\Omega(\mathbb{S}_\infty)$, and $\mathbb{A} \times \mathbb{S}_\infty$ with the product topology*

2. $\mathsf{tr}(x, X) = \mathsf{tr}(x, X \upharpoonright \mathsf{lev}(x))$

When tr is a layered valuation, we say that x is relatively true w.r.t. X, or just x true w.r.t. X, if $\mathsf{tr}(x, X) = \mathsf{true}$. By definition the truth of x w.r.t. X depends only on the atoms of lover level than $\mathsf{lev}(x)$; it follows that

$$\mathsf{lev}(x) = 0 \Rightarrow \mathsf{tr}(x, X) = \mathsf{tr}(x, \emptyset)$$

that is the truth value of atoms of level 0 is absolute, and depends just on the choice of tr. Let tr be some fixed truth predicate. We will now introduce a notion of knowledge state which is "maximal" in a sense.

Definition 3.2 (Sound and Complete States) *Let $X \in \mathbb{S}_\infty$, then:*

1. X sound $\Leftrightarrow \forall x \in X.\ \mathsf{tr}(x, X) = \mathsf{true}$
2. X complete $\Leftrightarrow \forall x \in \mathbb{A}.\ X \cap [x] = \emptyset \Rightarrow \mathsf{tr}(x, X) = \mathsf{false}$
3. *X is a* model *if and only if X is sound and complete.*

In words, X is sound if all its atoms are true w.r.t. X; X is a model if it is maximal among sound states. We may prove that there are models: this fact will play an important role in the study of the notion of learning.

Theorem 3.3 (Existence of Models) *There exists a model $X \in \mathbb{S}_\infty$.*

References

[1] F. Aschieri, S. Berardi, *Interactive Learning-Based Realizability Interpretation for Heyting Arithmetic with* $\mathtt{EM}_1$, Proceedings of TLCA 2009, Springer Lecture Notes in Computer Science, vol. 5608, 2009

[2] S. Berardi, T. Coquand, S. Hayashi, *Games with 1-Bactracking*, APAL 2010, to appear. 210-225.

[3] S. Berardi and U. de'Liguoro, *Toward the interpretation of non-constructive reasoning as non-monotonic learning*, Information and Computation, vol. 207, 1, pag. 63-81, (2009).

[4] S. Berardi and U. de'Liguoro, *Interactive realizers. A new approach to program extraction from non constructive proofs*, to appear in ACM-TOCL.

[5] S. Berardi and U. de'Liguoro, *Knowledge Spaces and Interactive Realizers*, available at: `http://www.di.unito.it/~deligu/papers/BdLKS.pdf`

[6] T. Coquand. A semantics of evidence for classical arithmetic. *J. Symb. Log.*, 60:325–337, 1995.

[7] E. M. Gold, *Limiting Recursion*, Journal of Symbolic Logic 30, pag. 28-48 (1965) Cambridge University Press (1989)

SAHLQVIST PRESERVATION FOR MODAL MU-ALGEBRAS

NICK BEZHANISHVILI AND IAN HODKINSON

Abstract. We define Sahlqvist fixed point formulas and present completeness and correspondence results for modal fixed point logics axiomatized by these formulas for LFP-definable classes of general μ-frames. We also define modal μ^*-algebras and show that Sahlqvist fixed point formulas are preserved under completions of conjugated modal μ^*-algebras. This work is based on [3] and [4].

1. Introduction

Modal fixed point logic is an extension of modal logic with fixed point operators that is very expressive, but still decidable, e.g., [6, Section 5]. Many expressive modal and temporal logics are fragments of it: e.g., [6, Section 4.1].

[1] studied non-classical, order-topological semantics of modal μ-calculus. Descriptive frames are order-topological structures extensively used in modal logic: e.g., [5, Chapter 5]. In [1] the authors define, what we call in this paper, descriptive μ-frames – those descriptive frames that admit a topological interpretation of the least fixed point operator. Unlike the classical semantics of fixed point logics, in descriptive μ-frames the least fixed point operator is interpreted as the intersection of not *all* pre-fixed points, but of all *clopen* pre-fixed points.

In this paper we extend this semantics to what we call *admissible semantics*, by evaluating fixed point operators as meets of pre-fixed points from some given subset $\mathcal{F}$ of a modal algebra $\mathcal{A}$. We call such algebras modal μ^*-algebras. In case $\mathcal{F} = \mathcal{A}$, we recover the semantics of [1]. There are (at least) three reasons for studying this semantics. First, admissible semantics gives completeness for all (normal) systems of modal fixed point logic, whereas completeness results for fixed point logics with the classical semantics are very sparse [13], [7]. Second, we can obtain positive results such as Theorems 3.5 and 4.1 for admissible semantics, whereas the analogues for the classical semantics do not hold. We are also able to obtain a workable notion of completion as opposed to the standard one which was shown to be 'badly behaved' [11]. Third, admissible semantics provides more 'realistic' logics for practical applications. For example, if we are dealing with the spatial logic of the real plane, we might want to assign formulas only to some practically realizable subsets such as, say, polygons — resulting in an admissible spatial semantics. If we take this seriously, then fixed point operators should also be interpreted relative to admissible semantics.

We give Sahlqvist completeness and correspondence results for admissible semantics. Moreover, for an arbitrary $\mathcal{F}$ we show that Sahlqvist fixed point formulas are preserved under completions of conjugated modal μ^*-algebras.

2. Modal algebras and descriptive frames

We recall that the language of modal μ-calculus consists of a countably infinite set of propositional variables (x, y, p, q, x_0, x_1, etc), constants $\bot$ and $\top$, connectives $\wedge$, $\vee$, $\neg$, modal operators $\Diamond$ and $\Box$, and formulas $\mu x\varphi$ for all formulas φ positive in x (i.e., x occurs under the scope of only an even number of negations). Formulas of modal μ-calculus will be called *modal μ-formulas.* A formula without μ-operators will be called a *modal formula.*

A *Kripke frame* is a pair (W, R), where W is a non-empty set and $R \subseteq W^2$ a binary relation. Let (W, R) be a Kripke frame. An *assignment* is a map h from the propositional variables into the powerset $\mathcal{P}(W)$ of W. For each modal μ-formula φ we denote by $[\![\varphi]\!]_h$ the set of points satisfying φ under h. Satisfiability and validity of a modal μ-formula in the Kripke model (W, R, h) and the frame (W, R) are defined in a standard way (see, e.g., [6]).

For each $w \in W$ we let $R(w) = \{v \in W : wRv\}$. For each $U \subseteq W$ we let $[R]U = \{v \in W : R(v) \subseteq U\}$ and $\langle R\rangle U = \{v \in W : R(v) \cap U \neq \emptyset\}$. A *Stone space* is a compact Hausdorff topological space with a basis of clopen sets. A *descriptive frame* is a pair (W, R) such that W is a Stone space and R a binary relation on W such that $R(w)$ is closed for each $w \in W$ and the set $\mathsf{Clop}(W)$ of clopen subsets of W is closed under the operations $[R]$ and $\langle R\rangle$.

A *modal algebra* is a pair $\mathfrak{B} = (B, \Diamond)$, where B is a Boolean algebra and $\Diamond$ a unary operation on B satisfying $\Diamond 0 = 0$ and $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$ for each $a, b \in B$. There is a well known duality between modal algebras and descriptive frames (see e.g., [5]).

Definition 2.1.

(1) *Let $\mathfrak{B} = (B, \Diamond)$ be a modal algebra and $\mathcal{F} \subseteq B$. A map h from propositional variables to B is called an* algebra assignment. *We define a (possibly partial) semantics for modal μ-formulas by the following inductive definition.*

$[\bot]_h^{\mathcal{F}} = 0$, $[\top]_h^{\mathcal{F}} = 1$, *and* $[x]_h^{\mathcal{F}} = h(x)$, *where x is a propositional variable,*
$[\varphi \wedge \psi]_h^{\mathcal{F}} = [\varphi]_h^{\mathcal{F}} \wedge [\psi]_h^{\mathcal{F}}$, $[\varphi \vee \psi]_h^{\mathcal{F}} = [\varphi]_h^{\mathcal{F}} \vee [\psi]_h^{\mathcal{F}}$, *and* $[\neg\varphi]_h^{\mathcal{F}} = \neg[\varphi]_h^{\mathcal{F}}$,
$[\Diamond\varphi]_h^{\mathcal{F}} = \Diamond[\varphi]_h^{\mathcal{F}}$ *and* $[\Box\varphi]_h^{\mathcal{F}} = \Box[\varphi]_h^{\mathcal{F}}$.

For $a \in B$ we denote by h_x^a the algebra assignment such that $h_x^a(x) = a$ and $h_x^a(y) = h(y)$ for each variable $y \neq x$. If φ is positive in x, let

$$[\mu x\varphi]_h^{\mathcal{F}} = \bigwedge\{a \in \mathcal{F} : [\varphi]_{h_x^a}^{\mathcal{F}} \leq a\},$$

if this meet exists; otherwise, the semantics for $\mu x\varphi$ is undefined.

(2) *A triple $(B, \Diamond, \mathcal{F})$ is called a* modal $\mathcal{F}$-μ-algebra *if $[\varphi]_h^{\mathcal{F}}$ is defined for any modal μ-formula φ and any algebra assignment h.*

(3) *A modal $\mathcal{F}$-μ-algebra $(B, \Diamond, \mathcal{F})$ is called a* modal μ-algebra *if $\mathcal{F} = B$. A* modal μ^*-algebra *is a modal $\mathcal{F}$-μ-algebra for some $\mathcal{F}$.*

Recall that a modal algebra $(B, \Diamond)$ is called *complete* if B is a complete Boolean algebra; that is, for each subset S of B the meet $\bigwedge S$ and the join $\bigvee S$ exist. Every complete modal algebra B is a modal μ-algebra. Locally finite modal algebras are other examples of modal μ-algebras (see [3] for details).

Definition 2.2. *Let (W, R) be a descriptive frame, $\mathfrak{F} \subseteq \mathcal{P}(W)$ and h an arbitrary assignment, that is, a map from the propositional variables to $\mathcal{P}(W)$. We define the semantics for modal μ-formulas by the following inductive definition.*

- $[\![\bot]\!]_h^{\mathfrak{F}} = \emptyset$, $[\![\top]\!]_h^{\mathfrak{F}} = W$, *and* $[\![x]\!]_h^{\mathfrak{F}} = h(x)$, *where x is a propositional variable,*
- $[\![\varphi \wedge \psi]\!]_h^{\mathfrak{F}} = [\![\varphi]\!]_h^{\mathfrak{F}} \cap [\![\psi]\!]_h^{\mathfrak{F}}$, $[\![\varphi \vee \psi]\!]_h^{\mathfrak{F}} = [\![\varphi]\!]_h^{\mathfrak{F}} \cup [\![\psi]\!]_h^{\mathfrak{F}}$, *and* $[\![\neg\varphi]\!]_h^{\mathfrak{F}} = W \setminus [\![\varphi]\!]_h^{\mathfrak{F}}$,
- $[\![\Diamond\varphi]\!]_h^{\mathfrak{F}} = \langle R\rangle[\![\varphi]\!]_h^{\mathfrak{F}}$ *and* $[\![\Box\varphi]\!]_h^{\mathfrak{F}} = [R][\![\varphi]\!]_h^{\mathfrak{F}}$.
- *For $U \in \mathcal{P}(W)$ we denote by h_x^U a new assignment such that $h_x^U(x) = U$ and $h_x^U(y) = h(y)$ for each propositional variable $y \neq x$. Let φ be positive in x. Then*

$$[\![\mu x\varphi]\!]_h^{\mathfrak{F}} = \bigcap\{U \in \mathfrak{F} : [\![\varphi]\!]_{h_x^U}^{\mathfrak{F}} \subseteq U\}.$$

Let (W, R) be a descriptive frame. We call a map h from the propositional variables to $\mathcal{P}(W)$ a *set-theoretic assignment.* If $\mathrm{rng}(h) \subseteq \mathsf{Clop}(W)$ then h is called a *clopen assignment.*

Definition 2.3. *A triple $(W, R, \mathfrak{F})$ is called a* general frame *if (W, R) is a Kripke frame and $\mathfrak{F} \subseteq \mathcal{P}(W)$. Elements of $\mathfrak{F}$ are called* admissible sets. *An assignment h from the propositional variables to $\mathfrak{F}$ is called an* admissible *assignment. $(W, R, \mathfrak{F})$ is called a* general μ-frame *if $[\![\varphi]\!]_h^{\mathfrak{F}} \in \mathfrak{F}$ for each modal μ-formula φ and admissible assignment h.*

A descriptive frame (W, R) is called a descriptive μ-frame *if for each clopen assignment h and modal μ-formula φ, the set $[\![\varphi]\!]_h^{\mathsf{Clop}(W)}$ is clopen.*

Clearly, a descriptive μ-frame (W, R) can be viewed as a general μ-frame $(W, R, \mathsf{Clop}(W))$.

Theorem 2.4. ([1]) *The correspondence between modal algebras and descriptive frames restricts to a one-to-one correspondence between modal μ-algebras and descriptive μ-frames.*

Also, every axiomatically defined system of modal fixed point logic is sound and complete with respect to modal μ-algebras, descriptive μ-frames and general μ-frames [1, 7].

3. Completeness and correspondence of Sahlqvist fixed point formulas

Definition 3.1. *We call a modal μ-formula φ* positive *if it does not contain any negations. For each $m \in \omega$ we let $\Box^0 x = x$ and $\Box^{m+1} x = \Box(\Box^m x)$.*

Definition 3.2. *A formula φ is called a* Sahlqvist fixed point formula *if it is obtained from formulas of the form $\neg\Box^m x$ (x a propositional variable, $m \in \omega$) and positive formulas (in the language with the μ-operator) by applying the operations $\vee$ and $\Box$.*

Remark 3.3. In the language without fixed point operators, the above definition of Sahlqvist formula is different from the 'standard' definition (see e.g., [5]), but any Sahlqvist formula of [5] is equivalent to a conjunction of Sahlqvist formulas in the aforementioned sense.

Theorem 3.4. *Let (W, R) be a descriptive frame,*[1] *$w \in W$ and φ a Sahlqvist fixed point formula. If $w \in [\![\varphi]\!]_f^{\mathsf{Clop}(W)}$ for each clopen assignment f, then $w \in [\![\varphi]\!]_h^{\mathsf{Clop}(W)}$ for each set-theoretic assignment h.*

In algebraic terminology, Theorem 3.4 means roughly that Sahlqvist fixed point formulas are preserved under canonical extensions of modal μ^*-algebras.

Now let LFP be the first-order language augmented with the least fixed point operator μ; see, e.g., [8]. We assume that μ is applied to unary predicates only. For each propositional variable p we reserve a unary predicate symbol P. An LFP-formula ξ is said to be an *LFP-frame condition* if it has no free variables or free unary predicate symbols.

Let $\mathfrak{M} = (W, R, \mathfrak{F})$ be a general μ-frame and h an admissible assignment. We view $\mathfrak{M}$ as an LFP-structure via $P^{\mathfrak{M}} = h(p) \subseteq W$, for each propositional variable p. The interpretation of LFP-formulas is standard (see, e.g., [8, Section 8]), except for expressions of the type $(\mu(Z, u)\,\xi(u, Z))(v)$, where Z is a unary predicate symbol and u and v first-order variables. This is interpreted in $(\mathfrak{M}, h, g)$ as stating that the value assigned to v is in the intersection of all admissible (elements of $\mathfrak{F}$) pre-fixed points of the map $F : \mathcal{P}(W) \to \mathcal{P}(W)$, where

$$F(U) = \{w \in W : (\mathfrak{M}, h_z^U, g_u^w) \models \xi(u, Z)\}, \tag{1}$$

in which g_u^w is a first-order assignment mapping variable u to the point $w \in W$. For an LFP frame condition ξ, the notion '$(\mathfrak{M}, h, g) \models \xi$' does not depend on h and g, so we drop them.

Theorem 3.5. (**Main Theorem 1**) *For each Sahlqvist fixed point formula φ there exists a constructible LFP-frame condition $\chi(\varphi)$ such that each modal fixed point logic axiomatized by a set Φ of Sahlqvist fixed point formulas is*

1. *sound and complete with respect to the class of general μ-frames satisfying the LFP-frame conditions $\{\chi(\varphi) : \varphi \in \Phi\}$,*
2. *sound and complete with respect to the class of descriptive μ-frames satisfying the LFP-frame conditions $\{\chi(\varphi) : \varphi \in \Phi\}$.*

Essential in the proofs of Theorems 3.5 and 3.4 is the analogue of Esakia's lemma for modal μ-formulas. See [3] for details and for examples of Sahlqvist fixed point formulas and their corresponding LFP-frame conditions. Also, [2] defines a larger class of Sahlqvist fixed point formulas and studies their LFP-correspondence for *classical* semantics of fixed points.

4. Preservation of Sahlqvist fixed point formulas under completions

Let A be a Boolean algebra. A subalgebra B of A is said to be a *dense subalgebra* if for each $0 < a \in A$ there exists $0 < b \in B$ such that $b \leq a$. The *MacNeille completion* $\overline{A}$ of A is a complete Boolean algebra $\overline{A}$ such that A is a dense subalgebra of $\overline{A}$ and all joins and meets that exist in A are preserved in $\overline{A}$. In other words, if $S \subseteq A$ is such that $\bigvee^A S$ exists, then $\bigvee^A S = \bigvee^{\overline{A}} S$, and similarly for meets. The completion of a Boolean algebra always exists and is unique up to isomorphism (see, e.g., [12]).

Let $f, g : A \to A$. g is called the *conjugate* of f if for all $a, b \in A$ we have $a \wedge f(b) = 0$ iff $b \wedge g(a) = 0$. A modal algebra $\mathcal{A} = (A, f_i)_{i \in I}$ with multiple diamond operators is called *conjugated* if the f_i fall into conjugate

[1] Note that we do not require that (W, R) is a descriptive μ-frame.

pairs. It is well known (see e.g., [10]) that if $\mathcal{A}$ is conjugated then $\overline{\mathcal{A}} = (\overline{A}, \overline{f}_i)_{i\in I}$ is also a conjugated modal algebra, where $\overline{f_i}$ is defined by: $\overline{f}_i(b) = \bigvee^{\overline{A}}\{f_i(a) : a \in A, a \leq b\}$. $\overline{\mathcal{A}}$ is called the *completion* of $\mathcal{A}$.

[11] shows *inter alia* that there exists a conjugated modal μ-algebra $\mathcal{A}$ that does not admit an embedding into a complete modal μ-algebra preserving fixed point formulas. This motivates the following, alternative definition of completions of modal μ-algebras. For a conjugated modal $\mathcal{F}$-μ-algebra $\mathcal{A} = (A, f_i, \mathcal{F})_{i\in I}$ (with multiple diamond operators), the modal $\mathcal{F}$-μ-algebra $\overline{\mathcal{A}} = (\overline{A}, \overline{f}_i, \mathcal{F})_{i\in I}$ is called its *$\mathcal{F}$-completion.* Fixed point formulas are preserved by the natural embedding of an $\mathcal{F}$-μ-algebra into its $\mathcal{F}$-completion. Moreover,

Theorem 4.1. (**Main Theorem 2**) *Any Sahlqvist fixed point formula valid in a conjugated modal $\mathcal{F}$-μ-algebra is also valid in its $\mathcal{F}$-completion.*

The proof of Theorem 4.1 is in a way similar to the proof of Theorem 3.4, but does not use the analogue of Esakia's lemma. Theorem 4.1 can be formulated in the more general setting of conjugated Boolean algebras with operators. See [4] for all details. Finally, we note that the following result of Givant and Venema [9] is an immediate corollary of Theorem 4.1:

Corollary 4.2. ([9]) *Let $\mathcal{A}$ be a conjugated modal algebra and $\overline{\mathcal{A}}$ its completion. Then any modal Sahlqvist formula valid in $\mathcal{A}$ is also valid in $\overline{\mathcal{A}}$.*

References

[1] S. Ambler, M. Kwiatkowska, and N. Measor. Duality and the completeness of the modal μ-calculus. *Theoretical Computer Science*, 151:3–27, 1995.

[2] J. van Benthem, N. Bezhanishvili, and I. Hodkinson. Sahlqvist correspondence for modal mu-calculus. 2011. submitted.

[3] N. Bezhanishvili and I. Hodkinson. Sahlqvist theorem for modal fixed point logic. 2010. submitted.

[4] N. Bezhanishvili and I. Hodkinson. Preservation of Sahlqvist fixed point equations in completions of Boolean algebras with operators. 2011. in preparation.

[5] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Cambridge University Press, 2001.

[6] J. Bradfield and C. Stirling. Modal mu-calculus. In P. Blackburn, J. van Benthem, and F. Wolter, editors, *Handbook of Modal Logic*, pages 721–756. Elsevier, 2007.

[7] B. ten Cate and G. Fontaine. An easy completeness proof for the modal μ-calculus on finite trees. In L. Ong, editor, *FOSSACS 2010*, volume 6014 of *LNCS*, pages 161 –175. Springer-Verlag, 20109.

[8] H-D. Ebbinghaus and J. Flum. *Finite model theory.* Springer-Verlag, Berlin, second edition, 1999.

[9] S. Givant and Y. Venema. The preservation of Sahlqvist equations in completions of Boolean algebras with operators. *Algebra Universalis*, 41(1):47–84, 1999.

[10] D. Monk. Completions of Boolean algebras with operators. *Math. Nachr.*, 46:47–55, 1970.

[11] L. Santocanale. Completions of μ-algebras. *Ann. Pure Appl. Logic*, 154(1):27–50, 2008.

[12] R. Sikorski. *Boolean algebras.* Springer-Verlag, Berlin, 1960.

[13] I. Walukiewicz. Completeness of Kozen's axiomatisation of the propositional μ-calculus. *Inform. and Comput.*, 157(1-2):142–182, 2000. LICS 1995 (San Diego, CA).

Nick Bezhanishvili: Department of Computing, Imperial College London, London SW7 2AZ, UK, nbezhani@doc.ic.ac.uk.
`http://www.doc.ic.ac.uk/~nbezhani/`

Ian Hodkinson: Department of Computing, Imperial College London, London SW7 2AZ, UK, i.hodkinson@imperial.ac.uk.
`http://www.doc.ic.ac.uk/~imh/`

Basic pseudo hoops and normal valued basic pseudo hoops

Michal Botur (join work with Anatolij Dvurečenskij and Tomasz Kowalski)

We recall that a *pseudo hoop* is an algebra $(M, \cdot, \to, \rightsquigarrow, 1)$ of type $\langle 2,2,2,2,0,0\rangle$ such that, for all $x, y, z \in M$,

i) $x \cdot 1 = x = 1 \cdot x$,

ii) $x \to x = 1 = x \rightsquigarrow x$,

iii) $(x \cdot y) \to z = x \to (y \to z)$

iv) $(x \cdot y) \rightsquigarrow z = y \rightsquigarrow (x \rightsquigarrow z)$

v) $(x \to y) \cdot x = (y \to x) \cdot y = x \cdot (x \rightsquigarrow y) = y \cdot (y \rightsquigarrow x)$.

If we set $x \leq y$ if and only if $x \to y = 1$ (this is equivalent to $x \rightsquigarrow y = 1$), then $\leq$ is a partial order such that $x \wedge y = (x \to y) \cdot x$ and M is a $\wedge$-semilattice.

We are saying that a pseudo hoop M satisfies *Riesz decomposition property* if $a \geq b \cdot c$ implies that there are two elements $b' \geq b$ and $c' \geq c$ such that $a = b' \cdot c'$.

Theorem. *Every pseudo hoom M satisfies Riesz decomposition property.*

We recall that filter F on pseudo hoop M is a non empty set closed over product and upper bounds. A filter F is called to be a *normal* if $x \to y \in F$ iff $x \rightsquigarrow y \in M$. The normal filters are just 1-kernels of the congruences.

A filter F of a pseudo hoop M is said to be *prime* if, for two filters F_1, F_2 on M, $F_1 \cap F_2 \subseteq F$ entails $F_1 \subseteq F$ or $F_2 \subseteq F$. We denote by $\mathcal{P}(M)$ the system of all prime filters of a pseudo hoop M.

Moreover, due to Riesz decomposition property we can show that:

Theorem. *The system of all filters $\mathcal{F}(M)$ of a pseudo hoop M is a distributive lattice under the set-theoretical inclusion. In addition $F \cap \bigvee_i F_i = \bigvee_i (F \cap F_i)$.*

A pseudo hoop M is said to be *basic* if, for all $x, y, z \in M$

(B1) $(x \to y) \to z \leq ((y \to x) \to z) \rightsquigarrow z$,

(B2) $(x \rightsquigarrow y) \rightsquigarrow z \leq ((y \rightsquigarrow x) \rightsquigarrow z) \rightsquigarrow z$.

We can state the following theorems:

Theorem. *The variety of basic pseudo hoops termwise equivalent to the variety of pseudo BL-algebras (thus it is lattice ordered set and the prelinearity identity $x \to y \vee y \to= x = 1 = x \rightsquigarrow y \vee y \rightsquigarrow x$ holds).*

Remark. Let M be a basic pseudo hoop. The value of an element $g \in M \setminus \{1\}$ is any filter V of M that is maximal with respect to the property $g \notin V$. A value V exists (more precisely, any filter F with $g \notin V$ is contained in some value V) and it is prime. Let $\mathrm{Val}(g)$ be the set of all values of $g < 1$. The filter V^* generated by a value V of g and by the element g is said to be the cover of V.

We say that a basic pseudo hoop M is *normal-valued* if every value V of M is normal in its cover v^*. According to Wolfenstein, an ℓ-group G is normal-valued iff every $a, b \in G^-$ satisfy

$$b^2 \cdot a^2 \leq a \cdot b. \tag{1}$$

Hence, every cancellative pseudo hoop M is normal-valued iff (1) holds for any $a, b \in M$. Moreover, every representable pseudo hoop satisfies (1). Similarly, a pseudo MV-algebra is normal-valued iff (1) holds. Generally we prove:

Theorem. *Let M be a basic pseudo hoop. Then M is normal-valued if and only if (1) holds and*

i) $((x \to y)^n \rightsquigarrow y)^2 \leq (x \rightsquigarrow y)^{2n} \to y$ *for any* $n \in \mathbb{N}$,

ii) $((x \rightsquigarrow y)^n \to y)^2 \leq (x \to y)^{2n} \rightsquigarrow y$ *for any* $n \in \mathbb{N}$,

holds.

In what follows, we present a variety of basic pseudo hoops satisfying a single equation such that the inequality (1) is a necessary and sufficient condition for M to be normal-valued.

We say that a bounded pseudo hoop M with a minimal element 0 is *good* if $(x \to 0) \rightsquigarrow 0 = (x \rightsquigarrow 0) \to 0$. For example, every pseudo MV-algebra is good as well as every representable pseudo hoop is good. Now we present a stronger equality:

$$(x \to y) \rightsquigarrow y = (x \rightsquigarrow y) \to y. \tag{2}$$

For example, every negative cone of an ℓ-group and the negative interval of an ℓ-group with strong unit satisfies (2). We can state:

Theorem. *Let M be a basic pseudo hoop satisfying (2). Then M is normal-valued if and only if* (1) *holds.*

References

[1] M.R. Darnel: *Theory of Lattice-Ordered Groups*, Marcel Dekker, Inc., New York, 1995.

[2] A. Di Nola, G. Georgescu, and A. Iorgulescu: *Pseudo-BL algebras I,*, Multiple Val. Logic **8** (2002), 673–714.

[3] A. Di Nola, G. Georgescu, and A. Iorgulescu: *Pseudo-BL algebras II,*, Multiple Val. Logic **8** (2002), 715–750.

[4] A. Dvurečenskij, *Pseudo MV-algebras are intervals in ℓ-groups*, J. Austral. Math. Soc. **70** (2002), 427–445.

[5] A. Dvurečenskij: *Aglianò-Montagna type decomposition of linear pseudo hoops and its applications*,J. Pure Appl. Algebra **211** (2007), 851–861.

[6] A. Dvurečenskij, R. Giuntini, and T. Kowalski,: *On the structure of pseudo BLalgebras and pseudo hoops in quantum logic*,Found. Phys. **40** (2010), 1519–1542. DOI:10.1007/s10701-009-9342-5

Author address:

Michal Botur
Department of Algebra and Geometry
Palacký University Olomouc, 17. listopadu 12
771 46 Olomouc, Czech Republic
e-mail: botur@inf.upol.cz

CLASSIFYING THE UNIFICATION PROBLEMS OF THE THEORY OF DISTRIBUTIVE LATTICES AND KLEENE ALGEBRAS

LEONARDO MANUEL CABRER AND SIMONE BOVA

Abstract. In this paper we present a procedure to determine if a unification problem in the equational theory of Distributive Lattices or Kleene algebras is unitary, finitary or nullary.

In [2], it was proven that the unification type of the theory of distributive lattices is nullary. In [3], it was proven that the decision problem of whether a the unification problem of the equational theory of for distributive lattices has a solution is decidable.

The nullarity of the unification problem for distributive lattices do not provide other information of the theory than saying that there is a unification problem with nullary type. In this paper we will provide a procedure to check whether a unification problem is unitary, finitary or nullary in the theory of distributive lattices, and we will prove that there are no infinitary unification problems. Given a unification problem U on the language of distributive lattice we will present first order conditions on the poset of evaluations into the lattice 0,1 that satisfy the equations of the given unification problem (, equivalently first order condition on the dual Priestley space of the finitely presented distributive lattice given by U) that completely characterize the unification type of the problem U.

In order to obtain similar results for the equational theory of Kleene algebra, our main tool will be the natural duality for Kleene algebras (see [1, Chp. 4]). We will first provide a characterization of the dual space of projective Kleene algebra. Using this characterization we will provide procedure to determine if a unification problem is unitary, finitary or nullary.

This paper is strongly based in the theory of algebraic unifiers develop in [2].

References

[1] D. M. Clark and B. A. Davey. *Natural Dualities for the Working Algebraist*, Cambridge University Press, 1998.

[2] S. Ghilardi, Unification through projectivity, *J. Logic Comput.* 7, 6, 733752 (1997).

[3] V. Sofronie-Stokkermans, On Unification for Bounded Distributive Lattices, *ACM Transactions on Computational Logic* 8, 2, Article 12 (2007).

BASIC ALGEBRAS AND THEIR APPLICATIONS

IVAN CHAJDA

Algebraic axiomatization of the classical propositional calculus is provided via Boolean algebras, i.e. bounded distributive lattices with complementation. However, not every reasoning may be described by means of the classical two-valued logic. In particular, two important cases can be mentioned. At first, it is the logic of quantum mechanics where the logical law of excluded middle fails. This caused that, contrary to the classical logic, the induced lattice need not be distributive although the negation is still a complementation. In 1940's, G. Birkhoff and J. von Neumann found out that the appropriate algebraic tool for axiomatizing this logic are orthomodular lattices.

Another important class of non-classical logics having numerous applications are many-valued logics. In particular, the Łukasiewicz many-valued logic is of interest due to the fact that it is a fuzzy logic which is applied in numerous technical devices (e.g. fuzzy regulators, fuzzy control systems etc.). It was recognized by C. C. Chang in the 1950's that the appropriate tool for axiomatizing many-valued Łukasiewicz logics are the so-called MV-algebras.

The mentioned examples of non-classical logics motivated us to find a basic tool which is common to all of them. As we have already mentioned, the logic of quantum mechanics is axiomatized by a lattice which is complemented but not necessarily distributive. It turns out that MV-algebras induce bounded lattices which are distributive and the logical connective negation is realized by an antitone involution which need not be a complementation. Hence, it is natural to search for a common algebraic structure among non necessarily distributive bounded lattices equipped with an antitone involution.

Having a bounded lattice $\mathcal{L} = (L; \vee, \wedge, 0, 1)$, for every element $a \in L$ the interval $[a, 1]$ is called a **section**. By an **antitone involution** on a lattice $\mathcal{L}$ is meant a mapping f of L into itself such that $f(f(a)) = a$ for each $a \in L$ and for $x, y \in L$ with $x \leq y$ we have $f(y) \leq f(x)$. We say that $\mathcal{L}$ is endowed by **section antitone involutions** if for every $a \in L$ there exists an antitone involution on $[a, 1]$. Hence, there exist so many antitone involutions as many elements the lattice $\mathcal{L}$ has. Due to this, we will denote an antitone involution on the section $[a, 1]$ by a superscript ^{a}, i.e. for $x \in [a, 1]$ its map is denoted by x^a. The fact that $\mathcal{L}$ has section antitone involutions is expressed by the notation $\mathcal{L} = (L; \vee, \wedge, (^a)_{a \in L}, 0, 1)$.

It is easy to check that if $\mathcal{L} = (L; \vee, \wedge, ^{\perp}, 0, 1)$ is an orthomodular lattice then $\mathcal{L}$ has section antitone involutions where for each $a \in L$ we have $x^a = x^{\perp} \vee a$ (then, due to (OML), $x \wedge x^a = x \wedge (x^{\perp} \vee a) = a$ and, evidently, $x \vee x^a = x \vee x^{\perp} \vee a = 1$ thus it is in fact a complementation in $[a, 1]$).

Now we turn our attention to MV-algebras. Having an MV-algebra $\mathcal{A} = (A; \oplus, \neg, 0)$, we denote by $1 := \neg 0$ and define an order on the underlying set as follows

$$x \leq y \quad \text{if and only if} \quad \neg x \oplus y = 1.$$

Then $0 \leq x \leq 1$ for each $x \in A$ and the ordered set $(A; \leq)$ is in fact a bounded lattice where

$$x \vee y = \neg(\neg x \oplus y) \oplus y, \quad x \wedge y = \neg(\neg x \vee \neg y).$$

Moreover, the mapping $x \to \neg x$ is an antitone involution and the so-called **induced lattice** $\mathcal{L}(A) = (A; \vee, \wedge, \neg, 0, 1)$ is, moreover, distributive. Although the lattice operations $\vee$ and $\wedge$ are in fact term operations of the MV-algebra $\mathcal{A}$, we cannot express $\oplus$ as a term operation in the induced lattice $\mathcal{L}(A)$. However, for every $a \in A$ the mapping $x \to x^a = \neg x \oplus a$ is an antitone involution on the section $[a, 1]$ (which need not be a complementation), in particular, $\neg x = x^0$. Hence, the induced lattice $\mathcal{L}(A) = (A; \vee, \wedge, (^a)_{a \in A}, 0, 1)$ is a distributive lattice with section antitone involutions. The advantage of this setting is that now we can express the operations of $\mathcal{A}$ as term operations of $\mathcal{L}(A)$, namely

$$\neg x = x^0 \quad \text{and} \quad x \oplus y = (x^0 \vee y)^y.$$

Comparing the induced lattice of an MV-algebra with an orthomodular lattice, we can see that both of them are lattices with section antitone involutions which differ in the properties that these involutions are complementations but the lattice need not be distributive in the first case but the lattice is distributive and the involutions need not be complementations in the second case. We conclude that the common base of algebraic axiomatization of the logic of quantum mechanics as well as of many-valued Łukasiewicz logic are bounded lattices with section antitone involutions. Although these lattices are very useful tools for computations, a formal disadvantage is that they are not of the same type since the number of unary operations (section antitone involutions) depends on the number of elements of the lattice. To avoid this difficulty, let us define the following.

By a **basic algebra** is meant an algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ of type (2,1,0) satisfying the following identities

(BA1) $x \oplus 0 = x$
(BA2) $\neg\neg x = x$
(BA3) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$
(BA4) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg 0.$

It is easy to show that every MV-algebra is a basic algebra.

A basic algebra $\mathcal{A}$ is **commutative** if $x \oplus y = y \oplus x$ for each $x, y \in A$.

Theorem 1. *Every finite commutative basic algebra is an MV-algebra.*

For infinite basic algebras it is not the case as it was proved by M. Botur:

Theorem 2. *There exists a commutative (even linearly ordered) infinite basic algebra which is not an MV-algebra.*

We can complete this relationship with a recent result by M. Kolařík:

Theorem 3. *A basic algebra is an MV-algebra if and only if it is associative (i.e. $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ for each $x, y, z \in A$).*

However, we are interested in the question how are the basic algebras related to bounded lattices with section antitone involutions. The answer is as follows:

Theorem 4. *(1) Let $\mathcal{L} = (L; \vee, \wedge, (^a)_{a \in A}, 0, 1)$ be a bounded lattice with section antitone involutions. If we define*

$$x \oplus y := (x^0 \vee y)^y \quad \text{and} \quad \neg x := x^0$$

then $\mathcal{A}(L) = (L; \oplus, \neg, 0)$ is a basic algebra. We have $x \vee y = \neg(\neg x \oplus y) \oplus y$, $x \wedge y = \neg(\neg x \vee \neg y)$ and $x^a = \neg x \oplus a$ for $x \in [a, 1]$.

(2) Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra and put

$$x \vee y := \neg(\neg x \oplus y) \oplus y \quad \text{and} \quad x \wedge y := \neg(\neg x \vee \neg y).$$

Let $x^a = \neg x \oplus a$ and $1 := \neg 0$. Then $\mathcal{L}(A) = (A; \vee, \wedge, (^a)_{a \in A}, 0, 1)$ is a bounded lattice with section antitone involutions where the lattice order is given by $x \leq y$ iff $\neg x \oplus y = 1$ and $\neg x = x^0$, $x \oplus y = (x^0 \vee y)^y$.

(3) The correspondence between bounded lattices with section antitone involutions and basic algebras thus established is one-to-one, i.e. $\mathcal{A}(\mathcal{L}(A)) = \mathcal{A}$ and $\mathcal{L}(\mathcal{A}(L)) = \mathcal{L}$.

As mentioned, every orthomodular lattice $\mathcal{L} = (L; \vee, \wedge, ^\perp, 0, 1)$ is in fact a bounded lattice with section antitone involutions and, due to Theorem 4, it can be organized into a basic algebra. Since $\neg x = x^\perp$ and $x^a = x^\perp \vee a = \neg x \vee a$ for $x \in [a, 1]$, we easily derive by Theorem 4

$$x \oplus y = (x^0 \vee y)^y = \neg(x^0 \vee y) \vee y = (x^\perp \vee y)^\perp \vee y = (x \wedge y^\perp) \vee y.$$

Due to the orthomodular law, it yields the quasi-identity

$$x \leq y \Rightarrow y \oplus x = y$$

which can be converted into the identity

(OMI) $$y \oplus (x \wedge y) = y.$$

It is an easy calculation to show the converse.

Theorem 5. *Orthomodular lattices form a subvariety of the variety of basic algebras $\mathcal{B}$ determined by the identity (OMI).*

The concept of effect algebra was introduced by D. J. Foulis and M. K. Bennett in the sake to axiomatize unsharp events in quantum mechanics.

By an **effect algebra** is meant a structure $\mathcal{E} = (E; +, 0, 1)$ such that 0 and 1 are elements of E and $+$ is a partial binary operation on E satisfying the following conditions:

(E1) $x + y$ is defined iff $y + x$ is defined and then $x + y = y + x$

(E2) $x + (y + z)$ is defined iff $(x + y) + z$ is defined and then $x + (y + z) = (x + y) + z$

(E3) for each $x \in E$ there is a unique $x' \in E$ such that $x + x'$ is defined and $x + x' = 1$

(E4) $x + 1$ is defined if and only if $x = 0$.

Having an effect algebra $\mathcal{E} = (E; +, 0, 1)$, we can define $x \leq y$ if and only if $y = x + z$ for some $z \in E$. It is elementary to prove that $\leq$ is a partial order on E such that $0 \leq x \leq 1$ for every $x \in E$. In the case when $(E; \leq)$ becomes a lattice, $\mathcal{E}$ is called a **lattice effect algebra**.

For our sake, it is important that every lattice effect algebra is a bounded lattice with section antitone involutions where for $x \in [a, 1]$ we have

$$x^a = x' + a.$$

The just revealed relationship can be formally described as follows:

Theorem 6. *Let $\mathcal{E} = (E; +, 0, 1)$ be a lattice effect algebra. Define $x \oplus y = (x \wedge y') + y$ and $\neg x = x'$. Then $\mathcal{B}(E) = (E; \oplus, \neg, 0)$ is a basic algebra.*

Since $x \wedge y' \leq y'$ in each case, it yields that the operation $\oplus$ as defined above is everywhere defined. One can easily recognized that

$$x + y = x \oplus y \quad \text{whenever} \quad x \leq \neg y.$$

The previous result gets rise a question what basic algebras are those derived from lattice effect algebras.

Theorem 7. *Let $\mathcal{E} = (E; +, 0, 1)$ be a lattice effect algebra. Then the derived basic algebra $\mathcal{B}(E)$ satisfies the quasiidentity*

$$\text{(BEA)} \quad x \leq \neg y \quad \textit{and} \quad x \oplus y \leq \neg z \quad \Rightarrow \quad x \oplus (z \oplus y) = (x \oplus y) \oplus z.$$

The converse is also valid, i.e.

Theorem 8. *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra satisfying the quasi-identity (BEA). Define $1 = \neg 0$ and $x + y$ is defined if $x \leq y'$ and then $x + y = x \oplus y$. Then $\mathcal{E}(A) = (A; +, 0, 1)$ is a lattice effect algebra.*

Department of Algebra and Geometry, Faculty of Sciences, Palacký University Olomouc, 17. listopadu 12, 771 46 Olomouc, Czech Republic

E-mail address: chajda@inf.upol.cz

Properties of relatively pseudocomplemented directoids

I. Chajda, **M. Kolařík**, F. Švrček

Faculty of Science, Palacký University Olomouc
17. listopadu 12, 771 46 Olomouc, Czech Republic

chajda@inf.upol.cz, **kolarik@inf.upol.cz**, svrcekf@inf.upol.cz

1 Introduction

Relatively pseudocomplemented lattices and semilattices play an important role in the investigation of intuitionistic logics and their reducts. They were intensively studied by G.T. Jones [6]. The operation of relative pseudocomplementation serves as an algebraic counterpart of the intuitionistic connective implication. We can refer to the compendium [4] where essential results on relatively pseudocomplemented semilattices and lattices are gathered.

To investigate some more general algebraic systems connected with non-classical logic (as e.g. BCK-algebras, BCI-algebras, etc.), we often study ordered sets which are not necessarily semilattices. However, a bit weaker structure was introduced by J. Ježek and R. Quackenbush [5] as follows.

By a **directoid** (a commutative directoid in [5]) we mean a groupoid $\mathcal{D} = (D; \sqcap)$ satisfying the identities

(D1) $x \sqcap x = x$ (idempotency),

(D2) $x \sqcap y = y \sqcap x$ (commutativity),

(D3) $x \sqcap ((x \sqcap y) \sqcap z) = (x \sqcap y) \sqcap z$ (weak associativity).

Of course, every $\wedge$-semilattice is a directoid but not vice versa. It can be shown that every directoid $\mathcal{D} = (D; \sqcap)$ can be converted into an ordered set $(D; \leq)$ via

$$x \leq y \quad \text{if and only if} \quad x \sqcap y = x$$

and every downward directed ordered set $(D; \leq)$ can be organized into a directoid taking

$$x \sqcap y = y \sqcap x \in L(x, y) = \{z \in D;\, z \leq x \text{ and } z \leq y\}$$

arbitrarily for non-comparable elements x, y and

$$x \sqcap y = y \sqcap x = x \text{ when } x \leq y,$$

see [5] or [4] for details. It is worth noticing that the operation $\sqcap$ is not isotone in general, in fact we have

$$x \leq y \quad \Rightarrow \quad x \sqcap z \leq y \sqcap z \text{ for all } x, y, z \in D$$

if and only if $(D; \sqcap)$ is an $\wedge$-semilattice where $\sqcap$ coincides with the infimum $\wedge$ (with respect to $\leq$).

A natural question arises if a directoid with a least element 0 can be equipped with pseudocomplementation. This task was investigated in [2] where an axiom system for pseudocomplementation on directoids was presented. Another problem is how to define and characterize relatively pseudocomplemented directoids. As mentioned in [3], x is a greatest element satisfying $a \wedge x \leq b$ if and only if x is a greatest element satisfying $a \wedge x = a \wedge b$ in any relatively pseudocomplemented semilattice. However, if for some a, b of an $\wedge$-semilattice $\mathcal{S}$ the relative pseudocomplement does not exist then the conditions need not coincide, see the following

Example 1. Let $\mathcal{S} = (S, \wedge)$ be an $\wedge$-semilattice where $S = \{0, a, b, c, 1\}$ whose diagram is depicted in Fig. 1.

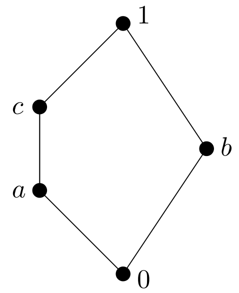


Fig. 1

Then there exists no pseudocomplement of c with respect to a since $c \wedge b = 0 \leq a$, $c \wedge a = a$ but there is no greatest $x \in S$ with $c \wedge x \leq a$. On the other hand, there exists a greatest $x \in S$ with $c \wedge x = c \wedge a$, namely $x = a$.

$\diamondsuit$

Of course, the aforementioned conditions are not equivalent in directoids either, see [3] for details. Hence we use the following definition of relative pseudocomplementation in directoids which was introduced in [3].

Definition. *Let $\mathcal{D} = (D; \sqcap)$ be a directoid and $a, b \in D$. An element x is called a **relative pseudocomplement of** a **with respect to** b if it is a greatest element of D such that $a \sqcap x = a \sqcap b$. It is denoted by $a * b$. A directoid $\mathcal{D}$ is **relatively pseudocomplemented** if there exists $a * b$ for every $a, b \in D$.*

The fact that $\mathcal{D}$ is a relatively pseudocomplemented directoid will be expressed by the notation $\mathcal{D} = (D; \sqcap, *)$. As shown in [3], every relatively pseudocomplemented directoid has a greatest element (which is denoted by 1).

Let us mention that if the above definition of relative pseudocomplementation is used for $\wedge$-semilattices, what we get is nothing else than the definition of the so-called sectional pseudocomplementation as defined in [1].

As already noticed, in a relatively pseudocomplemented semilattice our new definition of $a*b$ coincides with the usual relative pseudocomplement of a with respect to b. Hence, every relatively pseudocomplemented semilattice belongs to the class of relatively pseudocomplemented directoids.

Example 2. For the semilattice $\mathcal{S}$ from Example 1, the operation table for $*$ (defined by the above Definition) is

$*$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	c	c	1	c	1
c	b	a	b	1	1
1	0	a	b	c	1

Hence, if $\mathcal{S}$ is considered as a directoid then it is relatively pseudocomplemented. $\diamondsuit$

2 Axiom system

It was shown in [3] that the class of relatively pseudocomplemented directoids forms a variety which is determined by additional four identities as follows.

Proposition. *(See Theorem 2 in [3].) Let $(D; \sqcap)$ be a directoid and let $*$ be a binary operation on D. Then $\mathcal{D} = (D; \sqcap, *)$ is a relatively pseudocomplemented directoid if and only if it satisfies the following identities:*

(S1) $x \sqcap (x * y) = x \sqcap y$,

(S2) $(x * y) \sqcap y = y$,

(S3) $x * y = x * (x \sqcap y)$,

(S4) $x * x = y * y$.

A natural question is if the axioms (S1) – (S4) or, more generally, the axioms (D1) – (D3), (S1) – (S4) are independent. An immediate reflexion shows that it is not the case. In fact, we can prove the following result.

Theorem 1. *Let $\mathcal{D} = (D; \sqcap, *)$ be an algebra with two binary operations. Then $\mathcal{D}$ is a relatively pseudocomplemented directoid if and only if it satisfies the identities (D2), (D3), (S1), (S2) and (S3). The identities (D2), (D3), (S1), (S2) and (S3) are independent.*

3 Relative pseudocomplement as a residuum

It is well-known that relatively pseudocomplemented $\wedge$-semilattices can be considered alternatively as residuated structures where the relationship between the operations $\wedge$ and $*$ is established by the so-called **adjointness property**:

$$a \wedge x \leq b \quad \text{if and only if} \quad x \leq a * b. \tag{AP}$$

As mentioned above, this cannot be translated to directoids since the operation $\sqcap$ is not isotone. A natural question is if also (AP) can be modified for directoids to characterize relative pseudocomplementation as a residual operation. By replacing $a \wedge x \leq b$ by $a \sqcap x = a \sqcap b$, we can easily infer

$$a \sqcap x = a \sqcap b \quad \Rightarrow \quad x \leq a * b. \tag{I}$$

Unfortunately, the converse implication fails for relatively pseudocomplemented directoids, see the following

Example 3. Let $\mathcal{D} = (D, \sqcap, *)$ be a relatively pseudocomplemented directoid whose Hasse diagram is depicted in Fig. 2 (one can easily enumerate the operation $*$).

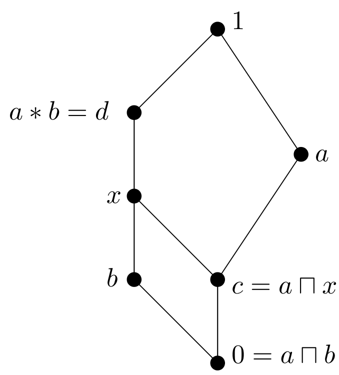


Fig. 2

Although $x \leq d = a * b$, we have $a \sqcap x = c \neq 0 = a \sqcap b$. $\diamondsuit$

So, the right hand side of implication (I) must be completed to reach the condition in the form of equivalence. Our solution follows.

Theorem 2. *Let $(D; \sqcap)$ be a directoid and let $*$ be a binary operation on D. Then $\mathcal{D} = (D; \sqcap, *)$ is a relatively pseudocomplemented directoid if and only if the following adjointness property holds:*

$$a \sqcap x = a \sqcap b \quad \textit{if and only if} \quad x \leq a * b \ \textit{ and } \ a \sqcap (a * b) = a \sqcap x. \tag{AD}$$

Although the condition (AD) is more complex than (AP), relatively pseudocomplemented directoids satisfy also a condition which is more similar to the adjointness property.

Theorem 3. *Let $\mathcal{D} = (D; \sqcap, *)$ be a relatively pseudocomplemented directoid. The following condition is satisfied in $\mathcal{D}$ for all $a, b, x \in D$:*

$$a \sqcap x \leq b \quad \textit{if and only if} \quad x \leq (a \sqcap x) * b. \tag{A}$$

4 Further

We get also two important congruence properties, namely congruence distributivity and 3-permutability valid in the variety $\mathcal{V}$ of relatively pseudocomplemented directoids. Then we show some basic results connected with subdirect irreducibility in $\mathcal{V}$. Finally, we show another way how to introduce pseudocomplementation on directoids via relative pseudocomplementation.

References

[1] Chajda I.: *An extension of relative pseudocomplementation to non-distributive lattices*, Acta Sci. Math. (Szeged) **69** (2003), 491–496.

[2] Chajda I.: *Pseudocomplemented directoids*, Comment. Math. Univ. Carol. **49** (2008), 533–539.

[3] Chajda I.: *Relatively pseudocomplemented directoids*, Comment. Math. Univ. Carol. **50** (2009), 349–357.

[4] Chajda I., Halaš R., Kühr J.: *Semilattice Structures*, Heldermann Verlag (Lemgo, Germany), 2007, 228pp, ISBN 978-3-88538-230-0.

[5] Ježek J., Quackenbush R.: *Directoids: Algebraic models of up-directed sets*, Algebra Universalis **27** (1990), 49–69.

[6] Jones G.T.: *Pseudo-complemented semi-lattices*, Ph.D. Thesis, Univ. of California, Los Angeles, 1972.

A final Vietoris coalgebra beyond compact spaces and a generalized Jónsson-Tarski duality

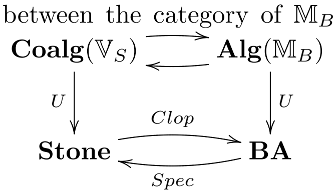
Liang-Ting Chen

School of Computer Science, University of Birmingham, United Kingdom
L.Chen@cs.bham.ac.uk

Abstract. "Always topologize!"[1]

1 Introduction

In a coalgebraic perspective, Kripke frames can be viewed as $\mathcal{P}$-coalgebras where $\mathcal{P}$ is the covariant powerset functor. Likewise, descriptive frames are $\mathbb{V}_S$-coalgebras over Stone spaces as shown in [4], and the Jónsson-Tarski duality is a duality between the category of $\mathbb{M}_B$-algebras and the category of $\mathbb{V}_S$-coalgebras ,i.e.



where $\mathbb{V}_S$ is the Vietoris construction and $\mathbb{M}_B$ is the modal algebra construction over Boolean algebras. The duality also holds for positive normal modal logic, i.e. the duality between modal algebras over distributive lattice and the Vietoris topology over Priestley spaces [5].

Following these dualities, we generalise it to a duality between **Coalg**($\mathbb{V}$) and **Alg**($\mathbb{M}_F$) over stably locally compact spaces where $\mathbb{V}$ is the Vietoris topology construction taking compact lenses (i.e. the topological counterpart of semi-fitted sublocales) and $\mathbb{M}_F$ is the modal algebra construction over frames. These construction can be found in [3].

As for the final $\mathbb{V}$-coalgebra, we start from the algebraic view instead of the coalgebraic view and obtain an initial $\mathbb{M}_F$-algebra concretely via the initial sequence. The calculation relies on the coherence preservation of $\mathbb{M}_F$.By Stone duality and the natural isomorphism between $\mathbb{V}\,\mathrm{Pt}$ and $\mathrm{Pt}\,\mathbb{M}_F$ for stably locally compact spaces, we find the final $\mathbb{V}$-coalgebra from the initial $\mathbb{M}_F$-algebra. We further generalise results to coherent Vietoris polynomial functors.

This duality enables us to study modal logic in a coalgebraic and topological perspective only with a very mild condition. It provides a duality for most of continuous multi-valued functions, topological transition systems e.g. the denotational semantics of nondeterministic systems, topological automata etc.

2 Background and Definitions

Most of the definitions can be found in [2] and [3], but we use slightly different symbols. Here we only briefly define notions used frequently.

Given a frame A, the *modal algebra construction over frames* $\mathbb{M}_F : \mathbf{Frm} \to \mathbf{Frm}$ is an endofunctor mapping A to a free frame generated by tokens $\Diamond a$ and $\Box a$, $a \in A$, subject to the following relations: (a) $\Box$ and $\Diamond$ preserves directed joins , (b) $\Box$ preserves finitary meets, (c) $\Diamond$ preserves finitary joins , (d) $\Diamond(a \wedge b) \geq \Diamond a \wedge \Box b$, and (e) $\Box(a \vee b) \leq \Box a \vee \Diamond b$; and mapping a morphism $f : A \to B$ to $\mathbb{M}_F f$ defined on the generating set by $\Diamond a \mapsto \Diamond b$ and $\Box a \mapsto \Box b$. $\mathbb{M}_D : \mathbf{DLat} \to \mathbf{DLat}$ is defined similarly without relation (a).

A frame A is (a) *locally compact* if it is a domain; (b) *stably locally compact* if it is locally compact and $a \ll b_i$ for $i = 1, 2$ implies $a \ll b_1 \wedge b_2$ where $\ll$ is the way-below relation; (c) *coherent* if it is isomorphic to the ideal completion of the *distributive lattice*[1] of its compact elements[2], i.e. $\mathrm{Idl}(\mathcal{K}A)$. For coherent frames, not every frame homomorphism $f : A \to B$ maps compact elements to compact elements, so we say f is *coherent* if f maps $\mathcal{K}A$ to $\mathcal{K}B$. The category **CohFrm** of coherent frames consists of coherent frames and coherent maps, and it is equivalent to the category **DLat** of distributive lattices.

The category **Loc** of *locales* is defined as the dual of **Frm**. We use the same definitions to call locales, e.g. coherent locale, locally compact locale etc. A locale morphism $f : A \to B$ comes from a frame homomorphism $f^* : B \to A$, so we say a locale morphism $f : A \to B$ is *coherent* if its dual $f^* : B \to A$ is coherent.

Note that (a) [2, VII.4.3] a (stably) locally compact locale is spatial , (b) [2, II.2.11] the forgetful functor from **Frm** to **DLat** has a left adjoint Idl; (c) [2, VII.4.6] a coherent locale is stably locally compact.

3 Outline of Results

3.1 An initial $\mathbb{M}_F$-algebra

Firstly, we are going to show that the initial $\mathbb{M}_F$-algebra exists. This fact relies on two key results: $\mathbb{M}_F$ preserves coherence, and the category of coherent frames is equivalent to the category of distributive lattices.

Theorem 1. *$\mathbb{M}_F$ preserves coherence, i.e.* $\mathbb{M}_F(\mathrm{Idl}A) \cong \mathrm{Idl}(\mathbb{M}_D A)$ *and* $\mathbb{M}_F f$ *is coherent if f is coherent.*

Note that to compute the F-initial algebra, we start from the initial object and apply F repeatedly to it. For $\mathbb{M}_F$, we notice that $\mathbb{M}_F^i 2$ are coherent for all i. Therefore we compute the initial $\mathbb{M}_D$-algebra instead.

Theorem 2. *$\mathbb{M}_D$ is finitary, i.e. $\mathbb{M}_D$ preserves filtered colimits.*

As a corollary, the initial sequence of $\mathbb{M}_D$ stabilises at ω, i.e. $\mathbb{M}_D$ has the least fixed point $\mathbb{M}_D^\omega 2 = \varinjlim_{i<\omega} \mathbb{M}_D^i 2$. In particular, $\mathbb{M}_D^\omega 2$ is the union $\bigcup_{i<\omega} \mathbb{M}_D^i 2$.

Lemma 1. *The unique morphism from* **2** *to any coherent frame is coherent.*

[1] Following Johnstone's convention, a distributive lattice is *bounded*.

[2] An element $a \in A$ is compact if $a \ll a$.

By applying the left adjoint Idl : **DLat** → **Frm** to the initial sequence of $\mathbb{M}_D$, we obtain a sequence isomorphic to the initial sequence of $\mathbb{M}_F$ as it preserves colimits.

Note that $\mathbb{M}_F\mathrm{Idl}(\mathbb{M}_D^\omega 2) \cong \mathrm{Idl}(\mathbb{M}_D\mathbb{M}_D^\omega 2) \cong \mathrm{Idl}(\mathbb{M}_D^\omega 2)$ by coherence preservation of $\mathbb{M}_F$ and the isomorphism between $\mathbb{M}_D^{\omega+1}2$ and $\mathbb{M}_D^\omega 2$. Hence we have:

Theorem 3. *The initial sequence of $\mathbb{M}_F$ stabilises at ω.*

Let α denote the isomorphism from $\mathbb{M}_F^{\omega+1}$ to $\mathbb{M}_F^\omega$.

3.2 A generalised Jónsson-Tarski duality

According to [3], given a spatial locale A, which comes from a topological space X, we may not have enough points in $\mathbb{M}_F A$ as well as sublocales of $\mathbb{M}_F A$. The necessary condition to preserve spatiality is stable local compactness.

Lemma 2 ([3, 1.6][3, 3.9]). *Let A be a stably locally compact locale. Then (a) $\mathbb{M}_F A$ is locally compact (and spatial); (b) any compact semi-fitted*[3] *sublocale of A is spatial.*

Since a stably locally compact locale is spatial, we call X a *stably locally compact space* if ΩX is stably locally compact.

Corollary 1 ([3, 3.10]). *If A is stably locally compact locale, then $\gamma_A : \mathbb{V}\,\mathrm{Pt}\,A \cong \mathrm{Pt}\,\mathbb{M}_F A$ where $\mathbb{V}$ is the Vietoris construction over spaces: $\mathbb{V}X$ is the space of compact lenses*[4] *K of X with the Vietoris topology generated by*

$$\Box U = \{K : K \subseteq U\} \text{ and } \Diamond U = \{K : K \cap U \neq \emptyset\}$$

for any open set U in X.

Provided above facts and that the category of sober spaces is dual to the category of spatial frames, we can conclude:

Corollary 2. *The category of $\mathbb{V}$-coalgebras is dual to the category of $\mathbb{M}_F$-algebras for stably locally compact spaces and frames.*

3.3 A final $\mathbb{V}$-coalgebra over stably locally compact spaces

The initial $\mathbb{M}_F$-algebra over frames is also the initial $\mathbb{M}_F$-algebra over stably locally compact frames since the initial object is stably locally compact and every object in the initial sequence is stably locally compact by construction. Following

[3] A sublocale is *semi-fitted* if it is an intersection of arbitrary many open sublocales and a closed sublocale.

[4] A lens is the topological counterpart of semi-fitted sublocale. Thus, a set S is a *lens* if it is an intersection of open sets and a closed set.

this, we can easily obtain the final $\mathbb{V}$-coalgebra for stably locally compact spaces from the $\mathbb{M}_F$-initial algebra as follows:

$$\begin{array}{ccc} \mathrm{Pt}\,\mathbb{M}_F^{\omega}2 & \xrightarrow{\mathrm{Pt}\,\alpha} & \mathrm{Pt}\,\mathbb{M}_F\mathbb{M}_F^{\omega}2 \\ & \searrow_{\gamma\circ(\mathrm{Pt}\,\alpha)} & \downarrow \gamma \\ & & \mathbb{V}(\mathrm{Pt}\,\mathbb{M}_F^{\omega}2) \end{array}$$

where $\gamma \circ (\mathrm{Pt}\,\alpha)$ is the desired final $\mathbb{V}$-coalgebra. It is also called the *canonical model* in modal logic.

3.4 The coherent Vietoris polynomial functors

The coherent Vietoris polynomial functors are defined in the following way:

$$T ::= \mathbb{I} \mid \mathbb{K}_C \mid T + T \mid T \times T \mid \mathbb{M}_F T$$

where $\mathbb{I}$ is the identity functor and $\mathbb{K}_C$ is the constant functor ranging over coherent frames. From the viewpoint of Lawvere theory, we can show that a Vietoris polynomial functor of distributive lattices has the least fixed point, and therefore it is easy to generalise above results of $\mathbb{V}$ to coherent Vietoris polynomial functors.

To generalise it, we apply the same technique: (a) we start from distributive lattices; (b) compute its initial algebra; (c) apply Idl to find the corresponding initial algebra for frames; (d) find a duality between algebras and coalgebras; (e) and apply Pt to obtain the final coalgebra. Since the left adjoint Idl preserves colimits and $\mathbb{V}$ preserves coherences, the only thing we have to check is the product of polynomial functors. However,

Lemma 3. $\mathrm{Idl}(A \times B) \cong \mathrm{Idl}A \times \mathrm{Idl}B$ *is natural in* A *and* B*.*

Secondly, products and coproducts of stably locally compact frames can be identified as coproducts and products of spaces respectively from the duality. It shows immediately the duality between T-algebras and L-coalgebras where T is a coherent Vietoris polynomial functor and L is the dual to T.

References

1. M. H. Stone, The representation of Boolean algebras. *Bulletin of the American Mathematical Society*, 44(12):807816, 1938.
2. P. T. Johnstone, *Stone spaces.* ,Cambridge Studies in Advanced Mathematics, no 3, Cambridge University Press 1982.
3. P. T. Johnstone, Vietoris Locales and Localic Semilattices. *Continuous lattices and their Applications* (R.-E. Hoffmann, ed.), Pure and Applied Mathematics, no.101, Marcel Dekker, 1985, pp. 155–18.
4. C. Kupke and A. Kurz and Y. Venema, Stone coalgeras. *Theoret. Comput. Sci.*, 327, 2004.
5. N. Bezhanishvili and A. Kurz, Free modal algebras: A coalgebraic perspective. In *CALCO 2007*, volume 4624 of *LNCS*, pp. 143-157. Springer-Verlag 2007.

Almost (MP)-based substructural logics

Petr Cintula, Institute of Computer Science, Czech Academy of Sciences, `cintula@cs.cas.cz`
Carles Noguera, Artificial Intelligence Research Institute, CSIC, `cnoguera@iiia.csic.es`

This paper is a contribution to the theory of substructural logics. We introduce the notions of (MP)-*based* and *almost* (MP)-*based* logics (w.r.t. a special set of formulae D), which leads to an alternative proof of the well-known forms of the local deduction theorems for prominent substructural logics (FL, FL_e, FL_{ew}, etc.). Roughly speaking, we decompose the proof of the local deduction theorem into the trivial part, which works almost classically, and the non-trivial part of determining with respect to which set (if any) the logic is almost (MP)-based. We can also show connection of (almost) (MP)-based condition and the proof by cases properties for generalized disjunctions and the description of (deductive) filters generated by some elements of a given algebra.

In order to provide as general theory as possible, i.e., to cover more logics than the usual Ono's definition of substructural logics [6] (i.e. axiomatic extensions of the logic of pointed residuated lattices) we propose a more general notion of substructural logic based on a very weak system lacking not only structural rules, but also associativity of multiplicative conjunction, and consider all its (even non-axiomatic) extensions, expansions (by new connectives), and well-behaved fragments thereof. This defines a wide family of logical systems containing pretty much all prominent substructural logics.

Our basic logic will be the non-associative variant for the Full Lambek Calculus [6, 7], here denoted as SL. Its language, $\mathcal{L}_{\mathrm{SL}}$, consists of residuated conjunction &, right $\searrow$ and left $\swarrow$ residual implications,[1] lattice conjunction $\wedge$ and disjunction $\vee$, and truth constants $\overline{0}, \overline{1}$. The logic SL is given by the following axiomatic system:

$$\vdash \varphi \searrow \varphi \qquad \varphi, \varphi \searrow \psi \vdash \psi \qquad \varphi \vdash (\varphi \searrow \psi) \searrow \psi \qquad \varphi \searrow \psi \vdash (\psi \searrow \chi) \searrow (\varphi \searrow \chi) \qquad \psi \searrow \chi \vdash (\varphi \searrow \psi) \searrow (\varphi \searrow \chi)$$

$$\vdash \varphi \searrow ((\psi \swarrow \varphi) \searrow \psi) \qquad \varphi \searrow (\psi \searrow \chi) \vdash \psi \searrow (\chi \swarrow \varphi) \qquad \psi \swarrow \varphi \vdash \varphi \searrow \psi$$

$$\vdash \varphi \wedge \psi \searrow \varphi \qquad \vdash \varphi \wedge \psi \searrow \psi \qquad \varphi, \psi \vdash \varphi \wedge \psi \qquad \vdash (\chi \searrow \varphi) \wedge (\chi \searrow \psi) \searrow (\chi \searrow \varphi \wedge \psi)$$

$$\vdash \varphi \searrow \varphi \vee \psi \qquad \vdash \psi \searrow \varphi \vee \psi \qquad \vdash (\varphi \searrow \chi) \wedge (\psi \searrow \chi) \searrow (\varphi \vee \psi \searrow \chi) \qquad \vdash (\chi \swarrow \varphi) \wedge (\chi \swarrow \psi) \searrow (\chi \swarrow \varphi \vee \psi)$$

$$\vdash \psi \searrow (\varphi \searrow \varphi \,\&\, \psi) \qquad \psi \searrow (\varphi \searrow \chi) \vdash \varphi \,\&\, \psi \searrow \chi$$

$$\vdash \overline{1} \qquad \vdash \overline{1} \searrow (\varphi \searrow \varphi) \qquad \vdash \varphi \searrow (\overline{1} \searrow \varphi)$$

Definition 1 *A logic* L *in a language* $\mathcal{L}$ *containing* $\searrow$ *is a* substructural logic *if*

- L *is the expansion of the* $\mathcal{L} \cap \mathcal{L}_{\mathrm{SL}}$*-fragment of* SL.
- *for each* n, $i < n$, *and each* n*-ary connective* $c \in \mathcal{L} \setminus \mathcal{L}_{\mathrm{SL}}$ *holds:*

$$\varphi \searrow \psi, p \searrow \varphi \vdash_{\mathrm{L}} c(\chi_1, \dots \chi_i, \varphi, \dots, \chi_n) \searrow c(\chi_1, \dots \chi_i, \psi, , \dots, \chi_n)$$

Note that the first condition implies that the second condition holds for connectives from $\mathcal{L}_{\mathrm{SL}}$. Any substructural logic is finitely equivalential [5], order algebraizable [9], weakly implicative [2, 3], and algebraizable in the sense of Blok and Pigozzi [1] in the presence of either $\vee$ or $\wedge$ in its language. The class of substructural logics as just defined contains:

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[1] In the literature on substructural logics, the implications are usually denoted by $/, \backslash$, whereas in the literature on non-commutative fuzzy logic are used the symbols $\to$ (with swapped arguments) and $\rightsquigarrow$. Here we use the signs $\swarrow, \searrow$ suggested by L.N. Stout, since besides indicating the side of conjoining the antecedent in the residuation law they also mark the direction of the implication from the antecedent to the succedent. In logics satisfying the exchange rule $\varphi \searrow (\psi \searrow \chi) \vdash \psi \searrow (\varphi \searrow \chi)$ both implications coincide and then we denote them by the usual symbol $\to$.

- substructural logics Ono's sense, including e.g. monoidal logic, uninorm logic, psBL, GBL, BL, Intuitionistic logic, (variants of) relevance logics, Łukasiewicz logic;
- expansion of the mentioned logics by additional connectives, e.g. (classical) modalities, exponentials in (variants of) Linear Logic and Baaz delta in fuzzy logics;
- fragments of the mentioned logics to languages containing implication, e.g. BCK, BCI, psBCK, BCC, hoop logics, etc.;
- non-associative logics recently developed by Buszkowski, Farulewski, Galatos, Ono, Halaš, Botur, etc.

What seems to be left aside is e.g. the logic $\mathrm{BCK}_\wedge$ of BCK-semilattices [8] (because it does not satisfy one of our axioms, namely: $(\chi \searrow \varphi) \wedge (\chi \searrow \psi) \searrow (\chi \searrow \varphi \wedge \psi)$). This observation calls for a comment on the postulative nature of our definition: when we claim that some logic is substructural and it has a connective $c \in \mathcal{L}_{\mathrm{SL}}$ we *postulate* how this connective should behave. Thus $\mathrm{BCK}_\wedge$ in the language $\{\searrow, \wedge\}$ is not a substructural logic ($\wedge$ does not behave as it should) but (!) if we would formulate $\mathrm{BCK}_\wedge$ in the language $\{\searrow, \overline{\wedge}\}$ it would indeed be a substructural logic (because then the only SL connective present in its language, implication, behaves as it should).

Definition 2 ((MP)-based substructural logic) *A substructural logic is* (MP)-based *if it has a presentation where* (MP) *is the only deduction rule.*

In substructural logics *with* & *and* $\overline{1}$ *in the language* we can introduce the following notation: $\varphi^0 = \{\overline{1}\}$, $\varphi^1 = \{\varphi\}$, $\varphi^{n+1} = \{\varphi \,\&\, \psi, \psi \,\&\, \varphi \mid \psi \in \varphi^n\}$ for every $n \geq 1$; note that if the logic is *associative*, we can identify φ^n just with any of its elements. The presence of $\overline{1}$ (or &) could be avoided at the price of more cumbersome formulations of the theorems (in case of & also we would also need to assume, implicationally expressed, associativity). The proof of the next theorem is almost trivial:

Theorem 3 (Implicational deduction theorem) *Let* L *be a substructural logics with* & *and* $\overline{1}$ *in the language. Then:* L *is* (MP)*-based iff* L *is finitary and for each set* $\Gamma \cup \{\varphi, \psi\}$ *of formulae the following holds:*

$$\Gamma, \varphi \vdash_{\mathrm{L}} \psi \qquad \textit{iff} \qquad \Gamma \vdash_{\mathrm{L}} \chi \searrow \psi \textit{ for some } n \geq 0 \textit{ and } \chi \in \varphi^n.$$

Clearly FL_{ew} is an example of (MP)-based logic, thus we have just shown that it enjoys this form of deduction theorem (and obviously the same holds for its axiomatic extensions). On the other hand, we can use the previous theorem to show that FL_e is not (MP)-based: indeed, $\varphi \vdash \varphi \wedge \overline{1}$ would entail provability of the theorem $\varphi^n \searrow \varphi \wedge \overline{1}$ for some n which can be refuted by a simple semantical counterexample.

Our next aim is to obtain some form of deduction theorem for FL_e, FL, and other substructural logics. To this end, we need to introduce three auxiliary notions. First, given a set S of formulae, we denote by $\prod S$ the smallest set of formulae containing $S \cup \{\overline{1}\}$ and closed under & (it can be seen as the free groupoid with unit generated by S). Second, we introduce a notion of (MP)-based companion for a given logic:

Definition 4 (Logic $\mathrm{L}^{(\mathrm{MP})}$) *Let* L *be a substructural logic. By* $\mathrm{L}^{(\mathrm{MP})}$ *we denote the logic axiomatized by all theorems of* L *and* modus ponens *as the only inference rule.*[2]

Note that L is (MP)-*based* iff $\mathrm{L} = \mathrm{L}^{(\mathrm{MP})}$ and that $\mathrm{L}^{(\mathrm{MP})}$ need not be a substructural logic in the sense of Definition 1. Not withstanding this, we are able to easily prove a deduction theorem for $\mathrm{L}^{(\mathrm{MP})}$ (we formulate already it in a stronger form needed for the next corollary), which in turn will allow to obtain a deduction theorem for L.

[2]Example: if L is the global variant of a normal modal logic, then $\mathrm{L}^{(\mathrm{MP})}$ is its local variant.

Lemma 5 *Let* L *be a substructural logic with* & *and* $\overline{1}$ *in its language. Then:*

$$\Gamma, S \vdash_{\mathrm{L(MP)}} \psi \qquad \textit{iff} \qquad \Gamma \vdash_{\mathrm{L(MP)}} \hat{\varphi} \searrow \psi \textit{ for some } \hat{\varphi} \in \prod S.$$

The third auxiliary, but crucial, notion is that of *almost* (MP)*-based substructural logic*:

Definition 6 (Almost (MP)**-based substructural logic)** *A substructural logic* L *is* almost (MP)-based *if there is a subset* $D(v, \vec{p}) \subseteq \mathrm{Th_L}(v)$ *(*v *is a variable and* $\vec{p}$ *are possibly present parameters) such that*

$$\Gamma \vdash_{\mathrm{L}} \varphi \qquad \textit{iff} \qquad \bigcup\{D(\psi, \vec{p}) \mid \psi \in \Gamma\} \vdash_{\mathrm{L(MP)}} \varphi.$$

Notice that each (MP)-based logic is almost (MP)-based (with $D = \{v\}$) and that without loss of generality we can assume that $v \in D$ (unless explicitly said otherwise). Also notice that any axiomatic extension of an almost (MP)-based logic is almost (MP)-based too. Finally, note that each almost (MP)-based logic can be axiomatized with (MP) as the only *non-unary* rule, the question whether the converse is true as well seems to be open. The previous lemma allows us to straightforwardly extend the scope of the implicational deduction theorem to *almost* MP-based logics.

Corollary 7 (Deduction theorem and almost (MP)**-based substructural logics)** *Let* L *be a substructural logic with* & *and* $\overline{1}$ *in the language and* $D(v, \vec{p}) \subseteq \mathrm{Th_L}(v)$*. Then:* L *is* almost (MP)-based *w.r.t. the set* $D(v, \vec{p})$ *if, and only if, for each set* $\Gamma \cup \{\varphi, \psi\}$ *of formulae holds:*

$$\Gamma, \varphi \vdash_{\mathrm{L}} \psi \qquad \textit{iff} \qquad \Gamma \vdash_{\mathrm{L}} \hat{\varphi} \searrow \psi \textit{ for some } \hat{\varphi} \in \prod D(\varphi, \vec{p}).$$

Thus the proof of the deduction theorem in a given logic was rather effortlessly reduced to the proof that the logic is almost (MP)-based. Recall the footnote 2 and observe that the well-known connection of global and local variants of modal logic K can, in our terminology, be formulated as 'global K is almost (MP)-based with the $D(v) = \{v, \Box v, \Box\Box v, \dots\}$', which immediately give us the deduction theorem of K. We show that the logics FL_e and FL (and so all their axiomatic extensions) are almost (MP)-based and determine the corresponding set D. As we can see, even the proof of the most complicated case is rather simple:

Theorem 8

- *The logic* FL_e *is almost* (MP)*-based with the set* $D_{\mathrm{FL}_e} = \{v \wedge \overline{1}\}$.
- *The logic* FL *is almost* (MP)*-based with the following set:*

$$D_{\mathrm{FL}} = \{\gamma(v) \mid \gamma(v) \textit{ an iterated conjugate}\}.$$

Proof: We know that FL can be axiomatized by (MP) and rules (con_l) and (con_r): $\varphi \vdash \lambda_\alpha(\varphi)$ and $\varphi \vdash \rho_\alpha(\varphi)$.

Let us denote the set $\bigcup\{D(\chi, \vec{p}) \mid \chi \in \Gamma\}$ as $\hat{\Gamma}$. We show that for each ψ in the proof of $\Gamma \vdash_{\mathrm{FL}} \varphi$ we have $\hat{\Gamma} \vdash_{\mathrm{FL(MP)}} \gamma(\psi)$, for each iterated conjugate γ. The claim then follow from taking $\gamma = \lambda_{\overline{1}}$ and the trivial fact that $\vdash_{\mathrm{FL}} \varphi \wedge \overline{1} \searrow \varphi$.

If ψ is an axiom or an element of Γ, the claim is trivial. Assume that ψ was proved using the rule (con_l). Then $\psi = \lambda_\alpha(\chi)$ for some formula χ appearing the proof before ψ. The induction assumption will give us $\hat{\Gamma} \vdash_{\mathrm{FL(MP)}} \gamma(\chi)$ for each iterated conjugate γ. Thus, in particular, for each iterated conjugate γ' we have $\hat{\Gamma} \vdash_{\mathrm{FL(MP)}} \gamma'(\lambda_\alpha(\chi))$. The proof for ($\mathrm{con}_r$) is completely analogous.

Finally, assume that $\Gamma \vdash_{\mathrm{FL}} \chi$ and $\Gamma \vdash_{\mathrm{FL}} \chi \searrow \psi$. Thus, by the induction assumption, for each iterated conjugate γ: $\hat{\Gamma} \vdash_{\mathrm{FL(MP)}} \gamma(\chi)$ and $\hat{\Gamma} \vdash_{\mathrm{FL(MP)}} \gamma(\chi \searrow \psi)$. The fact that $\vdash_{\mathrm{FL}} \gamma(\varphi \searrow \psi) \searrow (\gamma(\varphi) \searrow \gamma(\psi))$ and *modus ponens* complete the proof.

Interestingly enough, these deductions theorems yield a connection with a variant of the classical proof by cases property. Recall that classical logic enjoys the following metarule:

$$\frac{\Gamma, \varphi \vdash \chi \qquad \Gamma, \psi \vdash \chi}{\Gamma, \varphi \vee \psi \vdash \chi}$$

We will see now how a similar property can be obtained for associative substructural logics with a more complex form of disjunction built from the sets $D(v, \vec{p})$ used to show that these logics are almost (MP)-based.

Theorem 9 *Let* L *be an* associative *substructural logic with* & *and* $\overline{1}$ *in the language such that* L *is* almost (MP)*-based w.r.t. the set* $D(v, \vec{p})$*. Then each set* $\Gamma \cup \{\varphi, \psi, \chi\}$ *of formulae holds:*

$$\frac{\Gamma, \varphi \vdash \chi \qquad\qquad \Gamma, \psi \vdash \chi}{\Gamma \cup \{\alpha \vee \beta \mid \alpha \in D(\varphi, \vec{p}), \beta \in D(\psi, \vec{p})\} \vdash \chi}$$

Corollary 10 *The following meta-rule holds in* FL*:*

$$\frac{\Gamma, \varphi \vdash \chi \qquad\qquad \Gamma, \psi \vdash \chi}{\Gamma \cup \{\gamma_1(\varphi) \vee \gamma_2(\psi) \mid \gamma_1(v), \gamma_1(v) \textit{ iterated conjugates}\} \vdash \chi}$$

The following meta-rule holds in FL_e*:*

$$\frac{\Gamma, \varphi \vdash \chi \qquad \Gamma, \psi \vdash \chi}{\Gamma, (\varphi \wedge \overline{1}) \vee (\psi \wedge \overline{1}) \vdash \chi}$$

The following meta-rule holds in FL_{ew}*:*

$$\frac{\Gamma, \varphi \vdash \chi \qquad \Gamma, \psi \vdash \chi}{\Gamma, \varphi \vee \psi \vdash \chi}$$

There is also a clear relation of almost (MP)-basedness and the description of the (deductive) filters generated by a set (which is exactly what the deduction theorem says for filters in the Lindebaum algebra (theories), taking in account that implication defined the order). However due to the lack of space we will not go into details here.

References

[1] W. J. Blok and D. L. Pigozzi. *Algebraizable Logics*, volume 396 of *Memoirs of the American Mathematical Society*. American Mathematical Society, Providence, RI, 1989.

[2] P. Cintula. Weakly implicative (fuzzy) logics I: Basic properties. *Archive for Mathematical Logic*, 45(6):673–704, 2006.

[3] P. Cintula and C. Noguera. Implicational (semilinear) logics I: Basic notions and hierarchy. *Archive for Mathematical Logic*, 49(4):417–446, 2010.

[4] P. Cintula and C. Noguera. A general framework for fuzzy logics. In P. Cintula, P. Hájek, and C. Noguera, editors, *Handbook of Mathematical Fuzzy Logic*. College Publications, 201x. To appear.

[5] Janusz Czelakowski. *Protoalgebraic Logics*, volume 10 of *Trends in Logic*. Kluwer, Dordercht, 2001.

[6] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono. *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*, volume 151 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, Amsterdam, 2007.

[7] N. Galatos and H. Ono. Cut elimination and strong separation for substructural logics: An algebraic approach. *Annals of Pure and Applied Logic*, 161(9):1097–1133, 2010.

[8] P. Idziak. Lattice operations in BCK-algebras. *Mathematica Japonica*, 29(6):839–846, 1982.

[9] James G. Raftery. Order algebraizable logics, Manuscript, 2009.

Free algebras via a functor on partial algebras

Dion Coumans and Sam van Gool

In this paper we introduce a new setting, based on partial algebras, for studying constructions of finitely generated free algebras. We will give sufficient conditions under which the finitely generated free algebras for a variety **V** may be described as the colimit of a chain of finite partial algebras obtained by repeated application of a functor. In particular, our method encompasses the construction of finitely generated free algebras for varieties of algebras for a functor as in [2], Heyting algebras as in [1] and S4 algebras as in [5].

Free algebras are of particular interest for varieties $\mathbf{V}_{\mathcal{L}}$ arising from algebraizable logics $\mathcal{L}$. A thorough understanding of the finitely generated free algebras of such a variety $\mathbf{V}_{\mathcal{L}}$ can yield powerful results, for example related to questions about term complexity, decidability of logical equivalence, interpolation and normal forms, *i.e.*, problems in which one considers formulas whose variables are drawn from a finite set.

In case the class of algebras $\mathbf{V}_{\mathcal{L}}$ associated with the logic $\mathcal{L}$ is axiomatized by equations which are of rank 1 for an operation f,[1] the algebras for the logic can be represented as algebras for a functor $F_{\mathcal{L}}$ on the category of underlying algebra reducts without the operation f. This functor $F_{\mathcal{L}}$ enables a constructive description of the free $\mathbf{V}_{\mathcal{L}}$ algebras [2]. As many interesting logics are not axiomatized by rank 1 axioms, one would want to extend these existing techniques. However, as is shown in [6], non-rank 1 logics cannot be represented as algebras for a functor and therefore we cannot use the standard construction of free algebras in a straightforward way. Ghilardi pioneered the construction of free algebras for non-rank 1 varieties in [3]. Here he describes a method to incrementally build finitely generated free Heyting algebras by constructing a chain of distributive lattices, where, in each step, implications are freely added to the lattice, while keeping a specified set of implications which are already defined in the previous step. In a subsequent paper, Ghilardi extended these techniques to apply to modal logic [4], and used his algebraic and duality theoretic methods to derive normal forms for modal logics, notably S4.

Recently, this line of research has been picked up again. In [1] N. Bezhanishvili and Gehrke have re-analysed Ghilardi's incremental construction and have derived it by repeated application of a *functor*, based on the ideas of the coalgebraic approach to rank 1 logics and Birkhoff duality for finite distributive lattices. Shortly after, Ghilardi [5] gave a new construction of the free S4 algebra in the same spirit. However, the methods in [1] and [5] rely on specific properties

[1] An equation is of *rank* 1 *for an operation* f if every variable occurs under the scope of exactly one occurrence of f.

of Heyting algebras and S4 algebras respectively, and they do not directly apply in a general setting. Studying this work has led to the insight that the natural setting to consider is that of a functor on a category of partial algebras, together with a natural transformation from the identity to the functor.

We will now outline our *general functorial method* for constructing free algebras. Note that our focus here is on Boolean logics with one additional unary $\vee$-preserving connective $\Diamond$, and their associated algebras (modal algebras), but the general method applies in a much wider setting. However, assuming that operations are unary makes the notation less heavy, and assuming that $\Diamond$ is $\vee$-preserving will simplify matters when duality theoretic methods are illustrated in this context.

The notion of *rank* of a modal term is central in this paper and therefore we give a precise definition.

Definition. Let P be a set of variables. We denote the set of Boolean terms in P by $\mathbf{T_{BA}}(P)$. The sets $\mathbf{T}^n_{\mathbf{MA}}(P)$ of **modal terms in** P **of rank at most** n are defined inductively as follows.

$$\mathbf{T}^0_{\mathbf{MA}}(P) := \mathbf{T_{BA}}(P),$$
$$\mathbf{T}^{n+1}_{\mathbf{MA}}(P) := \mathbf{T_{BA}}(P \cup \{\Diamond t : t \in \mathbf{T}^n_{\mathbf{MA}}(P)\}).$$

The *(quasi-)equational class* $\mathbf{V}$ defined by a set of (quasi-)equations $\mathcal{E}$ is the class of algebras satisfying all (quasi-)equations in $\mathcal{E}$. A classical theorem of Birkhoff says that for every (quasi-)equational class $\mathbf{V}$ of algebras and set of variables P, the free $\mathbf{V}$ algebra over P, $F_{\mathbf{V}}(P)$, exists.

The notion of rank allows us to understand this free algebra in a layered manner as follows. For each $n \geq 0$, the (equivalence classes of) terms of rank at most n form a Boolean subalgebra B_n of $F_{\mathbf{V}}(P)$. Furthermore, for each n, the operator $\Diamond$ on $F_{\mathbf{V}}(P)$ yields a map $\Diamond_{n+1} \colon B_n \to B_{n+1}$. Hence, we have a chain of Boolean algebras with embeddings and operations between them:

$$B_0 \overset{\Diamond_1}{\rightrightarrows} B_1 \overset{\Diamond_2}{\rightrightarrows} B_2 \overset{\Diamond_3}{\rightrightarrows} \cdots$$

The Boolean reduct of $F_{\mathbf{V}}(P)$ is the colimit of the chain of Boolean algebras and embeddings and the operator $\Diamond$ is the unique extension of the functions $\Diamond_n$ to a function on $F_{\mathbf{V}}(P)$.

One aspect of the new perspective on this chain that we propose in this paper is the following. Instead of considering $\Diamond_{n+1}$ as a map $B_n \to B_{n+1}$, we propose to view it as an partial operator on B_{n+1} (which is only defined on elements in the subalgebra B_n). This leads to the notion of *partial modal algebra*, which we define to be a pair $(B, \Diamond^B)$, where $\Diamond^B$ is a partial diamond

$B \to B$, whose domain, $dom(\Diamond^B)$, is a Boolean subalgebra of B. A homomorphism from $(B, \Diamond^B)$ to $(C, \Diamond^C)$ is a Boolean algebra homomorphism $B \to C$ which maps $dom(\Diamond^B)$ into $dom(\Diamond^C)$, and preserves $\Diamond$ whenever it is defined.
The above chain may be described as a chain in the category of partial modal algebras

$$(B_1, \Diamond_1) \rightarrowtail (B_2, \Diamond_2) \rightarrowtail (B_3, \Diamond_3) \rightarrowtail \cdots,$$

and this will allow us to see the chain as produced by repeated application of a functor F.
The crucial point of our method is that we can prove that, in a fairly general setting, it is possible to obtain this approximating chain of $F_{\mathbf{V}}(P)$ by a uniform construction, using a notion of *free image-total functor* on a given category $\mathbf{pV}$ of partial algebras. The *total* algebras in $\mathbf{pV}$ form a full subcategory $\mathbf{V}$ of $\mathbf{pV}$. In the Theorem below, we give conditions on the functor so that repeated application of it yields the approximating chain of the free total $\mathbf{V}$ algebra over a given finite $\mathbf{pV}$ algebra. To obtain the approximating chain of the free $\mathbf{V}$ algebra over a given *set*, it then remains to describe the first $\mathbf{pV}$ algebra of the chain, which is often easy to do.
A set of quasi-equations $\mathcal{E}$ of rank at most 1 naturally gives rise to a free image-total functor $F_{\mathcal{E}}$ on the subcategory $\mathbf{pV}_{\mathcal{E}}$ of partial algebras satisfying the equations in $\mathcal{E}$.[2] This functor $F_{\mathcal{E}}$ is roughly defined as follows. Given a $\mathbf{pV}_{\mathcal{E}}$ algebra $(B, \Diamond^B)$, we want to extend it to a $\mathbf{pV}_{\mathcal{E}}$ algebra in which the operator is defined for all the elements of B. To do so, we will add formal elements $\blacklozenge b$ to B, for all $b \in B$, and turn the resulting set into a $\mathbf{pV}_{\mathcal{E}}$ algebra by taking an appropriate quotient. In this quotient, we force the newly defined operator $\blacklozenge$ to agree with the old $\Diamond^B$, whenever $\Diamond^B$ was defined. From this construction, we also get a natural transformation $\eta : 1 \to F_{\mathcal{E}}$, where η_B maps the $\mathbf{pV}_{\mathcal{E}}$ algebra $(B, \Diamond^B)$ into the $\mathbf{pV}_{\mathcal{E}}$ algebra $F_{\mathcal{E}}(B, \Diamond^B)$.
Given a particular finite $\mathbf{pV}_{\mathcal{E}}$ algebra $(B_0, \Diamond_0)$, we inductively define, for all $n \geq 0$,

$$(B_{n+1}, \Diamond_{n+1}) := F_{\mathcal{E}}(B_n, \Diamond_n),$$

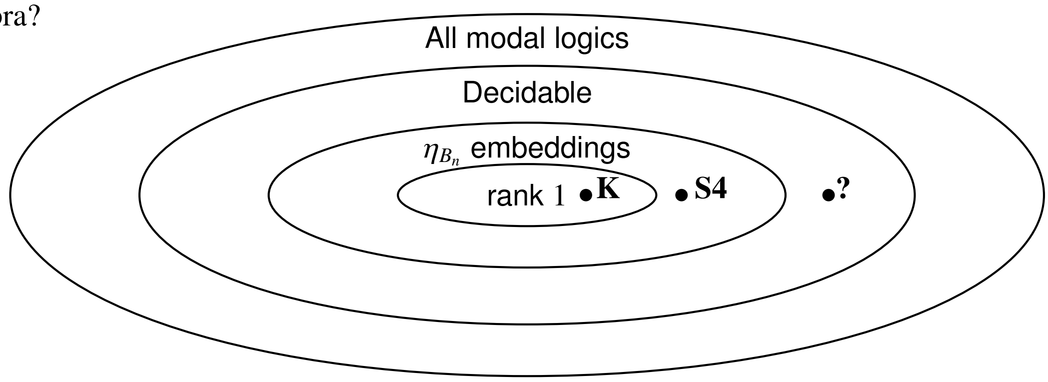
yielding a chain of partial $\mathbf{pV}_{\mathcal{E}}$ algebras with maps η_{B_n} between them. We then prove the following theorem.

Theorem. *If each of the maps $\eta_{B_n} : (B_n, \Diamond_n) \to (B_{n+1}, \Diamond_{n+1})$ is an embedding, then the chain of partial algebras $\{\eta_{B_n} : (B_n, \Diamond_n) \to (B_{n+1}, \Diamond_{n+1})\}_{n \geq 0}$ is the approximating chain of the free $\mathbf{V}_{\mathcal{E}}$ algebra over $(B_0, \Diamond_0)$.*

All known step-by-step constructions of free algebras that were mentioned above are special cases of this general result. Clearly, an important question at this point is how to determine in

[2] Note that *any* set of quasi-equations may be rewritten to a logically equivalent set of quasi-equations of rank at most 1 using flattening, see *e.g.* [5].

general, for a given set of quasi-equations, whether the maps η_{B_n} are embeddings. We illustrate that in the case of S4 modal algebras, duality theory is an useful tool to give a concrete description of the functor $F_{\mathcal{E}}$ so that one can check that each component of η is an embedding, using a Stone-type duality theory for partial modal algebras.
It follows from the existence of non-decidable logics that we cannot hope that, for every set of quasi-equations, the maps η_{B_n} are embeddings. We conjecture that there even exist decidable logics for which the maps η_{B_n} are not all embeddings. Finding examples of such logics is left for future work. Hence, the main contribution of our work is that we have provided a general framework for studying the following question: Given an algebraizable logic $\mathcal{L}$ with associated variety $\mathbf{V}_{\mathcal{L}}$, does there exist a functor which yields the approximating chain for the free $\mathbf{V}_{\mathcal{L}}$ algebra?



References

[1] Nick Bezhanishvili and Mai Gehrke, *Finitely generated free Heyting algebras via Birkhoff duality and coalgebra*, to appear in Logical Methods in Computer Science.

[2] Nick Bezhanishvili and Alexander Kurz, *Free modal algebras: a coalgebraic perspective*, in: Algebra and Coalgebra in Computer Science, Lecture Notes in Computer Science, vol. 4624, Springer, 2007, pp. 143-157.

[3] Silvio Ghilardi, *Free Heyting algebras as bi-Heyting algebras*, Math. Rep. Acad. Sci. Canada **XVI** (1992), no. 6, 240-244.

[4] Silvio Ghilardi, *An algebraic theory of normal forms*, Annals of Pure and Applied Logic **71** (1995), 189-245.

[5] Silvio Ghilardi, *Continuity, Freeness, and Filtrations*, Rapporto Interno 331-10, Dipartimento di Scienze dell'Informazione, Università degli Studi di Milano (2010).

[6] Alexander Kurz and Jiri Rosický, *The Goldblatt-Thomason theorem for coalgebras*, in: Algebra and Coalgebra in Computer Science, Lecture Notes in Computer Science, vol. 4624, Springer, 2007, pp. 342355.

On augmented posets and $(\mathcal{Z}_1, \mathcal{Z}_2)$-complete posets

Mustafa Demirci
Akdeniz University, Faculty of Sciences,
Department of Mathematics, 07058, Antalya/TURKEY
e-mail: demirci@akdeniz.edu.tr

The main objective of this talk is to compare the approach of Banaschewski and Bruns [2] and the subset selection-based approach to the classification of posets with specified joins and meets, and is to derive the concepts and results in the latter approach from the corresponding ones in the former.

Throughout this study, the letters $\mathcal{Z}$ and $\mathcal{Z}_i$ $(i = 1, ..., 4)$ always denote subset selections [4, 6, 8], and $\mathcal{Q}$ designates the quadruple $(\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4)$ where $\mathcal{Z}_3$ and $\mathcal{Z}_4$ are subset systems [4, 9, 13]. The concepts of $\mathcal{Z}_1$-join-completeness and $\mathcal{Z}_2$-meet-completeness suggest a useful classification of posets, and have many fruitful applications [1, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. $\mathcal{Z}$-join(meet)-complete poset is, by definition, a poset P with the property that each $M \in \mathcal{Z}(P)$ has the join (meet) in P [6, 13]. We use the term "$(\mathcal{Z}_1, \mathcal{Z}_2)$-complete poset" instead of a $\mathcal{Z}_1$-join-complete and $\mathcal{Z}_2$-meet-complete poset for the sake of shorthness. $(\mathcal{Z}_1, \mathcal{Z}_2)$-complete posets forms a construct whose morphisms are $(\mathcal{Z}_3, \mathcal{Z}_4)$-continuous maps, i.e. a map $f : P \to Q$ is $(\mathcal{Z}_3, \mathcal{Z}_4)$-continuous iff f is monotone, and preserves $\mathcal{Z}_3$-joins and $\mathcal{Z}_4$-meets [6]. This category that we denote by $\mathcal{Q}$-**CPos** provides a practically useful framework for many order-theoretic structures, e.g. various categories of semilattices, lattices, complete lattices, directed-complete posets, chain-complete posets and σ-lattices can be considered in the form of $\mathcal{Q}$-**CPos** for particular cases of $\mathcal{Q}$. The category $\mathcal{Q}$-**CPos** also enables us to define $\mathcal{Q}$-space a useful generalization of the notion of topological space: A $\mathcal{Q}$-space is a pair (X, τ) consisting of a set X and a subset τ (so-called a $\mathcal{Q}$-system on X) of $\mathcal{P}(X)$ such that the inclusion map $i_\tau : (\tau, \subseteq) \hookrightarrow (\mathcal{P}(X), \subseteq)$ is a $\mathcal{Q}$-**CPos**-morphism. Many familiar systems of subsets of a set are examples of $\mathcal{Q}$-systems, e.g., topology, pretopology, closure systems, algebraic closure systems, kernel systems. The category $\mathcal{Q}$-**SPC** of $\mathcal{Q}$-spaces and $\mathcal{Q}$-space-continuous maps extends the familiar category **Top** of topological spaces to the present setting.

As another approach to the classification of posets with specified joins and meets, augmented posets have been proposed by Banaschewski and Bruns [2]. An augmented poset here means a triple $U = (|U|, \mathfrak{J}U, \mathfrak{M}U)$ consisting of a poset $|U|$, a subset $\mathfrak{J}U$ of the power set $\mathcal{P}(U)$ such that each $S \in \mathfrak{J}U$ has the join in $|U|$ and a subset $\mathfrak{M}U$ of $\mathcal{P}(U)$ such that each $R \in \mathfrak{M}U$ has the meet in $|U|$. Augmented posets together with structure preserving maps (i.e. $h : U \to V$ is a structure preserving map iff $h : |U| \to |V|$ is a monotone map such that $h(S) \in \mathfrak{J}V$ and $h(\bigvee S) = \bigvee h(S)$ for all $S \in \mathfrak{J}U$, $h(R) \in \mathfrak{M}V$ and $h(\bigwedge R) = \bigwedge h(R)$ for all $R \in \mathfrak{M}U$) constitute a category **P** [2]. It was shown in [2] that **P** is dually adjoint to the category **S** of spaces. Recall that objects of **S** are the quadruples (the so-called spaces) $W = (|W|, \mathfrak{D}(W), \Sigma(W), \Delta(W))$, where $|W|$ is a set,

$\mathfrak{D}(W)$ is a subset of $\mathcal{P}(|W|)$, $\Sigma(W)$ is a subset of $\{\mathfrak{U} \subseteq \mathfrak{D}(W) \mid \cup\mathfrak{U} \in \mathfrak{D}(W)\}$ and $\Delta(W)$ is a subset of $\{\mathfrak{B} \subseteq \mathfrak{D}(W) \mid \cap\mathfrak{B} \in \mathfrak{D}(W)\}$, while its morphisms $f : W_1 \to W_2$ are functions $f : |W_1| \to |W_2|$ with the properties that $(f^{\leftarrow})^{\rightarrow}(\mathfrak{D}(W_2)) \subseteq \mathfrak{D}(W_1)$, $(f^{\leftarrow})^{\rightarrow}(\mathfrak{U}) \in \Sigma(W_1)$ and $(f^{\leftarrow})^{\rightarrow}(\mathfrak{B}) \in \Delta(W_1)$ for all $\mathfrak{U} \in \Sigma(W_2)$ and for all $\mathfrak{B} \in \Delta(W_2)$. The restriction of this adjunction to the full subcategory **SpaP** of **P** of all spatial objects and the full subcategory **SobS** of **S** of all sober objects gives a dual equivalence between **SpaP** and **SobS**.

Augmented posets and $(\mathcal{Z}_1, \mathcal{Z}_2)$-complete posets are seemingly two different approaches to the classification of posets with specified joins and meets. Given a poset P, let us denote the set of all $M \in \mathcal{Z}(P)$ with the join (meet) by $\mathcal{Z}^{\sup}(P)$ ($\mathcal{Z}^{\inf}(P)$). We will show in this presentation that the functors $G_{\mathcal{Q}} : \mathcal{Q}$-**CPos**$\to$**P** and $H_{\mathcal{Q}} : \mathcal{Q}$-**SPC**$\to$**S**, defined by $G_{\mathcal{Q}}(P) = \left(P, \mathcal{Z}_3^{\sup}(P), \mathcal{Z}_4^{\inf}(P)\right)$, $G_{\mathcal{Q}}(f) = f$, $H_{\mathcal{Q}}(X,\tau) = \left(X, \tau, \mathcal{Z}_3^{\sup}(\tau), \mathcal{Z}_4^{\inf}(\tau)\right)$ and $H_{\mathcal{Q}}(g) = g$ for each $(\mathcal{Z}_1, \mathcal{Z}_2)$-complete poset P, for each $\mathcal{Q}$-**CPos**-morphism f, for each $\mathcal{Q}$-space (X,τ) and for each $\mathcal{Q}$-**SPC**-morphism g, are two full embeddings, and so Banaschewski and Bruns's approach is more general than the subset selection-based approach. Despite this fact, categories in the subset selection-based approach provide a more direct formulation of various frequently used categories. The full embeddings $G_{\mathcal{Q}}$ and $H_{\mathcal{Q}}$ give rise to many nice results. One of them is that spatiality for $(\mathcal{Z}_1, \mathcal{Z}_2)$-complete posets and sobriety for $\mathcal{Q}$-spaces can be defined by means of the corresponding notions in augmented posets and spaces: A $(\mathcal{Z}_1, \mathcal{Z}_2)$-complete poset A is $\mathcal{Q}$-spatial iff $G_{\mathcal{Q}}(A)$ is spatial, a $\mathcal{Q}$-space (X,τ) is $\mathcal{Q}$-sober iff $H_{\mathcal{Q}}(X,\tau)$ is sober. Although $\mathcal{Q}$-spatiality and $\mathcal{Q}$-sobriety are generally different from the concepts of $\mathcal{Z}$-spatiality (alias being $\mathcal{Z}$-lattice) and $\mathcal{Z}$-sobriety in the sense of Erne [5, 6, 7, 8], we will point out that the latter concepts are particular cases of the former ones, and therefore give an answer to the question of how $\mathcal{Z}$-spatiality and $\mathcal{Z}$-sobriety relate to spatiality and sobriety in the sense of [2]. As another important implementation of the embeddings $G_{\mathcal{Q}}$ and $H_{\mathcal{Q}}$, we determine the suitable cases of the parameter $\mathcal{Q}$ making the full subcategory of $\mathcal{Q}$-**CPos** of all $\mathcal{Q}$-spatial objects dually equivalent to the full subcategory of $\mathcal{Q}$-**SPC** of all $\mathcal{Q}$-sober objects.

References

[1] Adámek, J., Koubek, V., Nelson E., Reiterman, J.: Arbitrarily large continuous algebras on one generator. Trans. Amer. Math. Soc. 291, 681-699 (1985)

[2] Banaschewski, B., Bruns, G.: The fundamental duality of partially ordered sets. Order 5, 61-74 (1988)

[3] Bandelt, H. -J., Erné, M.: The category of Z-continuous posets. J. Pure Appl. Algebra 30, 219-226 (1983)

[4] Erné, M: Bigeneration in complete lattices and principle separation in posets. Order 8, 197-221 (1991)

[5] Erné, M: The ABC of order and topology. In: Herrlich, H., Porst, H. E. (eds.) Category Theory at Work, pp.57-83. Heldermann, Berlin, Germany (1991)

[6] Erné, M: Algebraic ordered sets and their generalizations. In: Rosenberg, I., Sabidussi, G. (eds.) Algebras and Orders, Proc. Montreal, 1992, pp. 113-192, Kluwer Academic Publishers, Amsterdam, Netherlands (1993)

[7] Erné, M: $\mathcal{Z}$-continuous posets and their topological manifestation, Appl. Categ. Structures 7, 31-70 (1999)

[8] Erné, M: General Stone duality. Topology and its Applications 137, 125-158 (2004)

[9] Nelson, E.: Z-continuous algebras. In : Banaschewski, B., Hoffmann, R.E. (eds.) Continuous Lattices. Proc. Conf., Bremen 1979. Lect. Notes Math. 871, pp. 315-334. Springer-Verlag, Berlin (1981)

[10] Pasztor, A.: The epis of Pos(Z). Comment. Math. Univ. Carolin. 23, 285-299 (1982)

[11] Powers, R.C., Riedel, T.: Z-Semicontinuous posets. Order 20, 365-371 (2003)

[12] Venugopalan, G.: Z-continuous posets. Houston J. Math. 12, 275-294 (1986)

[13] Wright, J. B., Wagner, E. G., Thatcher, J. W.: A uniform approach to inductive posets and inductive closure. Theoret. Comput. Sci. 7, 57-77 (1978)

[14] Zhao, D.: On projective Z-frames. Can. Math. Bull. 40, 39-46 (1997)

ORDERED DOMAIN ALGEBRAS

ROB EGROT, ROBIN HIRSCH AND SZABOLCS MIKULÁS

Abstract. We give a finite axiomatisation to representable ordered domain algebras and show that finite algebras are representable on finite bases.

Domain algebras provide an elegant, one-sorted formalism for automated reasoning about program and system verification [DS08a, DS08b]. The algebraic behaviour of domain algebras have been investigated, e.g. in [DJS09a, DJS09b]. Their primary models are algebras of relations, viz. representable domain algebras. P. Jipsen and G. Struth raised the question whether the class $\mathbf{R}(;,\mathsf{dom})$ of representable domain algebras of the minimal signature $(;,\mathsf{dom})$ is finitely axiomatisable. To formulate the question precisely, let us recall the definition of representable domain algebras $\mathbf{R}(;,\mathsf{dom})$.

Definition 0.1. *The class* $\mathbf{R}(;,\mathsf{dom})$ *is defined as the isomorphs of* $\mathcal{A} = (A, ;, \mathsf{dom})$ *where* $A \subseteq \wp(U \times U)$ *for some base set* U *and*

$$x \,;\, y = \{(u, v) \in U \times U : (u, w) \in x \text{ and } (w, v) \in y \text{ for some } w \in U\}$$
$$\mathsf{dom}(x) = \{(u, u) \in U \times U : (u, v) \in x \text{ for some } v \in U\}$$

for every $x, y \in A$.

The signature $(;,\mathsf{dom})$ can be expanded to larger signatures τ by including other operations. For instance, we can define

$$\mathsf{ran}(x) = \{(v, v) \in U \times U : (u, v) \in x \text{ for some } u \in U\}$$
$$x^{\smile} = \{(v, u) \in U \times U : (u, v) \in x\}$$
$$1' = \{(u, v) \in U \times U : u = v\}$$

and the corresponding representation classes $\mathbf{R}(\tau)$ analogously to the definition of $\mathbf{R}(;,\mathsf{dom})$. We can also include bottom 0 and top 1 elements (interpreted as $\emptyset$ and $U \times U$, respectively) and the ordering $\subseteq$ to yield representable algebraic structures.

It turned out that the answer to the above problem is negative.

Theorem 0.2. *[*[HM11]*] Let* τ *be a similarity type such that* $(;,\mathsf{dom}) \subseteq \tau \subseteq (;,\mathsf{dom},\mathsf{ran},0,1')$. *The class* $\mathbf{R}(\tau)$ *of representable* τ*-algebras is not finitely axiomatisable in first-order logic.*

Note that the above theorem does not apply to signatures where the ordering $\subseteq$ is definable. In fact, D.A. Bredikhin proved [Bre77] that the class $\mathbf{R}(;,\mathsf{dom},\mathsf{ran},{}^{\smile},\subseteq)$ of representable algebraic structures is finitely axiomatisable. Our aim is to provide an alternative, and slightly more general, proof that $\mathbf{R}(;,\mathsf{dom},\mathsf{ran},{}^{\smile},0,1',\subseteq)$ is finitely axiomatisable. The advantage of our proof is that it uses a Cayley-type representation of abstract algebraic structures that also shows finite representability, i.e. that finite elements of $\mathbf{R}(;,\mathsf{dom},\mathsf{ran},{}^{\smile},0,1',\subseteq)$ can be represented on finite bases. In passing we note that if composition is not definable in τ, then $\mathbf{R}(\tau)$

has the finite representation property, but every signature containing $(\cap, ;, 1')$ or $(\cap, ;, ^\smile)$ fails to have the finite representation property.

Main result

Let **Ax** denote the following formulas.

Partial order: $\leq$ is reflexive, transitive and antisymmetric, with lower bound 0.

Monotonicity and normality: the operators $^\smile, ;, \mathsf{dom}, \mathsf{ran}$ are monotonic, e.g. $a \leq b \to a ; c \leq b ; c$ etc. and normal $0^\smile = 0 ; a = a ; 0 = \mathsf{dom}(0) = \mathsf{ran}(0) = 0$.

Involuted monoid: ; is associative, $1'$ is left and right identity for ;, $1'^\smile = 1'$ and $^\smile$ is an involution: $(a^\smile)^\smile = a$, $(a ; b)^\smile = b^\smile ; a^\smile$.

Domain/range axioms:

$$\mathsf{dom}(a) = (\mathsf{dom}(a))^\smile \leq 1' = \mathsf{dom}(1') \tag{1}$$

$$\mathsf{dom}(a) \leq a ; a^\smile \tag{2}$$

$$\mathsf{dom}(a^\smile) = \mathsf{ran}(a) \tag{3}$$

$$\mathsf{dom}(\mathsf{dom}(a)) = \mathsf{dom}(a) = \mathsf{ran}(\mathsf{dom}(a)) \tag{4}$$

$$\mathsf{dom}(a) ; a = a \tag{5}$$

$$\mathsf{dom}(a ; b) = \mathsf{dom}(a ; \mathsf{dom}(b)) \tag{6}$$

$$\mathsf{dom}(\mathsf{dom}(a) ; \mathsf{dom}(b)) = \mathsf{dom}(a) ; \mathsf{dom}(b) = \mathsf{dom}(b) ; \mathsf{dom}(a) \tag{7}$$

$$\mathsf{dom}(\mathsf{dom}(a) ; b) = \mathsf{dom}(a) ; \mathsf{dom}(b) \tag{8}$$

A model of these axioms is called an *ordered domain algebra*.

A consequence of axioms (4) and (5) is

$$\mathsf{dom}(a) ; \mathsf{dom}(a) = \mathsf{dom}(a) \tag{9}$$

Each of the axioms (1)–(8) has a dual axiom, obtained by swapping domain and range and reversing the order of compositions, and we denote the dual axiom by a ∂ superscript, thus for example, $(6)^\partial$ is $\mathsf{ran}(b ; a) = \mathsf{ran}(\mathsf{ran}(b) ; a)$. The dual axioms can be obtained from the axioms above, using the involution axioms and (3).

Our main result is the following.

Theorem 0.3. *The class $\mathbf{R}(;, \mathsf{dom}, \mathsf{ran}, ^\smile, 0, 1', \subseteq)$ is finitely axiomatisable:*

$$\mathcal{A} \in \mathbf{R}(;, \mathsf{dom}, \mathsf{ran}, ^\smile, 0, 1', \subseteq) \text{ iff } \mathcal{A} \models \mathbf{Ax}$$

and has the finite representation property.

Proof. First we extend the operations of a domain algebra to subsets of elements.

Definition 0.4. *Let $\mathcal{A}$ be an ordered domain algebra.*

1. *Write $D(\mathcal{A})$ for the set of domain elements of $\mathcal{A}$ — those elements $d \in \mathcal{A}$ such that $\mathsf{dom}(d) = d$. Observe that $(D(\mathcal{A}), ;)$ forms a lower semilattice ordered by $\leq$.*
2. *For $a \in \mathcal{A}$, let $a^\uparrow = \{b \in \mathcal{A} : a \leq b\}$ and more generally, for $X \subseteq \mathcal{A}$, let $X^\uparrow = \{b \in \mathcal{A} : (\exists a \in X) a \leq b\}$.*

(3) *We extend the operations so as to apply to sets of elements. If $X, Y \subseteq \mathcal{A}$, $a \in \mathcal{A}$, then*

$$X^{\smile} = \{x^{\smile} : x \in X\}^{\uparrow} \tag{10}$$

$$X ; Y = \{x ; y : x \in X,\ y \in Y\}^{\uparrow} \tag{11}$$

$$\mathsf{dom}(X) = \{\mathsf{dom}(x) : x \in X\}^{\uparrow} \tag{12}$$

$$\mathsf{ran}(X) = \{\mathsf{ran}(x) : x \in X\}^{\uparrow} \tag{13}$$

Note that these sets are all 'closed upwards' by definition.

(4) *A non-empty subset X of $\mathcal{A}$ is* closed *if*

$$\mathsf{dom}(X) ; X ; \mathsf{ran}(X) = X \tag{14}$$

Thus, for an ordered domain algebra $\mathcal{A}$, we can define another algebra on the subsets $\wp(\mathcal{A})$ of $\mathcal{A}$, and the partial order $\leq$ on $\wp(\mathcal{A})$ is given by $\supseteq$. We will denote this ordered algebra as $\mathcal{C}(\mathcal{A})$, the elements of $\wp(\mathcal{A})$ by upper case letters X, Y, Z etc. or by $a^{\uparrow}, b^{\uparrow}$ etc., and the elements of $\mathcal{A}$ with lower case letters a, b, c etc. It should be clear from this notational convention whether we evaluate a term in $\mathcal{A}$ or in $\mathcal{C}(\mathcal{A})$.

It is not difficult to check the following. Let $\tau \leq \sigma$ be an axiom of domain algebras such that every variable a occurs at most once in τ and at most once in σ. Then the inequality $\tau \supseteq \sigma$ is valid $\mathcal{C}(\mathcal{A})$. (Hint: use monotonicity and the validity of $\tau \leq \sigma$ in $\mathcal{A}$.) But axioms like (2) and (5) fail in general even in the subalgebra of upwards-closed elements of $\mathcal{C}(\mathcal{A})$.

Observe, from definition 0.4(3) and the transitivity of $\leq$, that $(\mathsf{dom}(X) ; X ; \mathsf{ran}(X))^{\uparrow} = \mathsf{dom}(X) ; X ; \mathsf{ran}(X)$, for any set $X \subseteq \mathcal{A}$. So every closed set is upwards closed. More equations are valid in $\mathcal{C}(\mathcal{A})$ if the variables are evaluated on closed elements, e.g. (5), but closed elements may not be closed under the operations, e.g. $X ; Y$ for closed X and Y is not closed in general, and (2) may still fail.

Let $Cl(\mathcal{A})$ be the set of all closed subsets of $\mathcal{A}$. Since $Cl(\mathcal{A}) \subseteq \wp(\mathcal{A})$, we have $|Cl(\mathcal{A})| \leq 2^{|\mathcal{A}|}$. Define a map F from $\mathcal{A}$ to a structure with base $Cl(\mathcal{A})$ as follows.

$$(X, Y) \in a^F \iff X ; a^{\uparrow} \subseteq Y \text{ and } Y ; (a^{\smile})^{\uparrow} \subseteq X \tag{15}$$

We claim that F yields a representation of $\mathcal{A}$. To this end let $0 \neq a \not\leq b$. It is easy to show that $(\mathsf{dom}(a))^{\uparrow}, a^{\uparrow}$ are closed. By monotonicity, (5) and (2), $(\mathsf{dom}(a))^{\uparrow} ; a^{\uparrow} \subseteq a^{\uparrow}$ and $a^{\uparrow} ; (a^{\smile})^{\uparrow} \subseteq (\mathsf{dom}(a))^{\uparrow}$, so $((\mathsf{dom}(a))^{\uparrow}, a^{\uparrow}) \in a^F$. Also, we cannot have $\mathsf{dom}(a) ; b \geq a$, by transitivity, monotonicity and (1), since $a \not\leq b$. Thus $((\mathsf{dom}(a))^{\uparrow}, a^{\uparrow}) \notin b^F$, whence F is faithful.

$0^F = \emptyset$, by normality and the partial order axioms. $\leq$ is correctly represented by the partial order axioms and monotonicity. $1'^F = \{(X, X) : X \in Cl(\mathcal{A})\}$ by the involuted monoid axioms. $^{\smile}$ is correctly represented by the involution axioms.

Next we check composition. If $(X, Y) \in a^F$ and $(Y, Z) \in b^F$, then $X ; a^{\uparrow} \subseteq Y$, $Y ; (a^{\smile})^{\uparrow} \subseteq X$, $Y ; b^{\uparrow} \subseteq Z$ and $Z ; (b^{\smile})^{\uparrow} \subseteq Y$. Hence $X ; (a ; b)^{\uparrow} \subseteq Z$ and $Z ; ((a ; b)^{\smile})^{\uparrow} = Z ; (b^{\smile} ; a^{\smile})^{\uparrow} \subseteq X$ by associativity and the involution axioms. So $(X, Z) \in (a ; b)^F$.

Conversely, assume that $(X, Z) \in (a;b)^F$. We need a closed Y such that $(X, Y) \in a^F$ and $(Y, Z) \in b^F$.

Claim 0.5. *The sets*

$$\alpha = X ; a^{\uparrow} ; \mathsf{ran}(Z ; (b^{\smile})^{\uparrow}) \text{ and } \beta = Z ; (b^{\smile})^{\uparrow} ; \mathsf{ran}(X ; a^{\uparrow})$$

and $\alpha \cup \beta$ are closed.

Thus we can define the closed element $Y = \alpha \cup \beta$. That ; is properly represented follows by the following claim.

Claim 0.6. $(X, Y) \in a^F$ *and* $(Y, Z) \in b^F$.

Finally, we show that dom and ran are properly represented. If $(X, Y) \in (\mathsf{dom}(a))^F$, then $X ; (\mathsf{dom}(a))^\uparrow \subseteq Y$. Since $\mathsf{dom}(a) \leq 1'$ by (1), we have that, for every $x \in X$, there is $y \in Y$ such that $x \geq x ; \mathsf{dom}(a) \geq y$. Since Y is (upwards) closed, we get $X \subseteq Y$. Similarly, we get $Y \subseteq X$ by $Y ; ((\mathsf{dom}(a))^\smile)^\uparrow \subseteq Y ; (\mathsf{dom}(a))^\uparrow \subseteq X$ (using (1)). Hence $X = Y$, i.e., $(X, X) \in (\mathsf{dom}(a))^F$. Note also that $\mathsf{dom}(a) \in \mathsf{ran}(X)$, since $\mathsf{dom}(a) \in \mathsf{ran}(Y ; (\mathsf{dom}(a))^\uparrow) \subseteq \mathsf{ran}(x)$.

Define the closed element $Z = X ; a^\uparrow$. Then $(X, Z) \in a^F$, since $X ; a^\uparrow \subseteq Z$ by definition, and

$$X ; a^\uparrow ; (a^\smile)^\uparrow \subseteq X ; (\mathsf{dom}(a))^\uparrow \subseteq X$$

by (2) and $\mathsf{dom}(a) \in \mathsf{ran}(X)$. Conversely, suppose $(X, Z) \in a^F$ (for some Z). Then $X ; a^\uparrow \subseteq Z$ and $Z ; (a^\smile)^\uparrow \subseteq X$. Since $Z ; (a^\smile)^\uparrow \subseteq X$, we have $\mathsf{dom}(a) = \mathsf{ran}(a^\smile) \in \mathsf{ran}(Z ; (a^\smile)^\uparrow) \subseteq \mathsf{ran}(X)$, whence $X ; (\mathsf{dom}(a))^\uparrow \subseteq X$, i.e. $(X, X) \in (\mathsf{dom}(a))^F$. So dom is correctly represented. Showing that ran is properly represented is similar. This finishes the proof of Theorem 0.3. □

References

[Bre77] D.A. Bredikhin, "An abstract characterization of some classes of algebras of binary relations", in *Algebra and Number Theory*, No. 2, pp. 3–19. Kabardino-Balkarsk. Gos. Univ., Nalchik, 1977. [In Russian.]

[DJS09a] J. Desharnais, P. Jipsen and G. Struth, "Domain and antidomain semigroups", in R. Berghammer et al. (eds.), *Relations and Kleene Algebra in Computer Science*, pages 73–87, Springer-Verlag, 2009.

[DJS09b] J. Desharnais, P. Jipsen and G. Struth, "Internal Axioms for Domain Semirings", presentation at *TACL'2009*, extended abstract available at `http://staff.science.uva.nl/~gfontain/tacl09-abstracts/tacl2009_submission_73.pdf`.

[DS08a] J. Desharnais and G. Struth, "Modal semirings revisited", in P. Audebaud and C. Paulin-Mohring (eds.), *MPC 2008*, pages 360–387, Springer-Verlag, 2008.

[DS08b] J. Desharnais and G. Struth, "Domain axioms for a family of near-semirings", in J. Meseguer and G. Roşu (eds.), *AMAST 2008*, pages 330–345, Springer-Verlag, 2008.

[HM11] R. Hirsch and Sz. Mikulás, "Axiomatizability of representable domain algebras", *Journal of Logic and Algebraic Programming*, Vol. 80(2), pp. 75–91, 2011.

Department of Computer Science
University College London
E-mail: R.Egrot@cs.ucl.ac.uk r.hirsch@cs.ucl.ac.uk

Department of Computer Science and Information Systems
Birkbeck College, University of London
E-mail: szabolcs@dcs.bbk.ac.uk

On the modal definability of topological simulation

David Fernández-Duque
Group for Logic, Language and Information,
Universidad de Sevilla

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1 Introduction

Simulation is a weak form of bisimulation satisfying the 'forth' but not necessarily the 'back' condition. In this paper, we are interested in the question of defining being simulated by a finite $\mathsf{S4}$-model over the class of all topological models. Our main results are that one cannot define simulability by finite $\mathsf{S4}$ models in basic modal logic, but we can remedy this with a small addition to the language. It is an operator on sets of formulas, very similar to one introduced by [1] under a different guise. We will call this the *tangle* operator.

We will use a basic modal language $\mathsf{L} = \mathsf{L}^{\Box}$ which is built from propositional variables in a *finite*[1] set PV using the Boolean connectives $\wedge$ and $\neg$ (all other connectives are to be defined in terms of these) and the unary modal operator $\Box$. As usual we write $\Diamond$ as a shorthand for $\neg\Box\neg$.

We are interested in interpreting L over the class of all $\mathsf{S4}$ frames and over the class of all topological models. We will focus particularly on models of the modal logic $\mathsf{S4}$, which is given by the axioms $\Box(\varphi \to \psi) \to \Box\varphi \to \Box\psi$, $\Box\varphi \to \Box\Box\varphi$ and $\Box\alpha \to \alpha$. The rules are Modus ponens and Necessitation.

The above axioms and rules are sound and complete for interpretations over transitive, reflexive Kripke models $\mathfrak{W} = \langle |\mathfrak{W}|, \preccurlyeq_{\mathfrak{W}}, [\![\cdot]\!]_{\mathfrak{W}} \rangle$, where we interpret $\Box\alpha$ by $w \in [\![\Box\alpha]\!]_{\mathfrak{W}}$ if and only if, whenever $v \preccurlyeq w$, $v \in [\![\alpha]\!]_{\mathfrak{W}}$.

We will also use the notation $w \prec v$ for $w \preccurlyeq v$ but $v \not\preccurlyeq w$ and $w \sim v$ for $w \preccurlyeq v$ and $v \preccurlyeq w$.

An more general interpretation of $\mathsf{S4}$ is that given by *topological models* $\mathfrak{X} = \langle |\mathfrak{X}|, \mathcal{T}_{\mathfrak{X}}, [\![\cdot]\!]_{\mathfrak{X}} \rangle$. Here we set $[\![\Box\alpha]\!]_{\mathfrak{X}} = [\![\alpha]\!]^{\circ}_{\mathfrak{X}}$ (the topological interior).

Kripke semantics can be seen as a special case of topological semantics if we let opens be the downsets.

[1]The reason we work with finite PV is that, evidently, one cannot define models up to simulation or bisimulation in the presence of infinitely many variables using a finite formula.

Definition 1.1. *A* state *is a triple* $\mathfrak{w} = \langle |\mathfrak{w}|, \preccurlyeq_{\mathfrak{w}}, 0_{\mathfrak{w}} \rangle$ *consisting of a finite* S4 *model equipped with a designated root* $0_{\mathfrak{w}}$*, that is, such that* $w \preccurlyeq_{\mathfrak{w}} 0_{\mathfrak{w}}$ *for all* $w \in |\mathfrak{w}|$.

Properly speaking, we are interested in defining simulability by states, i.e., by finite, pointed S4 models.

Definition 1.2. *If* $\mathfrak{X}$, $\mathfrak{Y}$ *are Kripke models, a* simulation *between* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *is a binary relation* $\chi \subseteq |\mathfrak{X}| \times |\mathfrak{Y}|$ *which preserves propositional variables and such that* $\chi^{-1}(U)$ *is open whenever* U *is (i.e.,* χ *is continuous).*

We will write $\langle \mathfrak{X}, x \rangle \trianglelefteq \langle \mathfrak{Y}, y \rangle$ if there exists a simulation χ between $\mathfrak{X}$ and $\mathfrak{Y}$ such that $x \; \chi \; y$.

2 Undefinability

If $\mathfrak{W}$ is a Kripke model, we will denote its *ultrafilter extension* by $\widetilde{\mathfrak{W}}$.

Theorem 2.1. *There exist a finite* S4 *model* $\mathfrak{W}$, $w \in |\mathfrak{W}|$ *and two* S4 *models*[2] $\mathfrak{X}$, $\mathfrak{Y}$ *with* $\langle \mathfrak{X}, x \rangle$ *and* $\langle \mathfrak{Y}, y \rangle$ *satisfying the same set of* $\mathsf{L}^{\Box}$*-formulas, such that* $\langle \mathfrak{W}, w \rangle \not\trianglelefteq \langle \mathfrak{X}, x \rangle$ *but* $\langle \mathfrak{W}, w \rangle \trianglelefteq \langle \mathfrak{Y}, y \rangle$.

Proof. We will begin by constructing a sort of 'hedgehog'. Let

$$|\mathfrak{X}| = \{0\} \cup \{\langle n, m \rangle : 0 < n < m\},$$

$x \preccurlyeq_{\mathfrak{X}} y$ if either $y = 0$ or $x = \langle n_x, m \rangle$, $y = \langle n_y, m \rangle$ and $n_y \leq n_x$.

Let $[\![p]\!]_{\mathfrak{X}}$ be the set $\{0\} \cup \{\langle n, m \rangle : n \text{ is even}\}$, $[\![q]\!]_{\mathfrak{X}}$ be $\{\langle n, m \rangle : n \text{ is odd}\}$. Define $\mathfrak{W}$ by setting $|\mathfrak{W}| = \{w_p, w_q\}$ with $[\![p]\!]_{\mathfrak{W}} = \{w_p\}$, $[\![q]\!]_{\mathfrak{W}} = \{w_q\}$ and $w_p \sim_{\mathfrak{W}} w_q$.

We claim that $\langle \mathfrak{W}, w_p \rangle \not\trianglelefteq \langle \mathfrak{X}, 0 \rangle$. Indeed, in order to have a simulation $\chi \subseteq |\mathfrak{W}| \times |\mathfrak{X}|$ with $w_p \; \chi \; 0$, we would need an infinite sequence of (possibly repeating) points $0 = y_0 \succcurlyeq_{\mathfrak{X}} y_1 \succcurlyeq_{\mathfrak{X}} y_2 \succcurlyeq_{\mathfrak{X}} \ldots$ with $y_i \in [\![p]\!]_{\mathfrak{X}}$ if i is even and $y_i \in [\![q]\!]_{\mathfrak{X}}$ if i is odd.

To see this, assume that χ is such a simulation. Because χ is continuous, $w_p \; \chi \; 0$ and $w_q \preccurlyeq_{\mathfrak{W}} w_p$, it follows that there is a world $y_1 \in [\![q]\!]_{\mathfrak{X}}$ such that $w_q \; \chi \; y_1$ and $y_1 \preccurlyeq_{\mathfrak{X}} 0$. By the same argument there is $y_2 \in [\![p]\!]_{\mathfrak{X}}$ such that $w_p \; \chi \; y_2$ and $y_2 \preccurlyeq_{\mathfrak{X}} y_1$, and this process can be continued indefinitely to construct $\langle y_n \rangle_{n<\omega}$.

However, such a sequence clearly does not exist in $\mathfrak{X}$, and therefore $\langle \mathfrak{W}, w_p \rangle \not\trianglelefteq \langle \mathfrak{X}, 0 \rangle$.

Now, consider the ultrafilter extension $\widetilde{\mathfrak{X}}$. The principal filter (0) satisfies the same set of $\mathsf{L}^{\Box}$-formulas as 0. However, we will show that $\langle \mathfrak{W}, w_p \rangle \trianglelefteq \left\langle \widetilde{\mathfrak{X}}, (0) \right\rangle$.

For each $i < \omega$ let $S_i = \{\langle i, m \rangle : i < m < \omega\}$; this is a 'horizontal slice' of the hedgehog. Note that $S_i \subseteq [\![p]\!]_{\mathfrak{X}}$ if i is even, $S_i \subseteq [\![q]\!]_{\mathfrak{X}}$ if i is odd.

[2]These will be infinite.

Let $\mathcal{U}$ be any non-principal ultrafilter[3] over ω. Set $\mathcal{Y}_0 = (0)$, and for $i > 0$ define $\mathcal{Y}_i$ to be the ultrafilter of all sets E such that $\{m < \omega : \langle i, m\rangle \in E\} \in \mathcal{U}$. Clearly $S_i \in \mathcal{Y}_i$ for all $i < \omega$, so it follows that $[\![p]\!]_{\mathfrak{X}} \in \mathcal{Y}_i$ if i is even, and hence $\mathcal{Y}_i \in [\![p]\!]_{\widetilde{\mathfrak{X}}}$; similarly, $\mathcal{Y}_i \in [\![q]\!]_{\widetilde{\mathfrak{X}}}$ if i is odd.

Also, note that $\mathcal{Y}_j \preccurlyeq_{\widetilde{\mathfrak{X}}} \mathcal{Y}_i$ whenever $i < j$. This is because if $E \in \mathcal{Y}_i$ and $F \in \mathcal{Y}_j$ we have that

$$E' = \{m < \omega : \langle i, m\rangle \in E\} \in \mathcal{U}$$

and

$$F' = \{m < \omega : \langle j, m\rangle \in F\} \in \mathcal{U},$$

therefore $E' \cap F' \neq \varnothing$ and for $m \in E' \cap F'$ we have that $\langle i, m\rangle \in E$, $\langle j, m\rangle \in F$ and $\langle j, m\rangle \preccurlyeq_{\mathfrak{X}} \langle i, m\rangle$.

Hence we have a simulation $\chi \subseteq |\mathfrak{W}| \times |\widetilde{\mathfrak{X}}|$ given by $w_p \;\chi\; y$ if $y = \mathcal{Y}_i$ with i even and $w_q \;\chi\; y$ if $y = \mathcal{Y}_i$ with i odd.

We have now shown that $\langle \mathfrak{X}, 0\rangle$ and $\left\langle \widetilde{\mathfrak{X}}, (0)\right\rangle$ satisfy the same set of $\mathsf{L}^{\Box}$-formulas, but $\langle \mathfrak{W}, w\rangle \ntrianglelefteq \langle \mathfrak{X}, 0\rangle$ and $\langle \mathfrak{W}, w_p\rangle \trianglelefteq \left\langle \widetilde{\mathfrak{X}}, (0)\right\rangle$, as desired. □

3 Definability

While simulability is undefinable in the basic modal language, this can be remedied by adding just a bit of expressive power. Namely, we need to be able to express an operation on sets which we will call their 'tangle'.

Definition 3.1 (Tangle)**.** *Let $\mathfrak{W}$ be a Kripke model and $\mathcal{S} \subseteq 2^{|\mathfrak{W}|}$. We define $\mathcal{S}^{\natural}$, the* tangle *of $\mathcal{S}$, to be the union of all sets $E \subseteq \bigcup \mathcal{S}$ such that, for all $S \in \mathcal{S}$ and $w \in E$, there is $v \preccurlyeq w$ with $v \in E \cap S$.*

Meanwhile, if $\mathfrak{X}$ is a topological space and $\mathcal{S} \subseteq 2^{|\mathfrak{X}|}$, we define $\mathcal{S}^{\natural}$ to be the union of all sets $E \subseteq \bigcup \mathcal{S}$ such that, for all $S \in \mathcal{S}$, $S \cap E$ is dense in E.

We wish to extend our language in order to incorporate the tangle operator into our semantics. Thus we consider the language $\mathsf{L}^{+} = \mathsf{L}^{\Box\natural}$, where $\natural$ acts on finite sets[4] of propositions of arbitrary size; that is, if $\varphi_0, ..., \varphi_n$ are formulas of L^{+} then so is $\natural\{\varphi_0, ..., \varphi_n\}$. We interpret $[\![\natural\{\varphi_0, ..., \varphi_n\}]\!]$ as $\{[\![\varphi_0]\!], ..., [\![\varphi_n]\!]\}^{\natural}$. This gives a system very close to ML^{*} introduced in [1]. Syntactically, this is a fragment of the μ-calculus [2] and can be classified as a flat fixpoint logic [3].

Although simulability by a state is not definable in the basic modal language, over L^{+} it is. We shall explicitly construct a formula $\mathrm{Sim}(\mathfrak{w})$ defining simulability by $\mathfrak{w}$ for any finite $\mathsf{S4}$ state $\mathfrak{w}$.

Below, $\tau(w)$ denotes the set of literals (propositional variables or their negation) which are true on w.

[3] These can be shown to exist by a well-known construction using Zorn's Lemma, considering a maximal extension of the filter of all cofinite sets.

[4] We limit ourselves to finite sets of formulas only because we wish for all formulas to be finite objects, but in principle one could consider a language where $\natural$ admits arbitrary sets of formulas without altering its definition.

Definition 3.2 (Simulability formula)**.** *Let* $\mathfrak{W}$ *be a finite* S4*-model. We will define a formula* $\mathrm{Sim}(\mathfrak{W}, w)$ *for* $w \in |\mathfrak{W}|$ *by induction on the height of* w*.*

First define, for $v \sim w$ *or* $v = w$*,* $\delta(v) = \bigwedge \tau(v) \wedge \bigwedge_{u \prec v} \Diamond \mathrm{Sim}(\mathfrak{W}, u)$*.*

Then, set $\mathrm{Sim}(\mathfrak{W}, w)$ *equal to* $\delta(w) \wedge \natural \{\delta(v)\}_{v \sim w}$*.*

Theorem 3.1. *Let* w *be a world on a finite* S4 *model* $\mathfrak{W}$*.*

Then, for every $x \in |\mathfrak{X}|$ *we have that* $\langle \mathfrak{W}, w \rangle \trianglelefteq \langle \mathfrak{X}, x \rangle$ *if and only if* $x \in [\![\mathrm{Sim}(\mathfrak{W}, w)]\!]_{\mathfrak{X}}$*.*

Our definability result can be extended to classes of finite models in certain settings. Let $\vec{p}$ be a finite set of propositional variables and $\mathcal{W}_{\vec{p}}$ denote the set of all states on the variables $\vec{p}$. L^{μ} denotes the language of the unimodal μ-calculus and $\mathcal{L}^{\mu}$ its set of theorems, while $\mathcal{L}^{\mu}(U)$ is the set of formulas valid on U. We define $\mathcal{L}^{+}$ analogously.

Theorem 3.2. *Any set* $U \subseteq \mathcal{W}_{\vec{p}}$ *which is closed under simulability is definable in* L^{+} *by a formula* $\varphi(U)$*, in the sense that* $\mathfrak{w} \models \varphi(U)$ *if and only if* $\mathfrak{w} \in U$*.*

Corollary 3.1. *If* $U \subseteq \mathcal{W}_{\vec{p}}$ *is closed under simulability, then* $\mathcal{L}^{\mu}(U)$ *is polynomially (in fact, linearly) reducible to* $\mathcal{L}^{\mu}$*.*

If further U *is open, then* $\mathcal{L}^{\mu}(U)$ *is axiomatized by* $\mathcal{L}^{\mu} + \varphi(U)$*.*

Analogous claims also hold for $\mathcal{L}^{+}$ *in place of* $\mathcal{L}^{\mu}$*.*

We can combine this with the fact that the validity problem for the μ-calculus is in EXPTIME [4] to show the following:

Corollary 3.2. *If* $U \subseteq \mathcal{W}_{\vec{p}}$ *is closed under simulability, then* $\mathcal{L}^{\mu}(U)$ *is decidable in* EXPTIME*.*

References

[1] A. Dawar and M. Otto. Modal characterisation theorems over special classes of frames. *Annals of Pure and Applied Logic*, 161:1–42, 2009. Extended journal version LICS 2005 paper.

[2] Dexter Kozen. Results on the propositional μ-calculus. *Theor. Comput. Sci.*, 27:333–354, 1983.

[3] Luigi Santocanale and Yde Venema. Completeness for flat modal fixpoint logics. *Annals of Pure and Applied Logic*, 162(1):55 – 82, 2010.

[4] Klaus Schneider. *Verification of reactive systems: formal methods and algorithms.* Springer, 2004.

LATTICE PSEUDOEFFECT ALGEBRAS AS DOUBLE RESIDUATED STRUCTURES

DAVID J. FOULIS, SYLVIA PULMANNOVÁ AND ELENA VINCEKOVÁ

1. Introduction

An effect algebra is a partial algebraic structure, originally introduced as an algebraic base for unsharp quantum measurements [5]. Recently, lattice effect algebras (LEAs) have been studied as possible algebraic models for the semantics of non-standard symbolic logics [6], just as MV-algebras (special kinds of LEAs) are algebraic models for Łukasiewicz many-valued logics, and orthomodular lattices (also special kinds of LEAs) are algebraic models for sharp quantum logical calculi. The interplay among conjunction, implication, and negation connectives on LEAs has been studied, the conjunction and implication connectives being related by a residuation law. As a result, a characterization of LEAs has been obtained in terms of so-called Sasaki algebras and a more general structures called CI (conjunction-implication) posets.

In [3, 4] pseudoeffect algebras (PEAs) were introduced as non-commutative generalizations of effect algebras. They are especially interesting for their relations with po groups [4, 2]. In the present talk based on the paper [7], we study lattice pseudoeffect algebras (LPEAs), focusing mainly on their logical aspects. In [9], it was shown that there are two analogues of the Sasaki product in LPEAs: the "right" and the "left". It turns out that all logical connectives are doubled, and we also have two residuation laws. As a main result, we obtain a characterization of lattice pseudoeffect algebras in terms of so-called pseudo Sasaki algebras. While LPEAs are partial algebraic structures, Sasaki algebras are total algebras.

Just as Sasaki algebras are a special subclass of CI-posets (see [6]), pseudo Sasaki algebras are a special subclass of so-called double CI-posets, which we introduce as bounded posets endowed with two pairs of binary operations $(\circ, \rightsquigarrow)$ and $(*, \rightarrow)$, again connected by residuation. In comparison with bounded residuated posets, see [8], it appears that the structure of double CI-posets is more general.

We introduce conditional double CI-posets as partially defined versions of double CI-posets, and we show that all pseudoeffect algebras can alternatively be described as conditional double CI-posets.

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2. Lattice pseudoeffect algebras and pseudo Sasaki algebras

Definition 2.1. A *pseudoeffect algebra* (PEA) is a partial algebra $(P; \oplus, 0, 1)$ of the type $(2, 0, 0)$ where the following axioms hold for any $a, b, c \in P$:

(PE1) $a \oplus b$ and $(a \oplus b) \oplus c$ exist iff $b \oplus c$ and $a \oplus (b \oplus c)$ exist and in this case $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
(PE2) There exists exactly one $d \in P$ and exactly one $e \in P$ such that $a \oplus d = e \oplus a = 1$.
(PE3) If $a \oplus b$ exists, there are elements $d, e \in P$ such that $a \oplus b = d \oplus a = b \oplus e$.
(PE4) If $a \oplus 1$ or $1 \oplus a$ exists, then $a = 0$.

In a pseudoeffect algebra, we may define a partial order in the following way:

$$a \leq b \text{ iff } c \oplus a = b \text{ for some } c \in P.$$

Equivalently,

$$a \leq b \text{ iff } a \oplus d = b \text{ for some } d \in P.$$

If a pseudoeffect algebra is a lattice under the partial order $\leq$, we call it a *lattice pseudoeffect algebra* (LPEA). A PEA admits two partial subtractions ("right" and "left") $\diagdown$ and $\diagup$ as follows: $b \diagdown a$ is defined and equals x iff $b = x \oplus a$, and $a \diagup b$ is defined and equals y iff $b = a \oplus y$. Thus both $b \diagdown a$ and $a \diagup b$ are defined iff $a \leq b$, and then $(b \diagdown a) \oplus a = b = a \oplus (a \diagup b)$. Moreover, for the elements d and e in axiom (PE2) we write $1 \diagdown a =: a^-$ (the "left" complement) and $a \diagup 1 := a^\sim$ (the "right" complement). Clearly, $0^\sim = 1 = 0^-$ and $1^\sim = 0 = 1^-$. Notice that $a \oplus b$ exists iff $a \leq b^-$, equivalently, iff $b \leq a^\sim$, also $a^{-\sim} = a^{\sim-} = a$ and $a \leq b \Rightarrow b^- \leq a^-, b^\sim \leq a^\sim$.

The following two operations on a lattice pseudoeffect algebra $(P; \oplus, 0, 1)$ were introduced in [9] as a generalization of the Sasaki product in LEAs:

Definition 2.2. $a \circ b := a \wedge b^- \diagup a$; $\quad a * b := a \diagdown a \wedge b^\sim$.

Definition 2.3. A structure $(P; \leq, ^-, ^\sim, \circ, *, 0, 1)$ of the type $(1, 1, 2, 2, 0, 0)$ will be called a *pseudo Sasaki algebra* if it satisfies the following axioms:

(PSA1) $a^{-\sim} = a^{\sim-} = a$ and $\;a \leq b \Rightarrow b^- \leq a^-,\ b^\sim \leq a^\sim$.
(PSA2) $a \circ 1 = 1 \circ a = a = 1 * a = a * 1$.
(PSA3) $a \circ b \leq c$ iff $a * c^- \leq b^-$ and
$a * b \leq c$ iff $a \circ c^\sim \leq b^\sim$.
(PSA4) $c \leq a, c \leq b \Rightarrow c \leq a \circ (a * b^-)^\sim = a * (a \circ b^\sim)^-$.
(PSA5) $a \leq b^-, c \leq a^\sim \circ b^\sim \Rightarrow (a^\sim \circ b^\sim) \circ c^\sim = a^\sim \circ (b^\sim \circ c^\sim)$ and $b \leq a^\sim, c \leq b^- * a^- \Rightarrow c^- * (b^- * a^-) = (c^- * b^-) * a^-$.
(PSA6) $a \leq b^- \Rightarrow (a^\sim \circ b^\sim)^- = (b^- * a^-)^\sim$.

The next two theorems show that lattice pseudoeffect algebras (as partial algebraic structures) are mathematically equivalent with pseudo Sasaki algebras (as total algebraic structures).

Theorem 2.4. *Let $(P; \oplus, 0, 1)$ be a lattice pseudoeffect algebra. With operations $\circ$ and $*$ as in Definition 2.2, $(P; \leq, ^-, ^\sim, \circ, *, 0, 1)$ becomes a pseudo Sasaki algebra.*

Theorem 2.5. *Let $(P; \leq, ^-, ^\sim, \circ, *, 0, 1)$ be a pseudo Sasaki algebra. With the operation $\oplus$ defined by $A \oplus b = (b^- * a^-)^\sim = (a^\sim \circ b^\sim)^-$ whenever $a \leq b^-$, $(P; \oplus, 0, 1)$ is an LPEA.*

Notice that a PEA is an effect algebra iff $a \oplus b$ exists whenever $b \oplus a$ exists, and the equality $a \oplus b = b \oplus a$ holds. It then follows that the right and left subtractions and left and right complements coincide. In this case we have $a \circ b = a * b$ for all a, b, and the pseudo Sasaki algebra $P(L)$ is reduced to a Sasaki algebra [6, Definition 4.1].

3. Double CI-posets

In [6], the notion of a conjunction/implication poset (CI-poset) was introduced. As a non-commutative generalization of a CI-poset, we will now consider a structure $(P; \leq, 0, 1, \circ, *, \rightarrow, \rightsquigarrow)$ satisfying the following two axioms:

(1) $1 \circ a = a \circ 1 = a = 1 * a = a * 1$ (unity),
(2) $a \circ b \leq c \Leftrightarrow b \leq a \rightsquigarrow c$ and $a * b \leq c \Leftrightarrow b \leq a \rightarrow c$ (residuation),

and with complements defined by $a^- := a \rightarrow 0$ and $a^\sim := a \rightsquigarrow 0$

Definition 3.1. The structure $(P; \leq, 0, 1, \circ, *, \rightarrow, \rightsquigarrow)$ satisfying axioms (1) and (2) will be called a *double CI-poset*. If a double CI-poset is a lattice, we call it a *double CI-lattice*.

Double CI-posets are very general algebraic structures, which include many other residuated algebras as special subalgebras, among them pseudo Sasaki algebras and residuated po monoids [8].

Definition 3.2. A double CI-poset P satisfies:

(i) the *pseudo-involution condition* if $a^{-\sim} = a = a^{\sim -}$ and $a \leq b \Rightarrow b^- \leq a^-$, $b^\sim \leq a^\sim$;
(ii) the *divisibility condition* if $c \leq a$, $c \leq b \Leftrightarrow c \leq a \circ (a \rightsquigarrow b) = a * (a \rightarrow b)$;
(iii) the *ortho-exchange condition* if
$a^- \circ b^- = 0$ and $c^\sim \leq a \circ b$ implies $b^- \leq a * c$
and
$a^\sim * b^\sim = 0$ and $c^- \leq a * b$ implies $b^\sim \leq a \circ c$;
(iv) the *self-adjointness condition* if $a \circ b \leq c \Leftrightarrow a * c^- \leq b^-$ and $a * b \leq c \Leftrightarrow a \circ c^\sim \leq b^\sim$.

for all $a, b, c \in P$.

Theorem 3.3. *Every pseudo Sasaki algebra is a double CI-lattice. Conversely, a double CI-poset is a pseudo Sasaki algebra iff it satisfies conditions* (i) (*pseudo-involution*), (ii) (*divisibility*), (iii) *resp.* (iv) (*self-adjoiness resp. ortho-exchange*) *of Definition 3.2, and conditions* (*PSA5*) *and* (*PSA6*) *in Definition 2.3.*

Remark 3.4. Recall that a *residuated partially ordered groupoid* is an algebra $(A; \cdot, \backslash, /, \leq)$ such that $(A; \leq)$ is a partially ordered set and the following law of residuation holds:

$$(res) \qquad x \cdot y \leq z \text{ iff } y \leq x\backslash z \text{ iff } x \leq z/y.$$

(For more information see [8]). Let us consider a CI-poset. Under the stipulation

$$x\backslash y = x \rightsquigarrow y, \qquad x/y = y \to x,$$

we observe that the residuation law (res) is satisfied iff $x \circ y = y * x = x \cdot y$. Hence a CI-poset is a residuated po-groupoid iff $x \circ y = y * x$ for all x, y. It is a residuated po-monoid iff in addition the operation $\circ$ is associative.

Remark 3.5. Up to now, we considered lattice ordered pseudoeffect algebras as double residuated structures. We note that in [1] it was shown that certain pseudoeffect algebras, so called "good" pseudoeffect algebras, can be characterized as conditionally residuated structures, where the residuated operations are partially defined. Now, if we restrict the implication and conjunction operations in double CI-posets to suitably defined partial operations, we obtain *conditional double CI-posets*, which give an equivalent description for all pseudoeffect algebras.

References

[1] I. Chajda, R. Halaš: *Effect algebras are conditionally residuated structures*, preprint.

[2] A. Dvurečenskij: *Pseudo MV-algebras are intervals in l-groups*, J. Austral. Math. Soc. **72** (2002), 427–445.

[3] A. Dvurečenskij, T. Vetterlein: *Pseudoeffect algebras. I. Basic properties* Inter. J. Theor. Phys. **40** (2001), 685–701.

[4] A. Dvurečenskij, T. Vetterlein: *Pseudoeffect algebras. II. Group representation.* Inter. J. Theor. Phys. **40** (2001), 703–726.

[5] D.J. Foulis, M.K. Bennett: *Effect algebras and unsharp quantum logics*, Found. Phys. **24** (1994), 1325–1346.

[6] D. J. Foulis, S. Pulmannová: *Logical connectives on lattice effect algebras.*, preprint.

[7] D. J. Foulis, S. Pulmannová, E. Vinceková: *Lattice pseudoeffect algebras as double residuated structures*, Soft. Comput., to appear.

[8] N. Galatos, P. Jipsen, T. Kowalski, H. Ono: *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*, in Studies in Logic, vol. 151, Elsevier 2007.

[9] S. Pulmannová: Generalized Sasaki projections and Riesz ideals in pseudoeffect algebras, Int. J. Theor. Phys. **42** (2003), 1413–1423.

Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA, USA; Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia

E-mail address: `foulis@math.umass.edu, pulmann@mat.savba.sk, vincek@mat.savba.sk`

CANONICAL EXTENSIONS AND UNIVERSAL PROPERTIES

MAI GEHRKE
JACOB VOSMAER

Abstract. The theory of canonical exensions is an important tool in algebraic logic. We discuss several universal properties of canonical extensions, which relate them to domain theory and topological algebra. In particular, we show that the canonical extension of a lattice can be given a dcpo presentation, and that canonical extensions have universal properties with respect to profinite and certain Boolean topological lattice-based algebras.

1. Introduction

Canonical extensions, which were introduced in 1951 by Jónsson and Tarski [9, 10] as part of the representation theory for relation algebras, are nowadays an important construction in algebraic logic. Inspired by the way any Boolean algebra embeds in the power set of its ultrafilters, canonical extension provide an algebraic, point-free perspective on Stone duality. In this extended abstract we discuss several mathematical properties of canonical extensions.

Using Stone duality, one can embed any Boolean algebra $\mathbb{A}$ in a power set algebra: simply take X, the space of all ultrafilters of $\mathbb{A}$, and embed $\mathbb{A}$ in $\mathcal{P}(X)$ by sending $a \mapsto \{x \in X \mid a \in x\}$. The embedding of $\mathbb{A}$ into $\mathcal{P}(X)$ can be seen as the starting point of the theory of canonical extensions. We study this embedding without explicitly using Stone duality. The advantage of this is that one can then define canonical extensions for bounded lattices [5] (or even posets [4]) rather than just for Boolean algebras.

Concretely, the canonical extension of a lattice $\mathbb{L}$ is defined up to isomorphism as a completion $\mathbb{C}$ of $\mathbb{L}$, that is as a lattice embedding $e\colon \mathbb{L} \to \mathbb{C}$ where $\mathbb{C}$ is a complete lattice, so that:

- for all $x, y \in \mathbb{C}$ such that $x \nleq y$, there exist $F, I \subseteq \mathbb{L}$ such that
  - F is a filter, $\bigwedge e[F] \leq x$ and $\bigwedge e[F] \nleq y$;
  - I is an ideal, $\bigvee e[I] \geq y$ and $\bigvee e[I] \ngeq x$;
- for every filter $F \subseteq \mathbb{L}$ and for every ideal $I \subseteq \mathbb{L}$, if $\bigwedge e[F] \leq \bigvee e[I]$ then $F \cap I \neq \emptyset$.

Since canonical extensions are unique up to isomorphism, we will speak of *the* canonical extension of any given lattice $\mathbb{L}$ and denote this extension as $e\colon \mathbb{L} \to \mathbb{L}^\delta$. This definition is the starting point of an extensive theory [7] of completions not only for lattices but also for lattice-based algebras.

In this extended abstract, we present two recently developed perspectives on canonical extensions, both of which can be seen in terms of universal properties.

- We show how canonical extensions of *lattices* can be viewed as dcpo's using techniques from domain theory in §2.
- We show that canonical extensions of *lattice-based algebras* have universal properties with respect to topogical algebras in §3.

2. Domain-theoretic universal properties

Usually, the canonical extension $e\colon \mathbb{L} \to \mathbb{L}^\delta$ of a lattice is understood in terms of the relation between the internal structure of $\mathbb{L}^\delta$ and the forward image of $\mathbb{L}$ under e; see above. In [6] and [13, §2.3], we showed that $\mathbb{L}^\delta$ can also be characterized *externally* in terms of of how $\mathbb{L}^\delta$ sits in the category of dcpo's and Scott-continuous functions, using a universal property. Since this characterization uses Scott-continuous functions, we propose to call it a *domain-theoretic* [1] characterization. We make use of technical results due to Jung, Moshier and Vickers [11].

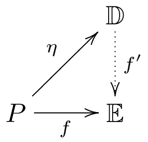
Definition 1. A *dcpo presentation* [11] is a triple $\langle P, \sqsubseteq, \lhd \rangle$ where

Gehrke: IMAPP, Radboud University Nijmegen; Vosmaer (corresponding author): ILLC, University of Amsterdam.

- $\langle P, \sqsubseteq\rangle$ is a pre-order;
- $\lhd \subseteq P \times \mathcal{P}(P)$ is a binary relation such that $a \lhd U$ only if $U \subseteq P$ is non-empty and directed.

An order-preserving map $f : P \to \mathbb{D}$ to a dcpo $\mathbb{D}$ is *cover-stable* if for all $a \lhd U$, $f(a) \leq \bigvee f[U]$.

In other words, a dcpo presentation consists of a pre-ordered set of generators $\langle P, \sqsubseteq\rangle$ together with set of relations of the form $a \leq \bigvee U$. A dcpo presentation $\langle P, \sqsubseteq, \lhd\rangle$ *presents* a dcpo $\mathbb{D}$ if there exists a cover-stable order-preserving map $\eta : P \to \mathbb{D}$ such that for all dcpos $\mathbb{E}$, if $f : P \to \mathbb{E}$ is a cover-stable order-preserving map then there exists a unique Scott-continuous $f' : \mathbb{D} \to \mathbb{E}$ such that $f' \circ \eta = f$. If this is the case, we say that $\langle P, \sqsubseteq, \lhd\rangle$ presents $\mathbb{D}$ via η.



We know from [11] that every dcpo presentation presents a dcpo.

We will now need some notation. Given a lattice $\mathbb{L}$, we denote its filter and ideal completion by $\mathcal{F}\mathbb{L}$ and $\mathcal{I}\mathbb{L}$, respectively. It is well known that $\mathcal{I}\mathbb{L}$ can be characterized externally as the free dcpo over $\mathbb{L}$; similarly $\mathcal{F}\mathbb{L}$ is the free co-dcpo over $\mathbb{L}$. Given two sets $X, Y \subseteq Z$ we define $X \between Y :\Leftrightarrow X \cap Y \neq \emptyset$. We can now define the dcpo presentation with which we characterize the canonical extension:

Definition 2. Given a lattice $\mathbb{L}$, we define a dcpo presentation $\Delta(\mathbb{L}) := \langle \mathcal{F}\mathbb{L}, \supseteq, \lhd_{\mathbb{L}}\rangle$, where for all $F \in \mathcal{F}\mathbb{L}$ and $S \subseteq \mathcal{F}\mathbb{L}$ directed,

$$F \lhd_{\mathbb{L}} S \text{ iff } \forall I \in \mathcal{I}\mathbb{L},\ \left[\forall F' \in S,\ F' \between I\right] \Rightarrow F \between I.$$

Observe that the generators of $\Delta(\mathbb{L})$ are the *filters* of $\mathbb{L}$, ordered by reverse inclusion. The basic cover relations use a universal quantification over all *ideals* of $\mathbb{L}$. The following theorem states that $\Delta(\mathbb{L})$ is indeed a presentation of $\mathbb{L}^\delta$. The map $e^{\mathcal{F}} : \mathcal{F}\mathbb{L} \to \mathbb{L}^\delta$ is defined as $e^{\mathcal{F}} : F \mapsto \bigwedge e[F]$.

Theorem 3. *Let $\mathbb{L}$ be a lattice and let $e : \mathbb{L} \to \mathbb{L}^\delta$ be its canonical extension. Then $\Delta(\mathbb{L})$ presents $\mathbb{L}^\delta$ via $e^{\mathcal{F}} : \mathcal{F}\mathbb{L} \to \mathbb{L}^\delta$.*

The abstract machinery of dcpo presentations [11] allows us to do more than just characterize the canonical extension of $\mathbb{L}$: we can also use it to extend operators on lattices to operators on canonical extensions. An operator $f : \mathbb{L}^n \to \mathbb{L}$ is a map which preserves binary joins in each coordinate. The following result, which is due to Gehrke and Harding [5], can now be seen as an instantiation of the dcpo presenation results of [11]. The condition "$\omega_{\mathbb{A}^\delta} = (\omega_{\mathbb{A}})^\triangledown$" means that we need to take the lower extension of each operator involved; we will not go into this technical detail here.

Theorem 4 ([5])**.** *Let $\mathbb{A}$ be a lattice-based algebra and let $s \preccurlyeq t$ be an inequation. If for each operation ω occurring in s or t, $\omega_{\mathbb{A}}$ is an operator and $\omega_{\mathbb{A}^\delta} = (\omega_{\mathbb{A}})^\triangledown$, then $\mathbb{A} \models s \preccurlyeq t$ implies $\mathbb{A}^\delta \models s \preccurlyeq t$.*

3. Universal properties with respect to topological algebras

Although the theory of canonical extensions can be viewed *sec* as being about completions of lattices, its real strength lies in providing completions for *lattice-based algebras*. In the previous section, we saw that the canonical extension $\mathbb{L}^\delta$ of any lattice $\mathbb{L}$ has a universal property with respect to dcpo's. Below we will see, that the canonical extension $\mathbb{A}^\delta$ of any lattice-based algebra $\mathbb{A}$ has one or more universal properties with respect to *topological algebras*, i.e. algebras with continuous operations. The results in this section come from [13, Ch. 3] and [7].

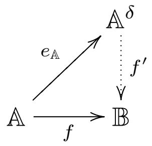
Below, we will consider two kinds of topological lattice-based algebras: *Boolean topological* algebras and *profinite* algebras. A Boolean topological algebra is simply a topological algebra whose topology is Boolean, i.e. compact, Hausdorff and zero-dimensional. Natural examples of such algebras are provided by profinite algebras: algebras with are projective limits of finite algebras. If we endow the finite algebras $\mathbb{A}_i$ in a projective limit $\varprojlim_I \mathbb{A}_i$ with the discrete topology, then $\varprojlim_I \mathbb{A}_i$ inherits a Boolean topology from the $\mathbb{A}_i$. Profinite algebras have been studied extensively in Galois theory [12], but the construction makes as much sense in a universal algebra setting as it does in the restricted case of group theory. Moreover, the property of being profinite (i.e. being the projective limit of a collection of finite algebras) can correspond to meaningful properties of an algebra. For instance:

- A Boolean algebra is profinite iff it is complete and atomic;
- A distributive lattice is profinite iff it is complete and bi-algebraic;
- A Heyting algebra is profinite iff it is complete, bi-algebraic and residually finite, iff it is (isomorphic to) the down-set lattice of an image-finite poset;
- A distributive lattice with operators is profinite iff it is (isomorphic to) the complex algebra of a hereditarily finite ordered Kripke frame.

The results above about Boolean algebras and distributive lattices have a long history and are described in [8, Ch. VI]. The result about Heying algebras was established more recently by Bezhanishvili & Bezhanishvili [2]; the result about distributive lattices with operators is discussed in [13, §4.1].

Interestingly, canonical extensions are strongly related to profinite lattice-based algebras, as we will see in the result below.

Theorem 5. *Let $f\colon \mathbb{A} \to \mathbb{B}$ be a homomorphism from a lattice-based algebra $\mathbb{A}$ to a profinite lattice-based algebra $\mathbb{B}$. Then f extends uniquely to the canonical extension $e_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{A}^\delta$ of $\mathbb{A}$: there exists a unique complete homomorphism $f'\colon \mathbb{A}^\delta \to \mathbb{B}$ such that $f' \circ e_{\mathbb{A}} = f$.*

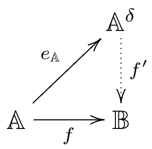


The result above should perhaps not be called a universal property in the strictest sense, because it need not be the case that $\mathbb{A}^\delta$ is itself a profinite algebra [13, Example 3.4.3]. In fact, one can prove the following:

Theorem 6. *Fix a lattice-based similarity type Ω and let $\mathcal{V}$ be a variety of lattice-based Ω-algebras.*
(1) *If $\mathcal{V}$ is finitely generated then for every $\mathbb{A} \in \mathcal{V}$, $\mathbb{A}^\delta$ is profinite;*
(2) *If Ω is finite and for every $\mathbb{A} \in \mathcal{V}$, $\mathbb{A}^\delta$ is profinite, then $\mathcal{V}$ is finitely generated.*

Remember that earlier we introduced profinite algebras as an example of Boolean topological algebras. All profinite algebras are Boolean topological algebras but the converse is not true; one example is provided by the variety of modal algebras [13, §4.3]: there exist Boolean topological modal algebras $\mathbb{B}$ which are not profinite (also see the Example below). For such modal algebras $\mathbb{B}$, our Theorems 5 and 6 need no longer hold. However, the *lattice reduct* of a Boolean topological modal algebra is always a profinite *lattice*. This allows us to prove the following result. Observe that we denote the lattice reduct of a lattice-based algebra $\mathbb{A}$ by $\mathbb{A}^l$, and that a monotone lattice-based algebra is simply an algebra for which each operation is either order-preserving or order-reversing in each coordinate.

Theorem 7. *Fix a lattice-based similarity type Ω and a canonical extension type β. Let $\mathbb{A}$ be a monotone lattice-based Ω-algebra. If $f\colon \mathbb{A} \to \mathbb{B}$ is an Ω-homomorphism to a Boolean topological monotone lattice-based Ω-algebra $\mathbb{B}$ and if $\mathbb{B}^l$, the lattice reduct of $\mathbb{B}$, is profinite, then there exists a unique complete Ω-algebra homomorphism $f'\colon \mathbb{A}^\delta \to \mathbb{B}$ such that $f' \circ e_{\mathbb{A}} = f$.*



We would like to point out that the condition in the theorem above that $\mathbb{B}^l$ be profinite is not superfluous: there exist examples of Boolean topological modular lattices which are not profinite [3]. The universal property introduced in Theorem 7 leads to the following interesting corollary:

Theorem 8. *Fix a lattice-based signature Ω and let $\mathbb{A}$ be a monotone lattice-based Ω-algebra such that $\mathbb{A}^l$, the lattice reduct of $\mathbb{A}$, is profinite. Then there exists a unique complete* lattice *homomorphism $g\colon (\mathbb{A}^\delta)^l \to \mathbb{A}^l$ such that $g \circ e_{\mathbb{A}} = \mathrm{id}_{\mathbb{A}}$. Moreover, the following are equivalent:*

(1) $g\colon \mathbb{A}^\delta \to \mathbb{A}$ *is an* Ω*-algebra homomorphism;*
(2) $\mathbb{A}$ *is a Boolean topological algebra.*

Example. In the case of modal algebras (or more generally, Boolean algebras with operators), Theorem 8 can be interpreted in terms of Kripke frames as follows. First, observe the following fact [13, §4.3.1]:

- A modal algebra admits a Boolean topology iff it is (isomorphic to) the complex algebra of an image-finite Kripke frame,

where we remind the reader that a Kripke frame is image-finite iff each of its point has a finite set of successors. Theorem 8 now dually corresponds to the well-known observation that a Kripke frame is image-finite iff it embeds in its ultrafilter extension; see [13, §4.3.2] for more details.

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References

[1] S. Abramsky and A. Jung. Domain theory. In *Handbook of logic in computer science*, volume 3, pages 1–168. Clarendon Press, Oxford, 1994.

[2] G. Bezhanishvili and N. Bezhanishvili. Profinite Heyting algebras. *Order*, 25(3):211–227, 2008.

[3] D. J. Clinkenbeard. Simple compact topological lattices. *Algebra Universalis*, 9(3):322–328, 1979.

[4] J. M. Dunn, M. Gehrke, and A. Palmigiano. Canonical extensions and relational completeness of some substructural logics. *J. Symbolic Logic*, 70:713–740, 2005.

[5] M. Gehkre and J. Harding. Bounded lattice expansions. *J. Algebra*, 238:345–371, 2001.

[6] M. Gehrke and J. Vosmaer. Canonical extension and canonicity via dcpo presentations, 2011. Theoretical Computer Science (in press).

[7] M. Gehrke and J. Vosmaer. A view of canonical extension, 2011. Proceedings of the Eighth International Tbilisi Symposium on Language, Logic and Computation.

[8] P. T. Johnstone. *Stone spaces*, volume 3 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1982.

[9] B. Jónsson and A. Tarski. Boolean algebras with operators I. *Amer. J. Math.*, 73:891–939, 1951.

[10] B. Jónsson and A. Tarski. Boolean algebras with operators II. *Amer. J. Math.*, 74:127–162, 1952.

[11] A. Jung, M. A. Moshier, and S. J. Vickers. Presenting dcpos and dcpo algebras. In *Proceedings of the 24th Conference on the Mathematical Foundations of Programming Semantics (MFPS XXIV)*, volume 218 of *Electronic Notes in Theoretical Computer Science*, pages 209–229, 2008.

[12] L. Ribes and P. Zalesskii. *Profinite groups*, volume 40 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2000.

[13] J. Vosmaer. *Logic, Algebra and Topology. Investigations into canonical extensions, duality theory and point-free topology.* PhD thesis, Institute for Logic, Language and Computation, December 2010. Promotores: M. Gehrke and Y. Venema.

E-mail address: `mgehrke@math.ru.nl, contact@jacobvosmaer.nl`

COMPLETIONS OF SEMILATTICES

M. J. GOUVEIA AND H. A. PRIESTLEY

The work we present here was motivated by recent results obtained for lattice-based algebras in finitely generated varieties which reconcile canonical, natural and profinite extensions of those algebras.

A canonical extension of a lattice-based algebra is a completion of its lattice reduct with special properties that allow one to lift to it the non-lattice basic operations. In [5] Gehrke and Harding proved that a canonical extension of a bounded lattice always exists and that it is unique up to isomorphism. Subsequently the ideas were extended to posets, and existence and uniqueness of canonical extensions established in this wider setting (see for example [3]). In particular canonical extensions of semilattices, and of the semilattice reducts of lattices, may be studied.

A profinite limit of an algebra is a particular subalgebra of a direct product whose factors are finite quotients of the algebra. When the algebra belongs to a residually finite variety, the algebra embeds into its profinite limit, and this then serves as a completion. The operations of the given algebra are naturally extended by the operations of the profinite completion, which are coordinatewise defined. In the particular case of a variety $\mathcal{V}$ of lattice-based algebras, $\mathcal{V}$ is residually finite whenever it is finitely generated and so both completions, canonical and profinite, are available for every algebra in $\mathcal{V}$ and they coincide (see [9] and [4]).

The third theme we wish to consider, that of a natural extension, arises out of ideas from natural duality theory (for background see the text of Clark and Davey [1]). In its simplest form this theory applies to suitable quasivarieties $\mathcal{D} = \mathbb{ISP}(\mathbf{M})$, where $\mathbf{M}$ is a finite algebra. One seeks $\underset{\sim}{\mathbf{M}}$ (some relational structure on M equipped with the discrete topology τ), so that $\mathcal{D}$ is dually equivalent to the category $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ of isomorphic copies of closed substructures of powers of $\underset{\sim}{\mathbf{M}}$. When this is possible, the alter ego $\underset{\sim}{\mathbf{M}}$ is said to yield a duality on $\mathcal{D}$. This is the case in particular when $\mathcal{D}$ is the variety $\mathcal{D}$ of bounded distributive lattices, which is representable as $\mathbb{ISP}(2)$, with $\underset{\sim}{\mathbf{M}} = (\{0,1\}; \leqslant, \tau)$ with $\leqslant$ the usual partial order. We arrive at Priestley duality, which (in an equivalent formulation) was used by Gehrke and Jónsson in [7] to build the canonical extension of any $\mathbf{L} \in \mathcal{D}$ as a lattice of order-preserving maps. In [2], Davey, Gouveia, Haviar and Priestley mimiced this construction in the setting of any quasivariety $\mathbb{ISP}(\mathbf{M})$, whether dualisable or not, and more generally for any residually finite variety (or merely prevariety). The natural extension so defined was shown always to coincide with the profinite completion [2, Theorem 3.6]. Moreover, as a consequence of the results of [2], the natural extension of an algebra $\mathbf{A}$ in a dualisable quasivariety $\mathbb{ISP}(\mathbf{M})$ is the algebra consisting of all the maps from the dual of $\mathbf{A}$ to M which preserve the relational structure of a dualising alter ego.

Now let us focus on the variety of central interest to us here. Let $\mathcal{S}$ be the category of $\vee$-semilattices with 0 where the morphisms are the maps that preserve $\vee$ and 0. (Since we shall consider only semilattices of this type, we shall henceforth use the term 'semilattice' to mean an element of $\mathcal{S}$.) It is well known that $\mathcal{S}$ is generated, as a variety and as a quasivariety, by the two-element semilattice $\mathbf{M} = \langle\{0,1\}; \vee, 0\rangle$. The classic duality for semilattices due to Hofmann, Mislove and Stralka [10] can with hindsight be seen as an instance of a natural duality. The discretely topologised semilattice $\underset{\sim}{\mathbf{M}} := \langle\{0,1\}; \vee, 0, \tau\rangle$ provides an alter ego for $\mathbf{M}$ whereby the category $\mathcal{S}$ is dually equivalent to the category $\mathcal{Z} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ of all Boolean topological join-semilattices with 0.

Since $\mathcal{S} = \mathbb{ISP}(\mathbf{M})$, the elements of any semilattice $A \in \mathcal{S}$ are separated by the morphisms in $\mathcal{S}(\mathbf{A}, \mathbf{M})$. Consequently the map $e\colon \mathbf{A} \to \mathbf{M}^{\mathcal{S}(\mathbf{A},\mathbf{M})}$ defined by evaluation

$$\forall a \in A\ \forall h \in \mathcal{S}(\mathbf{A}, \mathbf{M})\ e(a)(h) = h(a)$$

is an $\mathcal{S}$-embedding. The natural extension of $\mathbf{A}$ is, by definition, the topological closure of $e(A)$ in the direct power $\underset{\sim}{\mathbf{M}}^{\mathcal{S}(\mathbf{A},\mathbf{M})}$. Consider the forgetful functor $^\flat\colon \mathcal{Z} \to \mathcal{S}$ that assigns to each dual $\mathrm{D}(\mathbf{A})$ the semilattice $\mathrm{D}^\flat(\mathbf{A}) = \langle\mathcal{S}(\mathbf{A}, \mathbf{M}); \vee, 0\rangle$. Since $\mathrm{D}^\flat(\mathbf{A})$ is in $\mathcal{S}$ we may consider the semilattice $\mathrm{D}^\flat(\mathrm{D}^\flat(\mathbf{A}))$. The elements of this semilattice are all the maps from $\mathrm{D}^\flat(\mathbf{A})$ to $\{0,1\}$ that preserve $\{\vee, 0\}$. By applying Theorem 4.3 of [2], $\mathrm{D}^\flat(\mathrm{D}^\flat(\mathbf{A}))$ is the natural extension of $\mathbf{A}$. Moreover, since the variety $\mathcal{S}$ is the quasivariety generated by $\mathbf{M}$, we know that $\mathrm{D}^\flat(\mathrm{D}^\flat(\mathbf{A}))$ is also the profinite limit of $\mathbf{A}$.

Let us now investigate the canonical extension of a semilattice, viewed as a poset. Let $\mathbf{P}$ be a poset. A completion $\mathbf{C}$ of P, via an embedding ϵ is *compact* if, for each down-directed up-set F and each up-directed down-set I in P, the following condition is satisfied:

$$\bigwedge \epsilon(F) \leqslant \bigvee \epsilon(I) \Leftrightarrow F \cap I \neq \emptyset$$

The completion is *dense* if its copy in $\mathbf{C}$ is both Σ_1-dense and Π_1-dense. Here a subset Q of P is Σ_1-*dense* if any $u \in P$ is a join of meets of down-directed up-sets of Q and, order dually, is Π_1-*dense* if any $u \in P$ is a meet of joins of up-directed down-sets of Q.

The profinite limit $\widehat{\mathbf{A}}$ of a semilattice $\mathbf{A}$ is, by virtue of its definition, a semilattice with 0 which is closed under arbitrary joins. However it does not satisfy the required conditions to be a canonical extension of $\mathbf{A}$. Even so, the profinite limit of a semilattice is quite "close" to being a canonical extension.

Theorem *Let* $\mathbf{A}$ *in* $\mathcal{S}$ *and let* $e\colon \mathbf{A} \to \widehat{\mathbf{A}}$ *be the natural embedding of* $\mathbf{A}$ *into the profinite lattice* $\widehat{\mathbf{A}}$. *Then* $\widehat{A}$ *is a compact and* Π_1*-dense completion of* $\mathbf{A}$ *via the embedding* e.

In general the profinite limit $\widehat{\mathbf{A}}$ of $\mathbf{A}$ is not the canonical extension of $\mathbf{A}$: it may be too big for the Σ_1-density condition to be satisfied. To obtain a candidate for the canonical extension of $\mathbf{A}$ we therefore restrict attention to the subset $C_\vee(A)$ of $\widehat{A}$ whose elements *are* joins of directed meets of elements of $e(A)$. Note that any such join is also a meet of joins of elements of $e(A)$, so that the Π_1-density property will not be sacrificed by cutting down from $\widehat{\mathbf{A}}$ to $C_\vee(A)$ and that $e\colon \mathbf{A} \to C_\vee(A)$ is certainly an order-embedding.

If it is to serve as a completion of $\mathbf{A}$, we need $C_\vee(A)$ to be a complete lattice. First note that every directed meet of elements of $e(A)$ is defined pointwise in the lattice $\widehat{\mathbf{A}}$. Hence $C_\vee(A)$, endowed with the pointwise join, forms a subsemilattice of $\widehat{\mathbf{A}}$. This subsemilattice is also closed under non-empty arbitrary (pointwise) joins.

We can now obtain a complete lattice

$$\mathbf{C}_\vee(\mathbf{A}) := \langle C_\vee(A); \vee, \wedge, 0, 1\rangle$$

where the meet operation is obtained from $\vee$ as follows:

$$\bigwedge(\alpha_t)_{t\in T} := \bigvee\{\, \alpha \in C_\vee(A) \mid \forall t \in T\ \alpha \leqslant \alpha_t \,\},$$

for every non-empty family $(\alpha_t)_{t\in T}$ of elements of $C_\vee(\mathbf{A})$. (The meet of the empty set is the top element $1 = \bigvee e(A)$ of $\widehat{\mathbf{A}}$.)

Theorem *The complete lattice $\mathbf{C}_\vee(\mathbf{A})$ is the canonical extension of the semilattice $\mathbf{A}$ with respect to the semilattice embedding e.*

We shall denote the canonical extension $\mathbf{C}_\vee(\mathbf{A})$ of the $\vee$-semilattice $\mathbf{A}$ by $\mathbf{A}^\delta_\vee$.

Every bounded lattice $\mathbf{L}$ has a reduct $\mathbf{A}$ in $\mathcal{S}$ and so, as a semilattice, is embeddable via e in the compact and dense complete lattice $\mathbf{A}^\delta_\vee$. The embedding e turns out to be a lattice homomorphism and consequently $\mathbf{A}^\delta_\vee$ is revealed to be a compact and dense completion of the lattice $\mathbf{L}$. This leads to a theorem linking lattice and semilattice canonical extensions.

Theorem *Let $\mathbf{L}$ be a lattice with 0 and let $\mathbf{A}$ be its reduct in $\mathcal{S}$. The lattice $\mathbf{A}^\delta_\vee$ is the canonical extension of $\mathbf{L}$.*

The canonical extension of a bounded distributive lattice is completely distributive. In [5] Gehrke and Harding established a restricted form of complete distributivity valid in the canonical extension of any bounded lattice; this provides an important technical tool for the analysis of canonical extensions of lattice-based algebras. We are able to show that Gehrke and Harding's result extends to the semilattice case. Specifically we prove that the canonical extension $\mathbf{A}^\delta_\vee$ of a $\mathbf{A} \in \mathcal{S}$ satisfies the $\bigvee\bigwedge$-restricted distributive law:

$$\bigvee\{\, \bigwedge e(Y) \mid Y \in \mathcal{Y} \,\} = \bigwedge\{\, \bigvee e(Z) \mid Z \subseteq A, \forall Y \in \mathcal{Y}\ Z \cap Y \neq \emptyset \,\},$$

where $\mathcal{Y}$ is a family of down-directed subsets of $\mathbf{A}$. Order dually, the canonical extension $\mathbf{A}^\delta_\wedge$ of the meet-semilattice reduct $\mathbf{A}_\wedge$ of $\mathbf{L}$ is, up to isomorphism, the canonical extension of $\mathbf{A}$ and satisfies the $\bigwedge\bigvee$ restricted distributive law. By considering the two semilattice reducts of a bounded lattice, we recapture Gehrke and Harding's result.

We have already noted that the profinite limit and the canonical extension of a semilattice in $\mathcal{S}$ do not need to coincide. That is also true when the semilattice is a reduct of a bounded lattice. However many examples can be found of (infinite) lattices whose profinite and canonical completions do coincide. This happens in particular for any bounded lattice that satisfying the Ascending Chain Condition (ACC) and for any infinite chain. An infinite chain is a very special case of a bounded distributive lattice $\mathbf{L}$. Realising the canonical extension of $\mathbf{L}$ as the lattice of all order-preserving maps from $\mathcal{D}(\mathbf{L}, 2)$ to $\{0, 1\}$, we are able to prove that the canonical extension $\mathbf{L}^\delta$ is a semilattice retract of the profinite limit $\widehat{\mathbf{L}_\vee}$. In some particular cases we can go further and prove that $\mathbf{L}^\delta$ is isomorphic to $\widehat{\mathbf{L}_\vee}$; equivalently, the profinite limit of the lattice $\mathbf{L}$ is isomorphic to the profinite limit of its join-semilattice reduct.

Theorem *The canonical extension of a bounded distributive lattice* $\mathbf{A}$ *is isomorphic to the profinite limit* $\widehat{\mathbf{A}_\vee}$ *of its* $\vee$*-semilattice reduct* $\mathbf{A}_\vee$ *whenever the poset of prime ideals of* $\mathbf{A}$ *has finite width.*

This result applies, for example, to every bounded distributive lattice with finite width, which consequently has isomorphic profinite and canonical completions.

References

1. Davey, B.A. and Clark, D.M.: Natural Dualities for the Working Algebraist. Cambridge University Press, 1998
2. Davey, B.A., Gouveia, M.J., Haviar, M., Priestley, H.A.: Natural extensions and profinite completions of algebras. To appear in Algebra Universalis
3. Dunn, J.M., Gehrke, M. and Palmigiano, A.: Canonical extensions and relational completeness of some substructural logics. J. Symbolic Logic, **70(3)**, (2005) 713–740.
4. Gouveia, M.J: A note on profinite completions and canonical extensions. Algebra Universalis, **64** (2010), 21–23.
5. Gehrke, M. and Harding, J.: Bounded lattice expansions, J. Algebra. **55** (2001), 345–371.
6. Gierz, G., Hofmann, K., Keimel, K., Lawson, J. D., Mislove, M. and Scott, D. S.: Continuous lattices and domains. Cambridge University Press, (2003).
7. Gehrke, M., Jónsson, B.: Bounded distributive lattice expansions. Math. Scand. **94**, 13–45 (2004)
8. Gehrke, M. and Priestley, H. : Canonical extensions and completions of posets and lattices, Reports on Mathematical Logic, 43, (2008) 133–152.
9. Harding, J.: On profinite completions and canonical extensions. Algebra Universalis, **55** (2006), 293–296.
10. Hofmann, K.H., Mislove, M. and Stralka, A.R.: The Pontryagin duality of compact 0-dimensional semilattices and its applications. Springer-Verlag Lecture Notes in Math. **396** (1974).

Faculdade de Ciências da Universidade de Lisboa & CAUL, P-1749-016 Lisboa, Portugal

E-mail address: matjoao@ptmat.fc.ul.pt

Mathematical Institute, University of Oxford, 24/29 St Giles, Oxford OX1 3LB, United Kingdom

E-mail address: hap@maths.ox.ac.uk

Formulas of Finite Number Propositional Variables in the Intuitionistic Logic With the Solovay Modality

Leo Esakia [1] , Revaz Grigolia [2]

Abstract

A description of finitely generated free algebras over the variety of Solovay algebras, as well as over its pyramid locally finite subvarieties is given.

In [5] Robert Solovay, among other things, presented a set-theoretical translation of modal formulas by putting $\Box p$ to mean "p is true in every transitive model of Zermelo-Fraenkel Set Theory **ZF**". By defining an interpretation as a function s sending modal formulas to sentences of **ZF** which commutes with the Boolean connectives and putting $s(\Box p)$ to be equal to the statement "$s(p)$ is true in every transitive model of **ZF**", Solovay formulated a modal system, which we call here SOL, and announced its **ZF**-completeness.

SOL is the classical modal system which results from the Gödel-Löb system GL (alias, the provability logic) by adding the formula $\Box(\Box p \to \Box q) \vee \Box(\Box q \to \Box p \wedge p)$ as a new axiom.

ZF-completeness: *For any modal formula p, $SOL \vdash p$ iff* **ZF** $\vdash s(p)$ *for any Solovay's interpretation s. [1]*

Now we shall formulate a simple system $I.SOL$, which is an intuitionistic "companion" of SOL: the composition of the well-known Gödel's modal translation of Heyting Calculus and split-map (= splitting a formula $\Box p$ into the formula $p \wedge \Box p$) provides the needed embedding of $I.SOL$ into SOL.

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[1]Department of Mathematical Logic of A. Razmadze Mathematical Institute. e-mail: esakia@hotmail.com

[2]Institute of Cybernetics, Tbilisi State University, e-mail: grigolia@yahoo.com

Definition 1. *An intuitionistic modal system $I.SOL$ is an extension of the proof-intuitionistic logic KM obtained by postulating the formula $(\Box p \to \Box q) \vee (\Box q \to p)$ as a new axiom.*

We recall that the proof-intuitionistic logic KM (=Kuznetsov- Muravitsky [4]) is the Heyting propositional calculus HC enriched by $\Box$ as Prov modality satisfying the following conditions: $p \to \Box p$, $\Box p \to (q \vee (q \to p))$, $(\Box p \to p) \to p$.

The purpose of this paper is to investigate the set of formulas of m propositional variable of the system $I.SOL$. The algebraic semantics for the system $I.SOL$ is based on the notion of Heyting algebra with an operator.

Definition 2. *A Heyting algebra with an operator $\Box$ is called Solovay algebra, if the following conditions are satisfied: $p \leq \Box p$, $\Box p \leq q \vee (q \to p)$, $\Box p \to p = p$, $(\Box p \to \Box q) \vee (\Box q \to p) = \top$.*

The class of all Solovay algebras forms a variety, which we denote by **SA**. It is known that the variety **SA** is finitely approximated and that if $(H, \Box) \in$ **SA** then the Heyting algebra H is cascade Heyting algebra [2]. A Heyting algebra H is called a *cascade Heyting algebra*, if H belongs to the variety generated by the class of all finite Boolean cascades. A finite Heyting algebra H is a *Boolean cascade*, if there exist Boolean lattices $B_1, \dots, B_k$ such that $H = B_1 + \dots + B_k$, where each B_i is a convex sublattice of H and $B_i + B_{i+1}$ denotes the ordinal sum of B_i and B_{i+1} in which the smallest element of B_i and the largest element of B_{i+1} are identified.

- *The variety of Solovay algebras is generated by cascade Heyting algebras.*

A topological space X with binary relation R is said to be *GL-frame* if: 1) X is a Stone space (i. e. 0-dimensional, Hausdorf and compact topological space); 2) $R(x)$ and $R^{-1}(x)$ are closed sets for every $x \in X$ and $R^{-1}(A)$ is a clopen for every clopen A of X; 3) for every clopen A of X and every element $x \in A$ there is an element $y \in A \setminus R^{-1}(A)$ such that either xRy or $x \in A \setminus R^{-1}(A)$.

A map $f : X_1 \to X_2$ from a GL-frame X_1 to a GL-frame X_2 is said to be *strongly isotone* if $f(x)R_2y \Leftrightarrow (\exists z \in X_1)(xR_1z \& f(z) = y)$.

Let us denote by **G** the category of GL-frames and continuous strongly isotone maps.

An algebra $(A; \vee, \wedge, \diamond, -, 0, 1)$ is said to be *diagonalizable algebra* if $(A; \vee, \wedge, -, 0, 1)$ is Boolean algebra and $\diamond$ satisfies the following conditions : (1) $\diamond(a \vee b) = \diamond(a) \vee \diamond(b)$, (2)$\diamond(0) = 0$, (3)$\diamond(a) \leq \diamond(a \vee -\diamond(a))$.

A pair $(X; R)$ is said to be *S-frame* if : 1) $(X; R)$ is GL-frame; 2)$(X, R_\circ)$ is a poset (where $R_\circ$ denotes a reflexive closure of R); 3) for every $x, y, z, u \in X$ if uRx,uRz,xRy and $\neg(xRz)$, then zRy.

Let $\mathbf{S}$ be the category of S-frames and continuous strongly isotone maps. The duality between the category of Solovay algebras and the category of S-frames is obtained by specialization of the duality between the categories $\mathbf{D}$ (diagonalizable algebras) and $\mathbf{G}$ on the case of Solovay algebras .

Now we describe finitely generated free Solovay algebras by means of a description of corresponding frames using the coloring technics which is a generalization of one-generated case, but not having such nice visualization. We describe a frame $X(n)$ for $n > 1$, corresponding to n-generated free Solovay algebra $F_{\mathbf{SA}}(n)$, by level, i. e. by the elements of fixed depth. The set of elements of the first level (i. e. the set of elements with depth 1) $X_1(n)$ contains 2^n elements, every of which has a color $p \subset \{1, ..., n\}$ thereby that different elements have different colors. On $X_1(n)$ define the binary relation $R_1 \subset X_1^2(n)$: xR_1y is false for every $x, y \in X_1(n)$. It is clear that the Solovay algebra $F_{\mathbf{SA}_1}(n)$ of all subsets of $X_1(n)$ is the free n-generated algebra in $\mathbf{SA}_1$. Observe, that the algebra $F_{\mathbf{SA}_1}(n)$ is a diagonalizable algebra. Now we represent the elements of the second level. For every element $a \in X_1(n)$ there are $a_1, ..., a_k \in X_2(n)$ (= the set of all elements of the second level) with $Col(a_i) \subset Col(a)$, $i = 1, ..., k$, such that a_i is covered by only the element a. Further, for every set $\{u_1, ..., u_k\}$ of incomparable elements of $X_1(n)$ there exists an element u_p such that $p = Col(u_p) \subset \bigcap_1^k Col(u_i)$ and u_p is covered by only the elements $u_1, ..., u_k$. Let $X_2'(n)$ be the set of all elements of second level, i. e. the elements of depth 2. Let $X_2(n) = X_1(n) \cup X_2'(n)$ and R_2 be the binary relation defined on $X_2(n)$ by the construction with $x\overline{R}_2x$ for every $x \in X_2(n)$. Now let us suppose that $(X_m(n), R_m)$ is constructed and construct (X_{m+1}, R_{m+1}) in the following way. For every element $a \in X_m(n)$ there are $a_1, ..., a_k \in X_{m+1}(n)$ (= the set of all elements of the m-th level) such that $Col(a_i) \subset Col(a)$, $i = 1, ..., k$, and a_i is covered by only the element a. For every set $\{u_1, ..., u_k\}$ of incomparable elements of $X_m(n)$, such that every u_i is covered by a fixed set U of elements of $X_{m-1}(n)$, there exists an element u_p such that $p = Col(u_p) \subset \bigcap_{i=1}^k Col(u_i)$ for every $i = 1, ..., k$ and u_p is covered by only the elements $u_1, ..., u_k$. Hereby become exhausted all elements of $m + 1$ depth denoted by $X_{m+1}'(n)$. Let $X_{m+1}(n) =$

$X_m(n) \cup X'_{m+1}(n)$ and R_{m+1} the binary relation defined on $X_{m+1}(n)$ by this construction with $xR_{m+1}x$ for every $x \in X_{m+1}(n)$. Henceforth we suppose that R_m (and R_{m+1} as well) coincides with its transitive closure. The algebra $F_{\mathbf{SA}_m}(n)$ of all upper cones of $X_m(n)$ is a Solovay algebra from pyramid variety $\mathbf{SA}_m$. Let $(X(n), R) = \bigcup_{i=1}^{\infty}(X_i(n), R_i)$. In the Fig. 3 is depicted a part of $(X(n), R)$.

Notice, that the Heyting algebra of the frame $R(x)$, for every $x \in X(n)$, is cascade Heyting algebra and, hence, it is S-frame. Let $G_i = \{x \in X(n) : i \in Col(x)\}$, $i = 1, ..., n$. Observe, that G_i is an upper cone of $X(n)$. Let $F_{\mathbf{SA}}(n)$ be an algebra generated by the set $\{G_1, ..., G_n\}$ by operations $\cup, \cap, \rightarrow, \Box$, where $\Box Y = -R^{-1} - Y$.

Theorem 3. *1) The algebra $F_{\mathbf{SA}_m}(n)$ is generated by $\{G_1^{(m)}, ..., G_n^{(m)}\}$, where $G_i^{(m)} = G_i \cap X_m(n)$.*

2) The Solovay algebra $F_{\mathbf{SA}}(n)$ is a subdirect product of algebras $F_{\mathbf{SA}_m}(n)$, $m = 1, 2, ...$

Theorem 4. *The algebra $F_{\mathbf{SA}}(n)$ is n-generated free Solovay algebra for any positive integer n.*

One-generated free Solovay algebra is described in [3].

References

[1] Boolos, G., 'The Logic of Provability', Cambridge Univ. Press, 1993, XV, 276.

[2] Esakia, L.,'Provability interpretation of Intuitionistic Logic' , *Logical Investigations*, **5**, Moscow, Nauka, 1998, 19-24. (in Russian)

[3] Esakia, L., and R. Grigolia, R., 'Formulas of one propositional variale in Intuitionistic logic with Solovay modality', *Logic and Logical Philosophy*, Vol. 17 (2008), 111127.

[4] Kuznetsov, A., and A. Muravitsky, 'On superintuitionistic logics as fragments of Proof Logic extensions', *Studia logica*, **45** (1986), 77-99.

[5] Solovay, R., 'Provability interpretation of modal logics', *Israel Journal of Mathematics*, **25**(1976), 287-304.

CONTINUOUS METRICS

GONÇALO GUTIERRES AND DIRK HOFMANN

1. Introduction

Domain theory is generally concerned with the study of *ordered sets* admitting certain (typically up-directed) suprema and a notion of approximation, here the latter amounts to saying that each element is a (up-directed) supremum of suitably defined "finite" elements. From a different perspective, domains can be viewed as very particular *topological spaces*; in fact, in his pioneering paper [5] Scott introduced the notion of continuous lattice precisely as injective topological T_0 space. This interplay between topology and algebra is very nicely explained in [1] where, employing a particular property of monads of filters, the authors obtain "new proofs and [...] new characterizations of semantic domains and topological spaces by injectivity".

Since Lawvere's ground-breaking paper [3] it is known that an individual metric spaces X can be viewed as a category with objects the points of X, and the distance

$$d(x, y) \in [0, \infty]$$

plays the role of the "hom-set" of x and y. More modestly, one can think of a metric $d : X \times X \to [0, \infty]$ as an order relation on X with truth-values in $[0, \infty]$ rather than in the Boolean algebra $\mathsf{2} = \{\mathsf{false}, \mathsf{true}\}$. In fact, writing 0 instead of true, $\geqslant$ instead of $\Rightarrow$ and additon $+$ instead of and &, the reflexivity and transitivity laws of an ordered set become

$$0 \geq d(x, x) \qquad \text{and} \qquad d(x, y) + d(y, z) \geq d(x, z) \qquad (x, y, z \in X),$$

and in this talk we follow Lawvere's point of view and assume no further properties of d. As pointed out in [3], "this connection is more fruitful than a mere analogy, because it provides a sequence of mathematical theorems, so that enriched category theory can suggest new directions of research in metric space theory and conversely".

The concern of this talk is to contribute to the development of metric domain theory. Due to the many facets of domains, this can be pursued by either

(1) formulating order-theoretic concepts in the logic of $[0, \infty]$,
(2) considering injective "$[0, \infty]$-enriched topological spaces", or
(3) studying the algebras of "metric filter monads".

Inspired by [3], there is a rich literature employing the first point of view. However, in this talk we take a different approach and concentrate on the second and third aspect above. Our aim is to connect the theory of metric spaces with the theory of "$[0, \infty]$-enriched topological spaces" in a similar fashion as domain theory is supported by topology, where by "$[0, \infty]$-enriched topological spaces" we understand Lowen's approach spaces [4]. Here an approach space is to a topological space what a metric space is to an ordered set: it can be defined in terms of ultrafilter convergence where one associates to an ultrafilter $\mathfrak{x}$ and a point x a value of convergence $a(\mathfrak{x}, x) \in [0, \infty]$ rather then just saying that $\mathfrak{x}$ converges to x or not. In particular, injective T_0 approach spaces correspond bijectively to a class of metric spaces, henceforth thought of as "continuous metric spaces", and we show that injective T_0 approach spaces (aka "continuous metric spaces") can be equivalently described as continuous lattices equipped with an unitary and associative action of the continuous lattice $[0, \infty]$.

2. Main definitions and results

2.1. **Metric spaces.** In this talk we consider *metric spaces* in a more general sense: a metric $d : X \times X \to [0, \infty]$ on a set X is only required to satisfy

$$0 \geqslant d(x, x) \qquad \text{and} \qquad d(x, y) + d(y, z) \geqslant d(x, z).$$

For convenience we often also assume d to be *separated* meaning that $d(x, y) = 0 = d(y, x)$ implies $x = y$. A map $f : X \to X'$ between metric spaces $X = (X, d)$ and $X' = (X', d')$ is a *metric map* whenever $d(x, y) \geqslant d'(f(x), f(y))$ for all $x, y \in X$. The category of metric spaces and metric maps is denoted by Met. To every metric space $X = (X, d)$ one associates its *dual space* $X^{\mathrm{op}} = (X, d^\circ)$ where $d^\circ(x, y) = d(y, x)$, for all $x, y \in X$.

There is a canonical forgetful functor $(-)_p : \mathsf{Met} \to \mathsf{Ord}$: for a metric space (X, d), put $x \leq y$ if $0 \geqslant d(x, y)$, and every metric map preserves this order. The induced order of a metric space extends pointwise to metric maps making Met an *ordered category*, which enables us to talk about *adjunction*. Here metric maps $f : (X, d) \to (X', d')$ and $g : (X', d') \to (X, d)$ form an adjunction, written as $f \dashv g$, if $1_X \leq g \cdot f$ and $f \cdot g \leq 1_{X'}$; equivalently, $f \dashv g$ if and only if, for all $x \in X$ and $x' \in X'$,

$$d'(f(x), x') = d(x, g(x')).$$

The category Met is complete and, for instance, the product $X \times Y$ of metric spaces $X = (X, a)$ and (Y, b) is given by the Cartesian product of the sets X and Y equipped with the max-metric

$$d((x, y), (x', y')) = \max(a(x, x'), b(y, y')).$$

More interestingly to us is the plus-metric

$$d'((x, y), (x', y')) = a(x, x') + b(y, y')$$

on the set $X \times Y$, we write $a \oplus b$ for this metric and denote the resulting metric space as $X \oplus Y$. This operation is better behaved then the product $\times$ in the sense that, for every metric space X, the functor $X \oplus - : \mathsf{Met} \to \mathsf{Met}$ has a right adjoint $(-)^X : \mathsf{Met} \to \mathsf{Met}$.

In the sequel we consider the metric space $[0, \infty]$, with metric μ defined by $\mu(u, v) = v \ominus u := \max\{v - u, 0\}$.

A metric space $X = (X, d)$ is called *cocomplete* if every "down-set", i.e a metric map $\psi : X^{\mathrm{op}} \to [0, \infty]$, has a supremum. This is the case precisely if, for all $\psi \in [0, \infty]^{X^{\mathrm{op}}}$ and all $x \in X$,

$$d(\mathrm{Sup}_X(\psi), x) = \sup_{y \in X}(d(y, x) \ominus \psi(y)) = [\psi, y_X(x)];$$

hence X is cocomplete if and only if the Yoneda embedding $y_X : X \to [0, \infty]^{X^{\mathrm{op}}}$ has a left adjoint $\mathrm{Sup}_X : [0, \infty]^{X^{\mathrm{op}}} \to X$ in Met. More generally, one has:

Proposition 2.1. *For a metric space X, the following conditions are equivalent.*

(i) *X is injective (with respect to isometries).*
(ii) *$y_X : X \to [0, \infty]^{X^{\mathrm{op}}}$ has a left inverse.*
(iii) *y_X has a left adjoint.*
(iv) *X is cocomplete.*

Here a metric map $i : (A, d) \to (B, d')$ is called *isometry* if one has $d(x, y) = d'(i(x), i(y))$ for all $x, y \in A$, and X is injective if, for all isometries $i : A \to B$ and all $f : A \to X$ in Met, there exists a metric map $g : B \to X$ with $g \cdot i \simeq f$. Dually, an infimum of an "up-set" $\varphi : X \to [0, \infty]$ in $X = (X, d)$ is an element $x_0 \in X$ such that, for all $x \in X$,

$$d(x, x_0) = \sup_{y \in X}(d(x, y) \ominus \varphi(y)).$$

A metric space X is *complete* if every "up-set" has an infimum. By definition, an infimum of $\varphi : X \to [0, \infty]$ in X is a supremum of $\varphi : (X^{\mathrm{op}})^{\mathrm{op}} \to [0, \infty]$ in X^{op}, and everything said above can be repeated now in its dual form.

We are particularly interested in those metric spaces $X = (X, d)$ which admit suprema of metric "down-sets" of the form $\psi = d(-, x) + u$ where $x \in X$ and $u \in [0, \infty]$. We write $x + u$ for this suprema. The element $x + u \in X$ is characterised up to equivalence by

$$d(x + u, y) = d(x, y) \ominus u, \quad \text{for all } y \in X.$$

A metric map $f : (X, d) \to (X', d')$ preserves the supremum of $\psi = d(-, x) + u$ if and only if $f(x + u) \simeq f(x) + u$. Dually, an infimum of an "up-set" of the form $\varphi = d(x, -) + u$ is denote by $x \ominus u$, and it is characterised up to equivalence by

$$d(y, x \ominus u) = d(y, x) \ominus u.$$

One calls a metric space *tensored* if it admits all suprema $x + u$, and *cotensored* if X admits all infima $x \ominus u$.

Theorem 2.2. *Let $X = (X, d)$ be a metric space. Then the following assertions are equivalent.*

(i) *X is cocomplete*

(ii) *X has all order-theoretic suprema and is tensored and cotensored.*

(iii) *X is has all (order theoretic) suprema, is tensored and, for every $x \in X$, the monotone map $d(-, x) : X_p \to [0, \infty]$ preserves suprema.*

Under these conditions, the supremum of a "down-set" $\psi : X^{\mathrm{op}} \to [0, \infty]$ can be calculated as

$$\operatorname{Sup} \psi = \inf_{x \in X} (x + \psi(x)).$$

When $X = (X, d)$ is a tensored metric space, one has a metric map

$$X \oplus [0, \infty] \to X,\ (x, u) \mapsto x + u,$$

and one easily verifies the following properties.

(1) For all $x \in X$, $x + 0 \simeq x$.

(2) For all $x \in X$ and all $u, v \in [0, \infty]$, $(x + u) + v \simeq x + (u + v)$.

(3) $+ : X_p \times [0, \infty] \to X_p$ is monotone in the first and anti-monotone in the second variable.

(4) For all $x \in X$ and $(u_i)_{i \in I}$ in $[0, \infty]$, $x + \inf_{i \in I} u_i \simeq \bigvee_{i \in I} (x + u_i)$.

If X is separated, then the first three conditions just tell us that X_p is an algebra for the monad induced by the monoid $([0, \infty], \geqslant, +, 0)$ on $\mathsf{Ord}_{\mathrm{sep}}$. Hence, $X \mapsto X_p$ defines a forgetful functor

$$\mathsf{Met}_{\mathrm{sep},+} \to \mathsf{Ord}_{\mathrm{sep}}^{[0,\infty]},$$

where $\mathsf{Met}_{\mathrm{sep},+}$ denotes the category of tensored and separated metric spaces and tensor preserving metric maps, and $\mathsf{Ord}_{\mathrm{sep}}^{[0,\infty]}$ the category of separated ordered sets with an unitary and associative action of $([0, \infty], \geqslant, +, 0)$, $[0, \infty]$-algebras for short, and monotone maps which preserve this action.

Conversely, let now X be a $[0, \infty]$-algebra with action $+ : X \times [0, \infty] \to X$. We define

$$d(x, y) = \inf\{u \in [0, \infty] \mid x + u \leq y\}.$$

Theorem 2.3. *The category $\mathsf{Met}_{\mathrm{sep},+}$ is equivalent to the full subcategory of $\mathsf{Ord}_{\mathrm{sep}}^{[0,\infty]}$ defined by those $[0, \infty]$-algebras satisfying (4). Under this correspondence, (X, d) is a cocomplete separated metric space if and only if the $[0, \infty]$-algebra X has all suprema and $(-)+u : X \to X$ preserves suprema, for all $u \in [0, \infty]$.*

2.2. **Continuous metric spaces.** Approach spaces [4] can be introduced in many different ways, in this talk we think of them mainly as convergence structures, that is, as pairs (X, a) where X is a set and $a : UX \times X \to [0, \infty]$ satisfies

$$0 \geqslant a(\dot{x}, x) \qquad \text{and} \qquad Ua(\mathfrak{X}, \mathfrak{x}) + a(\mathfrak{x}, x) \geqslant a(m_X(\mathfrak{X}), x),$$

for $\mathfrak{X} \in UUX$, $\mathfrak{x} \in UX$, $x \in X$ and $Ua(\mathfrak{X}, \mathfrak{x}) = \sup_{\mathcal{A} \in \mathfrak{X}, A \in \mathfrak{x}} \inf_{\mathfrak{a} \in \mathcal{A}, x \in A} a(\mathfrak{a}, x)$. Approach spaces and approach maps (= convergence preserving maps) are the objects and morphisms of the category App.

By analogy with ordered sets and metric spaces, we think of an approach map $\psi : (UX)^{\mathrm{op}} \to [0, \infty]$ (with $(UX)^{\mathrm{op}}$ suitably defined) as a "down-set" of X. A point $x_0 \in X$ is a *supremum* of ψ if

$$a(\dot{x_0}, x) = \sup_{\mathfrak{x} \in UX} a(\mathfrak{x}, x) \ominus \psi(\mathfrak{x}),$$

for all $x \in X$. An approach map $f : (X, a) \to (Y, b)$ preserves the supremum of ψ if

$$b(\dot{f(x_0)}, y) = \sup_{\mathfrak{y} \in UY} b(Uf(\mathfrak{x}), y) \ominus \psi(\mathfrak{x}).$$

We call an approach space X *cocomplete* if every "down-set" $\psi : (UX)^{\mathrm{op}} \to [0, \infty]$ has a supremum in X, and we call an approach space X *totally cocomplete* if the Yoneda embedding $y_X : X \to PX$ (with $PX := [0, \infty]^{(UX)^{\mathrm{op}}}$) has a left adjoint in App. It is shown in [2] that

- the totally cocomplete approach spaces are precisely the injective ones, and that
- the category

$$\mathsf{ContMet}$$

of totally cocomplete approach T_0 spaces and supremum preserving (= left adjoint) approach maps is monadic over App, Met and Set. The construction $X \mapsto PX$ is the object part of the left adjoint $P : \mathsf{App} \to \mathsf{ContMet}$ of the inclusion functor $\mathsf{ContMet} \to \mathsf{App}$, and the maps $y_X : X \to PX$ define the unit y of the induced monad $\mathbb{P} = (P, y, m)$ on App. Composing this monad with the adjunction $(-)_d \dashv (-) : \mathsf{App} \leftrightarrows \mathsf{Set}$ gives the corresponding monad on Set.

This resembles very much well-known properties of injective topological T_0 spaces, which are known to be the algebras for the filter monad on Top, Ord and Set. These analogies make us confident that totally cocomplete approach T_0 spaces provide an interesting metric counterpart to continuous lattices. We write $\mathsf{ContLat}$ to denote the category of continuous lattices and maps preserving all infima and up-directed suprema.

Proposition 2.4. *The category* $\mathsf{ContLat}$ *admits a tensor product which represents bimorphisms. That is, for all* X, Y *in* $\mathsf{ContLat}$*, the functor*

$$\mathrm{Bimorph}(X \times Y, -) : \mathsf{ContLat} \to \mathsf{Set}$$

is representable by some object $X \otimes Y$ *in* $\mathsf{ContLat}$.

By unicity of the representing object, $1 \otimes X \simeq X \simeq X \otimes 1$ and $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$. Furthermore, $[0,\infty]$ is actually a monoid in $\mathsf{ContLat}$ since $+ : [0,\infty] \times [0,\infty] \to [0,\infty]$ is a bimorphism and therefore it is a morphism $+ : [0,\infty] \otimes [0,\infty] \to [0,\infty]$ in $\mathsf{ContLat}$, and so is $1 \to [0,\infty]$, $\star \mapsto 0$. We write

$$\mathsf{ContLat}^{[0,\infty]}$$

for the category whose objects are continuous lattices X equipped with a unitary and associative action $+ : X \otimes [0,\infty] \to X$ in $\mathsf{ContLat}$, and whose morphisms are those $\mathsf{ContLat}$-morphisms $f : X \to Y$ which satisfy $f(x + u) = f(x) + u$, for all $x \in X$ and $u \in [0,\infty]$. Our main result states now:

Theorem 2.5. $\mathsf{ContMet}$ *is equivalent to* $\mathsf{ContLat}^{[0,\infty]}$.

If an approach space (X, a) is totally cocomplete, then its underlying metric is tensored. Here, $X = (X, a)$ is sent to its underlying ordered set where $x \leq y \iff a(\dot{x}, y) = 0$ $(x, y \in X)$ equipped with the tensor product of X. A continuous lattice X with action $+$ is sent to the approach space (X, a) induced by the metric d and the topology α, with $d(x, y) = \inf\{u \in [0,\infty] \mid x + u \leq y\}$, $\alpha(\mathfrak{x}) = \bigwedge_{A \in \mathfrak{x}} \bigvee_{x \in A} x$ and $a(\mathfrak{x}, x) = d(\alpha(\mathfrak{x}), x)$.

Finally, note that a map $f : X \to Y$ between injective approach spaces is an approach map if and only if f preserves down-directed suprema and, for all $x \in X$ and $u \in [0,\infty]$, $f(x) + u \leq f(x + u)$.

References

[1] Martín Hötzel Escardó and Robert Flagg. Semantic domains, injective spaces and monads. Brookes, Stephen (ed.) et al., Mathematical foundations of programming semantics. Proceedings of the 15th conference, Tulane Univ., New Orleans, LA, April 28 - May 1, 1999. Amsterdam: Elsevier, Electronic Notes in Theoretical Computer Science. 20, electronic paper No.15 (1999)., 1999.

[2] Dirk Hofmann. Injective spaces via adjunction. *J. Pure Appl. Algebra*, 215(3):283–302, 2011.

[3] F. William Lawvere. Metric spaces, generalized logic, and closed categories. *Rend. Sem. Mat. Fis. Milano*, 43:135–166 (1974), 1973. Also in: Repr. Theory Appl. Categ. No. 1 (2002), 1–37.

[4] Robert Lowen. *Approach spaces.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1997. The missing link in the topology-uniformity-metric triad, Oxford Science Publications.

[5] Dana Scott. Continuous lattices. In *Toposes, algebraic geometry and logic (Conf., Dalhousie Univ., Halifax, N. S., 1971)*, pages 97–136. Lecture Notes in Math., Vol. 274. Springer, Berlin, 1972.

CMUC, Department of Mathematics, University of Coimbra, 3001-454 Coimbra, Portugal
E-mail address: ggutc@mat.uc.pt

CIDMA, Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal
E-mail address: dirk@ua.pt

METRIC COMPACT HAUSDORFF SPACES

GONÇALO GUTIERRES AND DIRK HOFMANN

1. Introduction

Motivated to the observation (due to Hausdorff, but see [5]) that a metric $d : X \times X \to [0,\infty]$ can be seen as a generalised order relation where one trades the Boolean algebra $2 = \{\mathsf{false}, \mathsf{true}\}$ for the quantale $[0,\infty]$, we study possible metric twins of order and domain-theoretic notions and show that they indeed look very much alike. The class of stably compact spaces, or equivalently *ordered compact Hausdorff spaces*, is the class of domains with arguably the most direct generalisation to metric spaces. The latter were introduced by [7] as triples $(X, \le, \mathcal{O})$ where $(X, \le)$ is an ordered set (we do not assume anti-symmetry here) and $\mathcal{O}$ is a compact Hausdorff topology on X so that $\{(x,y) \mid x \le y\}$ is closed in $X \times X$. A morphism of ordered compact Hausdorff spaces is a map $f : X \to Y$ which is both monotone and continuous. We denote as $\mathsf{OrdCompHaus}$ the resulting category of ordered compact Hausdorff space and morphisms.

In [8] it is shown that ordered compact Hausdorff spaces are precisely the Eilenberg-Moore algebras for the ultrafilter monad $\mathbb{U} = (U, e, m)$ suitably defined on Ord, the category of ordered sets. Here the functor $U : \mathsf{Ord} \to \mathsf{Ord}$ sends an ordered set $X = (X, \le)$ to the set UX of all ultrafilters on the set X equipped with the order relation

$$\mathfrak{x} \le \mathfrak{y} \quad \text{whenever} \quad \forall A \in \mathfrak{x}, B \in \mathfrak{y}\, \exists x \in A, y \in B \,.\, x \le y; \qquad (\mathfrak{x}, \mathfrak{y} \in UX)$$

and the maps

$$e_X : X \to UX \qquad\qquad m_X : UUX \to UX$$

$$x \mapsto \dot{x} := \{A \subseteq X \mid x \in A\} \qquad\qquad \mathfrak{X} \mapsto \{A \subseteq X \mid A^\# \in \mathfrak{X}\}$$

(where $A^\# := \{\mathfrak{x} \in UX \mid A \in \mathfrak{x}\}$) are monotone with respect to this order relation. Then, for $\alpha : UX \to X$ denoting the convergence of the compact Hausdorff topology $\mathcal{O}$, $(X, \le, \mathcal{O})$ is an ordered compact Hausdorff space if and only if $\alpha : U(X, \le) \to (X, \le)$ is monotone.

The presentation in [8] is even more general and gives also an extension of the ultrafilter monad $\mathbb{U}$ to Met. For a metric space $X = (X, d)$ and ultrafilters $\mathfrak{x}, \mathfrak{y} \in UX$, one defines a distance

$$Ud(\mathfrak{x}, \mathfrak{y}) = \sup_{A \in \mathfrak{x}, B \in \mathfrak{y}} \inf_{x \in A, y \in B} d(x, y)$$

and turns this way UX into a metric space. Then $e_X : X \to UX$ and $m_X : UUX \to UX$ are metric maps and $Uf : UX \to UY$ is a metric map if $f : X \to Y$ is so. Such a space can be described as a triple (X, d, α) where (X, d) is a metric space and α is (the convergence relation of) a compact Hausdorff topology on X so that $\alpha : U(X, d) \to (X, d)$ is a metric map. We denote the category of metric compact Hausdorff spaces and morphisms (i.e. maps which are both metric maps and continuous) as $\mathsf{MetCompHaus}$.

Example 1.1. The metric space $[0,\infty]$ with metric $\mu(u, v) = v \ominus u$ becomes a metric compact Hausdorff space with the Euclidean compact Hausdorff topology whose convergence is given by $\xi(\mathfrak{v}) = \sup_{A \in \mathfrak{v}} \inf_{v \in A} v$, for $\mathfrak{v} \in U[0,\infty]$.

2. Main definitions and results

2.1. **Stably compact topological spaces.** Anti-symmetric ordered compact Hausdorff spaces can be equivalently seen as special topological spaces. In fact, both structures of an ordered compact Hausdorff space $X = (X, \le, \mathcal{O})$ can be combined to form a topology on X whose opens are precisely those elements of $\mathcal{O}$ which are down-sets in $(X, \le)$. An ultrafilter $\mathfrak{x} \in UX$ converges to a point $x \in X$ with respect to this new topology if and only if $\alpha(\mathfrak{x}) \leq x$, where $\alpha : UX \to X$ denotes the convergence of $(X, \mathcal{O})$. Hence, $\leq$ is just the underlying order of $\mathcal{O}$ ($x \leq y$ if $\dot{x} \to y$) and $\alpha(\mathfrak{x})$ is a smallest convergence point of $\mathfrak{x} \in UX$ with respect to this order. Then, we can recover both $\leq$ and α from $\mathcal{O}$. A T_0 space $X = (X, \mathcal{O})$ comes from a anti-symmetric ordered compact Hausdorff space precisely if X is *stably compact* (see [4], for instance).

If we start with a metric compact Hausdorff space $X = (X, d, \alpha)$ instead, the construction above produces, for every $\mathfrak{x} \in UX$ and $x \in X$, the *value of convergence*

$$(*) \qquad a(\mathfrak{x}, x) = d(\alpha(\mathfrak{x}), x) \in [0, \infty],$$

which brings us into the realm of *approach spaces*.

2.2. **Stably compact approach spaces.** An approach space [6] is defined as a pair (X, δ) consisting of a set X and an *approach distance* δ on X, that is, a function $\delta : X \times 2^X \to [0, \infty]$ satisfying

(1) $\delta(x, \{x\}) = 0$,
(2) $\delta(x, \varnothing) = \infty$,
(3) $\delta(x, A \cup B) = \min\{\delta(x, A), \delta(x, B)\}$,
(4) $\delta(x, A) \leqslant \delta\left(x, A^{(\varepsilon)}\right) + \varepsilon$, where $A^{(\varepsilon)} = \{x \in X \mid \delta(x, A) \leqslant \varepsilon\}$,

for all $A, B \subseteq X$, $x \in X$ and $\varepsilon \in [0, \infty]$. A map $f : (X, \delta) \to (Y, \delta')$ is an *approach map* if $\delta(x, A) \geqslant \delta'(f(x), f(A))$, for every $A \subseteq X$ and $x \in X$. Approach spaces and approach maps are the objects and morphisms of the category App.

As in the case of topological spaces, approach spaces can be described in terms of many other concepts such as "closed sets" or convergence. For instance, every approach distance $\delta : X \times 2^X \to [0, \infty]$ defines a map

$$a : UX \times X \to [0, \infty],\ a(\mathfrak{x}, x) = \sup_{A \in \mathfrak{x}} \delta(x, A),$$

and vice versa, every $a : UX \times X \to [0, \infty]$ defines a function

$$\delta : X \times 2^X \to [0, \infty],\ \delta(x, A) = \inf_{A \in \mathfrak{x}} a(\mathfrak{x}, x).$$

Furthermore, a mapping $f : X \to Y$ between approach spaces $X = (X, a)$ and $Y = (Y, b)$ is an approach map if and only if $a(\mathfrak{x}, x) \geqslant b(Uf(\mathfrak{x}), f(x))$, for all $\mathfrak{x} \in UX$ and $x \in X$. The convergence maps ([2]) are precisely the functions $a : UX \times X \to [0, \infty]$ satisfying

$$0 \geqslant a(\dot{x}, x) \qquad \text{and} \qquad Ua(\mathfrak{X}, \mathfrak{x}) + a(\mathfrak{x}, x) \geqslant a(m_X(\mathfrak{X}), x),$$

where $\mathfrak{X} \in UUX$, $\mathfrak{x} \in UX$, $x \in X$ and

$$Ua(\mathfrak{X}, \mathfrak{x}) = \sup_{\mathcal{A} \in \mathfrak{X}, A \in \mathfrak{x}} \ \inf_{\mathfrak{a} \in \mathcal{A}, x \in A} a(\mathfrak{a}, x).$$

We can restrict $a : UX \times X \to [0, \infty]$ to principal ultrafilters and obtain a metric

$$a_0 : X \times X \to [0, \infty],\ (x, y) \mapsto a(\dot{x}, y) = \delta(y, \{x\})$$

on X. Certainly, an approach map is also a metric map, therefore this construction defines a functor from App to Met, the category of metric spaces and non-expansive maps, and then also to Ord.

Returning to metric compact Hausdorff spaces, one easily verifies that $(*)$ defines an approach structure on X (see [8], for instance). Since a homomorphism between metric compact Hausdorff spaces becomes an approach map with respect to the corresponding approach structures, one obtains a functor

$$K : \mathsf{MetCompHaus} \to \mathsf{App}.$$

The underlying metric of KX is just the metric d of the metric compact Hausdorff space $X = (X, d, \alpha)$, and $x = \alpha(\mathfrak{x})$ is a *generic convergence point* of $\mathfrak{x}$ in KX in the sense that

$$a(\mathfrak{x}, y) = d(x, y),$$

for all $y \in X$. The point x is unique up to equivalence since. We call an T_0 approach space *stably compact* if it is of the form KX, for some metric compact Hausdorff space X.

Lemma 2.1. *Let (X,d,α), (Y,d',β) be metric compact Hausdorff spaces with corresponding approach spaces (X,a) and (Y,b), and let $f:X\to Y$ be a map. Then f is an approach map $f:(X,a)\to(Y,b)$ if and only if $f:(X,d)\to(Y,d')$ is a metric map and $\beta\cdot Uf(\mathfrak{x})\leq f\cdot\alpha(\mathfrak{x})$, for all $\mathfrak{x}\in UX$.*

It is an important fact that K has a left adjoint

$$M:\mathsf{App}\to\mathsf{MetCompHaus}$$

which can be described as follows (see [3]). For an approach space $X=(X,a)$, MX is the metric compact Hausdorff space with underlying set UX equipped with the compact Hausdorff convergence $m_X:UUX\to UX$ and the metric

$$d:UX\times UX\to[0,\infty],\ (\mathfrak{x},\mathfrak{y})\mapsto\inf\{\varepsilon\in[0,\infty]\mid(\forall A\in\mathfrak{x})\ A^{(\varepsilon)}\in\mathfrak{y}\},$$

and $Mf:=Uf:UX\to UY$ is a homomorphism between metric compact Hausdorff spaces provided that $f:X\to Y$ is an approach map between approach spaces. The unit and the counit of this adjunction are given by

$$e_X:(X,a)\to(UX,d(m_X(-),-))\qquad\text{and}\qquad\alpha:(UX,d,m_X)\to(X,d,\alpha)$$

respectively, for (X,a) in App and (X,d,α) in $\mathsf{MetCompHaus}$.

Remark 2.2. All what was said here about metric compact Hausdorff spaces and approach space can be repeated, *mutatis mutandis*, for ordered compact Hausdorff spaces and topological spaces. For instance, the funcor $K:\mathsf{OrdCompHaus}\to\mathsf{Top}$ has a left adjoint $M:\mathsf{Top}\to\mathsf{OrdCompHaus}$ which sends a topological space X to $(UX,\leq,m_X)$, where

$$\mathfrak{x}\leq\mathfrak{y}\quad\text{whenever}\quad(\forall A\in\mathfrak{x})\ \overline{A}\in\mathfrak{y},$$

for all $\mathfrak{x},\mathfrak{y}\in UX$.

Example 2.3. The metric space $[0,\infty]$ with distance $\mu(x,y)=y\ominus x$ equipped with the Euclidean compact Hausdorff topology where $\mathfrak{x}$ converges to $\xi(\mathfrak{x}):=\sup_{A\in\mathfrak{x}}\inf A$ is a metric compact Hausdorff space (see Example 1.1) which gives the "Sierpiński approach space" $[0,\infty]$ with approach convergence structure $\lambda(\mathfrak{x},x)=x\ominus\xi(\mathfrak{x})$.

As any adjunction, $M\dashv K$ induces a monad on App (respectively on Top). Here, for any approach space X, the space $KM(X)$ is the set UX of all ultrafilters on the set X equipped with an apporach structure, and the unit and the multiplication are essentially the ones of the ultrafilter monad. Therefore we denote this monad also as $\mathbb{U}=(U,e,m)$. In particular, one obtains a functor $U:=KM:\mathsf{App}\to\mathsf{App}$ (respectively $U:=KM:\mathsf{Top}\to\mathsf{Top}$). Surprisingly or not, the categories of algebras are equivalent to the Eilenberg–Moore categories on Ord and Met:

$$\mathsf{Ord}^{\mathbb{U}}\simeq\mathsf{Top}^{\mathbb{U}}\qquad\text{and}\qquad\mathsf{Met}^{\mathbb{U}}\simeq\mathsf{App}^{\mathbb{U}}.$$

For any metric compact Hausdorff space (X,d,α) with corresponding approach space (X,a), $\alpha:U(X,a)\to(X,a)$ is an approach map; and for an approach space (X,a) with Eilenberg–Moore algebra structure $\alpha:U(X,a)\to(X,a)$, (X,d,α) is a metric compact Hausdorff space where d is the underlying metric of a and a is the approach structure induced by d and α.

If X is T_0, then an approach map $\alpha:UX\to X$ is an Eilenberg–Moore algebra structure on X if and only if $\alpha\cdot e_X=1_X$. Hence, a T_0 approach space $X=(X,a)$ is a $\mathbb{U}$-algebra if and only if

(1) every ultrafilter $\mathfrak{x}\in UX$ has a generic convergence point $\alpha(\mathfrak{x})$ meaning that $a(\mathfrak{x},x)=a_0(\alpha(\mathfrak{x}),x)$, for all $x\in X$, and
(2) the map $\alpha:UX\to X$ is an approach map.

It is observed already in [3] that the latter condition can be substituted by

(2') X is $+$-exponentiable.

Finally, the former condition can be splitted into the following two conditions:

(1a) for every ultrafilter $\mathfrak{x}\in UX$, $a(\mathfrak{x},-)$ is an approach prime element, and
(1b) X is sober (see [1]).

Hence, every stably compact approach space is sober. We call a +-exponentiable approach space X *stable* if X satisfies the condition (1a) above, and with this nomenclature one has

Proposition 2.4. *A T_0 approach space X is stably compact if and only if X is sober, +-exponentiable and stable.*

References

[1] B. Banaschewski, R. Lowen, and C. Van Olmen. Sober approach spaces. *Topology Appl.*, 153(16):3059–3070, 2006.

[2] Maria Manuel Clementino and Dirk Hofmann. Topological features of lax algebras. *Appl. Categ. Structures*, 11(3):267–286, 2003.

[3] Dirk Hofmann. Duality for distributive spaces. Technical report, 2010.

[4] Achim Jung. Stably compact spaces and the probabilistic powerspace construction. In J. Desharnais and P. Panangaden, editors, *Domain-theoretic Methods in Probabilistic Processes*, volume 87, 2004. 15pp.

[5] F. William Lawvere. Metric spaces, generalized logic, and closed categories. *Rend. Sem. Mat. Fis. Milano*, 43:135–166 (1974), 1973. Also in: Repr. Theory Appl. Categ. No. 1 (2002), 1–37.

[6] Robert Lowen. *Approach spaces*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1997. The missing link in the topology-uniformity-metric triad, Oxford Science Publications.

[7] Leopoldo Nachbin. *Topologia e Ordem*. Univ. of Chicago Press, 1950. In Portuguese, English translation: Topology and Order, Van Nostrand, Princeton (1965).

[8] Walter Tholen. Ordered topological structures. *Topology Appl.*, 156(12):2148–2157, 2009.

CMUC, Department of Mathematics, University of Coimbra, 3001-454 Coimbra, Portugal
E-mail address: ggutc@mat.uc.pt

CIDMA, Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal
E-mail address: dirk@ua.pt

Non-associative BL-algebras and quantum structures

Radomír Halaš

Various kinds of algebras have been treated in the last decades in connection with foundational questions about the formalism of quantum mechanics [12]. Among these algebras, which are in general referred to as quantum structures, we find for instance effect algebras [13], MV-algebras [10] and BCK-algebras [16]. Note that effect algebras were introduced by Foulis an Bennett in 1994 and the class of effect algebras includes orthomodular lattices and a subclass equivalent to MV-algebras. In other words, they are considered to be as an algebras of many-valued quantum logics.

On the other hand, the most celebrated fuzzy logic, Hájek's basic logic $\mathcal{BL}$ formalized by the class BL of BL-algebras, aims at formalizing in a quite general manner statements of fuzzy nature. It is a calculus of propositions which are true principly only to a certain degree, that is, to which in general no sharp yes or no is assigned.

We find that BL-algebras are relatively closely related to the quantum structures: BL-algebras generalize MV-algebras, which in turn are special cases of effect algebras; furthermore, as shown in [17], BL-algebras can be equivalently defined in the language of BCK-algebras.

In order to unify many-valued and quantum logics, in a recent paper [8] a new class BA of algebras, called basic algebras, was established.
Recall that a basic algebra is an algebra $\mathbf{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following identities

(BA1) $x \oplus 0 = x$;

(BA2) $\neg\neg x = x$ (double negation);

(BA3) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$ (Łukasiewicz axiom);

(BA4) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg 0$.

We emphasize that the classes BA and BL are quite different in the sense that each of them is not properly contained in the other one, and their intersection is just the class MV. Certain important subclasses of BA have been considered, which are somewhat more convenient to handle. The mentioned special classes play an important role in connection with the original motivation to study quantum structures, and they naturally appear in a different context in the context of fuzzy logic.

In [3] we offered a non-associative fuzzy logic $\mathcal{CBA}$, having as an equivalent algebraic semantics lattices with section antitone involutions satisfying the contraposition law, the so-called commutative basic algebras. The class (variety) CBA of commutative basic algebras has been intensively studied in series of papers (see e.g. [1], [4], [5], [8], [6]) and includes the class of MV-algebras. We have shown that the logic $\mathcal{CBA}$ is very close to the Łukasiewicz logic, both having the same finite models, and can be understood as its non-associative generalization. CBA can be understood as the algebraic model of a certain many-valued quantum logic, see [8].

Recently a non-associative version of BL-algebras, the so-called NABL-algebras, being an equivalent algebraic semantics of the logic $\mathcal{NABL}$, was presented by M. Botur, see [2]: recall that
an algebra $\mathbf{A} = (A, \vee, \wedge, \cdot, \rightarrow, 0, 1)$ of type $\langle 2, 2, 2, 2, 0, 0 \rangle$ is a non-associative residuated lattice if

(A1) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice,

(A2) $(A, \cdot, 1)$ is a commutative groupoid with the neutral element 1,

(A3) for any $x, y, z \in A$, $x \cdot y \leq z$ if and only if $x \leq y \rightarrow z$ (adjointness property).

Moreover, if $\mathbf{A} = (A, \vee, \wedge, \cdot, \rightarrow, 0, 1)$ satisfies both

$$(x \rightarrow y) \vee \alpha_b^a(y \rightarrow x) = 1, \qquad (\alpha\text{-prelinearity})$$

$$(x \rightarrow y) \vee \beta_b^a(y \rightarrow x) = 1, \qquad (\beta\text{-prelinearity})$$

then it is called an NABL-algebra.

Botur has shown that NABL-algebras are closely related to so-called copulas. Recall that copulas are a class of aggregation operators. Formally, they have most often been applied in probability theory where they play a central role in the process of joining marginal probability distributions to form multivariate probability distribution functions. Nowadays there are quite a lot of papers being devoted to their applications in multi-valued logic. One can mention e.g. [18], [15] etc.

Botur has shown that it $NABL$ forms a variety generated just by non-associative t-norms. Consequently, the logic $\mathcal{NABL}$ logic is the logic of non-associative t-norms and their residua [2].

Since the structure theory of most quantum structures is a rather difficult matter, several of their subclasses have been considered, which are somewhat convenient to handle. For example, the Riesz decomposition property was introduced for effect algebras. Whereas these special properties play basically no role in connection with the original motivation to study quantum structures, we see that they now naturally appear in a different context - in the context of fuzzy logic.

To see that some of them are actually the characteristic properties of NABL-algebras among certain very basic types of algebras, is the aim of our talk. We procced as follows: NABL-algebras have a conjunction-like and an implication-like operation, and each of these two operations is definable from each other. We shall view them as special bounded NAGs, where NAG means an abelian groupoid ordered in the natural manner. We consider NABL-algebras as special bounded BCK-like-algebras. The properties which single out NABL-algebras among both types of structures are those of the mentioned kind: the residuation property, the Riesz decomposition property, the strong cancellativity and the property of being mutual compatible. We characterize to which subclasses of NAGs and of BCK-like and effect-like-algebras certain important subvarieties of NABL-algebras correspond.

References

[1] M. Botur: *An example of a commutative basic algebra which is not an MV-algebra*, Math. Slovaca, **60**(2) (2010), 171–178.

[2] M. Botur: *A non-associative generalizaton of Hájek's BL algebras*, Fuzzy Sets and Systems, DOI 10.1016/j.fss.2011.02.015.

[3] M. Botur, R. Halaš: *Commutative basic algebras and non-associative fuzzy logics*, Archive for Math. Logic, **48**(3-4) (2009), 243–255.

[4] M. Botur, R. Halaš: *Finite commutative basic algebras are MV-algebras*, Mult. Val. Logic and Soft Comp. **14(1-2)** (2008), 69–80.

[5] M. Botur, R. Halaš: *Complete commutative basic algebras*, Order **24** (2007), 89-105.

[6] M. Botur, I. Chajda, R. Halaš: *Are basic algebras residuated lattices?*, Soft Comp. **14**(3) (2010), 251–255.

[7] I. Chajda, R. Halaš, J. Kühr: *Every effect algebra can be organized into a total algebra*, Algebra Universalis **60** (2009), 139–150.

[8] I. Chajda, R. Halaš, J. Kühr: *Many-valued quantum algebras*, Algebra Universalis **60** (2009), 63–90.

[9] I. Chajda, R. Halaš, J. Kühr: *Semilattice Structures*, Heldermann Verlag, Lemgo (Germany), 2007, 228pp, ISBN 978-3-88538-230-0.

[10] R.I.O. Cignoli, I.M.L. D'Ottaviano, D. Mundici: *Algebraic Foundations of Many-valued Reasoning*, Kluwer Acad. Publ., Dordrecht, 2000.

[11] R.I.O. Cignoli, E. Esteva, L. Godo, A. Torrens: *Basic fuzzy logic is the logic of continuous t-norms and their residua*, Soft Computing, **4** (2000), 106–112.

[12] A. Dvurečenskij, S. Pulmannová: *New Trends in Quantum Structures*, Kluwer Acad. Publ., Dordrecht/Ister Science, Bratislava 2000.

[13] D.J. Foulis, M.K. Bennett: *Effect algebras and unsharp quantum logics*, Found. Phys. **24** (1994), 1325–1346.

[14] P. Hájek: *Metamathematics of Fuzzy Logic*, Kluwer, Dordrecht, 1998.

[15] P. Hájek, R. Mesiar: *On copulas, quasicopulas and fuzzy logic*, Soft Comp. 12 (2008), 1239–1243.

[16] J. Meng, Y. B. Jun: *BCK-Algebras*, Kyung Moon Sa Co., Seoul (1994).

[17] T. Vetterlein: *BL-algebras and quantum structures*, Math. Slovaca **54**(2) (2004), 127–141.

[18] Yager R.R.: *Modelling holistic fuzzy implication using co-copulas*, Fuzzy Optim. Decis. Making 5, 207-226 (2006).

A Graphical Game Semantics for MLL

Florian Hatat
Université de Savoie
Florian.Hatat@univ-savoie.fr

Tom Hirschowitz
CNRS, Université de Savoie
Tom.Hirschowitz@univ-savoie.fr

Abstract

The graphical games of Delande and Miller [4] and Hirschowitz [5, 6] have been used in proof search and provability. This work obtains a first proof theoretical result using them. We define games for Multiplicative Linear Logic with units (MLL) over a set $\mathbb{A}$ of atomic propositions. We then construct a category of plays $\mathcal{G}(\mathbb{A})$, using a new method based on (i) a category $\mathbf{P}$ containing plays as particular morphisms, (ii) a stack (a weak variant of sheaves) of plays, with amalgamation in the stack implementing parallel composition, (iii) a factorisation system on $\mathbf{P}$, which we use to implement hiding. We finally prove that $\mathcal{G}(\mathbb{A})$ is the free split $\star$-autonomous category over $\mathbb{A}$ [7].

1 Context

Although game semantics has been a successful source of models for proof theories and programming languages, there is still no general, abstract account of what makes it work. A folklore intuition seems to be that strategies are a coinductive counterpart to traditional, inductive proof trees, where inference rules become moves. Total and finite innocent strategies should then correspond to proofs modulo cut elimination.

However, the standard technical approach to game semantics and composition of strategies in particular is very combinatorial and low-level, and seems hard to generalise. More recent work [9, 10] attempts to give a more abstract account of composition, by both structuring positions of the game using event structures, and exhibiting relevant structure in strategies.

Here, we follow the same route (in the context of propositional MLL), but structuring positions as particular graphs, and exhibiting different structure in plays. This allows us to handle both game semantical parallel composition and hiding, in a way that seems more amenable to generalisation.

2 Game

Positions of our game are partial directed graphs $E \underset{t}{\overset{s}{\rightrightarrows}} V$, where s and t are partial maps from edges to their source and target vertices, whose edges are labelled in MLL formulas over $\mathbb{A} = \{a, b, \dots\}$, defined by the grammar:

$$\begin{array}{rcccccc} A, B, \dots & ::= & \mathbb{A} & \mid & 1 & \mid & A \otimes B \\ & \mid & \mathbb{A}^{\perp} & \mid & \perp & \mid & A \mathbin{\rotatebox[origin=c]{180}{\&}} B. \end{array} \tag{1}$$

Positions form a category $\mathbf{P}$. Positions are considered equivalent modulo simultaneous linear negation and reversal of each edge, i.e., an edge $\xrightarrow{A}$ is the same as an edge $\xleftarrow{A^{\perp}}$.

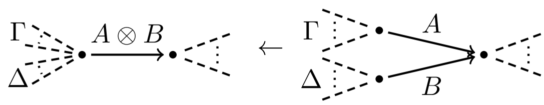
A morphism $f\colon (E, V) \to (E', V')$ of positions consists of a mapping $f\colon E + V \to E' + V'$, such that

- vertices are mapped to vertices,
- for any edge mapped to a vertex v, its source and target, when they exist, are also mapped to v,
- for any edge mapped to an edge e', its source and target, when they exist, are mapped to the source and target of e',

plus, for any edge e labelled A mapped to an edge e' labelled B, an occurrence of A in B (taking care in the right way of linear negation).

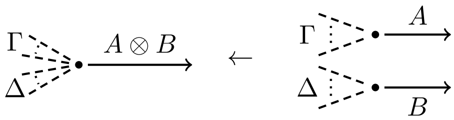
In particular, edges may be *collapsed* to vertices.

Moves are particular morphisms in **P**. An example move is the morphism:



(2)

which corresponds to the tensor rule of MLL. (A move $V \rightsquigarrow U$ is a morphism $V \leftarrow U$.) The morphism is more than a mere relation, e.g., it associates to the A edge the "left" occurrence (denoted by 0) in $A \otimes B$. Moves are stable under *restriction*, or *pullback* along a subposition, e.g., the morphism



(3)

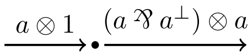
is still a move.

3 Parallel composition as amalgamation of plays

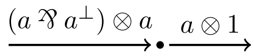
By passing from sequential plays

$$U \xleftarrow{m_1} U_1 \xleftarrow{m_2} U_2 \dots U_{n-1} \xleftarrow{m_n} U_n \tag{4}$$

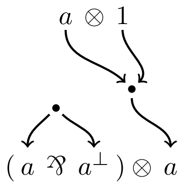
to *plays*, i.e., morphisms admitting a decomposition into moves, we obtain a *stack* (a weak version of sheaves). This means that compatible plays on parts of a given position canonically amalgamate into a play on the whole. To explain what this means, consider the positions

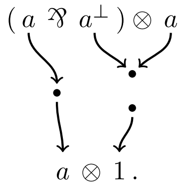


and

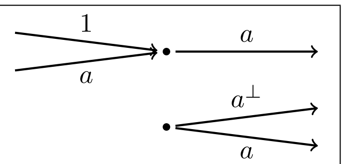
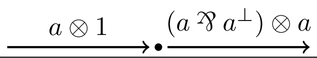


which we denote by $a \otimes 1 \vdash (a \parr a^\perp) \otimes a$ and $(a \parr a^\perp) \otimes a \vdash a \otimes 1$. Next consider the following plays from them:

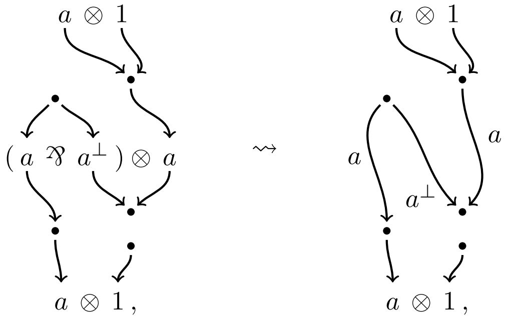




For readability, we here use a condensed semi-graphical representation of plays: the graphical part shows the final position, and each edge connects to the textual part through its occurrence in the initial position. It does not work in general, but does for plays on positions of the present form. For example, the left-hand picture really denotes the morphism

From	
To	

Amalgamating these plays along the $(a \parr a^\perp) \otimes a$ edge in the stack amounts to gluing them along the corresponding edges, as in:



(5)

yielding a play $p\colon U' \to U$ over $U = (a \otimes 1 \vdash (a \parr a^\perp) \otimes a \vdash a \otimes 1)$, i.e.,

$$\xrightarrow{a \otimes 1} \bullet \xrightarrow{(a \parr a^\perp) \otimes a} \bullet \xrightarrow{a \otimes 1} . \qquad (6)$$

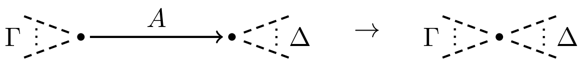
We restrict plays to an adequate notion of *won* plays, by asking that each vertex (also called *player*) holds a *won* neighbourhood, i.e., one of either shape:

By quotienting won plays by isomorphism of plays, we obtain a sheaf $\mathbb{S}$. Amalgamation in $\mathbb{S}$ corresponds to parallel composition in game semantics.

We finally mention that the proof that plays form a stack recasts Guerrini and Masini's [8] parsing criterion (P) and Danos and Regnier's [3] criterion (DR). P here appears as an obviously adequate criterion for detecting plays, but does not mix well with parallel composition, in the sense that locally satisfying P does not easily imply globally satisfying it. On the other hand, morphisms *locally* satisfying DR easily form a stack, and turn out to also coincide with plays (by reduction to the analogous question for proof nets).

4 Cut elimination as a factorisation system

We finally define composition by finding what corresponds to hiding, for which we use a factorisation system. Recall the play $p\colon U' \to U$ obtained by amalgamation, and depicted in 5. First, there is a morphism c from U to $V = (a \otimes 1 \vdash a \otimes 1)$, which merely collapses the middle edge e (labelled $(a \parr a^\perp) \otimes a$), as in:

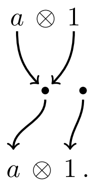


(7)

Our factorisation system provides the dashed arrows in

$$\begin{array}{ccc} U' & \xrightarrow{c'} & V' \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle q} \\ U & \xrightarrow[p]{} & V, \end{array} \qquad (8)$$

such that c' consists of some collapses of (non-dangling) edges, while q is a play. On our example, this collapses all edges above e, yielding the play q:



(9)

This defines a category $\mathcal{G}(\mathbb{A})$, where morphisms are (isomorphism classes of) such plays, and composition is by amalgamation + factorisation. Associativity of composition is a consequence of functoriality of factorisation, i.e., factoring along two cuts c_1, and then c_2, is the same as factoring along their composition.

We then show that $\mathcal{G}(\mathbb{A})$ is proof-theoretically relevant. As is well-known, the categorical counterpart of MLL is $\star$-autonomous structure [1, 2, 11]. Hughes [7] defines *split* $\star$-autonomous categories, where the usual structural unit isomorphisms $A \to A \otimes I$ and $I \otimes A \to A$ are only required to have left inverses, as opposed to two-sided inverses. He then shows that the free $\star$-autonomous category over $\mathbb{A}$ is a quotient of the free split $\star$-autonomous category $\mathcal{T}(\mathbb{A})$ over $\mathbb{A}$, under so-called Trimble *rewiring* [11]. We here show that $\mathcal{G}(\mathbb{A})$ is isomorphic to $\mathcal{T}(\mathbb{A})$ as a split $\star$-autonomous category (and hence that it is free).

5 Further work

A first direction for further work is to pursue our investigation of MLL, recovering the free $\star$-autonomous category. This will involve passing to (innocent) strategies, as opposed to plays. We could also consider extending our results to MALL.

We are also considering infinite settings, i.e., those corresponding to logics with contraction.

References

[1] M. Barr. *$\star$-Autonomous categories*, volume 752 of *Lecture Notes in Mathematics.* Springer-Verlag, 1979.

[2] M. Barr. $\star$-Autonomous Categories and Linear Logic. *Mathematical Structures in Computer Science*, 1(2):159–178, 1991.

[3] Vincent Danos and Laurent Regnier. The Structure of Multiplicatives. *Archives of Mathematical Logic*, 28:181–203, 1989.

[4] O. Delande and D. Miller. A neutral approach to proof and refutation in MALL. In *Logic in Computer Science, 2008. LICS'08. 23rd Annual IEEE Symposium on*, pages 498–508, 2008.

[5] André Hirschowitz, Michel Hirschowitz, and Tom Hirschowitz. Playing for truth on mutable graphs. Talk at the Colloquium in honor of Gérard Huet's 60th birthday, ENS, Paris, 2007. `http://www.lama.univ-savoie.fr/~hirschowitz/papers/H3.pdf`.

[6] André Hirschowitz, Michel Hirschowitz, and Tom Hirschowitz. Contraction-free proofs and finitary games for Linear Logic. In *Proceedings of the 25th Conference on Mathematical Foundations of Programming Semantics (MFPS 2009) MFPS 2009*, pages 287–305, 2009.

[7] Dominic J. D. Hughes. Simple free star-autonomous categories and full coherence. Draft, archived as math.CT/0506521 at arXiv.org., June 2005.

[8] Andrea Masini and Stefano Guerrini. Parsing MELL Proof Nets. *Theoretical Computer Science*, 254(1-2):317–335, 2001.

[9] Paul-Andre Melliès. Asynchronous games 4: A fully complete model of propositional linear logic. In Prakash Panangaden, editor, *Proceedings of the Twentieth Annual IEEE Symp. on Logic in Computer Science, LICS 2005*, pages 386–395. IEEE Computer Society Press, 2005.

[10] Samuel Mimram. *Sémantique des jeux asynchrones et réécriture 2-dimensionnelle.* PhD thesis, Université Paris 7, 2008.

[11] Todd H. Trimble. *Linear logic, bimodules, and full coherence for autonomous categories.* PhD thesis, Rutgers University, 1994.

On l-implicative-groups and associated algebras of logic

Afrodita Iorgulescu
Bucharest, ROMANIA
E-mail: afrodita.iorgulescu@ase.ro

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1 Introduction

Pseudo-MV algebras, the non-commutative generalizations of Chang's MV algebras, were introduced in 1999 and developed in [7]. Pseudo-MV algebras are intervals [6] in l-groups and pseudo-Wajsberg algebras are term equivalent [3], [4] to pseudo-MV algebras. Hence, pseudo-Wajsberg algebras had to be connected to a notion that is term equivalent to the l-group. That notion was introduced and studied in [8] under the name: the l-implicative-group.

$$\begin{array}{ccc} \mathbf{l-implicative-group} & \Longleftrightarrow & l-groups \\ \Updownarrow & & \Updownarrow \\ pseudo-Wajsberg\ algebras & \Longleftrightarrow & pseudo-MV\ algebras \end{array}$$

We recall the following definitions and results from [8]:

• An *implicative-group* is an algebra $\mathcal{G} = (G, \rightarrow, \rightsquigarrow, 0)$ of type $(2, 2, 0)$ such that the following axioms hold: for all $x, y, z \in G$,
(I1) $y \rightarrow z = (z \rightarrow x) \rightsquigarrow (y \rightarrow x), \quad y \rightsquigarrow z = (z \rightsquigarrow x) \rightarrow (y \rightsquigarrow x)$,
(I2) $y = (y \rightarrow x) \rightsquigarrow x, \quad y = (y \rightsquigarrow x) \rightarrow x$,
(I3) $x = y \Longleftrightarrow x \rightarrow y = 0 \Longleftrightarrow x \rightsquigarrow y = 0$,
(I4) $x \rightarrow 0 = x \rightsquigarrow 0$.

The implicative-group is said to be *commutative or abelian* if $x \rightarrow y = x \rightsquigarrow y$, for all $x, y \in G$.

The groups and the implicative-groups are termwise equivalent.

• A *partially-ordered implicative-group* or a *po-implicative-group* for short is a structure $\mathcal{G} = (G, \leq, \rightarrow, \rightsquigarrow, 0)$, where $(G, \rightarrow, \rightsquigarrow, 0)$ is an implicative-group and $\leq$ is a partial order on G compatible with $\rightarrow$, $\rightsquigarrow$, i.e. we have: for all $x, y, z \in G$,
(I5) $x \leq y$ implies $z \rightarrow x \leq z \rightarrow y$ and $z \rightsquigarrow x \leq z \rightsquigarrow y$.

• If the partial order relation $\leq$ is a lattice order relation, then $\mathcal{G}$ is a *lattice-ordered implicative-group* or an *l-implicative-group* for short, denoted $\mathcal{G} = (G, \vee, \wedge, \rightarrow, \rightsquigarrow, 0)$.

The l-groups and the l-implicative-groups are termwise equivalent.

2 Normal filters/ideals and compatible deductive systems

2.1 po-groups (po-implicative groups) and associated algebras on G^-, G^+

Recall the following definition: Let $\mathcal{G}_g = (G, \leq, +, -, 0)$ be a po-group. A convex po-subgroup S of $\mathcal{G}_g$ is *normal* if the following condition (N_g) holds: for any $g \in G$, $S + g = g + S$.

We introduce now the following definition: Let $\mathcal{G}_{ig} = (G, \leq, \rightarrow, \rightsquigarrow, 0)$ be a po-implicative-group. A deductive system S of $\mathcal{G}_{ig}$ is *compatible* if the following condition (C_{ig}) holds: for any $x, y \in G$, $x \rightarrow y \in S \Longleftrightarrow x \rightsquigarrow y \in S$.

Theorem 2.1 *Let $\mathcal{G}_{ig} = (G, \leq, \to, \rightsquigarrow, 0)$ be a po-implicative-group (or let $\mathcal{G}_g = (G, \leq, +, -, 0)$ be a po-group). Let S be a deductive system of $\mathcal{G}_{ig}$ (or, equivalently, a convex po-subgroup of $\mathcal{G}_g$).*
Then, S is compatible if and only if S is normal, i.e. $(C_{ig}) \Longleftrightarrow (N_g)$.

We introduce now the following definition:

(1) Let $\mathcal{M}^L = (M^L, \leq, \odot, 1)$ be a left-partially-ordered integral monoid (left-poim). A filter S^L of $\mathcal{M}^L$ is *normal* if the following condition (N^L) holds: for any $x \in M^L$, $S^L \odot x = x \odot S^L$.

(1') Let $\mathcal{M}^R = (M^R, \leq, \oplus, 0)$ be a right-poim. An ideal S^R of $\mathcal{M}^R$ is *normal* if the following condition (N^R) holds: for any $x \in M^R$, $S^R \oplus x = x \oplus S^R$.

Proposition 2.2 *Let $\mathcal{G} = (G, \leq, +, -, 0)$ be a po-group and S be a normal convex po-subgroup of $\mathcal{G}$. Then,*
(1) $S^L = S \cap G^-$ is a normal filter of the left-poim $\mathcal{G}^- = (G^-, \leq, \odot = +, \mathbf{1} = 0)$.
(1') $S^R = S \cap G^+$ is a normal ideal of the right-poim $\mathcal{G}^+ = (G^+, \leq, \oplus = +, \mathbf{0} = 0)$.

Recall the following definition (see [9], Definition 2.2.1):
Let $\mathcal{A}^L = (A^L, \leq, \to^L, \rightsquigarrow^L, 1)$ be a left-pseudo-BCK algebra. We say that a $(\to^L, \rightsquigarrow^L)$-deductive system S^L of $\mathcal{A}^L$ is *compatible* if the following condition (C^L) holds:

$$(C^L) \qquad for\ any\ x, y \in A^L,\ \ x \to^L y \in S^L \Longleftrightarrow x \rightsquigarrow^L y \in S^L.$$

2.2 l-groups (l-implicative groups) and associated algebras on G^-, G^+

Proposition 2.3 *Let $\mathcal{G} = (G, \vee, \wedge, \to, \rightsquigarrow, 0)$ be an l-implicative-group and S be a compatible deductive system of $\mathcal{G}$. Then,*

(1) $S^L = S \cap G^-$ is a compatible $(\to^L, \rightsquigarrow^L)$-deductive system of the left-pseudo-BCK(pP) lattice $\mathcal{G}^L = (G^-, \wedge, \vee, \to^L, \rightsquigarrow^L, \mathbf{1} = 0)$.

(1') $S^R = S \cap G^+$ is a compatible $(\to^R, \rightsquigarrow^R)$-deductive system of the right-pseudo-BCK(pS) lattice $\mathcal{G}^R = (G^+, \vee, \wedge, \to^R, \rightsquigarrow^R, \mathbf{0} = 0)$.

Theorem 2.4

(1) Let $\mathcal{A}^L = (A^L, \wedge, \vee, \to^L, \rightsquigarrow^L, 1)$ be a left-pseudo-BCK(pP) lattice verifying (pdiv) (or let $\mathcal{A}^L_m = (A^L, \wedge, \vee, \odot, 1)$ be a left-l-rim verifying (pdiv)). Let S^L be a $(\to^L, \rightsquigarrow^L)$-deductive system of $\mathcal{A}^L$ (or, equivalently, a filter of $\mathcal{A}^L_m$). Then S^L is compatible if and only if is normal, i.e. $(C^L) \Longleftrightarrow (N^L)$.

(1') Let $\mathcal{A}^R = (A^R, \vee, \wedge, \to^R, \rightsquigarrow^R, 0)$ be a right-pseudo-BCK(pS) lattice verifying (pdiv^d) (or let $\mathcal{A}^R_m = (A^R, \vee, \wedge, \oplus, 0)$ be a right-l-rim verifying (pdiv^d)). Let S^R be a $(\to^R, \rightsquigarrow^R)$-deductive system of $\mathcal{A}^R$ (or, equivalently, an ideal of $\mathcal{A}^R_m$). Then S^R is compatible if and only if is normal, i.e. $(C^R) \Longleftrightarrow (N^R)$.

Theorem 2.5 *Let $\mathcal{G}_{ig} = (G, \vee, \wedge, \to, \rightsquigarrow, 0)$ be an l-implicative-group (or let $\mathcal{G}_g = (G, \vee, \wedge, +, -, 0)$ be an l-group). Let S be a compatible deductive system of $\mathcal{G}_{ig}$ (or, equivalently, a normal convex l-subgroup of $\mathcal{G}_g$). Then,*

(1) $S^L = S \cap G^-$ is a compatible $(\to^L, \rightsquigarrow^L)$-deductive system of the left-pseudo-BCK(pP) lattice $\mathcal{G}^L = (G^-, \wedge, \vee, \to^L, \rightsquigarrow^L, \mathbf{1} = 0)$ (or, equivalently, S^L is a normal filter of the left-l-rim $\mathcal{G}^L_m = (G^-, \wedge, \vee, \odot = +, \mathbf{1} = 0)$), and S^L is compatible if and only if is normal, i.e. $(C^L) \Longleftrightarrow (N^L)$.

(1') $S^R = S \cap G^+$ is a compatible $(\to^R, \rightsquigarrow^R)$-deductive system of the right-pseudo-BCK(pS) lattice $\mathcal{G}^R = (G^+, \vee, \wedge, \to^R, \rightsquigarrow^R, \mathbf{0} = 0)$ (or, equivalently, S^R is a normal ideal of the right-l-rim $\mathcal{G}^R_m = (G^+, \vee, \wedge, \oplus = +, \mathbf{0} = 0)$), and S^R is compatible if and only if is normal, i.e. $(C^R) \Longleftrightarrow (N^R)$.

2.3 l-groups (l-implicative groups) and associated algebras on $[u', 0]$, $[0, u]$

We prove here that normality (compatibility) at l-group (l-implicative-group) G level is inherited by the algebras obtained by restricting the l-group (l-implicative-group) operations to any segment $[u', 0] \subset G^-$ and to any segment $[0, u] \subset G^+$. Also, that the equivalence $(C_{ig}) \Longleftrightarrow (N_g)$ (*compatible* if and only if *normal*), existing at l-group (l-implicative-group) level, is preserved by the algebras obtained by restricting the l-group (l-implicative-group) operations to intervals $[u', 0]$ and to $[0, u]$.

2.4 l-groups (l-implicative groups) and associated algebras on $\{-\infty\} \cup G^-$, $G^+ \cup \{+\infty\}$

We prove here that normality (compatibility) at l-group (l-implicative-group) G level is inherited by the algebras obtained by restricting the l-group (l-implicative-group) operations to $G^-_{-\infty} = \{-\infty\} \cup G^-$ and to $G^+_{+\infty} = G^+ \cup \{+\infty\}$. Also, that the equivalence $(C_{ig}) \Longleftrightarrow (N_g)$ (*compatible* if and only if *normal*), existing at l-group (l-implicative-group) level, is preserved by the algebras obtained by restricting the l-group (l-implicative-group) operations to $G^-_{-\infty}$ and to $G^+_{+\infty}$.

3 Representability

3.1 Representable l-groups, l-implicative-groups

Putting together the results from [1], Theorem 4.1.1 and our results, we obtained the following resuming:

Theorem 3.1 *Let $\mathcal{G} = (G, \vee, \wedge, +, -, 0)$ be an l-group or, equivalently, let $\mathcal{G} = (G, \vee, \wedge, \rightarrow, \rightsquigarrow, 0)$ be an l-implicative-group. The following are equivalent:*
(a) $\mathcal{G}$ is representable.

(b) For all $a, b \in G$, $2(a \wedge b) = 2a \wedge 2b$,
(b1) For all $a, b \in G$, $(b \rightarrow a) \wedge (a \rightsquigarrow b) \leq 0 \wedge [(b \rightsquigarrow a) \rightsquigarrow (b \rightarrow a)]$,
(b2) For all $a, b \in G$, $(b \rightsquigarrow a) \wedge (a \rightarrow b) \leq 0 \wedge [(b \rightarrow a) \rightarrow (b \rightsquigarrow a)]$.

(b^d) For all $a, b \in G$, $2(a \vee b) = 2a \vee 2b$,
($b1^d$) For all $a, b \in G$, $(b \rightarrow a) \vee (a \rightsquigarrow b) \geq 0 \vee [(b \rightsquigarrow a) \rightsquigarrow (b \rightarrow a)]$,
($b2^d$) For all $a, b \in G$, $(b \rightsquigarrow a) \vee (a \rightarrow b) \geq 0 \vee [(b \rightarrow a) \rightarrow (b \rightsquigarrow a)]$.

(c) For all $a, b \in G$, $a \wedge (-b - a + b) \leq 0$,
(c1) For all $x, y, z, w \in G$, $(x \rightsquigarrow y) \wedge (([((y \rightsquigarrow x) \rightsquigarrow z) \rightsquigarrow z] \rightarrow w) \rightarrow w) \leq 0$,
(c2) For all $x, y, z, w \in G$, $(x \rightarrow y) \wedge (([((y \rightarrow x) \rightarrow z) \rightarrow z] \rightsquigarrow w) \rightsquigarrow w) \leq 0$.

(c^d) For all $a, b \in G$, $a \vee (-b - a + b) \geq 0$,
($c1^d$) For all $x, y, z, w \in G$, $(x \rightsquigarrow y) \vee (([((y \rightsquigarrow x) \rightsquigarrow z) \rightsquigarrow z] \rightarrow w) \rightarrow w) \geq 0$,
($c2^d$) For all $x, y, z, w \in G$, $(x \rightarrow y) \vee (([((y \rightarrow x) \rightarrow z) \rightarrow z] \rightsquigarrow w) \rightsquigarrow w) \geq 0$.

(d) Each polar subgroup is normal.
(e) Each minimal prime subgroup is normal.
(f) For each $a \in G$, $a > 0$, $a \wedge (-b + a + b) > 0$, for all $b \in G$;
(f^d) For each $a \in G$, $a < 0$, $a \vee (-b + a + b) < 0$, for all $b \in G$.

3.2 Connections between the representability at l-implicative-group G level and the representability at G^-, G^+ level

Recall that in the **non-commutative case**, a non-commutative left-residuated lattice $\mathcal{A}^{\mathcal{L}} = (A^L, \wedge, \vee, \odot, \rightarrow^L, \rightsquigarrow^L, 1)$ or, equivalently, a left-pseudo-BCK(pP) lattice $\mathcal{A}^L = (A^L, \wedge, \vee, \rightarrow^L, \rightsquigarrow^L, 1)$ (with the pseudo-product $\odot$) is *representable* if it is a subdirect product of linearly-ordered ones. C.J. van Alten [2] proved that such non-commutative algebras are representable if and only if they satisfy the identity:

$$(x \rightsquigarrow^L y) \vee (([((y \rightsquigarrow^L x) \rightsquigarrow^L z) \rightsquigarrow^L z] \rightarrow^L w) \rightarrow^L w) = 1, \tag{1}$$

or the identity

$$(x \rightarrow^L y) \vee (([((y \rightarrow^L x) \rightarrow^L z) \rightarrow^L z] \rightsquigarrow^L w) \rightsquigarrow^L w) = 1. \tag{2}$$

Theorem 3.2 *Let $\mathcal{G} = (G, \vee, \wedge, \rightarrow, \rightsquigarrow, 0)$ be a representable l-implicative-group. Then,* *(1)* $\mathcal{G}^L = (G^-, \wedge, \vee, \to^L, \rightsquigarrow^L, \mathbf{1} = 0)$ *is a representable left-pseudo-BCK(pP) lattice (with the pseudo-product* $\odot = +$*).*

(1') $\mathcal{G}^R = (G^+, \vee, \wedge, \to^R, \rightsquigarrow^R, \mathbf{0} = 0)$ *is a representable right-pseudo-BCK(pS) lattice (with the pseudo-sum* $\oplus = +$*).*

Theorem 3.3 *Let* $\mathcal{G} = (G, \vee, \wedge, \to, \rightsquigarrow, 0)$ *be a representable l-implicative-group. Then,*

(1) the left-pseudo-BCK(pP) lattice $\mathcal{G}^L = (G^-, \wedge, \vee, \to^L, \rightsquigarrow^L, \mathbf{1} = 0)$ *(with the pseudo-product* $\odot = +$*) verifies also the following conditions: for all* $a, b \in G^-$*,*

(i) $(a \vee b)^2 = a^2 \vee b^2$*, i.e.* $(a \vee b) \odot (a \vee b) = (a \odot a) \vee (b \odot b)$*,*

(ii) Condition (i) is equivalent with condition

$$[b \to^L (a \rightsquigarrow^L (a \odot a))] \vee [a \rightsquigarrow^L (b \to^L (b \odot b))] = \mathbf{1}. \quad (3)$$

(iii) $(b \to^L a) \vee (a \rightsquigarrow^L b) = \mathbf{1}$*,*

(iv) Condition (iii) implies condition (3).

(1') the right-pseudo-BCK(pS) lattice $\mathcal{G}^R = (G^+, \vee, \wedge, \to^R, \rightsquigarrow^R, \mathbf{0} = 0)$ *(with the pseudo-sum* $\oplus = +$*) verifies also the following conditions: for all* $a, b \in G^+$*,*

(i') $2(a \wedge b) = 2a \wedge 2b$*, i.e.* $(a \wedge b) \oplus (a \wedge b) = (a \oplus a) \wedge (b \oplus b)$*,*

(ii') Condition (i') is equivalent with condition

$$[b \to^R (a \rightsquigarrow^R (a \oplus a))] \vee [a \rightsquigarrow^R (b \to^R (b \oplus b))] = \mathbf{0}. \quad (4)$$

(iii') $(b \to^R a) \wedge (a \rightsquigarrow^R b) = \mathbf{0}$*,*

(iv') Condition (iii') implies condition (4).

References

[1] M. Andersen, T. Feil, Lattice-Ordered Groups - An Introduction, D. Reidel Publishing Company, 1988.

[2] C.J. van Alten, Representable biresiduated lattices, *J. Algebra* 247, 2002, 672-691.

[3] R. Ceterchi, On Algebras with Implications, categorically equivalent to Pseudo-MV Algebras, *The Proceedings of the Fourth International Symposium on Economic Informatics*, INFOREC Printing House, Bucharest, Romania, May 1999, 912-916.

[4] R. Ceterchi, Pseudo-Wajsberg Algebras, *Mult. Val. Logic (A special issue dedicated to the memory of Gr. C. Moisil)*, 2001, Vol. 6, No. 1-2, 67-88.

[5] C. C. Chang, Algebraic analysis of many valued logics, *Trans. Amer. Math. Soc.* 88, 1958, 467-490.

[6] A. Dvurečenskij, Pseudo MV-algebras are intervals in l-groups, *Journal of Australian Mathematical Society*, 72, 2002, 427-445.

[7] G. Georgescu, A. Iorgulescu, Pseudo-MV algebras, *Mult. Val. Logic (A special issue dedicated to the memory of Gr. C. Moisil)*, Vol. 6, Nr. 1-2, 2001, 95-135.

[8] A. Iorgulescu, The implicative-group - a term equivalent definition of the group coming from algebras of logic - Part I, submitted.

[9] J. Kühr, Pseudo-BCK-algebras and related structures, Universita Palackého v Olomouci, 2007.

[10] D. Mundici, Interpretation of *AF* C^*-algebras in Lukasiewicz sentential calculus, *J. Funct. Anal.*, 65, 1986, 15-63.

On the number of extensions for fusions of modal logics above S4

Maxim Izmaylov, Moscow State University

Abstract

First, recall the definition of the fusion of logics. A fusion $\Lambda_1 \otimes \Lambda_2$ of two 1-modal logics Λ_1 and Λ_2 is the least 2-modal logic, which contains axioms of Λ_1 for the first modality and axioms of Λ_2 for the second.

A logic is called tabular, if it is determined with a finite Kripke frame. Maximal logic is a maximal by inclusion consistent logic. The lattice of extensions of a logic Λ is the set of modal logics $Ext(\Lambda) = \{\Xi | \Lambda \subseteq \Xi\}$, ordered by inclusion. One can easily check, that it is really a lattice.

Recall, that $S4$ is a modal logic $K + \{\Box p \to \Box\Box p\} + \{\Box p \to p\}$, which is determined with all reflexive transitive Kripke frames.

The main result of this work is the following theorem.

Theorem 1. *For any two logics Λ_1, Λ_2 over $S4$, if Λ_1 is not tabular and Λ_2 is not maximal, then the power of Ext($\Lambda_1 \otimes \Lambda_2$) is continuum.*

A logic is called pretabular, if it is not tabular, but every its extension is tabular. So, pretabular logics are the largest non-tabular logics.

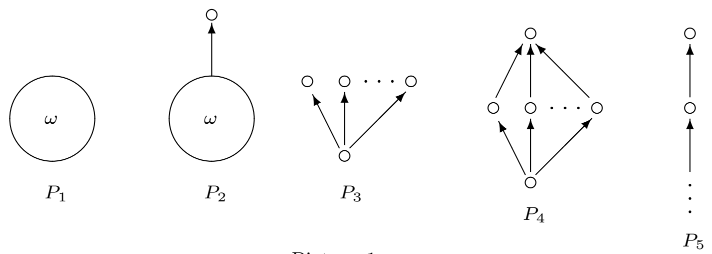
Lemma 1. *Every non-tabular logic is contained in pretabular logic.*

Lemma 2. *Every non-maximal logic over $S4$ is contained in one of following logics: Log($\circ\!\to\!\circ$) and Log($\circ\!\leftrightarrow\!\circ$).*

Here, $Log(F)$ means the logic of a Kripke frame F, all points mean to be reflexive. Because of these two lemmas, it is enough to prove Theorem 1 only for the case, when Λ_1 is pretabular and Λ_2 is of the form from lemma 2.

The rest of the proof is based on a theorem, which characterizes all pretabular logics over $S4$.

Theorem 2 (L. Esakia, V. Meskhi). *There are exactly five pretabular logics in $Ext(S4)$. They are determined by the following Kripke frames.*



Picture 1.

Here dots and ω mean a countable amount of points. A circle means a cluster, i.e. the equivalence class w.r.t. the relation $R \cap R^{-1}$. All the points are reflexive and all the arrows are transitive.

Thus, to prove Theorem 1, we should consider 10 cases: the combinations of 5 pretabular over S4 and 2 non-maximal over S4. In each case, we will show a continuum of extensions. To do it, we will need formulas Alt^1_n, b_n and ϕ_n, which are defined as follows:

$$Alt^1_n = \Box_1 p_1 \vee \Box_1(p_1 \to p_2) \vee \Box_1(p_1 \wedge p_2 \to p_3) \vee ... \vee \Box_1(p_1 \wedge ... \wedge p_n \to p_{n+1})$$
$$b_n = \bigwedge_{i=1}^{n+1} \Diamond_1 \Diamond_2 q_i \wedge \Box_1 \bigwedge_{i=1}^{n+1} \neg q_i \to \Diamond_1 \Diamond_2 \left(\bigvee_{1 \leqslant i < j \leqslant n+1} (q_i \wedge q_j) \right)$$
$$\phi_n = (\Box_1 \Box_2)^n b_2(q_1, q_2, q_3) \vee Alt^1_n(p_1, ..., p_{n+1})$$

The validity of the formula ϕ_n in a certain point of a Kripke frame corresponds to a quite complicated first order condition on this frame.

Lemma 3. *Suppose F is a $S4 \otimes S4$ Kripke frame. Formula ϕ_n is valid in a point x of F if and only if one of the following conditions is true:*

- *for any point y, accessible from x with a composition of relations $(R_1 \circ R_2)^n$, it is trus that $|R_2(R_1(y)) - R_1(y)| \leqslant 2$.*
- *$|R_1(x)| \leqslant n$.*

Then, for each case $Log(P_k) \otimes G$, where P_k is from pic. 1 and G is one of the logics $Log(\circ \!\to\! \circ)$ and $Log(\circ \!\leftrightarrow\! \circ)$, we construst logics Λ_I:

$$\Lambda_I = Log(P_k) \otimes G + \{\phi_i | i \in I\}, \text{ where } I \subseteq \mathbb{N} - \{1, 2, 3\}.$$

If we show that $I \neq J$ implies $\Lambda_I \neq \Lambda_J$, then the set $\{\Lambda_I | I \subseteq \mathbb{N} - \{1, 2, 3\}\}$ will be a continuum of extensions of a logic $Log(P_k) \otimes G$.

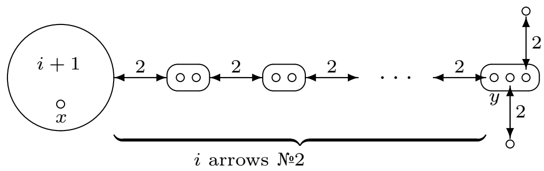
Thus, we should prove the fact that $I \neq J$ implies $\Lambda_I \neq \Lambda_J$ for all 10 cases. Here, each case should be considered separately, but the scheme is the same. We will construct Kripke frames $F_i \vDash Log(P_k) \otimes G$ with a condition:

$$F_i \nvDash \phi_i \text{ and } F_i \vDash \phi_j \text{ when } j \neq i$$

After we build these frames, if there exists $i \in I - J$, then $F_i \vDash \Lambda_J$ and $F_i \nvDash \Lambda_I$. So, $\Lambda_I \neq \Lambda_J$.

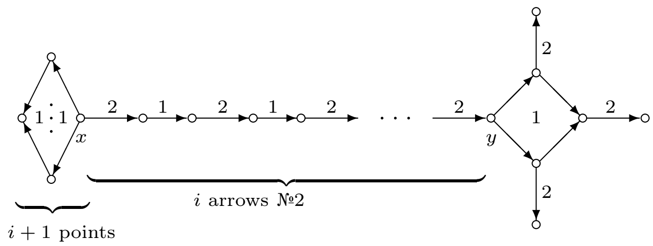
The form is the frames F_i is slightly different for all cases. We will show this form for a few cases. In the following pictures all the clusters mean relation 1.

Case $Log(P_1) \otimes Log(\circ\!\leftrightarrow\!\circ)$



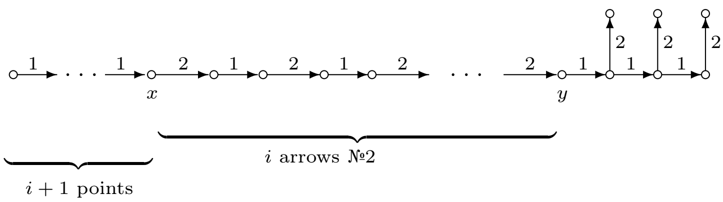
Picture 2

Case $Log(P_4) \otimes Log(\circ\!\rightarrow\!\circ)$



Picture 3

Case $Log(P_5) \otimes Log(\circ\!\rightarrow\!\circ)$



Picture 4

Bibliography.

1. A. Chagrov, M. Zakharyaschev, Modal Logic. Oxford, 1997

2. L. L. Esakia, V. Yu. Meskhi, Five critical systems. Theoria, 40:52-60, 1977

3. D.M. Gabbay, A. Kurucz, F. Wolter and M. Zakharyaschev: Many-dimensional modal logics: theory and applications. Studies in Logic and the Foundations of Mathematics, Volume 148. Elsevier, 2003

On Involutive FL_e-algebras

Sándor Jenei*†
Institute of Mathematics and Informatics
University of Pécs
Ifjúság u. 6, H-7624 Pécs, Hungary
e-mail: jenei@ttk.pte.hu
Department of Knowledge-Based Mathematical Systems
Johannes Kepler University
4040 Linz, Austria
e-mail: jenei@jku.at

Hiroakira Ono
Research Center for Integrated Science
Japan Advanced Institute of Science and Technology
1-1 Asahidai, Nomi, Ishikawa, 923-1292, Japan
e-mail: ono@jaist.ac.jp

1 Introduction

Commutative partially ordered monoids will be referred to as *uninorms*. Our aim is to investigate involutive uninorms. Our main question is the following: in an involutive FL_e-algebra, how far its uninorm (or its algebraic structure, in general) is determined by its "local behavior", i.e., its underlying t-norm and t-conorm. An answer to this question is presented for a particular case on $[0, 1]$ with $t = f$, which will illustrate our background idea. It says that the uninorm is determined uniquely by any of them, i.e., either by the t-norm or by the t-conorm [4]. In fact, the t-norm and the t-conorm are determined by each other, in this case. Then, a natural question is how far we can extend this, and when the uninorm is determined uniquely? Our main goal is to give an answer to this question: Uniqueness is guaranteed and moreover, the uninorm is represented by the twin-rotation construction whenever the algebra is conic. To have a closer look at the situation, then we consider involutive FL_e-algebras which are finite and linearly ordered. As a byproduct it follows that the logic **IUL** extended by the axiom $\mathbf{t} \leftrightarrow \mathbf{f}$ does not have the finite model property.

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†Corresponding author

$\mathcal{U} = \langle X, \circledast, \leq, t, f \rangle$ is called an *involutive FL_e-algebra* if $\mathcal{C} = \langle X, \leq \rangle$ is a poset, $\circledast$ is a uninorm over $\mathcal{C}$ with neutral element t, for every $x \in X$, $x \rightarrow_{\circledast} f = \max\{z \in X \mid x \circledast z \leq f\}$ exists, and for every $x \in X$, we have $(x \rightarrow_{\circledast} f) \rightarrow_{\circledast} f = x$. We will call $\circledast$ an *involutive uninorm.* For any uninorm $\circledast$ with neutral element t on the poset $\langle X, \leq \rangle$ define its positive and the negative cones by

$$X^+ = \{x \in X \mid x \geq t\} \qquad \text{and} \qquad X^- = \{x \in X \mid x \leq t\},$$

respectively. The algebra, and as well $\circledast$ is called *conic* if every element of X is comparable with t, that is, if $X = X^+ \cup X^-$. A moment's reflection shows that $\circledast$ restricted to X^+ (resp. X^-) is a t-conorm (resp. t-norm), call them the underlying t-conorm and t-norm of $\circledast$, respectively. Thus uninorms have a block-like structure; they have an underlying t-norm and t-conorm, that is, a t-norm and a t-conorm act on X^+ and on X^-, respectively. Now two questions arise naturally.

Q1. (Structural description) Given a t-norm and a t-conorm on X^+ and on X^-, respectively, how can one obtain a suitable extension on $X^+ \times X^-$, one which makes the resulted operation on $X^+ \cup X^-$ a conic involutive uninorm.

Q2. (Classification) Which pairs of a t-norm and a t-conorm have an appropriate extension on $X^+ \times X^-$ (and due to commutativity, on $X^- \times X^+$) such that the resulted operation on $X^+ \cup X^-$ is a conic involutive uninorm.

In the present talk we give answers to these questions. First, we show that for *involutive, conic* uninorms the extension in Q1 is unique, if exists. In other words, the underlying t-norm and t-conorm of an involutive uninorm determines the values of the involutive uninorm on $X^+ \times X^-$. This observation motivates the introducing of the twin-rotation construction.

The same questions have been investigated in [3, 2] for involutive uninorms under the condition that the underlying universe is a densely ordered, complete chain and $t = f$. In that setting it follows that the underlying t-norm and t-conorm uniquely determine (not only the whole uninorm operation but) one another. Moreover, the classification problem (Q2) was solved for involutive uninorms on the real unit interval $[0, 1]$ with $t = f$ provided that their underlying t-norm (or t-conorm) is continuous [4].

On finite chains a critical notion is the "rank" which measures how f differs from t. We establish a one-to-one correspondence between positive and negative rank algebras, a connection which is somewhat similar to the well-known de Morgan duals. This one-to-one correspondence defines what we call finite skew dualization. Finally, we solve Q2 for a few of the smallest and the largest possible non-positive ranks.

This research is motivated also by logic. As shown by Metcalfe and Montagna in [6], FL_e-algebras and involutive FL_e-algebras are equivalent algebraic semantics for the logics **UL** and **IUL**, respectively. They show, among other results, that **UL** is standard complete, that is, it is complete with respect to FL_e-algebras over $[0, 1]$. In addition, it is shown in [6] that **IUL** is chain-complete. Since whether **IUL** is standard complete remains open, the algebraic investigation of involutive uninorms have certain logical interest. We will show that **IUL** extended by the axiom $\mathbf{t} \leftrightarrow \mathbf{f}$ does not have the finite model property.

As for the details we present here only the twin-rotation construction:

Definition 1 (Twin-rotation construction) Let X_1 be a partially ordered set with top element t, and and X_2 be a partially ordered set with bottom element t such that the connected ordinal sum $os_c\langle X_1, X_2\rangle$ of X_1 and X_2 (that is putting X_1 under X_2, and identifying the top of X_1 with the bottom of X_2) has an order reversing involution $'$. Let $\otimes$ and $\oplus$ be commutative, residuated semigroups on X_1 and X_2, respectively, both with neutral element t. Assume, in addition, that

1. in case $t' \in X_1$ we have $x \to_\otimes t' = x'$ for all $x \in X_1$, $x \geq t'$, and
2. in case $t' \in X_2$ we have $x \to_\oplus t' = x'$ for all $x \in X_2$, $x \leq t'$.

Denote

$$\mathcal{U}_\otimes^\oplus = \langle os_c\langle X_1, X_2\rangle, \circledast, \leq, t, f\rangle$$

where $f = t'$ and $\circledast$ is defined as follows:

$$x \circledast y = \begin{cases} x \otimes y & \text{if } x, y \in X_1 \\ x \oplus y & \text{if } x, y \in X_2 \\ (x \to_\oplus y')' & \text{if } x \in X_2,\ y \in X_1, \text{ and } x \leq y' \\ (y \to_\oplus x')' & \text{if } x \in X_1,\ y \in X_2, \text{ and } x \leq y' \\ (y \to_\otimes (x' \wedge t))' & \text{if } x \in X_2,\ y \in X_1, \text{ and } x \not\leq y' \\ (x \to_\otimes (y' \wedge t))' & \text{if } x \in X_1,\ y \in X_2, \text{ and } x \not\leq y' \end{cases}. \quad (1)$$

Call $\circledast$ (resp. $\mathcal{U}_\otimes^\oplus$) the twin-rotation of $\otimes$ and $\oplus$ (resp. of the first and the second partially ordered monoid).

References

[1] N. Galatos, P. Jipsen, T. Kowalski, H. Ono, Residuated Lattices – An Algebraic Glimpse at Substructural Logics, *Elsevier*, 2007, 532 pp.

[2] S. Jenei, *Erratum "On the reflection invariance of residuated chains, Annals of Pure and Applied Logic 161 (2009) 220-227"*, Annals of Pure and Applied Logic 161 (2010), pp. 1603-1604

[3] S. Jenei, *On the reflection invariance of residuated chains*, Annals of Pure and Applied Logic, (2009) 220–227.

[4] S. Jenei, *On the relationship between the rotation construction and ordered abelian groups*, Fuzzy Sets and Systems, 161 (2010), 277–284.

[5] S. Jenei, *Structural description of a class of involutive uninorms via skew symmetrization, Journal of Logic and Computation* doi:10.1093/logcom/exp060

[6] G. Metcalfe, F. Montagna, *Substructural fuzzy logics*, Journal of Symbolic Logic, 72(3): 834–864, 2007.

Ordinal spaces for $\mathbf{GLB}_0$

Joost J. Joosten
University of Barcelona
jjoosten@ub.edu

Abstract

In this paper we define a bi-topological space on an interval of ordinals that is sound and complete for the closed fragment of **GLB**. This result sharply contrasts the existence of topological ordinal spaces for full **GLB** which is known to be independent of ZFC.

1 Introduction

The logic **GLB** is a normal modal logic with two modalities [0] and [1] both of which satisfy the axioms of Gödel Löb's logic **GL** together with the two axiom schemata that relate both modalities: $[0]\varphi \to [1]\varphi$ and $\langle 0\rangle\varphi \to [1]\langle 0\rangle\varphi$. It is known that **GLB** does not allow for Kripke semantics.

Topological semantics for **GLB** consists of a bi-topological space (X, τ_0, τ_1) where τ_0 is scattered, $\tau_0 \subset \tau_1$, and τ_1 contains each of the sets $d_0(A)$ with $A \subseteq X$. As usual, we denote by d_i is the derived set operator corresponding to τ_i. The modalities $\langle i\rangle$ are interpreted by d_i. Dually, the $[i]$ is interpreted by $\tilde{d}_i$ where $\tilde{d}_i(A) := -d_i(-A)$. It is an easy observation that both $d(A)$ and $\tilde{d}(A)$ are monotone in A.

One of the simplest and most natural examples of scattered spaces are ordinal spaces $([1, \kappa], \tau_0)$ where τ_0 denotes from now on the interval topology whose basis sets are $\{0\}$ and sets of the form $(\alpha, \beta]$. Topological semantics for **GLB** on $[1, k]$ exists under an infinite collection of instances of Jenssen's square principle which is known to be independent of ZFC. Moreover, the existence of a Mahlo cardinal implies non-existence of topological semantics for **GLB** on $[1, k]$. In this paper we shall exhibit a topology τ_1 so that the corresponding semantics on $[1, \kappa]$ is sound for the closed fragment of **GLB** and complete whenever $\kappa \geq \omega^{\omega^{\omega}}$. As we shall see, for the closed fragment we can work entirely within ZFC.

2 Related results and literature

In [4], Icard proved topological completeness for the closed fragment for all of **GLP**. He also used non-standard topologies similar to the one defined here and all of his argument is carried out in ZFC. However, his proof makes essential use of the existence of a universal Kripke model for the closed fragment of **GLP**.

The argument presented here does not hinge on such an assumption[1] and can thus be seen as a first exercise in defining topological semantics for logics[2] $\mathbf{GLP}_\Lambda$ where no Kripke semantics is yet known.

On the other hand, in a recent yet unpublished paper [3], Beklemishev and Gabelaia provide topological semantics for *full* **GLP** on ordinal spaces with non-standard topologies. Also their argument is done fully within ZFC. However, the topologies that they provide are obtained by using Zorn's lemma and therefore highly non-constructive. The topology exposed in this paper is explicitly definable and of constructive nature.

3 Completeness

With the closed fragment, the completeness part follows easily from the soundness part if an additional condition is satisfied. There is a fairly simple universal Kripke model for $\mathbf{GLB}_0$. If $\mathbf{GLB} \nvdash \varphi$ for some closed formula φ, this is witnessed by some point in that universal model. In the model $\langle 1 \rangle^n \top \to \langle 0 \rangle \neg \varphi \quad (\dagger)$ holds from some n, whence this is provable in **GLB**. If our topological space is sound for **GLB** and for any n has points where $\langle 1 \rangle^n \top$ holds, then it immediately follows from (†) that it is also complete for **GLB**.

4 Soundness

There are two ideas in defining τ_1. The first idea is to define τ_1 as tight as possible. For example, we want to make $\langle 1 \rangle \top$ true exactly and only in those points where we have $\langle 0 \rangle^n \top$ true for any $n \in \omega$ as $\mathbf{GLB} \vdash \langle 1 \rangle \top \to \langle 0 \rangle^n \top$.

The second idea reflects on the requirement that $d_0(A) \in \tau_1$ has to hold in order to have the axiom $\langle 0 \rangle \varphi \to [1] \langle 0 \rangle \varphi$ valid as each set A can be defined by a propositional variable in **GLB**. For the closed fragment however, it suffices to require $d_0(A) \in \tau_1$ for sets A definable in $\mathbf{GLB}_0$. We shall now show how these two ideas become implemented and are set to work.

If α is an ordinal, by $\mathsf{le}(\alpha)$ we denote the exponent of the last term in the Cantor Normal Form (CNF) expansion with base ω of α. For example, $\mathsf{le}(\omega^\omega + \omega^5) = 5$. Let X denote the entire space, that is, $X = [1, \kappa]$. One can easily proof by induction on $n \in \omega$ that $d_0^n(X) = \{\alpha \mid \mathsf{le}(\alpha) \geq n\}$. However, we can define $d^\alpha(X)$ also for ordinals α by stipulating $d^\lambda(X) = \cap_{\alpha < \lambda} d^\alpha(X)$ for limits λ and the corresponding generalization is readily proved by transfinite induction:

Lemma 4.1 $d_0^\alpha(X) = \{x \mid \mathsf{le}(x) \geq \alpha\}$

[1]In Section 4 we use the existence of a universal model for **GLP**. However, a purely syntactical proof can be given also.

[2]See [1] for a definition of the logics $\mathbf{GLP}_\Lambda$.

Thus, for example, the first point where $\langle 1\rangle\top$ could possibly be made true would be the point ω^ω and it turns out that the smallest topology doing so almost works for entire closed fragment of **GLB**. If we define $\tau_1' := \tau_0 \cup \{\{\alpha\} \mid \mathsf{le}(\alpha) \in \mathsf{Succ}\}$ where Succ is the class of successor ordinals then we can prove Lemma 4.2. Clearly, ω^ω is the first accumulation point of τ_1. However, with this definition of τ_1' one can never prove $\langle 0\rangle\alpha \to [1]\langle 0\rangle\alpha$. The smallest possible modification of τ' that leaves the accumulation points invariant works for our construction and we define $\tau_1 := \tau_0 \cup \{\overline{\{\omega^a \mid a \in A\}} \mid A \subseteq \mathsf{Succ}\}$, where $\overline{Y}$ denotes the closure of Y in τ_0.

Modal formulas that consist only of $\top$ preceded by a (possibly empty) sequence of consistency operators are called worms. From now on we shall often write just binary words $a_0a_1\ldots a_n$ instead of $\langle a_0\rangle\langle a_1\rangle\ldots\langle a_n\rangle\top$. The set of worms/words is denoted by S and the empty word is denoted by ϵ. An order $<_i$ on S is defined by $\alpha <_i \beta \ \Leftrightarrow\ \mathbf{GLB} \vdash \beta \to \langle i\rangle\alpha$ for $i \in \{0,1\}$. It turns out that, modulo provable equivalence in **GLB**, the order $<_0$ defines a well-order of type ω^ω. In particular, we can define an isomorphism o with the ordinal ω^ω as follows: $o(1^{n_0}0^{m_0}\ldots 0^{m_l}1^{n_{l+1}}) = \omega^{n_{l+1}} + m_l + \omega^{n_l} + n_l + \ldots + m_0 + \omega^{n_0}$.

If α is some worm, we denote by $d(\alpha, X)$ its topological interpretation, that is, $d(\epsilon, X) = X$ for the empty word ϵ and $d(i\alpha, X) = d_i(d(\alpha, X))$. Under our choice, we can define d_1 in terms of d_0. For words α we define the head $h(\alpha)$ and the remainder $r(\alpha)$ recursively: $h(\epsilon) = r(\epsilon) = \epsilon$, $h(x\beta) = x$ and, $r(x\beta) = \beta$.

Lemma 4.2 *For any worm α, we have that* $d(\alpha, X) = \{x \in X \mid o^{-1}(le(x)) \geq_{h(\alpha)} r(\alpha)\}$.

Proof. By induction on the length of α. The non-trivial induction steps are for $d(11\alpha, X)$, $d(10\alpha, X)$, and $d(01\alpha, X)$. In calculating the induction steps a couple of observations are very useful. Firstly, $\alpha \geq_0 \beta \leftrightarrow o(\alpha) \geq o(\beta)$. A generalization to the $\geq_1$ ordering reads $\alpha \geq_1 \beta \leftrightarrow [o(\alpha) \geq o(\beta) \wedge \mathsf{le}(o(\alpha)) \geq \mathsf{le}(o(\beta))]$. Using these two observations one can smoothly prove the lemma. □

Notice that our lemma yields in particular that $d(0\alpha, X) = d_0^{o(0\alpha)}(X)$.

Theorem 4.3 $([1, \kappa], \tau_0, \tau_1)$ *is sound for* $\mathbf{GLB}_0$.

Proof. By induction on the length of a derivation. It is known that any formula in the closed fragment of **GLB** can be proved by a proof that uses closed formulas only. Thus, it suffices to focus on the closed instantiations of the axioms of **GLB** (Modus Ponens and Necessitation are easy).

As τ_0 is scattered and $\tau_0 \subset \tau_1$, we have that τ_1 is also scattered and **Gl** holds for both [0] and [1]. Moreover, $\tau_0 \subset \tau_1$ enforces $[0]\varphi \to [1]\varphi$ to hold for any φ. The only axiom that remains to be verified is $\langle 0\rangle\varphi \to [1]\langle 0\rangle\varphi$ for closed formulas φ.

It is known that any such closed formula φ is equivalent to a Boolean combination of words. It is a straightforward exercise in normal forms (a direct consequence of Lemma 11 of [1]) to see that the $\langle 0\rangle$ of any such Boolean combination of words is equivalent to a disjunction of words all of which start with a 0.

Let us first see that for each of these disjuncts/words $0w_i$ we have that $\langle 0\rangle w_i \to [1]\langle 0\rangle w_i$. In other words and using Lemma 4.1, we need to show that $\{x \mid \mathsf{le}(x) \geq o(\alpha)+1\} \subseteq \tilde{d}_1(\{x \mid \mathsf{le}(x) \geq o(\alpha)+1\})$. But, any x with $\mathsf{le}(x) \geq o(\alpha)+1$ is either isolated in τ_1, in which case it is in $\tilde{d}_1(\varnothing)$ so by the monotonicity of $\tilde{d}_1$ certainly in $\tilde{d}_1(\{x \mid \mathsf{le}(x) \geq o(\alpha)+1\})$, or it is an accumulation point of points y with $\mathsf{le}(y) \geq o(\alpha)+1$ whence included in $\tilde{d}_1(\{x \mid \mathsf{le}(x) \geq o(\alpha)+1\})$.

Now that we have established the inclusion for one particular worm, the disjunction of finitely many such follows easily. As for each w_i we have that $v(\langle 0\rangle w_i) \subseteq \tilde{d}_1(v(\langle 0\rangle w_i))$, by the monotonicity of $\tilde{d}_1$ we have that $v(\langle 0\rangle w_i) \subseteq \tilde{d}_1(\cup_i v(\langle 0\rangle w_i)) = \tilde{d}_1(v(\bigvee_i \langle 0\rangle w_i))$. As this holds for any i we also have that $v(\bigvee_i \langle 0\rangle w_i) = \cup_i v(\langle 0\rangle w_i) \subseteq \tilde{d}_1(v(\bigvee_i \langle 0\rangle w_i))$. □

Corollary 4.4 $([1,\kappa],\tau_0,\tau_1)$ *is sound and complete for* $\mathbf{GLB}_0$ *whenever* $\kappa \geq \omega^{\omega^\omega}$.

Proof. From the previous theorem we obtain the soundness. From our observations in Section 3 we get completeness once for any $n \in \omega$ there are points around that validate $\langle 1\rangle^n \top$. By Lemma 4.2 we know that $d_1^n(X) = d_0^{o(1^n)}(X) = d_0^{\omega^n}(X)$. By Lemma 4.1 we know that points in $d_0^{\omega^n}(X)$ should have a last term of at least ω^{ω^n} in their CNF. The first ordinal where these elements are around for all $n \in \omega$ is ω^{ω^ω}. □

We note that the current proof seems amenable to generalizing to **GLP** once we realize that $\tau_1 = \tau_0 \cup \{\{\alpha\} \mid \mathsf{le}(\alpha) \in \mathsf{Succ}\} = \tau_0 \cup \{\{\alpha\} \mid \mathsf{le}(\alpha) = o(0\beta)$ for some worm $\beta\}$ which can be generalized in a straightforward way.

Using some independent set-theoretical assumptions, in [2] a model for full **GLB** is constructed. In that construction t_1 is the club topology. The current result does not seem to say anything about the soundness of the closed fragment of **GLB** when using the club topology and no further set-theoretical assumptions and we leave that as an open question.

References

[1] Beklemishev, L.D. (2005): Veblen hierarchy in the context of provability algebras. In: P. Hájek, L. Valdés-Villanueva, D. Westerstahl, eds., Logic, Methodology and Philosophy of Science, Proceedings of the Twelfth International Congress. Kings College Publications, 2005, p. 65-78.

[2] Beklemishev, L.D. (2010): Ordinal completeness of bimodal provability logic GLB. Department of Philosophy, Utrecht University, Logic Group Preprint Series 282, March 2010.

[3] Beklemishev, L.D., D. Gabelaia: Topological completeness of the provability logic GLP, yet unpublished, only available online, 2011.

[4] Icard, T. (2009): A topological study of the closed fragment of GLP, Journal of Logic and Computation, Advance Access published online, doi:10.1093/logcom/exp043, 2009.

On modal definability of Horn formulas

Stanislav Kikot

Abstract

In this short paper we give a criterion of modal definability of a first-order universal Horn sentence with exactly one positive atom in terms of its graph. As a consequence we obtain that every modal logic axiomatized by a single modal Horn formula (i.e. of the form $\mathbf{K} + \phi$ where ϕ is a modal Horn formula) is Kripke complete.

Modal definability of first-order formulas has been intensively studied in modal logic, and even applied to automatic reasoning [9]. On the one hand, it has a nice Goldblatt-Thomason characterization [4], on the other hand, the problem "decide whether a first-order formula is modally definable" is in general undecidable [2]. But the cause of this undecidability is in the undecidability of first-order logic, so when we restrict attention to a fragment with decidable implication, we are likely to obtain an algorithmic criterion for modal definability, as in this paper. Also this research is motivated by scrutinizing Theorem 5.9 of [3] saying that if L_1 and L_2 are Kripke complete and Horn axiomatizable unimodal logics, then $L_1 \times L_2 = [L_1, L_2]$ and studying whether Horn axiomatizability implies Kripke completeness. We give the positive answer to the last question for the case of a single universal Horn sentence with exactly one positive atom, but in general this problem seems to be open.

Consider the classical first-order language $\mathcal{L}f_\Lambda$ in the signature consisting of only binary predicates R_λ indexed by a finite set Λ. *An atom* is a formula of the form $x_i R_\lambda x_j$, where x_i and x_j are object variables and $\lambda \in \Lambda$. *Universal Horn sentences* (in short, *Horn formulas*) are closed (i.e. without free variables) formulas of the form $\forall x_1 \dots \forall x_n (\psi \to \phi)$, where ψ is a conjunction of atoms and ϕ is an atom. Allowing $\vee$ in ψ as in [3] is equivalent to considering conjunctions of such formulas. Universal Horn sentences can be represented by tuples of the form $D = (W^D, (R^D_\lambda : \lambda \in \Lambda), \alpha, \beta, \lambda_0)$, where $W^D = \{x_1, \dots, x_n\}$ is a finite set, R^D_λ are binary relations on W^D, $\alpha, \beta \in W^D$ and $\lambda_0 \in \Lambda$. Such a tuple D, called *a diagram*, gives rise to the Horn formula

$$E^D = \forall x_1 \dots \forall x_n \left(\bigwedge_{x_i R^D_\lambda x_j} x_i R_\lambda x_j \to \alpha R_{\lambda_0} \beta \right).$$

For a diagram D, define its size $|D| = \sum_{\lambda \in \Lambda} |R^D_\lambda|$, where $|R^D_\lambda|$ denotes the cardinality of R^D_λ. A diagram D is called *minimal* if there is no diagram D' of size less than $|D|$ such that $E^{D'} \equiv E^D$, where $\equiv$ denotes the predicate calculus equivalence. A diagram D is called *non-trivial* if E^D is not equivalent to $\top$.

We also consider the modal language $\mathcal{M}l_\Lambda$ with countably many propositional variables $p_1, p_2, \dots$, unary modalities $\Diamond_\lambda$ and their duals $\Box_\lambda$, where $\lambda \in \Lambda$ and boolean connectives $\wedge, \vee, \neg, \to$. A *Kripke frame* is an $\mathcal{L}f_\Lambda$-structure $F = (W, (R_\lambda : \lambda \in \Lambda))$. A *valuation of propositional variables in* F is a map θ assigning to any p_i a set $\theta(p_i) \subseteq W$. A *Kripke model* *built on a frame* F is a pair $M = (F, \theta)$ where θ is a valuation of propositional variables in F. The *truth of modal formula* ϕ in a point x of Kripke model M is defined in the standard way. A modal formula ϕ is *valid* on a Kripke frame F (denoted $F \models \phi$) if ϕ is true in every point of every model M built on F.

An $\mathcal{L}f_\Lambda$-sentence E is called *modally definable* if there exists a modal formula ϕ such that, for any Kripke frame F, $F \models E$ iff $F \models \phi$. Here $\models$ on the left-hand side means the classical truth of an $\mathcal{L}f_\Lambda$-formula in $\mathcal{L}f_\Lambda$-structure, while $\models$ on the right-hand side means the validity of a modal formula in a Kripke frame. If this equivalence holds and, in addition, E is a universal Horn sentence, then ϕ is called a *modal Horn formula*.

Consider a finite $\mathcal{L}f_\Lambda$-structure $F = (W, (R_\lambda : \lambda \in \Lambda))$, where, for all $\lambda \in \Lambda$, $R_\lambda \subseteq W \times W$. A sequence $x_1, \lambda_1, x_2, \lambda_2, \ldots x_n$, where $x_i \in W^T$, $\lambda_i \in \Lambda$ and $(x_i, x_{i+1}) \in R_{\lambda_i}$ for $1 \leq i \leq n-1$ is called *a directed path from x_1 to x_n in F*. The definition of *an undirected path from x_1 to x_n* is obtained by replacing $(x_i, x_{i+1}) \in R_{\lambda_i}$ with $(x_i, x_{i+1}) \in R_{\lambda_i} \cup R_{\lambda_i}^{-1}$. An $\mathcal{L}f_\Lambda$-structure $F = (W, (R_\lambda : \lambda \in \Lambda))$ is called *a directed tree* if there is a point $r \in W$ such that the following holds:

- $(R_\lambda)^{-1}(r) = \emptyset$ for all $\lambda \in \Lambda$,
- for every point $x \neq r$, there exists a unique directed path from r to x.

THEOREM 1. The Horn formula E^D corresponding to a minimal non-trivial diagram $D = (W^D, (R_\lambda^D : \lambda \in \Lambda), \alpha, \beta, \lambda_0)$ is modally definable iff the $\mathcal{L}f_\Lambda$-structure $(W^D, (R_\lambda^D : \lambda \in \Lambda))$ is a directed tree.

The proof of the 'if' direction is simple: if $(W^D, (R_\lambda^D : \lambda \in \Lambda))$ is a directed tree, then all points x_i except the root x_0 have a unique predecessor $x_{pr(i)}$ such that $x_{p(i)} R_{\lambda(i)} x_i$ for some $\lambda(i) \in \Lambda$. Assuming that the x are numbered in such a way that, for all i, $pr(i) < i$ and using the restricted universal quantifier

$$(\forall x_i \rhd_\lambda x_j) A \equiv \forall x_i (x_j R_\lambda x_i \to A),$$

we can rewrite E^D as

$$\forall x_0 (\forall x_1 \rhd_{\lambda(1)} x_0)(\forall x_2 \rhd_{\lambda(2)} x_{pr(2)}) \ldots (\forall x_n \rhd_{\lambda(n)} x_{pr(n)})(\alpha R_{\lambda_0} \beta).$$

This is obviously a Kracht formula [6], [7], so it is modally definable by a Sahlqvist formula. The proof of the 'only if' direction follows from lemmas 2 and 4 and the fact that all modally definable properties are preserved under disjoint unions and bounded morphisms (e.g. [1]). Together with the Sahlqvist completeness theorem it gives us that any modal logic axiomatizable by a single modal Horn formula is Kripke complete. The complexity of similar logics is studied in [5].

Consider two $\mathcal{L}f_\Lambda$-structures $F^1 = (W^1, (R_\lambda^1 : \lambda \in \Lambda))$ and $F^2 = (W^2, (R_\lambda^2 : \lambda \in \Lambda))$. A map $g : W^1 \to W^2$ is called a *homomorphism* from F^1 to F^2 if, for any $\lambda \in \Lambda$ and $a, b \in W^1$, $a R_\lambda^1 b$ implies $f(a) R_\lambda^2 f(b)$. For a finite $\mathcal{L}f_\Lambda$-structure $F = (W, (R_\lambda : \lambda \in \Lambda))$ and a (diagram of a) Horn formula $D = (W^D, (R_\lambda^D : \lambda \in \Lambda), \alpha, \beta, \lambda_0)$ we define a *Horn closure* F_D^* in the following way. Set $F_D^0 = F$. Let $F_D^{i-1} = (W, (R_\lambda^{i-1} : \lambda \in \Lambda))$ be already defined. Let $\mathcal{G}_i$ be the set of all homomorphisms from $(W^D, (R_\lambda^D : \lambda \in \Lambda))$ to F_D^{i-1}. Set $F_D^i = (W, (R_\lambda^i : \lambda \in \Lambda))$ where

$$R_{\lambda_0}^i = R_{\lambda_0}^{i-1} \cup \bigcup_{g \in \mathcal{G}_i} \{(g(\alpha), g(\beta))\}$$

and $R^i_\lambda = R^{i-1}_\lambda$ for $\lambda \neq \lambda_0$. Since W is finite, there exists n such that $F^n_D = F^{n+1}_D = F^{n+2}_D$, and so on. Then we set $F^*_D = F^n_D$ for such n. This construction generalizes the well-known transitive and symmetric closure.

An $\mathcal{L}f_\Lambda$-structure $F = (W, (R_\lambda : \lambda \in \Lambda))$ and a diagram $D = (F, \alpha, \beta, \lambda)$ are called *connected* if any two different points of W may be connected by an undirected way.

LEMMA 2. Take a minimal non-trivial diagram $D = (W^D, (R^D_\lambda : \lambda \in \Lambda), \alpha, \beta, \lambda_0)$. If an $\mathcal{L}f_\Lambda$-structure $G^D = (W^D, (R^D_\lambda : \lambda \in \Lambda))$ is not connected then E^D is not preserved under disjoint unions.

Proof. First suppose that α and β belong to different connected components of G^D. Then take $F = G^D$ and its Horn closure F^*_D. Thus we have $F^*_D \models E^D$ but $F^*_D \sqcup F^*_D \not\models E^D$, and the lemma is proved.

Then consider the case where G^D is split into connected components $K_1, \ldots, K_n$ and α and β belong to the same connected component, say, K_1. Note that since D is minimal, there is no homomorphism from K_2 to K_1, otherwise we can throw K_2 out of a diagram without affecting E^D semantically. Thus we have $K_1 \models E^D$, since there is no homomorphism from G^D in K_1 because of K_2. Put $D' = (K_1, \alpha, \beta, \lambda_0)$. Then $(G^D \setminus K_1)^*_{D'} \models E^D$ (since $E^{D'} \models E^D$) but $K_1 \sqcup (G^D \setminus K_1)^*_{D'} \not\models E^D$ (because of the identity homomorphism of G^D into itself and non-triviality of D). □

LEMMA 3. Consider two diagrams $D = (W^D, (R^D_\lambda : \lambda \in \Lambda), \alpha, \beta, \lambda_0)$ and $D' = (W^{D'}, (R^{D'}_\lambda : \lambda \in \Lambda), \alpha', \beta', \lambda'_0)$. Put $F = (W^D, (R^D_\lambda : \lambda \in \Lambda))$. Then $F^*_{D'} \models E^D$ implies $E^{D'} \models E^D$.

Proof. Take any $G = (W, (R_\lambda : \lambda \in \Lambda))$. Assume that $G \models E^{D'}$ and prove that $G \models E^D$. Take a homomorphism h from F to G. Now execute the process of construction of $F^*_{D'}$ and copy any its step by h into G, applying $G \models E^{D'}$ for each new edge. Finally we will obtain that $h(\alpha) R_{\lambda_0} h(\beta)$. □

LEMMA 4. Let D be a minimal non-trivial diagram. Then if $G^D = (W^D, (R^D_\lambda : \lambda \in \Lambda))$ contains a directed cycle or a point c with two incoming arrows then E^D is not preserved under bounded morphism.

Proof. First suppose that G^D contains a directed cycle. Then take $F = G^D$ and it unravelling $F^u = ((W^D)^u, ((R^D)^u_\lambda : \lambda \in \Lambda))$, where $(W^D)^u$ is the set of all directed pathes in F, with a natural bounded morphism $f : (W^D)^u \to W^D$, sending each path to its last point, and $(R^D)^u_\lambda$ defined in a standard way: for $x, y \in (W^D)^u$ $x(R^D)^u_\lambda y$ iff $y = x, \lambda, f(y)$. Since D is non-trivial, $F \not\models E^D$. But $F^u \models E^D$, since there is no homomorphism from G^D to the tree F^u because of a directed cycle in G^D, so the lemma is proved.

Now assume that G^B contains a vertex with at least two incoming edges. It means that there exist points $a, b, c \in W^D$ and $\lambda_1, \lambda_2 \in \Lambda$ such that $(a, c) \in R^D_{\lambda_1}$ and $(b, c) \in R^D_{\lambda_2}$. If $\lambda_1 \neq \lambda_2$, the same argument as for the directed cycle works: a point with two incoming arrows of different kinds cannot be embedded into the tree F^u.

But if $a \neq b$ and $\lambda_1 = \lambda_2$ it may still happen that there is a homomorphism h from G^B to F^u, in this case $h(a) = h(b)$. So we consider the set $\mathcal{T}$ of all directed trees T such that there exists a surjective homomorphism from G^B to T. We claim that there exists a directed tree $T_0 \in \mathcal{T}$ such that for all $T \in \mathcal{T}$ there exists a surjective homomorphism from T_0 to T.

Let $\sim$ be the smallest equivalence relation on W^D satisfying condition (cf. [8])

if there exists a, b, c, c' such that $aR_\lambda^D c$, $bR_\lambda^D c'$ and $c \sim c'$, then $a \sim b$.

Define $T_0 = (W^0, (R_\lambda^0 : \lambda \in \Lambda))$ where $W^0 = W^D/\sim$, and for equivalence classes $A, B \in W^0$ $AR_\lambda^0 B$ iff there exist $a \in A$ and $b \in B$ such that $aR_\lambda b$. In other words, T^0 is obtained from G^D by a sequence of following reductions: if there exist $a, b, c \in W^D$ such that $aR_\lambda^D c$ and $bR_\lambda^D c$, then join a and b into one point. The main property of $\sim$ is that for every homomorphism g from G^D to a directed tree T $a \sim b$ implies $g(a) = g(b)$, that is every such g factors through T_0.

Let h be the natural projection from G^B to T_0. Consider the diagram $D' = (T_0, h(\alpha), h(\beta), \lambda_0)$. In any case, a homomorphism from G_B to T_0 implies that $E_D \models E_{D'}$, and a vertex with two incoming edges in G^B implies that $|D'| < |D|$. Since D is minimal, $E^{D'} \not\models E^D$ and according to Lemma 3 it follows that $F_{D'}^* \not\models E^D$.

Now we can prove the lemma, since $(F^u)_{D'}^* \models E^D$ (use universal property of T_0), $F_{D'}^* \not\models E^D$ and f is a p-morphism not only from F^u to F, but also from $(F^u)_{D'}^*$ to $F_{D'}^*$. □

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References

[1] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Cambridge University Press, 2002.

[2] A. Chagrov and L. Chagrova. The truth about algorithmic problems in correspondence theory. In G. Governatori, I. Hodkinson, and Y. Venema, editors, *Advances in Modal Logic 6*, pages 121–138. King's College Publications, 2006.

[3] D. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyaschev. *Many-dimensional modal logics: theory and applications*. Studies in Logic and the Foundations of Mathematics, 148. Elsevier, 2003.

[4] R. Goldblatt and S. Thomason. Axiomatic classes in propositional modal logic. In *Algebra and Logic, Lecture Notes in Math., Vol. 450*, pages 163–173. Springer, Berlin, 1974.

[5] E. Hemaspaandra and H. Schnoor. On the complexity of elementary modal logics. In *Symposium on Theoretical Aspects of Computer Science*, pages 349–360, 2008.

[6] M. Kracht. How completeness and correspondence theory got married. In *M. de Rijke (Ed.), Diamonds and Defaults*, pages 175–214. Synthese Library, Kluwer, 1993.

[7] M. Kracht. *Tools and Techniques in Modal Logic*. Studies in Logic and the Foundations of Mathematics, 142. Elsevier, 1999.

[8] C. Lutz, D. Toman, and F. Wolter. Conjunctive query answering in the description logic EL using a relational database system. In *IJCAI*, pages 2070–2075, 2009.

[9] E. Zolin. Query answering based on modal correspondence theory. In *Proceedings of the 4th "Methods for modalities" Workshop (M4M-4)*, pages 21–37, 2005.

On interpolation in $NEXT(\mathbf{KTB})^*$

Zofia Kostrzycka

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The Brouwerian logic **KTB** is said to be non-transitive as it is characterized by the class of reflexive and symmetric (admitting non-transitive) frames. There are also considered in literature its extensions $\mathbf{T_n} = \mathbf{KTB} \oplus (4_n)$, with the so called axioms of $n-$transitivity:

$$(4_n) \qquad \Box^n p \to \Box^{n+1} p \quad \text{, for each } n \geq 1$$

They were defined by I. Thomas in 1964 (see [9]). Their completeness with respect to Kripke models was proven there.

Obviously, $\mathbf{T_1} = \mathbf{S5}$ and the following inclusions hold:

$$\mathbf{KTB} \subset ... \subset \mathbf{T_{n+1}} \subset \mathbf{T_n} \subset ... \subset \mathbf{T_2} \subset \mathbf{T_1} = \mathbf{S5}.$$

In contrast to the logics laying in the interval **S4**–**S5**, which are very well characterized, the logics between **KTB** and **S5** are, in some way, neglected. For example, it is known since the 1960's, that the logics **KTB** and $\mathbf{T_n}$ are Kripke complete, but for many years there was not known any Kripke incomplete logics in $NEXT(\mathbf{KTB})$. In 2006 Yutaka Miyazaki [7] constructed one Kripke incomplete logic in $NEXT(\mathbf{T_2})$ and continuum Kripke incomplete logics in $NEXT(\mathbf{T_5})$. Then the author defined in [4] and [5] appropriately a continuum Kripke incomplete logic and one finitely axiomatizable Kripke incomplete logic in $NEXT(\mathbf{T_2})$.

The logics from $NEXT(\mathbf{KTB})$ seem to be very diverse and worth studying. In this talk we pay attention to the Craig interpolation property (CIP) and Halldén completeness of the considered logics. Let us add that the logics

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from $NEXT(\mathbf{KTB})$ were not examined in this direction yet. Majority of L. Maximova's famous results concerns logics in $NEXT(\mathbf{S4})$.

Let us remind some definitions.

Definition 1. *A logic L has the Craig interpolation property (CIP) if for every implication $\alpha \to \beta$ in L, there exists a formula γ (interpolant for $\alpha \to \beta$ in L) such that $\alpha \to \gamma \in L$ and $\gamma \to \beta \in L$ and $Var(\gamma) \subseteq Var(\alpha) \cap Var(\beta)$.*

Definition 2. *A logic L is Halldén complete if*

$$\varphi \vee \psi \in L \text{ implies } \varphi \in L \text{ or } \psi \in L$$

for all φ and ψ containing no common variables.

Regarding (CIP), one may apply a very general method of construction of inseparable tableaux (see i.e. [1]) and get:

Theorem 1. *The logics* **KTB** *and* $\mathbf{T_n}$*, $n > 1$ have (CIP).*

There is an important connection between the Craig interpolation property and Halldén completeness of modal logics. It is presented in the following lemma due to G. F. Schumm [8]:

Lemma 1. *If L has only one Post-complete extension and is Halldén-incomplete, then interpolation fails in L.*

Although it is known that **KTB** is Halldén complete (see [6]), there are no results concerning this property in the case of logics $\mathbf{T_n}$, $n > 1$. From Theorem 2 and Lemma 1 we immediately obtain:

Corollary 1. *The logics* $\mathbf{T_n}$*, $n \geq 1$ are Halldén-complete.*

The next step in our investigation is to answer the question: 'how many logics in $NEXT(\mathbf{T_2})$ are Halldén-incomplete?'

In this case we take advantage of two constructions from [2] and [3]: the construction of infinite sequence of nonequivalent formulas in $\mathbf{T_2}$ and the construction of continuum of normal extensions distinguishable by these formulas.

Let us take $\alpha := p \wedge \neg\Diamond\Box p$ and define the following sequence of formulas:

$$
\begin{aligned}
A_1 &:= \neg p \wedge \Box\neg\alpha\,,\\
A_2 &:= \neg p \wedge \neg A_1 \wedge \Diamond A_1\,,\\
A_3 &:= \alpha \wedge \Diamond A_2 \wedge \neg\Diamond A_1\,,\\
A_{2n} &:= \neg p \wedge \Diamond A_{2n-1} \wedge \neg\Diamond A_{2n-2} \quad \text{for } \ n \geq 2\,,\\
A_{2n+1} &:= \alpha \wedge \Diamond A_{2n} \wedge \neg\Diamond A_{2n-1} \quad \text{for } \ n \geq 2\,.
\end{aligned}
$$

It is proven in [2], that

Lemma 2. *The formulas A_i, for $i \geq 1$, are not equivalent in the logic $\mathbf{T_2}$.*

On such a base we define more compound formulas:

$$
\begin{aligned}
\beta &:= \neg\Box p \wedge \Diamond\Box p, \quad \gamma := \beta \wedge \Diamond A_1 \wedge \neg\Diamond A_2 \wedge \neg\Diamond A_3,\\
\varepsilon &:= \beta \wedge \neg\Diamond A_1 \wedge \neg\Diamond A_2, \quad C_k := \Box^2[A_{k-1} \to \Diamond A_k] \quad \text{for } k \geq 2,\\
D_k &:= \Box^2[(A_k \wedge \neg\Diamond A_{k+1}) \to \Diamond\varepsilon], \quad E := \Box^2(\Box p \to \Diamond\gamma),\\
G_k &:= (\Box p \wedge \bigwedge_{i=2}^{k-1} C_i \wedge D_{k-1} \wedge E) \to \Diamond^2 A_k.
\end{aligned}
$$

Let $Prim := \{n \in \mathbb{N} :\ n+2 \text{ is prime } \wedge\ n \geq 5\}$. For any $X \subseteq Prim$ we define a logic L_X which is an axiomatic extension of the system $\mathbf{T_2}$:

$$L_X := \mathbf{T_2} \oplus \{G_k :\ k \in X\}.$$

It is proven in [3] that

Lemma 3. *Let $X, Y \subseteq Prim$ and $X \neq Y$. Then $L_X \neq L_Y$.*

Obviously, $card\,\{L_X :\ X \subseteq Prim\} = \mathfrak{c}$. We prove that:

Theorem 2. *There are uncountably many extensions of $\mathbf{T_2}$, which are Halldén-incomplete and hence - without (CIP).*

References

[1] A. Chagrow, M. Zakharyaschev, *Modal Logic*, Oxford Logic Guides 35, (1997).

[2] Z. Kostrzycka, *On formulas in one variable in* $NEXT(KTB)$, Bulletin of the Section of Logic, Vol. 35 (2/3), (2006), 119–131.

[3] Z. Kostrzycka, *On the existence of a continuum of logics in* $\mathrm{NEXT}(KTB \oplus \Box^2 p \to \Box^3 p)$, Bulletin of the Section of Logic, Vol. 36 (1), (2007), 1–7.

[4] Z. Kostrzycka, *On non-compact logics in* $NEXT(KTB)$, Math. Log. Quart., Vol. 54, No. 6, (2008), 582–589.

[5] Z. Kostrzycka, *On a finitely axiomatizable Kripke incomplete logic containing KTB*, J. Logic and Computation Advance access published March 2009.

[6] S.A. Kripke, *Semantical analysis of modal logic I*, Zeitschrift für Mathematische Logik und Grundlagen der Mathematik, Vol. 9, (1957), 176–186.

[7] Y. Miyazaki, *Kripke incomplete logics containing KTB*, Studia Logica, Vol. 85, (2007), 311–326.

[8] G. F. Schumm, *Some failures of interpolatin in modal logic*, Notre Dame Journal of Formal Logic, Vol. 27 (1), (1986), 108–110.

[9] I. Thomas, *Modal systems in the neighbourhood of T*, Notre Dame Journal of Formal Logic, Vol. 5 (1), (1964), 59–61.

Stably supported quantales with a given support

David Kruml
Masaryk University, Brno

In [2] P. Resende introduced a notion of stably supported quantale to describe relation between quantales and étale groupoids. One can think about stably supported quantales like about "quantales with enough projections". Another view is that stably supported quantales naturally generalize quantales of relations which make them interesting in other applications.

I intend to show that every frame (locale) is a support of some stably supported quantale. Even that we can request a way how the left and right-sided elements are multiplied. The construction of a quantale is a simple extension of that used in [1].

Let us recall the definition of stable supported quantale and its basic properties:

Definition 1. *A* supported quantale *is a unital involutive quantale Q with a specified sup-lattice endomorphism $\varsigma : Q \to Q$, called* support*, such that*

$$\varsigma a \leq e,$$
$$\varsigma a \leq a^* a,$$
$$\varsigma a \leq \varsigma a a$$

for any $a \in Q$.

The support is called stable *(and Q* stably supported*) if*

$$\varsigma a = e \wedge a$$

for every a.

1

It follows that a sub-lattice ςQ of supported quantale Q is a frame, and if Q is stably supported then ςQ is isomorphic to the sup-lattice of right-sided elements.

Recall that a frame homomorphism is called *open* if it has a left adjoint and satisfies the Frobenius reciprocity condition. If the homomorphism is a subframe inclusion $T \subseteq F$, then we denote the left adjoin by $|-|$ and call *a cover*. Then the Frobenius reciprocity condition is of the form

$$|a \wedge t| = |a| \wedge t \tag{1}$$

for $a \in F, t \in T$. At least, that provides a T-module structure on F and an "inner product" $F \times F \to T$.

Let us recall from [1] that F, T then make a symmetric triad (F, T, F), thus there is an involutive quantale for which F is isomorphic to the sup-lattice of right-sided elements and T to the sup-lattice of two-sided elelments. It is shown in the paper that there are two extremal solutions of the triad - the first is a tensor product $Q_0 = F \otimes_T F$, the second one Q_1 consists of pairs (f, g) of T-module endomorphisms $f, g : F \to F$ such that

$$|f(a) \wedge b| = |a \wedge g(b)|$$

for all $a, b \in F$.

In particular, $(id, id) \in Q_1$, hence it is a unital quantale. Since the elements are pairs of T-module endomorphisms, we obtain that the largest element is $(|-|, |-|)$. Indeed, $|a \wedge |b|| = |a| \wedge |b| = ||a| \wedge b|$ using twice (1). It follows that the right-sided elements are of the form $(a \wedge |-|, |a \wedge -|)$ and hence every pair $(a \wedge -, a \wedge -)$ could be a support element for Q_1. Indeed, by assigning $\varsigma(\phi, \psi)$ the largest such element less or equal to (ϕ, ψ) we obtain a well defined support on Q_1. Finally, $(id \wedge (a \wedge |-|))(b) \leq id(b) \wedge (a \wedge |b|) = a \wedge b$ proving that the support is stable. We conclude with the following theorem.

Theorem 1. *Let F be a frame and T its open subframe with cover $|-| : T \to F$. Then there exits a stably supported quantale Q such that*

(i) F is isomorphic to the sup-lattice of right-sided elements of Q,

(ii) T is isomorphic to the sup-lattice of two-sided elements of Q,

(iii) and $lr = |l \wedge r|$ for every left-sided l and right-sided r (after the identification provided by the two isomorphisms).

References

[1] D. Kruml, *Quantum triads: an algebraic approach*, arXiv:0801.0504v1, 2008.

[2] P. Resende, *tale groupoids and their quantales*, Advances in Mathematics 208 (2007), 147-209.

STONE DUALITY FOR SKEW BOOLEAN ALGEBRAS

GANNA KUDRYAVTSEVA

ABSTRACT. We present two refinements of the classical Stone duality between generalized Boolean algebras and locally compact Boolean spaces to dualities between left-handed skew Boolean algebras and étale spaces over locally compact Boolean spaces. We also apply our results to construct a series of adjunctions between the category of generalized Boolean algebras and the category of left-handed skew Boolean algebras.

Introduction. Skew Boolean algebras and skew Boolean ∩-algebras are non-commutative generalizations of Boolean algebras. They play an important role in the study of discriminator varieties and other generalizations of Boolean behavior both in universal algebra and in logic [2, 10, 11, 12, 13, 14].

The aim of the proposed talk is to present the results of [6, 7]. In [6] the classical Stone duality [15, 4] between generalized Boolean algebras and locally compact Boolean spaces is refined to the following two dualities between left-handed skew Boolean algebras and étale spaces over locally compact Boolean spaces.

Theorem 1. *The category* ESLCBS *of étale spaces over locally compact Boolean spaces whose morphisms are étale space cohomomorphisms over continuous proper maps is equivalent to the category* LSBA *of left-handed skew Boolean algebras whose morphisms are proper skew Boolean algebra homomorphisms.*

Theorem 2. *The category* ESLCBSE *of étale spaces with compact clopen equalizers over locally compact Boolean spaces whose morphisms are injective étale space cohomomorphisms over continuous proper maps is equivalent to the category* LSBIA *of left-handed skew Boolean ∩-algebras whose morphisms are proper skew Boolean ∩-algebra homomorphisms.*

These theorems generalize the classical Stone duality, looked at as the equivalence between the opposite category to the category of locally compact Boolean spaces and the category of generalized Boolean algebras.

In [7] we apply the results of [6] to construct a series of adjunctions between the category GBA of generalized Boolean algebras and the category LSBA of left-handed skew Boolean algebras, given by enriched Hom-functors. These adjunctions are based on the dual nature of finite sets, that can be looked at as discrete spaces and also as primitive skew Boolean algebras. Our construction provides the answer to the question posed in [14] to explicitly describe the left adjoint functor to the "twisted" functor ω.

Theorem 2 is closely related to the generalizations of Stone duality given in [5], since finite primitive skew Boolean ∩-algebras are quasi-primal. In particular, a partial case of Theorem 2 follows from the results of [5].

A different view of Stone duality for the category of skew Boolean ∩-algebras has recently appeared in [1]. Another related work is [8, 9], where the classical Stone duality is extended to non-commutative generalizations of Boolean algebras, called Boolean inverse monoids.

All skew Boolean algebras, considered in our talk, are left-handed.

From an étale space to a skew Boolean algebra. Recall that an *étale space* over X is a triple (E, π, X), where E, X are topological spaces and π a surjective local homeomorphism [3]. All étale spaces, considered in the sequel, are étale spaces over locally compact Boolean spaces.

If U is a compact clopen set in X then $E(U)$ denotes the set of all *sections* of E over U. The *stalks* of E are the equivalence classes induced by π. If $x \in X$ the stalk S in E such that $\pi(s) = x$ for all $s \in S$ will be denoted by E_x. We call an étale space (E, π, X) an *étale space with compact clopen equalizers*, provided that for every U, V compact clopen in X and any $A \in E(U)$, $B \in E(V)$, the intersection $A \cap B$ is a section, that is there is some compact clopen set $W \subseteq X$ such that $A \cap B \in E(W)$. The latter definition is natural because of its connection with the tools for studying discriminator varieties, such as Boolean products. It is easy to see that (E, π, X) is an étale space with compact clopen equalizers if and only if E is Hausdorff.

Let $(\mathcal{A}, g, X)$ and $(\mathcal{B}, h, Y)$ be étale spaces and $f : X \to Y$ be a continuous map. An *f-cohomomorphism* ([3, p. 14]) $k : \mathcal{B} \rightsquigarrow \mathcal{A}$ is a collection of maps $k_x : \mathcal{B}_{f(x)} \to \mathcal{A}_x$ for each $x \in X$ such that for every section $s \in \mathcal{B}(U)$ the function $x \mapsto k_x(s(f(x)))$ is a section of $\mathcal{A}$ over $f^{-1}(U)$.

Let X be a locally compact Boolean space and (E, f, X) be an étale space. Fix U and V to be compact clopen sets of X and let $A \in E(U)$, $B \in E(V)$. We define the *quasi-union* $A \underline{\cup} B$ of A and B to be the section in $E(U \cup V)$ given by

$$(A \underline{\cup} B)(x) = \begin{cases} B(x), & \text{if } x \in V, \\ A(x), & \text{if } x \in U \setminus V, \end{cases}$$

and the *quasi-intersection* $A \overline{\cap} B$ of A and B to be the section in $\mathcal{F}(U \cap V)$ given by

$$(A \overline{\cap} B)(x) = A(x) \text{ for all } x \in U \cap V.$$

We show that $(E, \underline{\cup}, \overline{\cap}, \varnothing)$ (where $\varnothing$ is the section of the empty set of X) is a left-handed Boolean skew lattice. Call $(E, \underline{\cup}, \overline{\cap}, \setminus, \varnothing)$ the *dual skew Boolean algebra* to the étale space $E = (E, f, X)$ and denote it by $E^\star = (E, f, X)^\star$.

The $\mathcal{D}$-classes of $(E, f, X)^\star$ are the stalks E_x, $x \in X$. The maximal generalized Boolean algebra image of $(E, f, X)^\star$ is isomorphic to $X^\star$, and the canonical projection $\delta : (E, f, X)^\star \to X^\star$ is given by $V \mapsto U$, whenever $V \in E(U)$, $U \in X^\star$.

Let (E, e, X) and (G, g, Y) be étale spaces over X and Y, respectively, $f : X \to Y$ be a continuous proper map and $k : G \rightsquigarrow E$ be an f-cohomomorphism. Let $A \in G(U)$. The section $k(A)$ is the image of the map sending $x \in f^{-1}(A)$ to $k_x(A(f(x)))$. We show that k is a proper skew Boolean algebra homomorphism from $G^\star$ to $E^\star$.

We obtain the functor $\mathbf{SB} : \mathsf{ESLCBS} \to \mathsf{LSBA}$ given by $\mathbf{SB}(E, f, X) = (E, f, X)^\star$ and $\mathbf{SB}(k) = k$.

From a skew Boolean algebra to an étale space. Let S be a skew Boolean algebra. Denote by $\alpha : S \to S/\mathcal{D}$ the canonical projection of S onto its maximal generalized Boolean algebra image $S/\mathcal{D}$. Call a nonempty subset U of S a *filter* provided that:

(1) for all $a, b \in S$: $a \in U$ and $b \geq a$ implies $b \in U$;
(2) for all $a, b \in S$: $a \in U$ and $b \in U$ imply $a \wedge b \in U$.

Call a subset U of S a *preprime filter* if U is a filter and there is a prime filter F of $S/\mathcal{D}$ such that $\alpha(U) = F$. Denote by $\mathcal{P}U_F$ the set of all preprime filters contained in $\alpha^{-1}(F)$. Call minimal elements of the sets $\mathcal{P}\mathcal{U}_F$ *prime filters* of S. Prime filters are exactly minimal nonempty preimages of 1 under the morphisms $S \to \mathbf{3}$. Denote the set of all prime filters U of S, such that $\alpha(U) = F$, by $\mathcal{U}_F$. We call prime filters of skew Boolean algebras SBA-prime filters.

Let $a \in S$, $a \neq 0$, and let F be a prime filter of $S/\mathcal{D}$, such that $\alpha(a) \in F$. Then the set

$$X_{a,F} = \{s \in S : s \geq t \text{ for some } t \in \alpha^{-1}(F) \text{ such that } t \leq a\}$$

is a SBA-prime filter of S, contained in $\mathcal{U}_F$ and containing a. Any element of $\mathcal{U}_F$ coincides with some $X_{a,F}$; and for any $U_1, U_2 \in \mathcal{U}_F$ we have either $U_1 = U_2$ or $U_1 \cap U_2 = \varnothing$.

Let $S^\star$ be the set of all SBA-prime filters of S. We call it the *spectrum* of S. Let $f : S^\star \to (S/\mathcal{D})^\star$ be the map, given by $U \mapsto F$, whenever $\alpha(U) = F$. In the classical Stone duality it is proved that $(S/\mathcal{D})^\star$ is a locally compact Boolean space, whose base constitute the compact clopen sets

$$M'(A) = \{F : F \text{ is a prime filter of } S/\mathcal{D} \text{ and } A \in F\}, \quad A \in S/\mathcal{D}.$$

Let $a \in S$. We define the set

$$M(a) = \{F : F \text{ is a SBA-prime filter of } S \text{ and } a \in F\}.$$

We topologize $S^\star$ so that the subbase of the topology is given by the sets $M(a)$, $a \in S$. We prove that $(S^\star, f, (S/\mathcal{D})^\star)$ is an étale space and we call it the *dual étale space* to the skew Boolean algebra S.

Let S, T be skew Boolean algebras, $k : T \to S$ be a proper homomorphism and $\overline{k} : T/\mathcal{D} \to S/\mathcal{D}$ be the induced proper homomorphism of generalized Boolean algebras. By the classical Stone duality, $\overline{k}^{-1}$ is a continuous proper map from $(S/\mathcal{D})^\star$ to $(T/\mathcal{D})^\star$. Set $g = \overline{k}^{-1}$. Let $S^\star = (S^\star, f_1, (S/\mathcal{D})^\star)$ and $T^\star = (T^\star, f_2, (T/\mathcal{D})^\star)$ to be the corresponding dual étale spaces. Let $F \in (S/\mathcal{D})^\star$ and $V \in S^\star_F = \mathcal{U}_F$. The set $k^{-1}(V)$, if nonempty, is some $U' \in \mathcal{PU}_{g(F)}$. We set $k_F(U) = V$, provided that U is a prime filter in $T^\star_{g(F)} = \mathcal{U}_{g(F)}$, $U \subseteq U'$ and $k^{-1}(V) = U'$. In this way we have defined a map k_F from $T^\star_{g(F)}$ to $S^\star_F$. The collection k_F, $F \in (S/\mathcal{D})^\star$, constitutes a $\overline{k}^{-1}$-cohomomorphism $\tilde{k} : T^\star \rightsquigarrow S^\star$.

We define the functor $\mathbf{ES} : \mathsf{LSBA} \to \mathsf{ESLCBS}$ by setting $\mathbf{ES}(S) = S^\star$ and $\mathbf{ES}(k) = \tilde{k}$.

On the proofs of Theorems 1 and 2. To prove Theorem 1 we show that the functors **ES** and **SB** establish the required equivalence of categories, where the natural isomorphism $\beta : 1_{\mathsf{LSBA}} \to \mathbf{SB} \cdot \mathbf{ES}$ and $\gamma : 1_{\mathsf{ESLCBS}} \to \mathbf{ES} \cdot \mathbf{SB}$ are given by

$$\beta_S(a) = M(a), S \in \mathrm{Ob}(\mathsf{LSBA}), a \in S; \tag{1}$$

$$\gamma_E(A) = N_A = \{N \in E^\star : A \in N\}, E \in \mathrm{Ob}(\mathsf{ESLCBS}), A \in E. \tag{2}$$

To prove Theorem 2 we first observe that (E, f, X) is an étale space with compact clopen equalizers if and only $E^\star$ has finite intersections. Let S, T be skew Boolean $\cap$-algebras and $k : S \to T$ be a proper homomorphism that preserves finite intersections. We show that $\tilde{k} : S^\star \rightsquigarrow T^\star$ is injective. Conversely, given étale spaces (E, e, X), (G, g, Y) and $k : E \rightsquigarrow G$, we observe that $k : E^\star \to G^\star$ preserves finite intersection and thus can be looked at as a skew Boolean $\cap$-algebra homomorphism.

Adjunctions between the categories LSBA and GBA. Let B be a generalized Boolean algebra and $X = B^\star$ be the dual locally compact Boolean space. Denote by $\omega_n(B)$, $n \geq 0$, the set of all continuous maps $f : X \to \{0, \dots, n+1\}$, such that $f^{-1}(1), \dots f^{-1}(n+1)$ are compact sets. We turn the sets $\omega_n(B)$ into left-handed skew Boolean algebras by defining the binary operations $\wedge$, $\vee$ and the nullary operation 0 as the induced operations of $\wedge$, $\vee$ and 0 on the primitive left-handed skew Boolean algebra $\mathbf{n+2}$. That is, for $f, g \in \omega_n(B)$ and $x \in X$ we set $(f \wedge g)(x) = f(x) \wedge g(x)$, $(f \vee g)(x) = f(x) \vee g(x)$ and the zero of $\omega_n(B)$ to be the zero function on X. This gives rise to the object-part of a series of functors from the category GBA to the category LCBS. It is convenient to identify $f \in \omega_n(B)$ with the $(n+1)$-tuple $(A_1, A_2, \dots, A_{n+1})$, where $A_i = f^{-1}(\{i, \dots, n+1\})$, $1 \leq i \leq n+1$.

The functor ω_1 was first studied in [14], where applying the Freyd's adjoint functor theorem it is proved that this functor has the left adjoint, and the question of describing this left adjoint was addressed. Below we provide the constructions of the functors Ω_n, which are enriched Hom-set functors and each Ω_n is the left adjoint to ω_n.

Let S be a skew Boolean algebra and $n \geq 0$. Let $\tilde{S}_n$ be the set of all non-zero SBA-homomorphisms from S to $\mathbf{n+2}$. We establish that there is a bijective correspondence between the elements of $\tilde{S}_n$ and the elements of the union of the sets $\{1, \dots, n+1\}^{\mathcal{U}_F}$ of all maps $f : \mathcal{U}_F \to \{1, \dots, n+1\}$, where F runs through $(S/\mathcal{D})^\star$. We identify elements of $\tilde{S}_n$ and of $\cup_{F \in (S/\mathcal{D})^\star}\{1, \dots, n+1\}^{\mathcal{U}_F}$. Note that the sets $\tilde{S}_n$ can be looked at as skew Boolean counterparts of the spectrum $B^\star$ of a generalized Boolean algebra B.

Let $x \in S^\star$ and let $F \in (S/\mathcal{D})^\star$ be such that $x \in \mathcal{U}_F$. For $1 \leq i \leq n+1$ we set

$$p_i(x) = \{f \in \{1, \dots, n+1\}^{\mathcal{U}_F} : f(x) = i\}.$$

For $s \in S$ and $i \in \{1, \dots, n+1\}$ we set $L(s, i) = \cup_{x \in M(s)} p_i(x)$.

We turn $\tilde{S}_n$ into a topological space by proclaiming the sets $L(s, i)$, $s \in S$, $1 \leq i \leq n+1$, to form a subbase of the topology. This topology naturally merges the product topologies on $\{1, \dots, n+1\}^{\mathcal{U}_F}$ and the locally compact Boolean space topology on $(S/\mathcal{D})^\star$. We prove that $\tilde{S}_n$ is a locally compact Boolean space and in the case when $(S/\mathcal{D})^\star$ is a Boolean space, then so is $\tilde{S}_n$. We set $\Omega_n(S) = (\tilde{S}_n)^\star$.

Theorem 3. *Let $n \geq 0$. The functor $\Omega_n : \mathsf{LSBA} \to \mathsf{GBA}$ is a left adjoint to the functor $\omega_n : \mathsf{GBA} \to \mathsf{LSBA}$. The unit of the adjunction $\eta : 1_{\mathsf{LSBA}} \to \omega_n\Omega_n$ is given by $\eta_S(a) = (\cup_{i=1}^{k} L(a, i))_{1 \leq k \leq n+1}$, $S \in \mathrm{Ob}(\mathsf{LSBA})$, $a \in S$.*

References

[1] A. Bauer, K. Cvetko-Vah, Stone duality for skew Boolean algebras with itersections, arXiv:1106.0425v1.

[2] R. J. Bignall, J. E. Leech, Skew Boolean algebras and discriminator varieties, *Algebra Universalis* **33** (1995), 387–398.

[3] G.E. Bredon, *Sheaf Theory*, Grad. Texts in Math., vol. 170, Springer-Verlag, 1997.

[4] H. P. Doctor, The categories of Boolean lattices, Boolean rings and Boolean spaces. *Canad. Math. Bull.* **7** (1964), 245-252.

[5] K. Keimel, H. Werner, Stone duality for varieties generated by quasi-primal algebras, *Mem. Amer. Math. Soc.* **148** (1974), 59–85.

[6] G. Kudryavtseva, A refinement of Stone duality to skew Boolean algebras, arXiv:1102.1242v2.

[7] G. Kudryavtseva, Adjunctions between Boolean and skew Boolean algebras, in preparation.

[8] M. V. Lawson, A non-commutative generalization of Stone duality, arXiv:0911.2863v2.

[9] M. V. Lawson, Non-commutative Stone duality: inverse semigroups, topological groupoids and C*-algebras, arXiv:1104.1054v1.

[10] J. Leech, Skew lattices in rings, *Alg. Universalis* **26** (1989), 48–72.

[11] J. Leech, Skew Boolean algebras, *Alg. Universalis* **27** (1990), 497–506.

[12] J. Leech, Normal skew lattices, *Semigroup Forum* **44** (1992), 1-8.

[13] J. Leech, Recent developments in the theory of skew lattices, *Semigroup Forum* **52** (1996), 7–24.

[14] J. Leech, M. Spinks, Skew Boolean algebras derived from generalized Boolean algebras, *Algebra Universalis* **58** (2008), no. 3, 287-302.

[15] M. H. Stone, Applications of the theory of Boolean rings to gneral topology. *Trans. Amer. Math. Soc*, **41** (1937), 375–481.

G. K.: University of Ljubljana, Faculty of Computer and Information Science, Tržaška cesta 25, SI-1001, Ljubljana, SLOVENIA.

E-mail address: `ganna.kudryavtseva@fri.uni-lj.si`

On n-potent and divisible pseudo-BCK-algebras

Jan Kühr

Department of Algebra and Geometry
Faculty of Science, Palacký University in Olomouc
17. listopadu 12, CZ-77146 Olomouc, Czech Republic
jan.kuhr@upol.cz

A porim (partially ordered residuated integral monoid) is (i) *divisible* if it is naturally ordered, in the sense that $a \leq b$ iff $a = x \cdot b = b \cdot y$ for some x, y, and (ii) *n-potent* (where $n \in \mathbb{N}$) if it satisfies the identity $x^n = x^{n+1}$. Divisible porims are also known as *pseudo-hoops* (see [5]), and divisible integral residuated lattices as *integral GBL-algebras* (see [7], [8]). We deal with divisibility and n-potency in the setting of pseudo-BCK-algebras (or biresiduation algebras) that are the residuation subreducts of porims (or integral residuated lattices). We attempt to generalize some structural results proved by Blok and Ferreirim [1] for hoops, and by Jipsen and Montagna [7], [8] for integral GBL-algebras.

By a *pseudo-BCK-algebra* we mean an algebra $\mathbf{A} = \langle A, \to, \rightsquigarrow, 1\rangle$ of type $\langle 2, 2, 0\rangle$ together with a partial order $\leq$ under which 1 is the top element, satisfying the following axioms: (i) $x \to y \leq (y \to z) \rightsquigarrow (x \to z)$, $x \rightsquigarrow y \leq (y \rightsquigarrow z) \to (x \rightsquigarrow z)$, (ii) $1 \to x = x = 1 \rightsquigarrow x$, and (iii) $x \leq y$ iff $x \to y = 1$ iff $x \rightsquigarrow y = 1$. BCK-algebras coincide with pseudo-BCK-algebras satisfying $x \to y = x \rightsquigarrow y$. For the original definition, see [4] or [6]. Every pseudo-BCK-algebra can be embedded into the $\{\to, \rightsquigarrow, 1\}$-reduct of an integral residuated lattice (we use $\to, \rightsquigarrow$ to denote the residuals, so the law of residuation has the form $x \cdot y \leq z$ iff $x \leq y \to z$ iff $y \leq x \rightsquigarrow z$), hence pseudo-BCK-algebras are equivalent to *biresiduation algebras* (see [9]). It is easy to show that a porim is divisible iff it satisfies the identities

$$\begin{aligned} (x \to y) \to (x \to z) &= (y \to x) \to (y \to z), \\ (x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z) &= (y \rightsquigarrow x) \rightsquigarrow (y \rightsquigarrow z), \end{aligned} \tag{1}$$

and it is therefore natural to call such pseudo-BCK-algebras *divisible*. A seemingly stronger version of divisibility was considered by Vetterlein [10] for pseudo-BCK-algebras that are join-semilattices with respect to the underlying order; specifically, in addition to (1) he required that the identities $(x \to y) \to (x \to z) = x \to ((x \rightsquigarrow y) \to z)$ and $(x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z) = x \rightsquigarrow ((x \to y) \rightsquigarrow z)$ be satisfied, but we prove that these two equations are derivable from (1).

A *deductive system* of a pseudo-BCK-algebra $\mathbf{A}$ is a subset X such that $1 \in X$ and, for all $a, b \in A$, whenever $a, a \to b \in X$ (or $a, a \rightsquigarrow b \in X$), then $b \in X$. If, moreover, for all $a, b \in A$ we have $a \to b \in X$ iff $a \rightsquigarrow b \in X$, then we say that X is a *normal* deductive system. The normal deductive systems of $\mathbf{A}$ are exactly the 1-classes of the relative congruences of $\mathbf{A}$, i.e. those θ for which $\mathbf{A}/\theta$ is a pseudo-BCK-algebra. Following [8], we call a pseudo-BCK-algebra *normal* when all its deductive systems are normal.

For $a, b \in A$ and $n \in \mathbb{N}_0$, the expression $a^n \to b$ (and analogously $a^n \rightsquigarrow b$) means: $a^0 \to b = b$ and $a^n \to b = a \to (a^{n-1} \to b)$ if $n \geq 1$. Normal pseudo-BCK-algebras can be characterized as follows: $\mathbf{A}$ is normal iff for all $a, b \in A$ there exist $m, n \in \mathbb{N}$ such that $a \to b \leq a^m \rightsquigarrow b$ and $a \rightsquigarrow b \leq a^n \to b$.

Given $n \in \mathbb{N}$, we say that a pseudo-BCK-algebra $\mathbf{A}$ is *n-potent* if, for all $a, b \in A$, we have

$$a^n \to b = 1 \quad \text{iff} \quad a^{n+1} \to b = 1.$$

This is the same as the condition that $a^n \rightsquigarrow b = 1$ iff $a^{n+1} \rightsquigarrow b = 1$. A porim is n-potent exactly if it is n-potent as a pseudo-BCK-algebra. We prove that a pseudo-BCK-algebra $\mathbf{A}$ is n-potent if and only if $\mathbf{A}$ satisfies either of the identities

$$x^n \to y = x^{n+1} \to y, \qquad x^n \rightsquigarrow y = x^{n+1} \rightsquigarrow y.$$

Cornish [3] studied BCK-algebras satisfying such identities. Every finite pseudo-BCK-algebra is n-potent, for some $n \in \mathbb{N}$, and the class of n-potent pseudo-BCK-algebras is a variety.

In accordance with the terminology that is used for BCK-algebras, by a *commutative* pseudo-BCK-algebra we mean an algebra that fulfills the identity

$$(x \to y) \rightsquigarrow y = (y \rightsquigarrow x) \to x.$$

In this case, the underlying poset of $\mathbf{A}$ is a join-semilattice in which $x \vee y = (x \to y) \rightsquigarrow y$ for all $x, y \in A$. Moreover, if $\mathbf{A}$ is both commutative and divisible, then it is a *cone algebra* in the sense of Bosbach [2], i.e., $\mathbf{A}$ can be represented as a subalgebra of the algebra $\mathbf{G}^- = \langle G^-, \to, \rightsquigarrow, 1\rangle$, where $\mathbf{G}$ is a lattice-ordered group and G^- its negative cone equipped with the operations $x \to y = y \cdot (x \vee y)^{-1}$ and $x \rightsquigarrow y = (x \vee y)^{-1} \cdot y$. Bounded cone algebras are termwise equivalent to pseudo-MV-algebras.

For any divisible pseudo-BCK-algebra $\mathbf{A}$, we prove the following: (i) If $\{1\}$ and A are the only deductive systems of $\mathbf{A}$, then $\mathbf{A}$ is a linearly ordered cone algebra. (ii) If $\mathbf{A}$ is an n-potent linearly ordered cone algebra, then it is an MV-algebra which has at most $n + 1$ elements. (iii) If $\mathbf{A}$ is n-potent, then it is normal and satisfies the identity $x^n \to y = x^n \rightsquigarrow y$.

Let $\mathbf{A}_1$, $\mathbf{A}_2$ be two pseudo-BCK-algebras such that $A_1 \cap A_2 = \{1\}$. Their *ordinal sum* is the pseudo-BCK-algebra $\mathbf{A}_1 \oplus \mathbf{A}_2 = \langle A_1 \cup A_2, \to, \rightsquigarrow, 1\rangle$, where

$$x \to y = \begin{cases} x \to_i y & \text{if } x, y \in A_i, \\ 1 & \text{if } x \in A_1 \setminus \{1\} \text{ and } y \in A_2, \\ y & \text{if } x \in A_2 \text{ and } y \in A_1 \setminus \{1\}, \end{cases}$$

and $\rightsquigarrow$ is defined in the same way. Obviously, both $\mathbf{A}_1$ and $\mathbf{A}_2$ are subalgebras of $\mathbf{A}_1 \oplus \mathbf{A}_2$, and $\mathbf{A}_1$ is “below” $\mathbf{A}_2$, i.e., $x < y$ for all $x \in A_1 \setminus \{1\}$ and $y \in A_2$.

Theorem 1. *Let $\mathbf{A}$ be a non-trivial subdirectly irreducible normal divisible pseudo-BCK-algebra. Then $\mathbf{A} = \mathbf{B} \oplus \mathbf{C}$, where $\mathbf{C}$ is a non-trivial subdirectly irreducible linearly ordered cone algebra.*

As a corollary we obtain:

Theorem 2. *Every n-potent divisible pseudo-BCK-algebra is a BCK-algebra. In particular, every finite divisible pseudo-BCK-algebra is a BCK-algebra.*

References

[1] W.J. Blok and I.M.A. Ferreirim: *On the structure of hoops*, Algebra Univers. 43 (2000), 233–257.

[2] B. Bosbach: *Concerning cone algebras*, Algebra Univers. 15 (1982), 58–66.

[3] W.H. Cornish: *Varieties generated by finite BCK-algebras*, Bull. Austral. Math. Soc. 22 (1980), 411–430.

[4] G. Georgescu and A. Iorgulescu: *Pseudo-BCK algebras: An extension of BCK algebras*, in: Proceedings of DMTCS01: Combinatorics, Computability and Logic, Springer, London, 2001, pp. 97–114.

[5] G. Georgescu, L. Leuştean, and V. Preoteasa: *Pseudo-hoops*, J. Mult.-Val. Log. Soft Comput. 11 (2005), 153–184.

[6] A. Iorgulescu: *Algebras of logic as BCK algebras*, Editura ASE, Bucharest, 2008.

[7] P. Jipsen and F. Montagna: *On the structure of generalized BL-algebras*, Algebra Univers. 55 (2006), 226–237.

[8] P. Jipsen and F. Montagna: *The Blok-Ferreirim theorem for normal GBL-algebras and its application*, Algebra Univers. 60 (2009), 381–404.

[9] C.J. van Alten: *On varieties of biresiduation algebras*, Studia Logica 83 (2006), 425–445.

[10] T. Vetterlein: *Pseudo-BCK algebras as partial algebras*, Inf. Sci. 180 (2010), 5101–5114.

Finite Embeddability Property of Distributive Lattice-ordered Residuated Groupoids with Modal Operators

Zhe Lin

Institute of Logic and Cognition, Sun Yat-sen University, Guangzhou, China
Faculty of Mathematics and Computer Science Adam Mickiewicz University, Poznań, Poland
pennyshaq@gmail.com

Abstract. We prove Finite Embeddability Property (FEP) of the class of residuated groupoids with modal operators $\Diamond$, $\Box^{\downarrow}$, satisfying standard S4-axioms (4, T), and distributive lattice operations $\wedge$, $\vee$. The key tool is an interpolation lemma in the style of Buszkowski [1],[2], adapted to S4-systems.

1 Introduction

Farulewski [6] proves FEP of the class of all residuated groupoids, solving an open problem from Blok and van Alten [5]. An essential tool was an interpolation lemma for a sequent system of Nonassociative Lambek Calculus (NL), following some ideas of Buszkowski [1]. Buszkowski [2] obtains similar results for residuated algebras, supplied with distributive lattice operations and a boolean or Heyting negation (also see [3]). He also proves the context-freeness of the corresponding grammars.

NL can be supplied with modal operations $\Diamond$, $\Box^{\downarrow}$, satisfying the adjoint law: $\Diamond A \Rightarrow B$ iff $A \Rightarrow \Box^{\downarrow}B$. Such systems were studied by e.g. Moortgat [11] and Plummer [12] in connection with Type-Logical Grammar. Plummer [11] assumes S4-axioms $\Diamond\Diamond A \Rightarrow A$, $A \Rightarrow \Diamond A$, replacing them by some structural rules; his results concern the context-freeness of such systems.

In [8], we extend Plummer's results to $\mathrm{NL}_{\mathrm{S4}}$ with assumptions and prove the polynomial time decidability of the consequence relation of $\mathrm{NL}_{\mathrm{S4}}$ and the context-freeness of the corresponding grammars. In [9], we prove FEP of the class of residuated groupoids with S4-modal operations. (We also admit an additional axiom $\Diamond(a \cdot b) \leq \Diamond a \cdot \Diamond b$).

The present paper extends the latter results to S4-modal residuated groupoids with operations of a distributive lattice, (but we omit the extra axiom $\Diamond(a \cdot b) \leq \Diamond a \cdot \Diamond b$), which enables us to obtain a simpler, more elegant form of the interpolation lemma.

We present a sequent system, complete with respect to the class of distributive lattice ordered S4-modal residuated groupoids. This system is an extension of Full Nonassociative Lambek Calculus (FNL). Since the distributive law $A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$ is added as a new axiom, this system does not admit cut-elimination (like in [2],[3]).

The proof of the interpolation lemma here is finer than in earlier papers (e.g. [8],[12]) which yields a simpler description of the set of possible intepolants.

Our interest in modal postulates and non-logical assumptions can be partly motivated by linguistics application. Firstly, Non-logical assumptions are useful in linguist, especially when we need some sequents, which can not be derivable in the logic system. For instance, in the Lambek calculus we can not transform $(s\backslash(s/s))$ (the type of sentence conjunction) to $vp\backslash(vp/vp)$ (the type of verb phrase conjunction). However, we can add the sequent $(s\backslash(s/s)) \Rightarrow vp\backslash(vp/vp)$ as an assumption. Secondly, S4-modalities might also be of some use for linguist purpose. There are many evidences for the usefulness of S4-modalities with associative rule in linguistics application (see Morrill [11] and Hepple [7]). A simple example can be found in [9]. Last but not least, although categorial grammars with additives are not popular in the linguistic literature. There are, nonetheless, good reasons for studying them. For example, in [12], Kanazawa proposes feature decomposition of basic categories by using conjunction and disjunction: *walks* is assigned type $(np \wedge sing)\backslash s$, *walk* type $(np \wedge pl)\backslash s$, *walked* type $np\backslash s$, *John* type $np \wedge sing$, *the Beatles* type $np \wedge pl$, *the Chinese* type $np \wedge (sing \wedge pl)$ and *became* type $(np\backslash s)/(np \vee ap)$, where ap stands for adjective phrase.

2 Main definition and results

Let us recall the sequent system of NL$\Diamond$. Formulae (types) are formed out of atomic types p, q, r ... by means of three binary operation symbols $\bullet$, $\backslash$, $/$ and two unary operation symbols $\Diamond$, $\Box^{\downarrow}$. Formula trees (formula-structures) are recursively defined as follow: (i) every formula is a formula-tree, (ii) if Γ, Δ are formula-trees, then $(\Gamma \circ \Delta)$ is a formula-tree, (iii) if Γ is a formula-tree, then $\langle \Gamma \rangle$ is a formula-tree. Sequents are of the form $\Gamma \Rightarrow A$ such that Γ is a formula tree and A is a formula.

One admits the axioms:

$$\text{(Id)} \quad A \Rightarrow A$$

and the inference rules

$$(\backslash\text{L}) \ \frac{\Delta \Rightarrow A; \quad \Gamma[B] \Rightarrow C}{\Gamma[\Delta \circ (A\backslash B)] \Rightarrow C} \quad (\backslash\text{R}) \ \frac{A \circ \Gamma \Rightarrow B}{\Gamma \Rightarrow A\backslash B} \quad (/\text{L}) \ \frac{\Gamma[A] \Rightarrow C; \quad \Delta \Rightarrow B}{\Gamma[(A/B) \circ \Delta] \Rightarrow C} \quad (/\text{R}) \ \frac{\Gamma \circ B \Rightarrow A}{\Gamma \Rightarrow A/B}$$

$$(\cdot\text{L}) \ \frac{\Gamma[A \circ B] \Rightarrow C}{\Gamma[A \cdot B] \Rightarrow C} \quad (\cdot\text{R}) \ \frac{\Gamma \Rightarrow A; \quad \Delta \Rightarrow B}{\Gamma \circ \Delta \Rightarrow A \cdot B} \quad (\text{CUT}) \ \frac{\Delta \Rightarrow A; \quad \Gamma[A] \Rightarrow B}{\Gamma[\Delta] \Rightarrow B}$$

The following are sequent rules for the unary modalities:

$$(\Diamond\text{L}) \ \frac{\Gamma[\langle A \rangle] \Rightarrow B}{\Gamma[\Diamond A] \Rightarrow B} \quad (\Diamond\text{R}) \ \frac{\Gamma \Rightarrow A}{\langle \Gamma \rangle \Rightarrow \Diamond A} \quad (\Box^{\downarrow}\text{L}) \ \frac{\Gamma[A] \Rightarrow B}{\Gamma[\langle \Box^{\downarrow} A \rangle] \Rightarrow B} \quad (\Box^{\downarrow}\text{R}) \ \frac{\langle \Gamma \rangle \Rightarrow A}{\Gamma \Rightarrow \Box^{\downarrow} A}$$

Distributive Full Nonassociative Lambek Calculus enriched with unary modalities DFNL$\Diamond$ employs operations $\cdot$, $\backslash$, $/$, $\wedge$ and $\vee$. One admits the following rules

$$(\wedge\text{L}) \ \frac{\Gamma[A_i] \Rightarrow B}{\Gamma[A_1 \wedge A_2] \Rightarrow B} \quad (\wedge\text{R}) \ \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$(\vee\text{L}) \ \frac{\Gamma[A_1] \Rightarrow B \quad \Gamma[A_2] \Rightarrow B}{\Gamma[A_1 \vee A_2] \Rightarrow B} \quad (\vee\text{R}) \ \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}$$

and the distributive axiom: (D) $\quad A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$.

By DFNL$_{4\text{T}}$, we denote the system DFNL$\Diamond$ enriched with the following structural rules 4 and T (corresponding to axioms 4, T)

$$(\text{T}) \ \frac{\Gamma[\langle \Delta \rangle] \Rightarrow A}{\Gamma[\Delta] \Rightarrow A} \quad (4) \ \frac{\Gamma[\langle \Delta \rangle] \Rightarrow A}{\Gamma[\langle\langle \Delta \rangle\rangle] \Rightarrow A}$$

Let Φ be a set of assumptions of the form $A \Rightarrow B$. Hereafter, we assume that Φ is finite. Let T denote a set of formulas. By a T-sequent we mean a sequent such that all formulas occurring in it belong to T. We write $\Phi \vdash_S \Gamma \Rightarrow_T A$ if $\Gamma \Rightarrow A$ has a deduction from Φ (in the given calculus S) which consists of T-sequents only (called a T-deduction). Two formulae A and B are said to be T-*equivalent* in calculus S, if and only if $\vdash_S A \Rightarrow_T B$ and $\vdash_S B \Rightarrow_T A$.

Lemma 1 *Let T be a set of formulae closed under $\vee$, $\wedge$. If $\Phi \vdash_{\text{DFNL}_{4\text{T}}} \Gamma[\langle \Delta \rangle] \Rightarrow_T A$ then there exists a $D \in T$ such that $\Phi \vdash_{\text{DFNL}_{4\text{T}}} \langle \Delta \rangle \Rightarrow_T D$, $\Phi \vdash_{\text{DFNL}_{4\text{T}}} \langle D \rangle \Rightarrow_T D$, and $\Phi \vdash_{\text{DFNL}_{4\text{T}}} \Gamma[D] \Rightarrow_T A$.*

Lemma 2 *Let T be a set of formulae closed under $\vee$, $\wedge$. If $\Phi \vdash_{\text{DFNL}_{4\text{T}}} \Gamma[\Delta] \Rightarrow_T A$ then there exists a $D \in T$ such that $\Phi \vdash_{\text{DFNL}_{4\text{T}}} \Delta \Rightarrow_T D$ and $\Phi \vdash_{\text{DFNL}_{4\text{T}}} \Gamma[D] \Rightarrow_T A$.*

Lemma 3 *If T is set of formulas generated from a finite set and closed under $\wedge$, $\vee$, then T is finite up to the relation of T-equivalence in* DFNL$_{4\text{T}}$.

A distributive lattice-ordered residuated groupoid with 4T-operators (4T-DLRG) is a structure $(G, \wedge, \vee, \cdot, \backslash, /, \Diamond, \Box^{\downarrow})$ such that $(G, \wedge, \vee)$ is a distributive lattice and $(G, \cdot, \backslash, /, \Diamond, \Box^{\downarrow})$ is a structure such that $\cdot, \backslash, /$ and $\Diamond, \Box^{\downarrow}$ are binary and unary operations on G, respectively, satisfying the following conditions:

$$a \cdot b \leq c \quad \text{iff} \quad b \leq a\backslash c \quad \text{iff} \quad a \leq c/b \tag{1}$$

$$\Diamond a \leq b \quad \text{iff} \quad a \leq \Box^{\downarrow} b \tag{2}$$

$$4: \quad \Diamond\Diamond a \leq \Diamond a \quad T: \quad a \leq \Diamond a \tag{3}$$

for all $a, b, c \in G$, where $\leq$ is the lattice ordering. It is easy to prove that $\mathrm{DFNL_{4T}}$ is strongly complete with respect to 4T-DLRGs. Besides by 4T-LRG, we denote a lattice-ordered residuated groupoid with 4T-operators. Let $\mathcal{G}$ be a 4T-DLRG. We recall some basic notions. A valuation μ in $\mathcal{G}$ is a homomorphism from the formula algebra into $\mathcal{G}$. A sequent $\Gamma \Rightarrow A$ is true in the model ($\mathcal{G}$, μ), if $\mu(\Gamma) \leq \mu(A)$. The strong completeness means the following: $\Phi \vdash_{\mathrm{DFNL_{4T}}} \Gamma \Rightarrow A$ if and only if, for any model ($\mathcal{G}$, μ). if all sequents from Φ are true, then $\Gamma \Rightarrow A$ is true.

By $\mathcal{G}$ and G, we denote a groupoid and its universe, respectively. Let $\mathcal{G} =(G, \cdot, \diamond)$ be a groupoid, where $\Diamond$ is a unary operation on G. On the powerset $P(G)$ one defines operations: $U \odot V = \{a \cdot b \in G : a \in U, b \in V\}$, $\Diamond U = \{\diamond a \in G : a \in U\}$, $U \backslash V = \{z \in G : U \odot \{z\} \subseteq V\}$, $V/U = \{z \in M; \{z\} \odot U \subseteq V\}$, $\Box^{\downarrow} U\{z \in G : \diamond z \in U\}$, $U \vee V = U \cup V$, $U \wedge V = U \cap V$. $P(G)$ with operations $\odot$, $\Diamond$, $\backslash$, $/$, $\Box^{\downarrow}$, $\vee$ and $\wedge$ is a distributive lattice-ordered residuated groupoid satisfying (1) and (2) (it is a complete lattice). The order is $\subseteq$. An operator C : $P(G) \to P(G)$ is called a 4T-closure operator (or: a $4T$-nucleus) on $\mathcal{G}$, if it satisfies the following conditions: (C1) $U \subseteq C(U)$, (C2) if $U \subseteq V$ then $C(U) \subseteq C(V)$, (C3) $C(C(U)) \subseteq C(U)$, (C4) $C(U) \odot C(V) \subseteq C(U \odot V)$, (C5) $\Diamond C(U) \subseteq C(\Diamond U)$, (C6) $C(\Diamond C(\Diamond C(U))) \subseteq C(\Diamond U)$, (C7) $C(U) \subseteq C(\Diamond U)$. For $U \subseteq P(G)$, U is called $C - closed$ if $U = C(U)$. By $C(G)$, we denote the family of $C - closed$ subsets of G. Let $U \otimes V = C(U \odot V)$, $\blacklozenge U = C(\Diamond U)$, $U \vee_C V = C(U \vee V)$, and $\backslash$, $/$, $\Box^{\downarrow}$, $\wedge$, be defined as above. By (C1)-(C5), $C(\mathcal{G}) = (C(G), \wedge, \vee_C, \otimes, \backslash, /, \blacklozenge, \Box^{\downarrow})$ is a complete lattice-ordered residuated groupoid [3]; it need not be distributive. The order is $\subseteq$. Using (C6)-(C8), it is easy to prove $\blacklozenge\blacklozenge U \subseteq \blacklozenge U$, $U \subseteq \blacklozenge U$. It follows that $C(\mathcal{G})$ is a 4T-LRG.

Let T be a nonempty set of formulae containing all subformulae of formulae in Φ. By T^*, we denote the set of all formula structures form out of formulae in T. Similarly $T^*[\circ]$ denotes the set of all contexts in which all formulae belong to T. Let $\Gamma[\circ] \in T^*$ and $A \in T$, we define:

$$[\Gamma[\circ], A] = \{\Delta : \Delta \in T^* \quad \text{and} \quad \Phi \vdash_{\mathrm{DFNL_{4T}}} \Gamma[\Delta] \Rightarrow_T A\}$$

We define $B(T)$ as the family of all sets $[\Gamma[\circ], A]$, defined above. One defines C_T as follows:

$$C_T(U) = \bigcap \{[\Gamma[\circ], A] \in B(T) : U \subseteq [\Gamma[\circ], A]\}$$

We prove the following proposition.

Proposition 1 *C_T is a* 4T*-modal closed operator.*

T donotes a set of formulae containing all formulae in Φ, which is closed under subformulae, $\wedge$ and $\vee$. $\mathcal{G}(\mathcal{T}^*) = (T^*, \circ, \langle\rangle)$ is a groupoid such that $\langle\rangle$ is an unary operation on T^*. Accordingly, $C_T(\mathcal{G}(\mathcal{T}^*))$ is a 4T-LRG. Let μ be a valuation such that $\mu(p) = [p]$ for all propositional variables in T. The following equations are true in $C(\mathcal{G}(\mathcal{T}^*))$ provided that all formulas appearing in them belong to T.

$$[A] \otimes [B] = [A \cdot B], \quad [A] \backslash [B] = [A \backslash B], \quad [A]/[B] = [A/B] \tag{4}$$

$$\blacklozenge [A] = [\Diamond A] \quad \Box^{\downarrow}[A] = [\Box^{\downarrow} A] \tag{5}$$

$$[A] \cap [B] = [A \wedge B] \quad [A] \vee_C [B] = [A \vee B] \tag{6}$$

By Lemma 3, there exists a finite set $R \subseteq T$ such that every formula from T is T-equivalent to some formula from R.

Lemma 4 *For any $U \in C_T(G(T^*))$, there exists a formula $A \in R$ such that $U = [A]$.*

Lemma 5 *$C_T(\mathcal{G}(\mathcal{T}^*))$ is a* $4\mathrm{T} - \mathrm{DLRG}$.

Now we are ready to prove SFMP for $\mathrm{DFNL_{4T}}$.

Lemma 6 *T denotes a set of formulae, containing all formulae in Φ and closed under $\wedge$, $\vee$, and subformulae. Let μ be a valuation in $C_T(\mathcal{G}(\mathcal{T}^*))$ such that $\mu(p) = [p]$. For any T-sequent $\Gamma \Rightarrow A$, this sequent is true in $(C_T(\mathcal{G}(\mathcal{T}^*)), \mu)$ if and only if $\Phi \vdash_{\mathrm{DFNL_{4T}}} \Gamma \Rightarrow_T A$.*

Proof: Assume a T-sequent $\Gamma \Rightarrow A$ be true in $(C(\mathcal{G}(\mathcal{T}^*)), \mu)$. Then $\mu(\Gamma) \subseteq \mu(A)$. Since $\Gamma \in \mu(\Gamma)$, we get $\Gamma \in \mu(A) = [A]$. Hence $\Phi \vdash_{\mathrm{DFNL_{4T}}} \Gamma \Rightarrow_T A$. Assume $\Phi \vdash_{\mathrm{DFNL_{4T}}} \Gamma \Rightarrow_T A$. We prove that $\Gamma \Rightarrow A$ is true in $(C(\mathcal{G}(\mathcal{T}^*)), \mu)$, by induction on T-deductions. The axioms (Id), (D) and the assumptions from Φ, restricted to T-sequents, are of the form $E \Rightarrow F$. By (4), (5) and (6) , we get $\mu(E) = [E]$ and $\mu(F) = [F]$. Assume $\Delta \in [E]$, we get $\Phi \vdash_{\mathrm{DFNL_{4T}}} \Delta \Rightarrow_T E$. Hence $\Phi \vdash_{\mathrm{DFNL_{4T}}} \Delta \Rightarrow_T F$, by (CUT), which yields $[E] \subseteq [F]$. Since $C(\mathcal{G}(\mathcal{T}^*))$ is a 4T-DLRG, all rules of $\mathrm{DFNL_{4T}}$ preserve the truth in $(C(\mathcal{G}(\mathcal{T}^*)), \mu)$, whence $\Gamma \Rightarrow A$ is true in $C(\mathcal{G}(\mathcal{T}^*))$. □

Theorem 7 *Assume that $\Phi \vdash_{\mathrm{DFNL_{4T}}} \Gamma \Rightarrow A$ does not hold. Then there exist a finite distributive lattice ordered residuated groupoid with* 4T*-operators $\mathcal{G}$ and a valuation μ such that all sequents from* Φ *are true but $\Gamma \Rightarrow A$ is not true in $(\mathcal{G}, \mu)$.*

Proof: Let T be the set of all formulas appearing in Φ and $\Gamma \Rightarrow A$. $\overline{T}$ denote the closure of T under $\wedge, \vee$. Hence $C(\mathcal{G}(\overline{\mathcal{T}}^*))$ is a finite distributive lattice ordered residuated groupoid, by Lemma 5. Assume $\Phi \nvdash_{\mathrm{DFNL_{4T}}} \Gamma \Rightarrow A$, which yields $\Phi \nvdash_{\mathrm{DFNL_{4T}}} \Gamma \Rightarrow_T A$. Let $\mu(p) = [p]$. By lemma 6, all sequents from Φ are true in $(C(\mathcal{G}(\mathcal{T}^*)), \mu)$ but $\Gamma \Rightarrow A$ is not true. □

Corollary 8 4T − DLRGs *has FEP.*

This results can also be extended to boolean residuated groupoids with S5-modal operators. We defer a detailed discussion to a forthcoming paper.

References

1. F. Belardinelli, P. Jipsen, and H. Ono, Algebraic Aspects of Cut Elimination, Studia Logica 77:209-240, 2004.
2. W. Buszkowski, Lambek Calculus with Nonlogical Axioms. In C. Casadio, P. J. Scott, and R. A. G. Seely (eds.), Language and Grammar, Studies in Mathematical Linguistics and Natural Language, CSLI, Lecture Notes 168:77-93, Stanford 2005
3. W. Buszkowski, Interpolation and FEP for Logic of Residuated Algebras, Logic Journal of the IGPL, doi:10.1093/jigpal/jzp094.
4. W. Buszkowski, and M. Farulewski, Nonassociative Lambek Calculus with Addi- tives and Context-Free Languages. In: O. Grumberg et al. (eds.), Languages: From Formal to Natural, LNCS 5533:45-58, 2009.
5. W. J. Blok, C. J. Van Alten On the finite embeddability property for residuated ordered groupoids, Transactions of the AMS 357 (10) (2005).
6. M. Farulewski, Finite embeddability property for residuated groupoids, Reports on Mathematical Logic, 43:25-42, 2008.
7. M. Hepple, The Grammer and Processing of Order and Denpendency: A Categorial Approach Phd Dissertation, Edinburgh, 1990.
8. J. Lambek, The mathematics of sentence structure. American Mathematical Monthly 65:154-170, 1958.
9. Z. Lin, Modal Nonassociative Lambek Calculus with Assumptions: Complexity and Context-freeness, In: A.H. Dediu, H. Fernau, C. Martin-Vide (eds.), Language and Automata Theory and Applications, Fourth International Conference, LATA 2010, LNCS 6031, 414-425, 2010.
10. Z. Lin, Finite Embeddability Property of S4 modal residuated groupoids, reports on Algebra and Sub-structural Logics - take four, Kanazawa, Japan, 06, 2010.
11. Morrill. G, Intensionality and boundness L&P13. 699-726, 1990.
12. Kanazawa, M.: The Lambek Calculus Enriched with Additional Connectives. Journal of Logic, Language and Information 1(2), 141C171 1992.
13. M. Okada, and K. Terui, The Finite Model Property for Various Fragments of Intuitionistic Linear Logic, Journal of Symbolic Logic 64:790-802, 1999.
14. M. Moortgat, Multimodal linguistic inference. Journal of Logic, Language and Information 5:349-385, 1996.
15. A.R. Plummer, S4 enriched multimodal categorial grammars are context-free. The- oretical Computer Science 388:173-180, 2007. Corrigendum. Theoretical Computer Scinece 403:406-408, 2008.

Algebraic semantics and model completeness for Intuitionistic Public Announcement Logic (Extended abstract)

Minghui Ma, Alessandra Palmigiano, Mehrnoosh Sadrzadeh

In this paper, we start studying epistemic updates using the standard toolkit of duality theory. We focus on public announcements, which are the simplest epistemic actions, and hence on single-agent[1] Public Announcement Logic (PAL) without the common knowledge operator. As is well known, the epistemic action of publicly announcing a given proposition is semantically represented as a process of relativization of the model encoding the current epistemic setup of the given agents; from the given model to its submodel relativized to the announced proposition. We give the dual characterization of the corresponding submodel-injection map, as a certain pseudo-quotient map between the complex algebras respectively associated with the given model and with its relativized submodel. As is well known, these complex algebras are complete atomic BAOs (Boolean algebras with operators). The dual characterization we provide naturally generalizes to much wider classes of algebras, which include, but are not limited to, arbitrary BAOs and arbitrary modal expansions of Heyting algebras (HAOs). In this way, we access the benefits and the wider scope of applications given by a point-free, intuitionistic theory of epistemic updates. As an application of this dual characterization, we axiomatize the intuitionistic analogue of PAL, which we refer to as IPAL, and prove soundness and completeness of IPAL w.r.t. both algebraic and relational models.

The logic of public announcements

Let AtProp be a countable set of proposition letters. The formulas of (single-agent) public announcement logic PAL are built by the following inductive rule:

$$\varphi ::= p \in \mathsf{AtProp} \mid \neg\varphi \mid \varphi \vee \psi \mid \Diamond\varphi \mid \langle\alpha\rangle\varphi.$$

Models for PAL are Kripke models $M = (W, R, V)$ such that R is an equivalence relation. The evaluation of the static fragment of the language is standard. Formulas of form $\langle\alpha\rangle\varphi$ are evaluated as follows:

$$M, w \Vdash \langle\alpha\rangle\varphi \quad \text{iff} \quad M, w \Vdash \alpha \text{ and } M^\alpha, w \Vdash \varphi,$$

where $M^\alpha = (W^\alpha, R^\alpha, V^\alpha)$ is defined as follows: $W^\alpha = [\![\alpha]\!]_M$, $R^\alpha = R \cap (W^\alpha \times W^\alpha)$ and for every $p \in \mathsf{AtProp}$, $V^\alpha(p) = V(p) \cap W^\alpha$.

Proposition 1 ([1, Theorem 27]). *PAL is axiomatized completely by the axioms for the modal logic S5 plus the following axioms:*

$$\langle\alpha\rangle p \leftrightarrow (\alpha \wedge p); \quad \langle\alpha\rangle\neg\varphi \leftrightarrow (\alpha \wedge \neg\langle\alpha\rangle\varphi); \quad \langle\alpha\rangle(\varphi \vee \psi) \leftrightarrow (\langle\alpha\rangle\varphi \vee \langle\alpha\rangle\psi); \quad \langle\alpha\rangle\Diamond\varphi \leftrightarrow (\alpha \wedge \Diamond(\alpha \wedge \langle\alpha\rangle\varphi)).$$

The intuitionistic modal logic MIPC

Introduced by Prior with the name MIPQ [5], the intuitionistic modal logic MIPC is largely considered the intuitionistic analogue of S5. The logic MIPC has been studied by many authors, viz. [2, 3] and the references therein. In this section we briefly review the notions and facts needed for the purposes of the present paper, and we refer to [2, 3] for their attribution. The formulas of MIPC are built by the following inductive rule:

$$\varphi ::= \bot \mid p \in \mathsf{AtProp} \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \to \psi \mid \Diamond\varphi \mid \Box\varphi.$$

Let $\top$ be defined as $\bot \to \bot$ and, for every formula φ, let $\neg\varphi$ be defined as $\varphi \to \bot$. The logic MIPC is the minimal set of formulas in this language which contains all the axioms of intuitionistic propositional logic, the following modal axioms:

$$\Box p \to p,\ p \to \Diamond p, \qquad (\Box p \wedge \Box q) \to \Box(p \wedge q),\ \Diamond(p \vee q) \to (\Diamond p \vee \Diamond q),$$
$$\Diamond p \to \Box\Diamond p,\ \Diamond\Box p \to \Box p, \quad \Box(p \to q) \to (\Diamond p \to \Diamond q),$$

[1] The results straightforwardly extend to the multi-agent setting.

and is closed under substitution, modus ponens and necessitation ($\varphi/\Box\varphi$).

The relational structures for MIPC, called *MIPC-frames*, are triples $\mathcal{F} = (W, \leq, R)$ such that $(W, \leq)$ is a nonempty poset and R is a binary equivalence relation such that $(R \circ \leq) \subseteq (\leq \circ R)$. *MIPC-models* are structures $M = (\mathcal{F}, V)$ such that $\mathcal{F}$ is a relational structure as specified above and $V : \mathsf{AtProp} \to \mathcal{P}^{\uparrow}(W)$ is a function mapping proposition letters to upward-closed subsets of W[2]. For any such model, its associated extension map $[\![\cdot]\!]_M \to \mathcal{P}^{\uparrow}(W)$ is defined recursively as follows:

$$\begin{array}{rclrcl} [\![p]\!]_M & = & V(p) & [\![\bot]\!]_M & = & \varnothing \\ [\![\varphi \vee \psi]\!]_M & = & [\![\varphi]\!]_M \cup [\![\psi]\!]_M & [\![\varphi \wedge \psi]\!]_M & = & [\![\varphi]\!]_M \cap [\![\psi]\!]_M \\ [\![\Diamond\varphi]\!]_M & = & R^{-1}[[\![\varphi]\!]_M] & [\![\Box\varphi]\!]_M & = & ((\leq \circ R)^{-1}[[\![\varphi]\!]^c_M])^c \\ [\![\varphi \to \psi]\!]_M & = & ([\![\varphi]\!]_M \cap [\![\psi]\!]^c_M){\downarrow}^c, & & & \end{array}$$

where $(.)^c$ is the complement operation. For any model M and any MIPC formula φ, we write:

$$M, w \Vdash \varphi \text{ if } w \in [\![\varphi]\!]_M; \quad M \Vdash \varphi \text{ if } [\![\varphi]\!]_M = W; \quad \mathcal{F} \Vdash \varphi \text{ if } [\![\varphi]\!]_M = W \text{ for any model } M \text{ based on } \mathcal{F}.$$

Proposition 2. *MIPC is sound and complete with respect to the class of MIPC-frames.*

The algebraic semantics for MIPC is given by a variety of Heyting algebras with operators (HAOs) which are called monadic Heyting algebras:

Definition 3. The algebra $\mathbb{A} = (A, \wedge, \vee, \to, \bot, \Diamond, \Box)$ is a *monadic Heyting algebra* (MHA) if $(A, \wedge, \vee, \to, \bot)$ is a Heyting algebra and the following inequalities hold:

$$\begin{array}{ll} \Box x \leq x,\ x \leq \Diamond x; & \Box x \to \Box y \leq \Box(x \to y),\ \Diamond(x \vee y) \leq (\Diamond x \vee \Diamond y); \\ \Diamond x \leq \Box\Diamond x,\ \Diamond\Box x \leq \Box x; & \Box(x \to y) \leq \Diamond x \to \Diamond y. \end{array}$$

Clearly, any formula in the language $\mathcal{L}$ of MIPC can be regarded as a term in the algebraic language of MHAs. Therefore, for any MHA $\mathbb{A}$ and any interpretation $V : \mathsf{AtProp} \to \mathbb{A}$, an MIPC formula φ is *true* in $\mathbb{A}$ under the interpretation V (notation: $(\mathbb{A}, V) \models \varphi$) if the unique homomorphic extension of V, which we denote $[\![\cdot]\!]_V : \mathcal{L} \to \mathbb{A}$, maps φ to $\top^{\mathbb{A}}$. An MIPC formula is *valid* in $\mathbb{A}$ (notation: $\mathbb{A} \models \varphi$), if $(\mathbb{A}, V) \models \varphi$ for every interpretation V. In the remainder, we will refer to the tuples $(\mathbb{A}, V)$ s.t. $\mathbb{A}$ is a MHA and V is an interpretation as *algebraic models*.
MIPC-frames give rise to complex algebras, just as Kripke frames do: for any MIPC-frame $\mathcal{F}$, the *complex algebra* of $\mathcal{F}$ is

$$\mathcal{F}^+ = (\mathcal{P}^{\uparrow}(W), \cap, \cup, \Rightarrow, \varnothing, \langle R\rangle, [\leq \circ R]),$$

where for every $X, Y \in \mathcal{P}^{\uparrow}(W)$,

$$\langle R\rangle X = R^{-1}[X],\ \ [\leq \circ R]X = ((\leq \circ R)^{-1}[X^c])^c,\ \ X \Rightarrow Y = (X \cap Y^c){\downarrow}^c.$$

Clearly, given a model $M = (\mathcal{F}, V)$, the extension map $[\![\cdot]\!]_M : \mathcal{L} \to \mathcal{F}^+$ is the unique homomorphic extension of $V : \mathsf{AtProp} \to \mathcal{F}^+$.

Proposition 4. *For every MIPC-model $(\mathcal{F}, V)$ and every MIPC formula φ,*

1. $(\mathcal{F}, V) \Vdash \varphi$ *iff* $(\mathcal{F}^+, V) \models \varphi$.
2. $\mathcal{F}^+$ *is a monadic Heyting algebra.*

Intuitionistic PAL

Let AtProp be a countable set of propositional letters. The formulas of the (single-agent) *intuitionistic public announcement logic* IPAL are built according to the following syntax rule:

[2]For every poset $(W, \leq)$, a subset Y of W is *upward-closed* if for every $x, y \in W$, if $x \leq y$ and $x \in Y$ then $y \in Y$.

$$\varphi ::= p \in \mathsf{AtProp} \mid \bot \mid \top \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \varphi \to \psi \mid \Diamond\varphi \mid \Box\varphi \mid \langle\alpha\rangle\varphi \mid [\alpha]\varphi.$$

IPAL is axiomatically defined by the axioms for MIPC plus the following axioms:

Preservation of logical constants	**Preservation of facts**
$\langle\alpha\rangle\bot = \bot$	$\langle\alpha\rangle p = \alpha \wedge p$
$[\alpha]\top = \top$	$[\alpha]p = \alpha \to p$
Interaction with disjunction	**Interaction with conjunction**
$\langle\alpha\rangle(\varphi \vee \psi) = \langle\alpha\rangle\varphi \vee \langle\alpha\rangle\psi$	$\langle\alpha\rangle(\varphi \wedge \psi) = \langle\alpha\rangle\varphi \wedge \langle\alpha\rangle\psi$
$[\alpha](\varphi \vee \psi) = \alpha \to (\langle\alpha\rangle\varphi \vee \langle\alpha\rangle\psi)$	$[\alpha](\varphi \wedge \psi) = [\alpha]\varphi \wedge [\alpha]\psi$
Interaction with implication	
$\langle\alpha\rangle(\varphi \to \psi) = \alpha \wedge (\langle\alpha\rangle\varphi \to \langle\alpha\rangle\psi)$	
$[\alpha](\varphi \to \psi) = \langle\alpha\rangle\varphi \to \langle\alpha\rangle\psi$	
Interaction with $\Diamond$	**Interaction with $\Box$**
$\langle\alpha\rangle\Diamond\varphi = \alpha \wedge \Diamond\langle\alpha\rangle\varphi$	$\langle\alpha\rangle\Box\varphi = \alpha \wedge \Box[\alpha]\varphi$
$[\alpha]\Diamond\varphi = \alpha \to \Diamond\langle\alpha\rangle]\varphi$	$[\alpha]\Box\varphi = \alpha \to \Box[\alpha]\varphi$

Algebraic models, and updates as pseudo quotients

Definition 5. An *algebraic model* is a tuple $M = (\mathbb{A}, V)$ s.t. $\mathbb{A}$ is an MIPC algebra and $V : \mathsf{AtProp} \to \mathbb{A}$.

For every $\mathbb{A}$ and every $a \in \mathbb{A}$, define the following equivalence relation $\equiv_a$ on $\mathbb{A}$: for every $b, c \in \mathbb{A}$,

$$b \equiv_a c \text{ iff } b \wedge a = c \wedge a.$$

Let $[b]_a$ (abbreviated as $[b]$ when no confusion arises) be the equivalence class of $b \in \mathbb{A}$. Let

$$\mathbb{A}^a := \mathbb{A}/{\equiv_a}$$

denote the quotient set. Clearly, $\mathbb{A}^a$ is an ordered set by putting $[b] \leq [c]$ iff $b' \leq_{\mathbb{A}} c'$ for some $b' \in [b]$ and some $c' \in [c]$. Let $\pi = \pi_a : \mathbb{A} \to \mathbb{A}^a$ be the canonical projection given by $b \mapsto [b]$.

Fact 6. *1. $\equiv_a$ is a congruence if $\mathbb{A}$ is a Boolean algebra, a Heyting algebra, a bounded distributive lattice or a frame.*

2. $\equiv_a$ is not a congruence w.r.t. modal operators.

3. For every $b \in \mathbb{A}$ there exists a unique $c \in \mathbb{A}$ s.t. $c \in [b]_a$ and $c \leq a$.

The fact above implies that each $\equiv_a$-equivalence class has a canonical representant, namely the only element in that class that is less than or equal to a. Hence, the map $i' = i'^a : \mathbb{A}^a \to \mathbb{A}$ given by $[b] \mapsto b \wedge a$ is well defined and injective.

Fact 7. *1. $\pi \circ i' = id_{\mathbb{A}^a}$.*

2. If $\mathbb{A} = \mathcal{F}^+$ for some Kripke frame $\mathcal{F}$, then $i'(b) = i[b]$ for every $b \in \mathbb{A}$.

Let $(\mathbb{A}, \Diamond, \Box)$ be a HAO. Define, for every $b \in \mathbb{A}$,

$$\Diamond^a[b] := [\Diamond(b \wedge a) \wedge a] = [\Diamond(b \wedge a)] \quad \text{and} \quad \Box^a[b] := [a \to \Box(a \to b)] = [\Box(a \to b)].$$

(The second equality holds because of Heyting inequalities $a \wedge (a \to \Box(a \to b)) \leq \Box(a \to b)$ and $a \wedge \Box(a \to b) \leq a \to \Box(a \to b)$.)

Fact 8. *For every HAO $(\mathbb{A}, \Diamond, \Box)$ and every $a \in \mathbb{A}$,*

1. *$\Diamond^a$, $\Box^a$ are normal modal operators. Hence $(\mathbb{A}^a, \Box^a, \Diamond^a)$ is a HAO.*
2. *If $\mathbb{A} = \mathcal{F}^+$ for some Kripke frame $\mathcal{F}$, then $\mathbb{A}^a \cong_{BAO} \mathcal{F}^{a+}$.*

Interpreting dynamic modalities in algebraic models

The following classical satisfaction condition for the dynamic diamond

$$M, w \Vdash \langle\alpha\rangle\varphi \quad \text{iff} \quad M, w \Vdash \alpha \text{ and } M^\alpha, w \Vdash \varphi$$

can be equivalently written as follows:

$$w \in [\![\langle\alpha\rangle\varphi]\!]_M \quad \text{iff} \quad \exists w' \in W^\alpha \text{ s.t. } i(w') = w \in [\![\alpha]\!]_M \text{ and } w' \in [\![\varphi]\!]_{M^\alpha}.$$

Because the map $i : M^\alpha \to M$ is injective, we have $w' \in [\![\varphi]\!]_{M^\alpha}$ iff $w = i(w') \in i[[\![\varphi]\!]_{M^\alpha}]$. Hence, we obtain:

$$w \in [\![\langle\alpha\rangle\varphi]\!]_M \quad \text{iff} \quad w \in [\![\alpha]\!]_M \cap i[[\![\varphi]\!]_{M^\alpha}],$$

from which we get that

$$[\![\langle\alpha\rangle\varphi]\!]_M = [\![\alpha]\!]_M \cap i[[\![\varphi]\!]_{M^\alpha}]. \tag{1}$$

Reasoning analogously in the case of the dynamic box, we obtain:

$$[\![[\alpha]\varphi]\!]_M = [\![\alpha]\!]_M \Rightarrow i[[\![\varphi]\!]_{M^\alpha}], \tag{2}$$

where the operation $X \Rightarrow Y$ is defined as $(W \setminus X) \cup Y$. The clauses (1) and (2) above, together with Fact 7.2, motivate the following

Definition 9. For every algebraic model $M = (\mathbb{A}, V)$, the *extension map* $[\![\cdot]\!]_M : Fm \to \mathbb{A}$ is defined recursively as follows:

$$\begin{array}{rclrcl}
[\![p]\!]_M & = & V(p) & [\![\bot]\!]_M & = & \bot^{\mathbb{A}} \\
[\![\top]\!]_M & = & \top^{\mathbb{A}} & [\![\varphi \vee \psi]\!]_M & = & [\![\varphi]\!]_M \vee^{\mathbb{A}} [\![\psi]\!]_M \\
[\![\varphi \wedge \psi]\!]_M & = & [\![\varphi]\!]_M \wedge^{\mathbb{A}} [\![\psi]\!]_M & [\![\varphi \to \psi]\!]_M & = & [\![\varphi]\!]_M \to^{\mathbb{A}} [\![\psi]\!]_M \\
[\![\Diamond\varphi]\!]_M & = & \Diamond^{\mathbb{A}}[\![\varphi]\!]_M & [\![\Box\varphi]\!]_M & = & \Box^{\mathbb{A}}[\![\varphi]\!]_M \\
[\![\langle\alpha\rangle\varphi]\!]_M & = & [\![\alpha]\!]_M \wedge^{\mathbb{A}} i'([\![\varphi]\!]_{M^\alpha}) & [\![[\alpha]\varphi]\!]_M & = & [\![\alpha]\!]_M \to^{\mathbb{A}} i'([\![\varphi]\!]_{M^\alpha}).
\end{array}$$

Here, $M^\alpha = (\mathbb{A}^\alpha, V^\alpha)$ s.t. $\mathbb{A}^\alpha = \mathbb{A}^{[\![\alpha]\!]_M}$ and $V^\alpha : \mathsf{AtProp} \to \mathbb{A}^\alpha$ is $\pi \circ V$, i.e. for every $p \in \mathsf{AtProp}$,

$$[\![p]\!]_{M^\alpha} = V^\alpha(p) = \pi(V(p)) = \pi([\![p]\!]_M).$$

Soundness and completeness

Theorem 10. *IPAL is complete wrt MIPC algebraic/relational models.*

Proof. Completeness is analogous to the proof of completeness of classical PAL wrt Kripke models, and follows from the reducibility of IPAL to MIPC via the interaction axioms. □

References

[1] J. van Benthem. *Modal Logic for Open Minds*. CSLI publications, 2010.

[2] G. Bezhanishvili. Varieties of Monadic Heyting Algebras. Part I. *Studia Logica* 61: 367-402, 1998.

[3] G. Bezhanishvili. Varieties of Monadic Heyting Algebras Part II: Duality theory. *Studia Logica* 62: 21-48, 1999.

[4] J. Plaza. Logics of Public Communications. In: *Proceedings 4th International Symposium on Methodologies for Intelligent Systems*, 201-216, 1989

[5] A. Prior. *Time and Modality*. Clarendon Press, 1957.

Well-composed J-logics and interpolation*

Larisa Maksimova

Sobolev Institute of Mathematics,
Siberian Branch of Russian Academy of Sciences,
630090, Novosibirsk, RUSSIA
Email: lmaksi@math.nsc.ru

Abstract

Extensions of the Johansson minimal logic are investigated. Representation theorems for well-composed logics with the Graig interpolation property CIP, restricted interpolation property IPR and projective Beth property PBP are stated. It is proved that PBP is equivalent to IPR for any well-composed logic, and there are only finitely many well-composed logics with CIP, IPR or PBP.

1 Propositional J-logics

Modal and superintuitionistic (s.i.) logics, and varieties of their associated algebras were at the centre of investigations by Leo Esakia [1].

In this paper we consider extensions of the Johansson minimal logic J; this family extends the class of s.i. logics. The main variants of the interpolation property are studied. It is known that the weak interpolation property is decidable over J [5]. There are only finitely many s.i.logics with CIP, IPR or PBP, all of them are fully described [2, 4]. Here we extend these results to the class of well-composed J-logics.

The language of J contains $\&, \vee, \to, \bot$ as primitive; negation is defined by $\neg A = A \to \bot$. The logic J can be given by the calculus, which has the same axiom schemes as the positive intuitionistic calculus Int^+, and the only rule of inference is modus ponens. By a J-*logic* we mean an arbitrary set of formulas containing all the axioms of J and closed under modus ponens and substitution rules. We denote

$\mathrm{Int} = \mathrm{J} + (\bot \to A)$, $\mathrm{Neg} = \mathrm{J} + \bot$, $\mathrm{Gl} = \mathrm{J} + (A \vee \neg A)$,

$\mathrm{Cl} = \mathrm{Int} + (A \vee \neg A)$, $\mathrm{JX} = \mathrm{J} + (\bot \to A) \vee (A \to \bot)$.

A J-logic is *superintuitionistic* if it contains the intuitionistic logic Int, and *negative* if contains Neg. A J-logic is *well-composed* if it contains JX. For a J-logic L, the family of J-logics containing L is denoted by $E(L)$.

If $\mathbf{p}$ is a list of variables, let $A(\mathbf{p})$ denote a formula whose all variables are in $\mathbf{p}$, and $\mathcal{F}(\mathbf{p})$ the set of all such formulas.

Let L be a logic. *The Craig interpolation property CIP, the restricted interpolation property IPR* and *the weak interpolation property WIP* are defined as follows (where the lists $\mathbf{p}, \mathbf{q}, \mathbf{r}$ are disjoint):

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CIP. If $\vdash_L A(\mathbf{p},\mathbf{q}) \to B(\mathbf{p},\mathbf{r})$, then there is a formula $C(\mathbf{p})$ such that $\vdash_L A(\mathbf{p},\mathbf{q}) \to C(\mathbf{p})$ and $\vdash_L C\mathbf{p}) \to B(\mathbf{p},\mathbf{r})$.

IPR. If $A(\mathbf{p},\mathbf{q}), B(\mathbf{p},\mathbf{r}) \vdash_L C(\mathbf{p})$, then there exists a formula $A'(\mathbf{p})$ such that $A(\mathbf{p},\mathbf{q}) \vdash_L A'(\mathbf{p})$ and $A'(\mathbf{p}), B(\mathbf{p},\mathbf{r}) \vdash_L C(\mathbf{p})$.

WIP. If $A(\mathbf{p},\mathbf{q}), B(\mathbf{p},\mathbf{r}) \vdash_L \bot$, then there exists a formula $A'(\mathbf{p})$ such that $A(\mathbf{p},\mathbf{q}) \vdash_L A'(\mathbf{p})$ and $A'(\mathbf{p}), B(\mathbf{p},\mathbf{r}) \vdash_L \bot$.

Suppose that $\mathbf{p}$, $\mathbf{q}$, $\mathbf{q}'$ are disjoint lists of variables that do not contain x and y, $\mathbf{q}$ and $\mathbf{q}'$ are of the same length, and $A(\mathbf{p},\mathbf{q},x)$ is a formula. We define *the projective Beth property*:

PBP. If $A(\mathbf{p},\mathbf{q},x), A(\mathbf{p},\mathbf{q}',y) \vdash_L x \leftrightarrow y$, then $A(\mathbf{p},\mathbf{q},x) \vdash_L x \leftrightarrow B(\mathbf{p})$ for some $B(\mathbf{p})$.

The weaker *Beth property BP* arises from PBP by omitting $\mathbf{q}$ and $\mathbf{q}'$.

All J-logics satisfy BP, and for these logics the following hold:

- CIP $\Rightarrow$ PBP $\Rightarrow$ IPR $\Rightarrow$ WIP, PBP $\not\Rightarrow$ CIP, WIP $\not\Rightarrow$ IPR.

It is proved in [5] that WIP is decidable over J, i.e. there is an algorithm which, given a finite set Ax of axiom schemes, decides if the logic J+Ax has WIP. The families of J-logics with WIP and of J-logics without WIP have the continuum cardinality.

The logics J, Int, Neg, Gl, Cl and JX possess CIP and hence all other above-mentioned properties. It is known [4] that

- IPR $\Leftrightarrow$ PBP over Int and Neg.

It is known that there are only finitely many s.i. and negative logics with CIP, IPR and PBP [2, 4]. Here we extend this result to all well-composed logics. Also we prove that IPR is equivalent to PBP in any well-composed logic.

2 Algebraic interpretation

The considered properties have natural algebraic equivalents. There is a duality between J-logics and varieties of J-algebras.

Algebraic semantics for J-logics is built via J-*algebras*, i.e. algebras $\mathbf{A} =< A; \&, \vee, \to, \bot, \top >$ such that A is a lattice w.r.t. $\&, \vee$ with the greatest element $\top$, $\bot$ is an arbitrary element of A, and

$z \leq x \to y \iff z\&x \leq y.$

A J-algebra $\mathbf{A}$ is a *Heyting algebra* if $\bot$ is the least element of A, and *a negative algebra* if $\bot$ is the greatest element of A; the algebra is *well-composed* if every its element is comparable with $\bot$. For any well-composed J-algebra $\mathbf{A}$, the set $\mathbf{A}^l = \{x|\ x \leq \bot\}$ forms a negative algebra, and the set $\mathbf{A}^l = \{x|\ x \geq \bot\}$ forms a Heyting algebra. If $\mathbf{B}$ is a negative algebra and $\mathbf{C}$ is a Heyting algebra, we denote by $\mathbf{B} \uparrow \mathbf{C}$ a well-composed algebra $\mathbf{A}$ such that $\mathbf{A}^l$ is isomorphic to $\mathbf{B}$ and $\mathbf{A}^u$ to $\mathbf{C}$. For a negative algebra $\mathbf{B}$, we denote by $\mathbf{B}^\Lambda$ a J-algebra arisen from $\mathbf{B}$ by adding a new greatest element $\top$.

A J-algebra $\mathbf{A}$ is *finitely indecomposable* if for all $x, y \in \mathbf{A}$:

$x \vee y = \top \Leftrightarrow (x = \top$ or $y = \top)$.

If A is a formula, $\mathbf{A}$ a J-algebra, then A is *valid in* $\mathbf{A}$ (in symbols, $\mathbf{A} \models A$) if the identity $A = \top$ is valid in $\mathbf{A}$. We write $\mathbf{A} \models L$ instead of $(\forall A \in L)(\mathbf{A} \models A)$. Let $V(L) = \{\mathbf{A}|\mathbf{A} \models L\}$. Each J-logic L is characterized by the variety $V(L)$.

We recall the definitions. A class V has *Amalgamation Property* if it satisfies

AP: For each $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V$ such that $\mathbf{A}$ is a common subalgebra of $\mathbf{B}$ and $\mathbf{C}$, there exist an algebra $\mathbf{D}$ in V and monomorphisms $\delta : \mathbf{B} \to \mathbf{D}$ and $\epsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta(x) = \epsilon(x)$ for all $x \in \mathbf{A}$.

Super-Amalgamation Property (SAP) is AP with extra conditions:

$$\delta(x) \le \epsilon(y) \Leftrightarrow (\exists z \in \mathbf{A})(x \le z \text{ and } z \le y),$$

$$\delta(x) \ge \epsilon(y) \Leftrightarrow (\exists z \in \mathbf{A})(x \ge z \text{ and } z \ge y).$$

Restricted Amalgamation Property (RAP) and *Weak Amalgamation Property (WAP)* are defined as follows:

RAP: for any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V$ such that $\mathbf{A}$ is a common subalgebra of $\mathbf{B}$ and $\mathbf{C}$, there exist an algebra $\mathbf{D}$ in V and homomorphisms $g : \mathbf{B} \to \mathbf{D}$ and $h : \mathbf{C} \to \mathbf{D}$ such that $g(x) = h(x)$ for all $x \in \mathbf{A}$ and the restriction of g onto $\mathbf{A}$ is a monomorphism.

WAP: For each $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V$ such that $\mathbf{A}$ is a common subalgebra of $\mathbf{B}$ and $\mathbf{C}$, there exist an algebra $\mathbf{D}$ in V and homomorphisms $\delta : \mathbf{B} \to \mathbf{D}$ and $\epsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta(x) = \epsilon(x)$ for all $x \in \mathbf{A}$, and $\bot \ne \top$ in $\mathbf{D}$ whenever $\bot \ne \top$ in $\mathbf{A}$.

A class V has *Strong Epimorphisms Surjectivity* if it satisfies

SES: For each $\mathbf{A}, \mathbf{B}$ in V, for every monomorphism $\alpha : \mathbf{A} \to \mathbf{B}$ and for every $x \in \mathbf{B} - \alpha(\mathbf{A})$ there exist $\mathbf{C} \in V$ and homomorphisms $\beta : \mathbf{B} \to \mathbf{C}$, $\gamma : \mathbf{B} \to \mathbf{C}$ such that $\beta\alpha = \gamma\alpha$ and $\beta(x) \ne \gamma(x)$.

Theorem 2.1. [3] *For any* J*-logic L:*
(1) L has CIP iff V(L) has SAP iff V(L) has AP,
(2) L has IPR iff V(L) has RAP, (3) L has WIP iff V(L) has WAP,
(4) L has PBP iff V(L) has SES.

In varieties of J-algebras: SAP $\iff$ AP $\Rightarrow$ SES $\Rightarrow$ RAP $\Rightarrow$ WAP.

3 Interpolation in well-composed J-logics

For $L_1 \in E(\text{Neg})$, $L_2 \in E(\text{Int})$ we denote by $L_1 \uparrow L_2$ a logic characterized by all algebras of the form $\mathbf{A} \uparrow \mathbf{B}$, where $\mathbf{A} \models L_1$, $\mathbf{B} \models L_2$; a logic characterized by all algebras $\mathbf{A} \uparrow \mathbf{B}$, where $\mathbf{A}$ is a finitely decomposable algebra in $V(L_1)$ and $\mathbf{B} \in V(L_2)$, is denoted by $L_1 \Uparrow L_2$.

In [3] an axiomatization was found for logics $L_1 \uparrow L_2$ and $L_1 \Uparrow L_2$, where L_1 is a negative and L_2 an s.i. logic.

It is known that there are only finitely many s.i. and negative logics with CIP, IPR and PBP [2, 3, 4]. We give the list of all negative logics with CIP:

$$\text{Neg},\ \text{NC} = \text{Neg} + (p \to q) \vee (q \to p),\ \text{NE} = \text{Neg} + p \vee (p \to q),\ \text{For} = \text{Neg} + p.$$

For any J-logic L define

$$L_{neg} = L + \bot.$$

The following theorem describes all well-composed logics with CIP.

Theorem 3.1. *Let L be a well-composed logic. Then L has CIP if and only if L coincides with one of the logics:*

(1) $L_1 \cap L_2$, where $L_1 = L_{neg}$ is a negative logic with CIP and L_2 is a superintuitionistic logic with CIP;

(2) $L_1 \cap (L_3 \Uparrow L_2)$*, where* $L_1 = L_{neg}$ *is a negative logic with CIP,* L_2 *is a consistent s.i. logic with CIP and* $L_3 \in \{\mathrm{Neg}, \mathrm{NC}, \mathrm{NE}\}$*;*

(3) $L_1 \cap (L_3 \uparrow L_2)$*, where* L_1*,* L_2*,* L_3 *are the same as in (2).*

The following two theorems give a full description of well-composed logics with IPR and PBP.

It is proved in [5] that WIP is decidable over J, i.e. there is an algorithm which, given a finite set Ax of axiom schemes, decides if the logic J+Ax has WIP. A crucial role in the description of J-logics with WIP [5] belongs to the following list of eight logics:

$$\mathrm{SL} = \{\mathrm{For}, \mathrm{Cl}, (\mathrm{NE} \uparrow \mathrm{Cl}), (\mathrm{NC} \uparrow \mathrm{Cl}), (\mathrm{Neg} \uparrow \mathrm{Cl}), (\mathrm{NE} \Uparrow \mathrm{Cl}), (\mathrm{NC} \Uparrow \mathrm{Cl}), (\mathrm{Neg} \Uparrow \mathrm{Cl})\}.$$

Let $\Lambda(L) = \{\mathbf{B}^{\Lambda} | \ \mathbf{B}^{\Lambda} \in V(L)\}$.

Theorem 3.2. *Let* L *be a well-composed logic, the logic* L_{neg} *have IPR and*

$$L = L_{neg} \cap L_0 \cap L_1,$$

where $L_0 \in SL$*,* $\Lambda(L_0) \supseteq \Lambda(L_1)$*,* $L_1 \in \{\mathrm{For}, (L_2 \uparrow L_3), (L_2 \Uparrow L_3)\}$*,* L_2 *is a negative logic with CIP, and* L_3 *is a superintuitionistic logic with IPR. Then* L *has IPR and, moreover,* L *has PBP.*

Theorem 3.3. *Let a well-composed logic* L *have IPR. Then the logic* L_{neg} *has IPR, and* L *is representable as*

$$L = L_{neg} \cap L_0 \cap L_1,$$

where $L_0 \in SL$*,* $\Lambda(L_0) \supseteq \Lambda(L_1)$*,* $L_1 \in \{\mathrm{For}, (L_2 \uparrow L_3), (L_2 \Uparrow L_3)\}$*,* L_2 *is a negative logic with CIP, and* L_3 *is a superintuitionistic logic with IPR.*

Corollary 3.4. *1. There are only finitely many well-composed logics with IPR; all of them are finitely axiomatizable.*

2. IPR and PBP are equivalent on the class of well-composed logics.

Problem 1. How many J-logics have CIP, IPR or PBP?
Problem 2. Are IPR and PBP equivalent over J?
Problem 3. Are CIP, IPR and/or PBP decidable over J? The same question for the class of well-composed logics.

References

[1] *L.L.Esakia*. Heyting algebras. I. Duality theory. "Metsniereba", Tbilisi, 1985. 105 pp. (in Russian)

[2] *D.M.Gabbay, L.Maksimova*. Interpolation and Definability: Modal and Intuitionistic Logics. Oxford University Press, Oxford, 2005.

[3] *L.L.Maksimova*. Interpolation and definability in extensions of the minimal logic. Algebra and Logic, 44 (2005), 726-750.

[4] *L.Maksimova*. Problem of restricted interpolation in superintuitionistic and some modal logics. Logic Journal of IGPL, 18 (2010), 367-380.

[5] *L.L.Maksimova* . Decidability of the weak interpolation property over the minimal logic. Algebra and Logic, 50, no. 2 (2011).

Towards a good notion of categories of logics

Hugo Luiz Mariano *
Caio de Andrade Mendes †

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1 Introduction

In this work we consider (finitary, propositional) logics through the original use of Category Theory: the study of the " sociology of mathematical objects", aligning us with a recent, and growing, trend of study logics through its relations with other logics (e.g., combinations of logics, fibrings, etc.). So will be objects of study the classes *of* logics, i.e. categories whose objects are logical systems (i.e., a signature with a Tarskian consequence relation) and the morphisms are related to (some concept of) translations between these systems.

The present work provides the first steps of a project of considering categories of logical systems satisfying *simultaneously* certain natural requirements such as:
(i) If they represent the majority part of the usual logical systems;
(ii) If they have good categorial properties (e.g., if they are a complete and/or cocomplete category, if they are accessible categories);
(iii) If they allow a natural notion of *algebraizable* logical system (as in the concept of Blok-Pigozzi algebraizable logic ([**BP**]) or Czelakowski's proto-algebraizability ([**Cze**]));
(iv) If they provide a satisfactory treatment of the *identity problem* of logical systems (when logics can be considered "the same"? ([**Bez**], [**CG**])).

In [**AFLM**] (and other works), was considered a **s**imple (but too **s**trict) notion of morphism of signatures, where are founded some categories of logics that satisfy simultaneously the first three requirements, but not the last one; here we will denote by $\mathcal{S}_s$ and $\mathcal{L}_s$ the category of signatures and of logics therein. In the papers [**BCC**] and [**CG**] (and others), is developed a more **f**lexible notion of morphism of signatures based on **f**ormulas as connectives (our notation for the associated category of signatures will be $\mathcal{S}_f$ and $\mathcal{L}_f$ will denote the associated category of logics), it encompass itens (i) and (iii) and allows some treatment of item (iv), but does not satisfy (ii).

In what follows, $X = \{x_0, x_1, \ldots, x_n, \ldots\}$ will denote a fixed enumerable set (written in a fixed order).

2 Known facts about categories of signatures and of logics

2.1 The categories $\mathcal{S}_s$ and $\mathcal{L}_s$

The category $\mathcal{S}_s$ is the category of signatures and *strict* morphisms of signatures.

The objects of $\mathcal{S}$ are signatures. A signature Σ is a sequence of sets $\Sigma = (\Sigma_n)_{n\in\omega}$ such that $\Sigma_i \cap \Sigma_j = \emptyset$ for all $i < j < \omega$. We write $|\Sigma| := \bigcup_{n\in\omega} \Sigma_n$ for the *support of* Σ and we denote by $F(\Sigma)$, the *formula algebra of* Σ, i.e. the set of all (propositional) formulas built with signature Σ over the variables in the set X. For all $n \in \mathbb{N}$ let $F(\Sigma)[n] = \{\varphi \in F(\Sigma) : var(\varphi) = \{x_0, x_1, \ldots, x_{n-1}\}\}$. The notion of complexity $compl(\varphi)$ of the formula φ is, as usual, the number of occurences of connectives in φ.

*Research supported by FAPESP, under the Thematic Project LOGCONS: number 2010/51038-0.
†Instituto de Matemática e Estatística, University of São Paulo, Brazil. Emails: hugomar@ime.usp.br, caio.mendes@usp.br.

If Σ, Σ' are signatures then a *strict* morphism $f : \Sigma \longrightarrow \Sigma'$ is a sequence of functions $f = (f_n)_{n\in\omega}$, where $f_n : \Sigma_n \longrightarrow \Sigma'_n$. Composition and identities in $\mathcal{S}_s$ are componentwise.

For each morphism $f : \Sigma \longrightarrow \Sigma'$ in $\mathcal{S}_s$ there is only one function $\widehat{f} : F(\Sigma) \longrightarrow F(\Sigma')$, called the *extension of* f, such that: (i) $\widehat{f}(x) = x$, if $x \in X$; (ii) $\widehat{f}(c_n(\psi_0, \ldots, \psi_{n-1}) = f_n(c_n)(\widehat{f}(\psi_0), \ldots, \widehat{f}(\psi_{n-1}))$, if $c_n \in \Sigma_n$. Then:
(0) $compl(\widehat{f}(\theta)) = compl(\theta)$, for all $\theta \in F(\Sigma)$.
(1) If $var(\theta) \subseteq \{x_{i_0}, \ldots, x_{i_{n-1}}\}$, then $\widehat{f}(\theta(\vec{x})[\vec{x} \mid \vec{\psi}]) = (\widehat{f}(\theta(\vec{x}))[\vec{x} \mid \widehat{f}(\vec{\psi})]$. Moreover $var(\widehat{f}(\theta)) = var(\theta)$ and then $\widehat{f}$ restricts to maps $\widehat{f}\restriction_n: F(\Sigma)[n] \longrightarrow F(\Sigma')[n]$, $n \in \mathbb{N}$.
(2) The extension to formula algebras of a composition is the extension's composition. The extension of an identity is the identity function on the formula algebra.

Observe that $\mathcal{S}_s$ is equivalent to the functor category $\mathbf{Set}^{\mathbf{N}}$, where $\mathbf{N}$ is the discrete category with object class $\mathbb{N}$; then $\mathcal{S}$ has all small limits and colimits and they are componentwise. Moreover, the category $\mathcal{S}_s$ is a finitely locally presentable category, i.e., $\mathcal{S}_s$ is a finitely accessible category that is cocomplete and/or complete. The finitely presentable signatures are precisely the signatures of finite support.

(Sub) For any *substitution* function $\sigma : X \longrightarrow F(\Sigma)$, there is only one *extension* $\widetilde{\sigma} : F(\Sigma) \longrightarrow F(\Sigma)$ such that $\widetilde{\sigma}$ is an "homomorphism": $\widetilde{\sigma}(x) = \sigma(x)$, for all $x \in X$ and $\widetilde{\sigma}(c_n(\psi_0, \ldots, \psi_{n-1}) = c_n(\widetilde{\sigma}(\psi_0)), \ldots, \widetilde{\sigma}(\psi_{n-1}))$, for all $c_n \in \Sigma_n$, $n \in \mathbb{N}$; it follows that for any $\theta(x_0, \ldots, x_{n-1}) \in F(\Sigma)$ $\widetilde{\sigma}(\theta(x_0, \ldots, x_{n-1})) = \theta(\sigma(x_0), \ldots, \sigma(x_{n-1}))$. The *identity substitution* induces the identity homomorphism on the formula algebra; the *composition substitution* of the substitutions $\sigma', \sigma : X \longrightarrow F(\Sigma)$ is the substitution $\sigma'' : X \longrightarrow F(\Sigma)$, $\sigma'' = \sigma' \star \sigma := \widetilde{\sigma'} \circ \sigma$ and $\widetilde{\sigma''} = \widetilde{\sigma' \star \sigma} = \widetilde{\sigma'} \circ \widetilde{\sigma}$.

(3) Let $f : \Sigma \longrightarrow \Sigma'$ be a $\mathcal{S}_s$-morphism. Then for any substitution $\sigma : X \longrightarrow F(\Sigma)$ there is another substitution $\sigma' : X \longrightarrow F(\Sigma')$ such that $\widetilde{\sigma'} \circ \widehat{f} = \widehat{f} \circ \widetilde{\sigma}$.

The category $\mathcal{L}_s$ is the category of propositional logics and *strict* translations as morphisms. This is a category "built above" the category $\mathcal{L}_s$, that is, there is an obvious forgetful functor $U_s : \mathcal{L}_s \longrightarrow \mathcal{S}_s$.

The objects of $\mathcal{L}_s$ are logics. A logic is an ordered pair $l = (\Sigma, \vdash)$ where Σ is an object of $\mathcal{S}_s$ and $\vdash$ codifies the "consequence operator" on $F(\Sigma)$: $\vdash$ is a binary relation, a subset of $Parts(F(\Sigma)) \times F(\Sigma)$, such that $Cons(\Gamma) = \{\varphi \in F(\Sigma) : \Gamma \vdash \varphi\}$, for all $\Gamma \subseteq F(\Sigma)$, gives a structural finitary closure operator on $F(\Sigma)$: **(a)** *inflationary*: $\Gamma \subseteq Cons(\Gamma)$; **(b)** *increasing*: $\Gamma_0 \subseteq \Gamma_1 \Rightarrow Cons(\Gamma_0) \subseteq Cons(\Gamma_1)$; **(c)** *idempotent*: $Cons(Cons(\Gamma)) \subseteq Cons(\Gamma)$; **(d)** *finitary*: $Cons(\Gamma) = \bigcup\{Cons(\Gamma') : \Gamma' \subseteq_{fin} \Gamma\}$; **(e)** *structural*: $\widetilde{\sigma}(Cons(\Gamma)) \subseteq Cons(\widetilde{\sigma}(\Gamma))$, for each substitution $\sigma : X \to F(\Sigma)$.

If $l = (\Sigma, \vdash), l' = (\Sigma', \vdash')$ are logics then a *strict translation morphism* $f : l \longrightarrow l'$ in $\mathcal{L}_s$ is a *strict* signature morphism $f : \Sigma \longrightarrow \Sigma'$ in $\mathcal{S}_s$ such that "preserves the consequence relation", that is, for all $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$, if $\Gamma \vdash \psi$ then $\widehat{f}[\Gamma] \vdash' \widehat{f}(\psi)$. Composition and identities are similar to $\mathcal{S}_s$.

$\mathcal{L}_s$ has natural notions of direct and inverse image logics under a $\mathcal{S}_s$-morphism and they have good properties. The category $\mathcal{L}_s$ is a finitely locally presentable category, i.e., $\mathcal{L}_s$ is a finitely accessible category that is cocomplete and/or complete.

2.2 The categories $\mathcal{S}_f$ and $\mathcal{L}_f$

The category $\mathcal{S}_f$ is the category of signatures and *flexible* morphisms of signatures.

We introduce the following notations:
If $\Sigma = (\Sigma_n)_{n\in\mathbb{N}}$ is a signature, then $T(\Sigma) := (F(\Sigma)[n])_{n\in\mathbb{N}}$ is a signature too.
We have the inverse bijections (just notations): $h \in \mathcal{S}_f(\Sigma, \Sigma') \leftrightsquigarrow h^\sharp \in \mathcal{S}_s(\Sigma, T(\Sigma'))$; $f \in \mathcal{S}_s(\Sigma, T(\Sigma')) \leftrightsquigarrow f^\flat \in \mathcal{S}_f(\Sigma, \Sigma')$.
For each signature Σ and $n \in \mathbb{N}$, let the function: $(j_\Sigma)_n : \Sigma_n \longrightarrow F(\Sigma)[n] \quad : \quad c_n \mapsto c_n(x_0, \ldots, x_{n-1})$.

For each morphism $f : \Sigma \longrightarrow \Sigma'$ in $\mathcal{S}_f$ there is only one function $\check{f}$: $F(\Sigma) \longrightarrow F(\Sigma')$, called the *extension of* f, such that: (i) $\check{f}(x) = x$, if $x \in X$; (ii) $\check{f}(c_n(\psi_0, \ldots, \psi_{n-1})) = (f_n(c_n)(x_0, \ldots, x_{n-1}))[x_0 \mid \check{f}(\psi_0), \ldots, x_{n-1} \mid \check{f}(\psi_{n-1})]$ if $c_n \in \Sigma_n$. The composition in $\mathcal{S}_f$ is given by $(f' \bullet f)^\sharp := (\check{f'}\restriction_n \circ f^\sharp_n)_{n\in\mathbb{N}}$. The identity id_Σ in $\mathcal{S}_f$ is given by $(id_\Sigma)^\sharp := ((j_\Sigma)_n)_{n\in\mathbb{N}}$.

The notion of extension of $\mathcal{S}_f$-morphism to formula algebras shares many properties with notion of extension of $\mathcal{S}_s$-morphism to formula algebras: e.g., the properties **(1)**, **(2)**, **(3)**.

The category $\mathcal{L}_f$ is the category of propositional logics and *flexible* translations as morphisms. This is a category "built above" the category $\mathcal{L}_f$, that is, there is an obvious forgetful functor $U_f : \mathcal{L}_f \longrightarrow \mathcal{S}_f$.

If $l = (\Sigma, \vdash), l' = (\Sigma', \vdash')$ are logics then a *flexible translation morphism* $f : l \longrightarrow l'$ in $\mathcal{L}_f$ is a *flexible* signature morphism $f : \Sigma \longrightarrow \Sigma'$ in $\mathcal{S}_f$ such that "preserves the consequence relation", that is, for all $\Gamma \cup \{\psi\} \subseteq F(\Sigma)$, if $\Gamma \vdash \psi$ then $\overset{\vee}{f}[\Gamma] \vdash' \overset{\vee}{f}(\psi)$. Composition and identities are similar to $\mathcal{S}_f$. It is known that $\mathcal{L}_f$ has weak products.

3 New results on categories of signatures and of logics

The notion of extension of $\mathcal{S}_f$-morphism to formula algebras shares many properties with notion of extension of $\mathcal{S}_s$-morphism to formula algebras, however:

(0)' If $f \in \mathcal{S}_f(\Sigma, \Sigma')$, then: $compl(\overset{\vee}{f}(\theta)) \geq compl(\theta)$, any $\theta \in F(\Sigma)$ iff $f(c_1) \neq x_0$, all $c_1 \in \Sigma_1$.

A (non full) subcategory of $\mathcal{S}_f$ with the same objects has a *strict* initial object iff all the morphisms $f : \Sigma \longrightarrow \Sigma'$ are such that $f(c_1) \neq x_0$, all $c_1 \in \Sigma_1$.

$\mathcal{S}_f$ has weak terminal object but do not have terminal object. $\mathcal{S}_f$ has coproducts but it only has "trivial" (i.e. "almost" in $\mathcal{S}_s$) coequalizers, idempotents, sections, retractions, isomorphisms.

We have the (faithful) functors:

$$(+) : \mathcal{S}_s \longrightarrow \mathcal{S}_f \quad : \quad (\Sigma \xrightarrow{f} \Sigma') \mapsto (\Sigma \xrightarrow{f^\sharp} \Sigma');$$

$$(-) : \mathcal{S}_f \longrightarrow \mathcal{S}_s \quad : \quad (\Sigma \xrightarrow{h} \Sigma') \mapsto ((F(\Sigma)[n])_{n\in\mathbb{N}} \xrightarrow{(\overset{\vee}{h}\restriction_n)_{n\in\mathbb{N}}} (F(\Sigma')[n])_{n\in\mathbb{N}}).$$

For each $f \in \mathcal{S}_s(\Sigma, \Sigma')$, we have $(\overset{\vee}{f^+}) = \widehat{f} \in Set(F(\Sigma), F(\Sigma'))$.

We have the *natural transformations*: $\eta : Id_{\mathcal{S}_s} \longrightarrow (-) \circ (+)$: $(\eta_\Sigma)_n := (j_\Sigma)_n$; $\varepsilon : (-) \circ (+) \longrightarrow Id_{\mathcal{S}_s}$: $(\varepsilon_\Sigma)^\sharp_n := id_{F(\Sigma)[n]}$ and we write $\mu = (+)\varepsilon(-)$. The functor $(+)$ is a *left adjoint* of $(-)$: η and ε are, respectively, the unit and the counit of the adjunction.

We have a (endo)functor $T : \mathcal{S}_s \longrightarrow \mathcal{S}_s$: $(\Sigma \xrightarrow{f} \Sigma') \mapsto ((F(\Sigma)[n])_{n\in\mathbb{N}} \xrightarrow{(\widehat{f}\restriction_n)_{n\in\mathbb{N}}} (F(\Sigma')[n])_{n\in\mathbb{N}})$. The functor T is faithful, reflects isomorphisms and preserves directed colimits.

We have $T = (-)\circ(+) : \mathcal{S}_s \longrightarrow \mathcal{S}_s$ and the monad $\mathcal{T} = (T, \eta, \mu)$ associated to the adjunction $(\eta, \varepsilon) : \mathcal{S}_s \overset{(+)}{\underset{(-)}{\rightleftarrows}} \mathcal{S}_f$ is such that $Kleisli(\mathcal{T}) = \mathcal{S}_f$.

As in $\mathcal{L}_s$, $\mathcal{L}_f$ has natural notions of direct and inverse image logics under a $\mathcal{S}_f$-morphism and they have good properties.

The forgetful functor $U_F : \mathcal{L}_f \longrightarrow \mathcal{S}_f$ has left and right adjoints and if a diagram in $\mathcal{L}_f$ is such that, its associated diagram by U_F has a limit/colimit in $\mathcal{S}_f$, then the original diagram has a limit/colimit in $\mathcal{L}_f$. Then a "shape type"' of limit/colimit exists in $\mathcal{L}_f$ iff it exists in $\mathcal{S}_s$. $\mathcal{L}_f$ has weak terminal object but do not have terminal object. $\mathcal{L}_f$ has coproducts but it only has "trivial" (i.e. "almost" in $\mathcal{L}_s$) coequalizers, idempotents, sections, retractions, isomorphisms.

In [**CG**] is shown that $\mathcal{L}_f$ solves the identity problem for the presentations of classical logic in terms of the (weak) concept of *equipollence of logics*[1]. But $\mathcal{L}_f$ does not solve problem of identity for the presentations of classical logic in terms of $\mathcal{L}_f$-isomorphisms.

We will write $\mathcal{Q}_f$ for the **q**uotient category of $\mathcal{L}_f$ by the relation of interprovability: the objects of $\mathcal{Q}_f$ are the logics $l = (\Sigma, \vdash)$ and $\mathcal{Q}_f((\Sigma, \vdash), (\Sigma', \vdash')) := \{[f] : f \in \mathcal{L}_f((\Sigma, \vdash), (\Sigma', \vdash'))\}$, where $[f] := \{g \in \mathcal{L}_f((\Sigma, \vdash), (\Sigma', \vdash')) : f \sim g\}$ and $f \sim g$ iff $(\overset{\vee}{f}/\dashv\vdash) = (\overset{\vee}{g}/\dashv\vdash) : F(\Sigma)/\dashv\vdash \longrightarrow F(\Sigma')/\dashv'\vdash'\}$; clearly, the relation $\sim$ is a congruence relation in the category $\mathcal{L}_f$ and we can take $\mathcal{Q}_f := \mathcal{L}_f/\sim$.

$\mathcal{Q}_f$ has terminal object, coequalizers, *weak* products and *weak* coproducts. The problem of identity for the presentations of classical logic is solved in terms of $\mathcal{Q}_f$-isomorphisms.

[1] We thank professor Marcelo Coniglio for that reference.

4 The appropriate categories of logics

A logic $(\Sigma, \vdash)$ is *congruential* if, for each $c_n \in \Sigma_n$ and each $\{(\varphi_0, \psi_0), \ldots, (\varphi_{n-1}, \psi_{n-1})\}$ such that $\varphi_0 \dashv\vdash \psi_0, \ldots, \varphi_{n-1} \dashv\vdash \psi_{n-1}$, then $c_n(\varphi_0, \ldots, \varphi_{n-1}) \dashv\vdash c_n(\psi_0, \ldots, \psi_{n-1})$. It follows that if $\vartheta_0, \vartheta_1 \in F(\Sigma)$ are such that $var(\vartheta_0) = var(\vartheta_1) = \{x_{i_0}, \ldots, x_{i_{n-1}}\}$ and $\vartheta_0 \dashv\vdash \vartheta_1$ then $\vartheta_0[\vec{x} \mid \vec{\varphi}] \dashv\vdash \vartheta_1[\vec{x} \mid \vec{\psi}]$. Clearly, the presentations of classical logic are congruential logics.

Denote $\mathcal{L}_f^c$ the full subcategory of $\mathcal{L}_f$ whose objects are the congruential logics. This is a reflective subcategory $i : \mathcal{L}_f^c \hookrightarrow \mathcal{L}_f$ has a left adjoint $c : \mathcal{L}_f \to \mathcal{L}_f^c$ such the underlying signatures of the logics l and $c(l)$ coincide. $\mathcal{L}_f^c$ has coproducts: it is the "congruential closure" of the coproduct in $\mathcal{L}_f$ of a discrete diagram in $\mathcal{L}_f^c$.

If $\mathcal{Q}_f^c$ denote the full subcategory of $\mathcal{Q}_f$ whose objects are the congruential logics and $i : \mathcal{Q}_f^c \hookrightarrow \mathcal{Q}_f$ is the inclusion functor, then i has a left adjoint $c : \mathcal{Q}_f \to \mathcal{Q}_f^c$. As in $\mathcal{Q}_f$, the problem of identity for the presentations of classical logic is solved in terms of $\mathcal{Q}_f^c$-isomorphisms.

$\mathcal{Q}_f^c$ is a cocomplete category. The *coproducts* in $\mathcal{Q}_f^c$ are obtained taking first a cone coproduct in $\mathcal{L}_f$: the vertex in $\mathcal{Q}_f^c$ is the congruential closure of the vertex in $\mathcal{L}_f$ and the cocone arrows in $\mathcal{Q}_f^c$ are the classes of equivalence of the cocone arrows in $\mathcal{L}_f$ (the congruential property is decisive in proof of uniqueness). The *coequalizers* in $\mathcal{Q}_f^c$ are obtained taking first a cone coequalizer in $\mathcal{Q}_f$ and then taking the induced cone in $\mathcal{Q}_f^c$ obtained by the reflection functor $c : \mathcal{Q}_f \to \mathcal{Q}_f^c$.

A congruential logics $(\Sigma, \vdash)$ is of *finite type* if it has a finite support signature ($card(\bigcup_{n \in \mathbb{N}} \Sigma_n) < \omega$) and is the congruential closure of a consequence relation over Σ that is generated by substitutions with a finite set of axioms and a finite set of (finitary) inference rules. There is only an enumerable set of classes of $\mathcal{Q}_f^c$-isomorphism of finite type congruential logics. Any congruential logic is a colimit in $\mathcal{Q}_f^c$ of a directed diagram of congruential logics of finite type. In $\mathcal{Q}_f^c$: if a congruential logic is *finitely presentable*, then it is a retract of a congruential logic of finite type.

A possible notion of (Lindenbaum) algebrized logic is given by the triples $(\Sigma, \vdash, \Delta/\dashv\vdash)$ where $l = (\Sigma, \vdash)$ is a logic and $\Delta \subseteq_{fin} F(\Sigma)[2]$ is a set of "equivalence formulas in the Lindenbaum sense" i.e.: **(a)** $\vdash \varphi\Delta\varphi$ [2]; **(b)** $\varphi\Delta\psi \vdash \psi\Delta\varphi$; **(c)** $\varphi\Delta\psi, \psi\Delta\vartheta \vdash \varphi\Delta\vartheta$; **(d)** $\varphi_0\Delta\psi_0, \ldots, \varphi_{n-1}\Delta\psi_{n-1} \vdash c_n(\varphi_0, \ldots, \varphi_{n-1})\Delta c_n(\psi_0, \ldots, \psi_{n-1})$; **(e)** $\varphi \dashv\vdash \psi$ iff $\vdash \varphi\Delta\psi$. Clearly, the underlying logic of $(l, \Delta/\dashv\vdash)$ is congruential.

The corresponding category $\mathcal{A}$ of algebrized logics has as morphims $f : (l, \Delta/\dashv\vdash) \longrightarrow (l', \Delta'/\dashv'\vdash')$ the $\mathcal{Q}_f^c$-morphisms $[f] : l \longrightarrow l'$ such that $\Delta' \vdash' \check{f}[\Delta]$. Composition and identities are as in $\mathcal{Q}_f^c$.

References

[**AFLM**] P. Arndt, R. A. Freire, O. O. Luciano, H. L. Mariano, *A global glance on categories in Logic*, Logica Universalis **1** (2007), 3-39.

[**Bez**] J.-Y. Béziau, *From Consequence Operator to Universal Logic: A Survey of General Abstract Logic*, in **Logica Universalis: Towards a General Theory of Logic** (J.-Y. Beziau, ed.), Birkhäuser, 2007, pp. 3–17.

[**BCC**] J. Bueno, M.E. Coniglio, W.A. Carnielli, Finite algebraizability via possible-translations semantics, *Proceedings of CombLog'04 — Workshop on Combination of Logics: Theory and Applications*, Departamento de Matemática, Instituto Superior Técnico), Lisboa, Portugal, 2004.

[**BP**] W. J. Blok, D. Pigozzi, **Algebraizable logics**, Memoirs of the AMS **396**, American Mathematical Society, Providence, USA, 1989.

[**CG**] C. Caleiro, R. Gonçalves, *Equipollent Logical Systems*, in **Logica Universalis: Towards a General Theory of Logic** (J.-Y. Beziau, ed.), Birkhäuser, 2007, pp. 97–109.

[**Cze**] J. Czelakowski, **Protoalgebraic logic**, Trends in Logic, Studia Logica Library, Kluwer Academic Publishers, 2001.

[**MM1**] H.L. Mariano, C.A. Mendes, *Analysing categories of signatures*, in preparation.

[**MM2**] H.L. Mariano, C.A. Mendes, *Towards a good notion of categories of logics*, in preparation.

[2] That is, if $\Delta = \{\Delta_u : u < v\}$, then $\vdash \varphi\Delta_u\varphi$, for all $u < v$.

Modal and Intuitionistic Natural Dualities via the Concept of Structure Dualizability

Yoshihiro Maruyama
Department of Humanistic Informatics
Graduate School of Letters
Kyoto University, Japan
maruyama@i.h.kyoto-u.ac.jp
http://researchmap.jp/ymaruyama/

Abstract Based on the concept of structure dualizability, we extend the theory of natural dualities (see [4]) so as to encompass Jónsson-Tarski's topological and Kupke-Kurz-Venema's coalgebraic dualities for all modal algebras (see [10]), and Esakia duality for all Heyting algebras (see [5]). Key notions are $\mathbb{ISP}_{\mathbb{M}}$ and $\mathbb{IS}_{\mathbb{R}}\mathbb{P}$, which allow us to generate all modal algebras and all Heyting algebras respectively from a single algebra. Since natural duality theory is closely related to many-valued logics, we provide applications of our theory to modal and intuitionistic many-valued logics.

The theory of natural dualities The theory of natural dualities is a general theory of Stone-type dualities based on the machinery of universal algebra. It basically discusses duality theory for the quasi-variety $\mathbb{ISP}(M)$ generated by a finite algebra M. It is useful for obtaining new dualities and actually encompasses many known dualities, including Stone duality for Boolean algebras (see [8]), Priestley duality for distributive lattices (see [1]), and Cignoli duality for MV_n-algebras, i.e., algebras of Łukasiewicz n-valued logic (see [3]), to name but a few (for more instances, see [4]).

A problem on the theory of natural dualities But natural duality theory has not subsumed the above-mentioned dualities for all modal algebras and all Heyting algebras. This seems to be mainly because the class of all modal algebras and the class of all Heyting algebras cannot be expressed as the quasi-variety generated by a single algebra, in contrast to the fact that any of Boolean algebras, distributive lattices and MV_n-algebras can be expressed as the quasi-variety generated by a single finite algebra, which works as a so-called "schizophrenic" object.

How to generate modal algebras and Heyting algebras In this work, we remedy the problem above by introducing new methods to generate a class of algebras from a single algebra, namely $\mathbb{ISP}_{\mathbb{M}}(\text{-})$ and $\mathbb{IS}_{\mathbb{R}}\mathbb{P}(\text{-})$ (the definitions will be given later). Crucial facts are as follows:

- The class of all modal algebras coincides with $\mathbb{ISP}_{\mathbb{M}}(\mathbf{2})$ for the two-element Boolean algebra $\mathbf{2}$;
- The class of all Heyting algebras coincides with $\mathbb{IS}_{\mathbb{R}}\mathbb{P}(\mathbf{2})$ for the two-element distributive lattice $\mathbf{2}$.

Moreover, we have the following facts: for $\mathbf{n} = \{0, 1/(n-1), 2/(n-1), ..., 1\}$ with the usual operations of MV-algebras, $\mathbb{ISP}_{\mathbb{M}}(\mathbf{n})$ coincides with the class of all algebras of Łukasiewicz n-valued modal logic (for this logic, see [2, 13]; a similar thing holds also for algebras of a version of Fitting's many-valued modal logic in [11]); for $\mathbf{n}$ with suitable operations, $\mathbb{IS}_{\mathbb{R}}\mathbb{P}(\mathbf{n})$ coincide with the class of all algebras of Łukasiewicz n-valued intuitionistic logic, which is defined by n-valued Kripke semantics. Thus, the notions of $\mathbb{ISP}_{\mathbb{M}}(\text{-})$ and $\mathbb{IS}_{\mathbb{R}}\mathbb{P}(\text{-})$ are natural and useful for our goal. In order to achieve our goal, it is enough to develop duality theory for $\mathbb{ISP}_{\mathbb{M}}(L)$ and $\mathbb{IS}_{\mathbb{R}}\mathbb{P}(L)$ for a sufficiently general algebra L.

The concept of structure dualizability Let L be a finite algebra. Suppose that we want to obtain duality for $\mathbb{ISP}(L)$. Since we want L to be a schizophrenic object in order to develop duality, we should look for a geometric structre on L that is in harmony with the algebraic structure of L. Our definition of harmony is as follows. Let $\mathfrak{X}$ be given topological and relational structures on L (relations may not be binary). We say that L is dualizable with respect to $\mathfrak{X}$ iff, for any $n \in \omega$,

$$\mathrm{Hom}_{\mathbf{Alg}}(L^n, L) = \mathrm{Hom}_{\mathbf{Sp}}(L^n, L)$$

where $\mathrm{Hom}_{\mathbf{Alg}}(L^n, L)$ is the set of homomorphisms, and $\mathrm{Hom}_{\mathbf{Sp}}(L^n, L)$ is the set of maps preserving the topological and relational structures. From this point of view, developing a duality for $\mathbb{ISP}(L)$ amounts to discovering a suitable structure $\mathfrak{X}$ on L such that L is dualizable with respect to $\mathfrak{X}$. The concept of structure dualizability is due to us, but similar notions have been known among duality theorists (see [4, 6]). For example, primality (see [7]; this is functional completeness in algebraic terms) can be seen as a special instance of structure dualizability.

Duality for $\mathbb{ISP}_{\mathbb{M}}(L)$ We first define $\mathbb{ISP}_{\mathbb{M}}(L)$. For simplicity, we assume that L has a bounded lattice reduct. For a Kripke frame (S, R), a modal power of L with respect to (S, R) is defined as $L^S \in \mathbb{ISP}(L)$ equipped with a unary operation $\Box_R$ on L^S defined by

$$(\Box_R f)(w) = \bigwedge \{f(w') \,;\, wRw'\}$$

where $f \in L^S$ and $w \in S$. Then, a modal power of L is defined as a modal power of L with respect to (S, R) for a Kripke frame (S, R). $\mathbb{ISP}_{\mathbb{M}}(L)$ denotes the class of all isomorphic copies of subalgebras of modal powers of L. $\mathbb{ISP}_{\mathbb{M}}(\mathbf{2})$ is the class of all modal algebras ($\mathbf{2}$ is the two-element Boolean algebra). We can obtain duality for $\mathbb{ISP}_{\mathbb{M}}(L)$ in the following way.

We assume: L is dualizable with respect to $\{M \mid M \text{ is a subalgebra of } L\}$ where note that M is a unary relation on L. Then, Keimel-Werner's duality theorem (see [9]) gives us a topological duality for $\mathbb{ISP}(L)$. We can then show that the duality for $\mathbb{ISP}(L)$ is lifted to a topological duality for $\mathbb{ISP}_{\mathbb{M}}(L)$. The topological duality for $\mathbb{ISP}_{\mathbb{M}}(L)$ can also be formulated in coalgebraic terms via the Vietoris space construction.

These results extend both Jónsson-Tarski's topological and Kupke-Kurz-Venema's coalgebraic dualities for all modal algebras, and also imply new coalgebraic dualities for algebras of Łukasiewicz n-valued modal logic and algebras of Fitting's many-valued modal logic as well as known topological dualities for them in [13, 11]. The details of the results can be found in [12].

Duality for $\mathbb{IS}_{\mathbb{R}}\mathbb{P}(L)$ We first define $\mathbb{IS}_{\mathbb{R}}\mathbb{P}(L)$. Assume that L has a bounded lattice reduct and has a binary operation $*$. For an ordered algebra A with a binary operation $*$, A is called $*$-residuated iff, for all $x, y \in A$,

$$\{z \in A \mid x * z \leq y\}$$

has a greatest element, which is denoted by $x \to y$. Then, $\mathbb{IS}_{\mathbb{R}}\mathbb{P}(L)$ is defined as the class of all isomorphic copies of $*$-residuated subalgebras of direct powers of L. $\mathbb{IS}_{\mathbb{R}}\mathbb{P}(\mathbf{2})$ is the class of all Heyting algebras if $*$ above is $\wedge$ ($\mathbf{2}$ is the two-element distributive lattice).

We assume that L is dualizable with respect to the Alexandrov topology (this is intuitionistic primality in some sense). Then we can show that $\mathbb{ISP}(L)$ is dually equivalent to coherent spaces in the sense of [8] (or spectral spaces or Esakia spaces). If L is the two-element distributive lattice, this is Stone (or Priestley) duality for distributive lattices. By restriction, we can obtain a duality for $\mathbb{IS}_{\mathbb{R}}\mathbb{P}(L)$ from the duality for $\mathbb{ISP}(L)$. If L is the two-element distributive lattice, this is Esakia duality for Heyting algebras. The duality for $\mathbb{IS}_{\mathbb{R}}\mathbb{P}(L)$ may be seen as an intuitionistic analogue of Hu's primal duality theorem (see [7]).

The duality for $\mathbb{IS}_{\mathbb{R}}\mathbb{P}(L)$ implies that if L and L' satisfy the assumptions of the duality, then $\mathbb{IS}_{\mathbb{R}}\mathbb{P}(L)$ and $\mathbb{IS}_{\mathbb{R}}\mathbb{P}(L')$ are categorically equivalent, which seems to contradict our intuition. We can define Łukasiewicz n-valued intuitionistic logic via n-valued Kripke semantics, and then a duality for algebras of the logic follows from the above-mentioned duality for $\mathbb{IS}_{\mathbb{R}}\mathbb{P}(L)$.

References

[1] G. Bezhanishvili, N. Bezhanishvili, D. Gabelaia, and A. Kurz, Bitopological duality for distributive lattices and Heyting algebras, *Math. Struct. Comput. Sci.* 20 (2010) 359-393.

[2] F. Bou, F. Esteva and L. Godo, On the minimum many-valued modal logic over a finite residuated lattice, to appear in *Journal of Logic and Computation*.

[3] R. L. O. Cignoli, E. J. Dubuc, and D. Mundici, Extending Stone duality to multisets and locally finite MV-algebras, *J. Pure Appl. Algebra*, 189 (2004) 37-59.

[4] D. M. Clark and B. A. Davey, *Natural dualities for the working algebraist*, CUP, 1998.

[5] L. Esakia, Topological Kripke models, *Soviet Math. Dokl.* 15 (1974) 147-151.

[6] D. Hofmann, On a generalization of the Stone-Weierstrass theorem, *Appl. Categ. Structures* 10 (2002) 569-592.

[7] T. K. Hu, Stone duality for primal algebra theory, *Math. Z.* 110 (1969) 180-198.

[8] P. T. Johnstone, *Stone spaces*, CUP, 1986.

[9] K. Keimel and H. Werner, Stone duality for varieties generated by a quasi-primal algebra, *Mem. Amer. Math. Soc.* 148 (1974) 59-85.

[10] C. Kupke, A. Kurz and Y. Venema, Stone coalgebras, *Theoretical Computer Science* 327 (2004) 109-134.

[11] Y. Maruyama, Dualities for algebras of Fitting's many-valued modal logics, *Fundamenta Informaticae* 106 (2011) 273-294.

[12] Y. Maruyama, Natural duality, modality and coalgebra, preprint (available in http://researchmap.jp/ymaruyama/).

[13] B. Teheux, A duality for the algebras of a Łukasiewicz $n+1$-valued modal system, *Studia Logica* 87 (2007) 13-36.

Categorical Duality between Point-Free and Point-Set Spaces

Yoshihiro Maruyama
Department of Humanistic Informatics
Graduate School of Letters
Kyoto University, Japan
maruyama@i.h.kyoto-u.ac.jp
http://researchmap.jp/ymaruyama/

Introduction We attempt to reveal the mathematical relationships between two aspects of the notion of space, namely ontological and epistemological aspects of it. On the one hand, there are set-theoretic, point-based concepts of space (e.g., topological space, convexity space [8], and measurable space), which we consider represents the ontological aspect of the notion of space. On the other hand, there are algebraic, point-free concepts of space (e.g., frame [4], continuous lattice [7], and σ-complete Boolean algebra), which we consider represents the epistemological aspect of the notion of space. Point-free and point-set spaces are often related in a dual way, and thus, in this work, we focus on the dual relationships between them.

We first develop a moderately general theory of dual adjunctions, then apply it to the case of Scott's continuous lattices and convexity spaces, and analyze in detail the resulting dual adjunction between them to obtain deeper insights that cannot be derived from a neat, general theory. In the process, we exploit the concrete-categorical concepts of a functor-structured category and topological axioms in it (see [1]) in order to obtain a general concept of a point-set space. A general concept of a point-free space is defined as an algebra for an endofunctor on **Sets** or an Eilenberg-Moore algebra of a monad on **Sets**.

In our theory of dual adjunctions, we intend to capture the practice of duality theory for point-free and point-set spaces, and so we stick to concrete ideas applied in practice rather than high-level abstractions (especially, we equip Hom-sets in an algebraic category with generalized topologies similar to Stone and Zariski topologies; Hom-sets in a topological category are endowed with pointwise operations). Our theory is not fully general for this reason. Nevertheless, most structures of our interest fall into the concrete **Sets**-based framework, including the above-mentioned ones. Moreover, the concrete features of our theory weaken relevant assumptions and make results more effective in practice. We also note that our result can be expressed without any category theory. In our theory of dual adjunctions, the category of "algebras" and the category of "spaces" involved are non-symmetric, while they are symmetric in some categorical developments of duality theory (see, e.g., [6]). We consider that they should not be symmetric, since they are indeed non-symmetric in practice. In practice, one category is of algebraic nature, and the other category is of spatial nature. We make explicit the difference between the two categories in this work (this also simplifies assumptions).

We see continuous lattices as point-free convexity spaces, since continuous lattices coincide with meet-complete posets with directed joins distributing over arbitrary meets (see [2, Theorem I-2.7]), where recall that a convexity space is a set equipped with a collection of subsets of it that is closed under arbitrary intersections and directed unions (see [8]). Our theory of dual adjunctions gives us a dual adjunction between continuous lattices and convexity spaces. We can refine this into a dual equivalence between "spatial" continuous lattices and "sober" convexity spaces (in fact, there is another duality between continuous lattices and convexity spaces that remedies a certain deficiency of this duality). The notion of polytope plays an important role in defining sober convexity spaces (some familiar convexity spaces such as $\mathbb{R}$ are not sober in contrast to the situation in topology). These results on continuous lattices are reformulations of results in the author's previous work [5]. We hope that these results lead us to interactions between domain theory and convex geometry (for instance, we can show that the concept of compactness in domain theory coincides with the concept of polytope in convex geometry).

Dual Adjunction via Harmony Condition We refer to [1] for the definitions of a functor-(co)structured category $\mathbf{Spa}(U)$ and topological (co)axiom in it where U is a faithful functor from a category to **Sets**. Given topological (co)axioms in a functor-(co)structured category, we can consider a full subcategory of the functor-(co)structured category that is definable by the topological (co)axioms. It is known that we can obtain almost all topological categories in this way (see [1]).

We introduce a new concept of Boolean topological coaxioms in $\mathbf{Spa}(U)^{\mathrm{op}}$. A Boolean topological coaxiom in $(\mathbf{Spa}(U))^{\mathrm{op}}$ is defined as a topological coaxiom $p : (C, \mathcal{O}) \to (C', \mathcal{O}')$ in $(\mathbf{Spa}(U))^{\mathrm{op}}$ such that

- Any element of $\mathcal{O} \setminus \mathcal{O}'$ can be expressed as a (possibly infinitary) Boolean combination of elements of $\mathcal{O}'$.

Let $\mathcal{Q}$ denote the contravariant powerset functor on **Sets**. Then:

- Any of the category of topological spaces, the category of convexity spaces, and the category of measurable spaces can be expressed as a full subcategory of $\mathbf{Spa}(\mathcal{Q})^{\mathrm{op}}$ that is definable by a class of Boolean topological coaxioms.

Hence, we consider such a category as a general concept of point-set spaces.

Let **Alg** denote a full subcategory of $\mathbf{Alg}(T)$ for an endofunctor T on **Sets** (we later consider the case that T is a monad and **Alg** is the Eilenberg-Moore category of T), and **Spa** a full subcategory of $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$ that is definable by a class of Boolean topological coaxioms in $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$. We assume the following.

- there is an object Ω in both **Alg** and **Spa**, i.e., there is $\Omega \in \mathbf{Sets}$ both with a structure map $h_\Omega : T(\Omega) \to \Omega$ such that $(\Omega, h_\Omega) \in \mathbf{Alg}$ and with a "generalized topology" $\mathcal{O}_\Omega \subset \mathcal{Q}(\Omega)$ such that $(\Omega, \mathcal{O}_\Omega) \in \mathbf{Spa}$;
- $(\mathbf{Alg}, \mathbf{Spa}, \Omega)$ satisfies the following harmony condition.

$(\mathbf{Alg}, \mathbf{Spa}, \Omega)$ is said to satisfy the harmony condition iff, for each $S \in \mathbf{Spa}$,

$$(\ \mathrm{Hom}_{\mathbf{Spa}}(S, \Omega), \ h_S : T(\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)) \to \mathrm{Hom}_{\mathbf{Spa}}(S, \Omega) \ )$$

is an object in **Alg** such that, for any $s \in S$ (let p_s be the corresponding projection from

$\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ to Ω), the following diagram commutes:

$$
\begin{array}{ccc}
T(\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)) & \xrightarrow[h_S]{} & \mathrm{Hom}_{\mathbf{Spa}}(S, \Omega) \\
\downarrow{\scriptstyle T(p_s)} & & \downarrow{\scriptstyle p_s} \\
T(\Omega) & \xrightarrow[h_\Omega]{} & \Omega
\end{array}
.
$$

We have to equip $\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ with an algebraic structure and $\mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$ with a geometric structure in order to make Hom-functors available. The algebraic structure of $\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ is provided by h_S above. The geometric structure of $\mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$ can be provided in the following way. We can equip $\mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$ with the generalized topology generated in **Spa** by

$$\{\langle a \rangle_O \ ; \ a \in A \text{ and } O \in \mathcal{O}_\Omega\}$$

where

$$\langle a \rangle_O := \{v \in \mathrm{Hom}_{\mathbf{Alg}}(A, \Omega) \ ; \ v(a) \in O\},$$

which is intuitively the region of the spectrum $\mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$ of valuations in which the truth value of a formula a is in a generalized open set O. The induced contravariant Hom-functors

$$\mathrm{Hom}_{\mathbf{Alg}}(\text{-}, \Omega) : \mathbf{Alg} \to \mathbf{Spa} \text{ and } \mathrm{Hom}_{\mathbf{Spa}}(\text{-}, \Omega) : \mathbf{Spa} \to \mathbf{Alg}$$

can be shown to be well defined and form a dual adjunction between categories **Alg** and **Spa**. If T is a monad and **Alg** is the Eilenberg-Moore category of T, then we do not need the assumption that h_S is in **Alg**. In this case, it suffices to verify only that the diagram above commutes.

We emphasize that the harmony condition is easy to verify in concrete cases as we can see in the example of continuous lattices and convexity spaces (in a similar way, we can also obtain a dual adjunction between σ-complete Boolean algebras and measurable spaces). Stone-type adjunctions in many-valued logics follow from this adjunction theorem as well as those in classical logics.

Dual adjunction theorem in [6] is described in a more abstract setting, and is more general than ours. In [6], the key assumptions for proving adjointness are the two initial lifting conditions on the two categories concerned. Our theory is more involved in the mechanism of how the initial liftings become possible. Although the two initial lifting conditions are symmetric in [6], in practice, the initial lifting on the algebraic side and the initial lifting on the topological side can be seen as based on different mechanisms, which are clarified in our adjunction theorem. Thus, although the scope is more limited, our theorem contains more details and is more effective in certain specific situations, including the following case of continuous lattices and convexity spaces.

Duality between Scott's Continuous Lattices and Convexity Spaces We first recall that a morphism of convexity spaces is defined like a continuous map, i.e., it is a map such that the inverse image of a convex set under it is also convex. A homomorphism of continuous lattices is defined as a map preserving arbitrary meets and directed joins.

The category of continuous lattices is the Eilenberg-Moore category of the filter monad on **Sets**. The category of convexity spaces can be expressed as a full subcategory of $\mathbf{Spa}(Q)^{\mathrm{op}}$ that is definable by suitable Boolean topological coaxioms. $\mathbf{2}$ $(= \{0, 1\})$ can be naturally seen as both a continuous lattice (with the obvious lattice operations) and a convexity space (with the convexity $\{\emptyset, \{1\}, \mathbf{2}\}$).

The harmony condition just means that the lattice of convex sets is closed under intersections and directed unions, and forms a continuous lattice, which is trivial. Thus, the general theory above gives us a dual adjunction between continuous lattices and convexity spaces.

The dual adjunction is refined into a dual equivalence as follows. We say that a continuous lattice is spatial iff it has enough principal Scott-open filters and that a convexity space is sober iff any polytope of the space (i.e., the convex hull of finitely many points) can be generated by a unique point. Then we obtain a dual equivalence between spatial continuous lattices and sober convexity spaces. Note that spatial continuous lattices coincide with algebraic lattices. This gives the convexity-theoretic understanding of Hofmann-Mislove-Stralka duality in [3].

A problem of the duality is that some familiar convexity space such as $\mathbb{R}$ are not sober, in contrast to the case of topology, and do not fall into the scope of the duality. In other words, we cannot recover the points of those spaces from their lattices of convex sets according to the duality. But this never means that the points of those spaces cannot be recovered in any way. Indeed, it is easy to recover the points of $\mathbb{R}$ from the lattice of convex sets of it.

This point of view leads us to a dual equivalence between the category of m-spatial continuous lattices and m-homomorphisms and the category of T_1 convexity spaces and morphisms of convexity spaces. Here, m-spatiality means having enough maximal meet-complete filters, and an m-homomorphisms is a homomorphism such that the inverse image of a maximal meet-complete filter under it is also a maximal meet-complete filter. A convexity space is T_1 iff any singleton is convex. The problem of the duality above is remedied in this duality, since familiar convexity spaces (including vector spaces over $\mathbb{R}$, n-spheres, real projective spaces, etc.) are usually T_1. At the same time, however, there seems to be no proper adjunction behind this duality, in contrast to the case of the former duality.

Thus we may say that there are two dualities between continuous lattices and convexity spaces and that each duality has its own advantage over the other.

References

[1] J. Adámek, H. Herrlich, and G. E. Strecker, *Abstract and Concrete Categories*, John Wiley and Sons, Inc., 1990.

[2] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove, and D. S. Scott, *Continuous Lattices and Domains*, CUP, 2003.

[3] K. H. Hofmann, M. Mislove and A. Stralka, *The Pontryagin Duality of Compact* 0*-Dimensional Semilattices and its Applications*, Lecture Notes in Math. 394, Springer, 1974.

[4] P. T. Johnstone, *Stone Spaces*, CUP, 1986.

[5] Y. Maruyama, Fundamental results for pointfree convex geometry, *Ann. Pure Appl. Logic* 161 (2010) 1486-1501.

[6] H.-E. Porst and W. Tholen, Concrete dualities, *Category Theory at Work* (1991) 111-136.

[7] D. Scott, Continuous lattices, *Lecture Notes in Mathematics* 274 (1972) 97-136.

[8] M. L. J. van de Vel, *Theory of Convex Structures*, North-Holland, 1993.

Categorically axiomatizing the classical quantifiers

Richard McKinley

IAM, Universität Bern, Switzerland

Abstract. We give a categorical axiomatization of cut elimination in classical logic, interpreting proofs in a hyperdoctrine-like framework in which the fibres are Führmann and Pym's classical categories: categories enriched with a partial order structure interpreting cut-elimination in propositional classical logic

One approach to understanding the identity of proofs (when two syntactic proofs are essentially the same) is to consider *interpretations* of proofs in a category $\mathcal{C}$ (where formulae are interpreted by objects and a proofs by arrows). For example, if we interpret natural deduction proofs in a cartesian-closed category, we obtain a theory on those proofs extending beta-eta equivalence. However, naïvely extending that result to *classical* logic, by adding a dualizing object to the cartesian-closed category, is fruitless: every such category is a poset. Pym and Führmann have defined *Classical categories*, which are order-enriched categories in which one can interpret classical proofs. The interpretation is sound with respect to cut-elimination: whenever a proof Φ cut-reduces to Ψ, we have that the interpretation of Φ is smaller than the interpretation of Ψ.

To obtain an axiomatization of classical categories, one starts with a model of multiplicative linear logic: a $*$-autonomous category. This is a symmetric monoidal category $(\mathcal{C}, 1, \otimes)$ together with a functor $(-)^* : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ such that $A^{**} = A$ and there is a natural isomorphism

$$Hom(A \otimes B, C^*) \cong Hom(A, (B \otimes C)^*).$$

A classical category has in addition, for each object A, a *commutative comonoid* structure, with structure morphisms

$$\Delta_A : A \to A \otimes A \qquad \langle\,\rangle_A : A \to 1.$$

We also assume that $\mathcal{C}$ *has comonoids*: the structure morphisms on $A \otimes B$ are built from those on A and those on B. The structure morphisms cannot natural: this would also lead to a collapse. If we assume instead that they are *lax*-natural with respect to the ordering on the hom-sets:

$$\Delta \circ f \leq (f \otimes f) \circ \Delta \qquad \langle\,\rangle \circ f \leq \langle\,\rangle.$$

and if the monoidal and closed structure of $\mathcal{C}$ preserves the order on hom-sets, then $\mathcal{C}$ is a classical category. Classical categories provide a setting for interpreting proofs in the multiplicatively formulated sequent calculus for propositional classical logic: the ordering on hom-sets interprets cut-reduction as formulated by Gentzen: if a proof Φ cut-reduces to a proof Ψ, then the interpretation of Φ in any classical category is smaller (in the hom-set ordering) than Ψ.

The work described in this abstract concerns an extension of these results to first-order classical logic. There is a standard approach for providing models of first-order logics from the propositional fragment, called the *hyperdoctrine* construction: a hyperdoctrine is an indexed category $D : \mathcal{B}^{op} \to \text{Cat}$ in which $\mathcal{B}$ is a category representing the first-order terms of the logic; an object of $\mathcal{B}$ is a sequence of free variables, and a morphism of $\mathcal{B}$ is a term. For a sequence $\bar{x}$ of variables, $D(\bar{x})$ is a category modelling the propositional logic: in our case, a classical category.

The operations of substitution and quantification arise in this setting as functors between the fibres. For example, the image t^* of a term t under D is a functor representing the substitution of that term t. The interesting fact about hyperdoctrines for intuitionistic logic (noticed first by Lawvere) is that the quantifiers are adjoint to the substitutions arising from projections in the base category $\mathcal{B}$: the existential quantifier emerges as the left adjoint, and the universal as the right adjoint.

We can immediately begin to formulate an axiomatization of the universal quantifier in classical logic: it should be a monoidal functor Π (the notion of structure-preserving functor for $*$-autonomous categories) which, in addition, preserves the comonoids interpreting the structural rules, and the order interpreting cut-reduction. The functor interpreting the existential quantifier will the arise from by duality. Rather than adjunctions, we will require that the quantifiers are related to substitution by "lax" adjunctions, in which the units and co-units are not natural transformations, but are lax in the same sense as the morphisms interpreting the structural rules.

One nice property of hyperdoctrines for natural deduction is that one can immediately infer that, since the existential quantifier is a left adjoint, it preserves coproducts, which model disjunctions, and similarly for the universal quantifier and products. We can find partial analogue of that result in this setting: by making some assumptions on the the partial order on hom-sets, we can derive that Π is monoidal from the fact that the lift of the projection is *strong* monoidal.

This setting actually stronger than what is needed to capture cut-reduction in first-order logic. The original definition of classical categories started, not from $*$-autonomous categories, but from the equivalent *symmetric linearly distributive categories (SLDCs) with negation*. The difference between the two axiomatizations is a categorical analogue of two different ways to present the negation of a formula in logic: either as a connective, or as a defined operation using De Morgan duality. In an SLDC, there are two monoidal structures $\otimes$ and $\invamp$, and a natural transformation

$$\delta : A \otimes (B \invamp C) \to (A \otimes B) \invamp C$$

mediating them (plus some coherence conditions). Such a category has negation if, for each object A, we have an object $\bar{A}$ with morphisms we have morphisms

$$\gamma : A \otimes \bar{A} \to 0 \qquad \tau : 1 \to A \oplus \bar{A},$$

with appropriate commuting diagrams. The notion of structure preserving functor for SLDCs is that of a *linear functor*: a pair of functors (F, G), with F monoidal wrt $\otimes$ and G comonoidal wrt $\invamp$, with coherence conditions that G is the dual of F in the corresponding $*$-autonomous category. In this setting, we can axiomatize the quantifiers without the assumption that they arise as duals, and study conditions forcing them to be dual.

The basic notion of *Classical Doctrine*, axiomatized using linearly distributive categories, is the following

(a) A classical doctrine is an indexed category $\mathcal{C} : \mathcal{B}^{op} \to \mathfrak{CAT}$ in which the base has finite products and each fibre is a classical category, such that:
(b) The functor a^* (defined for each morphism a in the base category) defines a linear functor (a^*, a^*): that is, a^* is self-dual.
(c) There exists, for each projection π in $\mathcal{B}$, a lax left adjoint Σ_π, and a right oplax-adjoint Π_π, to a^*;
(d) Σ_a is comonoidal w.r.t. $\oplus$ and Π_a is monoidal w.r.t $\otimes$ and the above lax adjunctions is symmetric monoidal in both tensors;
(e) The linear distributivity δ is strong with respect to the units and co-units of both adjunctions: the relevant naturality diagrams hold up to equality.
(f) **Beck condition**
if

$$\begin{array}{ccc} A & \xrightarrow{\;t\;} & B \\ {\scriptstyle r}\downarrow & \lrcorner & \downarrow{\scriptstyle s} \\ C & \xrightarrow{\;u\;} & D \end{array}$$

is a pullback in $\mathcal{B}$, and Σ_s, Σ_r are lax left adjoints to s^* and r^* respectively, the diagram

$$\begin{array}{ccc} \mathcal{C}^A & \xleftarrow{\;t^*\;} & \mathcal{C}^B \\ {\scriptstyle \Sigma_r}\downarrow & & \downarrow{\scriptstyle \Sigma_s} \\ \mathcal{C}^C & \xleftarrow{\;u^*\;} & \mathcal{C}^D \end{array}$$

commutes;
(g) **Frobenius strengths** The morphism

$$(\mathbf{id} \otimes \varepsilon_B^{\Sigma_a}) \circ \mu_{A,a^*B}^{\Sigma_a} : \Sigma_a(A \otimes a^*B) \to \Sigma A \otimes B$$

has a left adjoint $\mathbf{frob}^{\Sigma_a}_{A,B}$ such that

$$(\mathbf{id} \otimes \varepsilon^{\Sigma_a}_B) \circ \mu^{\Sigma_a}_{A,a^*B} \circ \mathbf{frob}^{\Sigma_a}_{A,B} = \mathbf{id}$$

and

$$\mathbf{frob}^{\Sigma_a}_{A,B} \circ (\mathbf{id} \otimes \varepsilon^{\Sigma_a}_B) \circ \mu^{\Sigma_a}_{A,a^*B} \leq \mathbf{id}$$

and the morphism

$$\mu^{\Pi_a}_{A,a^*B} \circ (\mathbf{id} \otimes \eta^{\Pi_a}_B) : \Pi_a A \parr B \to \Pi_a(A \parr a^*B)$$

has a right adjoint $\mathbf{frob}^{\Pi_a}_{A,B}$, such that

$$\mu^{\Pi_a}_{A,a^*B} \circ (\mathbf{id} \otimes \eta^{\Pi_a}_B) \circ \mathbf{frob}^{\Pi_a}_{A,B} \leq \mathbf{id}$$

and

$$\mathbf{frob}^{\Pi_a}_{A,B} \circ \mu^{\Pi_a}_{A,a^*B} \circ (\mathbf{id} \otimes \eta^{\Pi_a}_B) = \mathbf{id}$$

This is enough to obtain the following theorem:

Theorem 1. *If $\lfloor - \rfloor$ is an interpretation of proofs into a classical doctrine $\mathcal{C}$, and Φ cut reduces to Ψ, then $\lfloor \Phi \rfloor \leq \lfloor \Psi \rfloor$.*

In the talk, we will give some flavour of how this result is obtained, and additionally describe how, by considering the structure of proofs from $\exists x.(A \vee B)$ to $\exists x.A \vee \exists x.B$, from $\exists x.(A \wedge B)$ to $\exists x.A \wedge \exists x.B$, and from $\forall x.A$ to $\exists x.A$, we can find sufficient conditions for the pair (Σ, Π) of functors to be a linear functor: i.e. Σ arises as the dual of Π.

Admissible Multiple-Conclusion Rules

George Metcalfe and Christoph Röthlisberger

Mathematics Institute, University of Bern
Sidlerstrasse 5, Bern 3012, Switzerland

{george.metcalfe,roethlisberger}@math.unibe.ch

The *admissible rules* of a logic (understood as a consequence relation) are the rules that can be added to the logic without producing any new theorems (see [8] for a comprehensive study). In algebraic terms, the admissible rules of a quasivariety $\mathcal{Q}$ (which may correspond to a logic via algebraizability) are the quasiequations that hold in $\mathbf{F}_{\mathcal{Q}}$, the free algebra of $\mathcal{Q}$ on countably many generators. Moreover, $\mathcal{Q}$ is *structurally complete* (which for a logic means that every admissible rule is derivable) if for all finite sets of equations $\Sigma \cup \{s \approx t\}$ in the language of $\mathcal{Q}$:

$$\Sigma \vDash_{\mathcal{Q}} s \approx t \quad \Leftrightarrow \quad \Sigma \vDash_{\mathbf{F}_{\mathcal{Q}}} s \approx t.$$

That is, $\mathcal{Q}$ is structurally complete if and only if $\mathcal{Q}$ is generated as a quasivariety by $\mathbf{F}_{\mathcal{Q}}$: in symbols, $\mathcal{Q} = \mathbb{Q}(\mathbf{F}_{\mathcal{Q}})$. Equivalently, $\mathcal{Q}$ is structurally complete if every proper subquasivariety of $\mathcal{Q}$ generates a proper subvariety of the variety generated by $\mathcal{Q}$ (see [1] for details). For example, the classes of Boolean algebras, lattice-ordered abelian groups, and Gödel algebras (semilinear Heyting algebras) are structurally complete, but the classes of lattices, MV-algebras, and Heyting algebras are not (see, e.g., [8, 7, 2]).

In this work, we investigate a broader notion of admissibility encompassing *multiple-conclusion rules*, understood algebraically as ordered pairs of finite sets of equations, and written $\Sigma \Rightarrow \Delta$. Let us write $\Sigma \models_{\mathcal{K}} \Delta$ (or $\Sigma \models_{\mathbf{A}} \Delta$ when $\mathcal{K} = \{\mathbf{A}\}$) to denote that the universal formula $\bigwedge \Sigma \to \bigvee \Delta$ holds in all members of a class of algebras $\mathcal{K}$ in the same language. Then the rule $\Sigma \Rightarrow \Delta$ is said to be *admissible* in a quasivariety $\mathcal{Q}$ if $\Sigma \models_{\mathbf{F}_{\mathcal{Q}}} \Delta$. Moreover, we will say that $\mathcal{Q}$ is *universally complete* if

$$\Sigma \vDash_{\mathcal{Q}} \Delta \quad \Leftrightarrow \quad \Sigma \vDash_{\mathbf{F}_{\mathcal{Q}}} \Delta.$$

That is, $\mathcal{Q}$ is universally complete if and only if $\mathcal{Q}$ is generated as a universal class by $\mathbf{F}_{\mathcal{Q}}$: in symbols, $\mathcal{Q} = \mathbb{U}(\mathbf{F}_{\mathcal{Q}})$.

Let us illustrate the relevance of multiple-conclusion rules (investigated from a logical perspective by Shoesmith and Smiley in [9]) with some examples. Observe first that important properties of a logic or quasivariety may be formulated as the admissibility of a multiple-conclusion rule; e.g., the following rule corresponding to the *disjunction property* is admissible in Heyting algebras:

$$\{x \vee y \approx \top\} \quad \Rightarrow \quad \{x \approx \top, y \approx \top\}.$$

Similarly, in MV-algebras, the following rule, expressing a weaker disjunction property, is admissible:

$$\{x \vee \neg x \approx \top\} \quad \Rightarrow \quad \{x \approx \top, x \approx \bot\}.$$

Multiple-conclusion rules often provide a more natural and flexible framework for obtaining axiomatizations for the admissible rules of logics or quasivarieties, facilitating the derivation of more complicated axiomatizations using single-conclusion rules (e.g., for modal logics [4], MV-algebras [5], fragments of intermediate logics [3]). Indeed, a basis is defined in [6] for De Morgan algebras that makes use of the disjunction property rule $\{x \vee y \approx \top\} \Rightarrow \{x \approx \top, y \approx \top\}$ where no (finite) single-conclusion axiomatization has yet been obtained.

We remark also that multiple-conclusion rules (or universal formulas) play a fundamental role in the investigation of free algebras of certain classes of algebras. Notably, "Whitman's condition"

$$\{x \wedge y \leq z \vee w\} \;\; \Rightarrow \;\; \{x \leq z \vee w,\ y \leq z \vee w,\ x \wedge y \leq z,\ x \wedge y \leq w\}$$

is admissible in the variety of lattices (i.e., holds in all free lattices) [10].

A key aim of the work reported here is to obtain (syntactic or algebraic) characterizations of admissible (multiple-conclusion) rules and the notions of structural completeness and universal completeness. Let us first consider the following helpful lemma, which relates the structural completeness of a quasivariety to the existence of certain embeddings into the free algebra on countably many generators.

Lemma 1 ([2]). *Let $\mathcal{K}$ be a class of algebras in the same language. If each $\mathbf{A} \in \mathcal{K}$ embeds into $\mathbf{F}_{\mathbb{Q}(\mathcal{K})}$, then $\mathbb{Q}(\mathcal{K})$ is structurally complete.*

This lemma (and variants thereof) are used to establish structural completeness results for various (fragments of) logics and quasivarieties in [2], including the varieties of product algebras, cancellative hoops, and implicational subreducts of MV-algebras and BL-algebras. Let us consider here, however, a simple example from [6] based on Kleene algebras (in a language with binary connectives $\wedge$ and $\vee$, a unary connective $\neg$, and constants $\bot$ and $\top$):

$$\mathbf{C_3} = \langle\{-1, 0, 1\}, \min, \max, -, -1, 1\rangle$$
$$\mathbf{C_4} = \langle\{-2, -1, 1, 2\}, \min, \max, -, -2, 2\rangle.$$

It is well known that $\mathbf{C_3}$ generates the class of Kleene algebras KA as a quasivariety; i.e., $\mathsf{KA} = \mathbb{Q}(\mathbf{C_3})$. However, this quasivariety is not structurally complete, since the rule (or quasiequation)

$$\{x \approx \neg x\} \quad \Rightarrow \quad \{x \approx y\}$$

is admissible (there is no term t such that $t \approx \neg t$ holds in all Kleene algebras), but fails in $\mathbf{C_3}$. On the other hand, the quasivariety $\mathbb{Q}(\mathbf{C_4})$ is structurally complete. By Lemma 1, it is enough to observe that $e : \mathbf{C_4} \to \mathbf{F}_{\mathbb{Q}(\mathbf{C_4})}$ is an embedding where $g(1) = [x \vee \neg x]$, $g(-1) = [x \wedge \neg x]$, $g(2) = [\top]$, and $g(-2) = [\bot]$. Moreover, since $\mathbb{Q}(\mathbf{C_4})$ is axiomatized relative to KA by the admissible (in KA) quasiequation

$$\{\neg x \leq x,\ x \wedge \neg y \leq \neg x \vee y\} \quad \Rightarrow \quad \{\neg y \approx y\}$$

we obtain an axiomatization of the admissible rules of KA. Moreover, following a similar strategy, we obtain an axiomatization of the admissible rules of De Morgan lattices that makes use of the additional rule $\{x \approx \neg x\} \Rightarrow \{x \approx y\}$.

Lemma 1 suggests a close connection between structural completeness and the existence of certain embeddings. Indeed, restricting to quasivarieties generated by a single finite algebra, we obtain the following general characterization:

Theorem 1. *The following are equivalent for any finite algebra* $\mathbf{A}$*:*

(1) $\mathbb{Q}(\mathbf{A})$ *is structurally complete.*
(2) *Each finite* $\mathbf{B} \in \mathbb{Q}(\mathbf{A})$ *embeds into a product of* $\mathbf{F}_{\mathbb{Q}(\mathbf{A})}$.

An obvious question that arises is whether the converse of Lemma 1 holds (at least) when $\mathcal{K}$ contains just one finite algebra $\mathbf{A}$. In other words, can we restrict condition (2) in the previous theorem to embeddings into $\mathbf{F}_{\mathbb{Q}(\mathbf{A})}$? To see that this is not the case, consider an algebra $\mathbf{A} = \langle\{a, b, c, d\}, f\rangle$ with the unary function f defined by $f(a) = f(c) = f(d) = b$, $f(b) = a$. Then (easily) there cannot exist an embedding of $\mathbf{A}$ into $\mathbf{F}_{\mathbb{Q}(\mathbf{A})}$ since there is no element in $\mathbf{F}_{\mathbb{Q}(\mathbf{A})}$ which is the f-image of three different elements. On the other hand, the map $e : \mathbf{A} \to \mathbf{F}_{\mathbb{Q}(\mathbf{A})} \times \mathbf{F}_{\mathbb{Q}(\mathbf{A})}$ given by $e(a) = \langle f(f(x)), f(f(x))\rangle$; $e(b) = \langle f(x), f(x)\rangle$; $e(c) = \langle x, f(f(x))\rangle$; $e(d) = \langle f(f(x)), x\rangle$ is an embedding. So by the previous theorem, $\mathbb{Q}(\mathbf{A})$ is structurally complete.

However, $\mathbb{Q}(\mathbf{A})$ is not universally complete. The multiple-conclusion rule

$$\{f(x) \approx f(y)\} \quad \Rightarrow \quad \{x \approx y,\ f(f(x)) \approx y,\ f(f(y)) \approx x\}$$

is admissible in $\mathbb{Q}(\mathbf{A})$ (holds in $\mathbf{F}_{\mathbb{Q}(\mathbf{A})}$) but does not hold in $\mathbb{Q}(\mathbf{A})$. Indeed, for universal completeness, we obtain a stronger correspondence:

Theorem 2. *The following are equivalent for any finite algebra* $\mathbf{A}$*:*

(1) $\mathbb{Q}(\mathbf{A})$ *is universally complete.*
(2) *Each finite* $\mathbf{B} \in \mathbb{Q}(\mathbf{A})$ *embeds into* $\mathbf{F}_{\mathbb{Q}(\mathbf{A})}$.

We also consider extensions of the latter theorem to arbitrary quasivarieties via the notion of partial embeddability and investigate the relationship between embeddings into free algebras and projective (finitely presented) algebras.

References

1. C. Bergman, Structural completeness in algebra and logic, In H. Andréka, J. Monk, I. Nemeti (eds.) Algebraic Logic, Colloquia Mathematica Societatis János Bolyai 53 (1991) 59–73.
2. P. Cintula, G. Metcalfe, Structural completeness in fuzzy logics, Notre Dame Journal of Formal Logic 50 (2) (2009) 153–183.
3. P. Cintula, G. Metcalfe, Admissible Rules in the Implication-Negation Fragment of Intuitionistic Logic, Annals of Pure and Applied Logic 162 (10) (2010) 162–171.
4. E. Jeřábek, Admissible rules of modal logics, Journal of Logic and Computation 15 (4) (2005) 411–431.
5. E. Jeřábek, Bases of admissible rules of Łukasiewicz logic, Journal of Logic and Computation 20 (6) (2010) 1149–1163.
6. G. Metcalfe, C. Röthlisberger, Admissibility in De Morgan algebras, submitted.
7. J. S. Olson, J. G. Raftery, C. J. V. Alten, Structural completeness in substructural logics, Logic Journal of the IGPL 16 (5) (2008) 453–495.
8. V. Rybakov, Admissibility of Logical Inference Rules, Elsevier, 1997.
9. D. J. Shoesmith,T. J. Smiley, Multiple Conclusion Logic, Cambridge University Press, 1978.
10. P. Whitman, Free Lattices, Annals of Mathematics 42 (1941) 325–329.

Enrichable Elements in Heyting algebras

Alexei Y. Muravitsky
Louisiana Scholars' College
Northwestern State University
alexeim@nsula.edu

1 Logic KM and KM-algebras

In what follows we discuss an algebraic aspect of logic **KM**. This logic was formulated as calculus $I^{\triangle}$ by A.V. Kuznetsov in [3, 4], but it had appeared earlier in [5], (see a footnote on p. 224) as the variety of $\triangle$-pseudoboolean algebras, i.e., Heyting algebras enriched with a unary operation $\triangle$. Namely from that observation one could see a possibility of an embedding of **KM** into provability logic **GL**, which was confirmed and further developed in [7, 9].

We define **KM** in the monomodal propositional language with the infinite set of propositional variables $p, q, \ldots$ (with or without indices) and the connectives: $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (conditional), $\neg$ (negation) – *assertoric connectives* – and $\Box$ (necessity, if you wish). Arbitrary formulas of the assertoric fragment of this language are denoted by letters $A, B, \ldots$.

Then, **KM** is axiomatized via the two rules of inference, substitution and detachment (*modus ponens*), the axioms of intuitionistic propositional calculus, **Int**, and the following modal formulas:

$$\begin{array}{ll} (a) & p \rightarrow \Box p, \\ (b) & (\Box p \rightarrow p) \rightarrow p, \\ (c) & \Box p \rightarrow (q \vee (q \rightarrow p)). \end{array}$$

In [4], Kuznetsov formulated the theorem:

For any assertoric formulas A and B,

$$\mathbf{Int} + A \vdash B \;\Leftrightarrow\; \mathbf{KM} + A \vdash B. \qquad (\textit{Kuznetsov's Theorem})$$

Although a sketch of the proof was given in the Kuznetsov's paper, a detailed proof of this theorem has been remaining unknown. Also, there remains open the direct proof of the following corollary of the theorem, which is actually equivalent to the theorem.

> *Any Heyting algebra is embedded into the Heyting reduct of such a* **KM**-*algebra that the latter Heyting algebra and the initial one generate the same variety.* (Cf. [4], Corollary 1.)

It is clear that the last statement consists of the two parts: 1) the first part states that given a Heyting algebra $\mathfrak{A}$, there are a **KM**-algebra $\mathfrak{A}'$ and a Heyting embedding of the former into the latter; 2) the second part claims that that the former algebra and the Heyting reduct of the latter generate individually the same variety.

Although the first part, regardless of the second, had been proved long ago in [8], it has still been unclear, whether the construction defined there is suitable for the second part. However, this construction in conjunction with a property explained below allows one to arrive at the following conclusion: Given a Heyting algebra $\mathfrak{A}$, there is, in a sense, a "largest" and "smallest" algebra, both of which contain $\mathfrak{A}$ as their subalgebra and both have a designated element of $\mathfrak{A}$ enriched in the sense of modal **KM**-axioms or the identities $(a)-(b)$ below.

Definition 1.1 [1] *An algebra* $(\mathcal{A}, \wedge, \vee, \to \neg, \mathbf{0}, \mathbf{1}, \Box)$ *is called a* **KM-*algebra*** *if its assertoric reduct* $\mathfrak{A} = (\mathcal{A}, \wedge, \vee, \to \neg, \mathbf{0}, \mathbf{1})$ *is a Heyting algebra and a unary operation* $\Box$ *is subject to the following inequalities:*

$$\begin{array}{ll} (a) & x \leq \Box x, \\ (b) & \Box x \to x \leq x, \\ (c) & \Box x \leq y \vee (y \to x). \end{array}$$

A **KM**-*algebra is called a* **KM**-*expansion of its assertoric* (*or Heyting*) *reduct.*

Given two algebras $\mathfrak{A}$ and $\mathfrak{B}$ of the same type, we write $\mathfrak{A} \preccurlyeq \mathfrak{B}$ if $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$. We write $\mathfrak{A} \precsim \mathfrak{B}$ to say that $\mathfrak{A}$, not necessarily an algebra, is a relative subalgebra of algebra $\mathfrak{B}$ in the sense of [2].

2 Relation $\mathcal{E}$, enriched elements

Definition 2.1 ($\mathcal{E}$-pair, enriched element, relation $\mathcal{E}$) *Let* $\mathfrak{A}$ *be a Heyting algebra. Given two elements* $a, a^* \in \mathfrak{A}$*, the pair* (a, a^*) *is called an* $\mathcal{E}$*-pair of/in* $\mathfrak{A}$*, if the following holds:*

$$\begin{array}{ll} (1) & a \leq a^*; \\ (2) & a^* \to a = a; \\ (3) & a^* \leq x \vee (x \to a), \ \textit{for all } x \in \mathfrak{A}. \end{array}$$

An element $a \in \mathfrak{A}$ *is called enriched by an element* $a^* \in \mathfrak{A}$ *if* (a, a^*) *is an* $\mathcal{E}$*-pair in* $\mathfrak{A}$*.*

Then, we define

$$\mathcal{E} = \{(a, a^*) \mid (a, a^*) \textit{ is an } \mathcal{E}\textit{-pair in } \mathfrak{A}\}.$$

Where necessary, we will write $\mathcal{E}_{\mathfrak{A}}$ *to emphasize that relation* $\mathcal{E}$ *is associated with algebra* $\mathfrak{A}$*.*

[1] The term **KM**-*algebra* is due Esakia [1]. It replaces the old term, $\triangle$-*pseudoboolean algebra*, [6].

We note that for any Heyting algebra, its relation $\mathcal{E}$ is never empty, for $(\mathbf{1}, \mathbf{1})$ is an $\mathcal{E}$-pair. Also, if ω is the pre-top element of a subdirectly irreducible algebra, then $(\omega, \mathbf{1})$ is an $\mathcal{E}$-pair of this algebra.

It follows from [8] that an element $a \in \mathfrak{A}$ can be enriched by only one, if any, element $a^* \in \mathfrak{A}$.

We will be focusing on a possibility to enrich only one element of a Heyting algebra. For this purpose, a given Heyting algebra $\mathfrak{A}$, we select an element $a \in \mathfrak{A}$ and introduce it into the signature as a 0-ary operation τ. Thus we will be working with algebras $(\mathfrak{A}, \tau)$, where $\mathfrak{A}$ is a Heyting algebra.

3 Main results

Definition 3.1 *We say that algebra* $(\mathfrak{A}, \tau)$ *is packed in* $(\mathfrak{B}, \tau)$ *if* $(\mathfrak{A}, \tau) \precsim (\mathfrak{B}, \tau)$*, the element* τ *is enriched by an element* τ^* *in* $(\mathfrak{B}, \tau)$ *and algebra* $\mathfrak{B}$ *is generated by* $\mathfrak{A} \cup \{\tau^*\}$*.*

We use constructions of [8] and [2], §28, to prove the following.

Theorem 3.1 *Given an algebra* $(\mathfrak{A}, \tau)$*, there is an algebra* $(\mathfrak{A}_0, \tau)$ *such that* $(\mathfrak{A}, \tau)$ *is packed in the former and if* $(\mathfrak{A}, \tau)$ *is packed in* $(\mathfrak{B}, \tau)$*, then there is a surjective homomorphism* $\phi : (\mathfrak{A}_0, \tau) \to (\mathfrak{B}, \tau)$ *which is an isomorphism on the elements of* $\mathfrak{A}$*.*

Now consider the following abstract class of algebras:

$$\mathcal{S} = \{(\mathfrak{A}_i, \tau) \mid (\mathfrak{A}, \tau) \text{ is packed in } (\mathfrak{A}_i, \tau)\}$$

and define the following relation on $\mathcal{S}$:

$(\mathfrak{A}_i, \tau) \leq (\mathfrak{A}_j, \tau) \quad \Leftrightarrow \quad$ there is a surjective homomorphism of the latter onto the former, which is determined by the homomorphisms $\phi_k : (\mathfrak{A}_0, \tau) \to (\mathfrak{A}_k, \tau)$, $k \in \{i, j\}$, of Theorem 3.1.

Theorem 3.2 $(\mathcal{S}, \leq)$ *is a well-founded partially ordered set with the top element* $(\mathfrak{A}_0, \tau)$*.*

We conclude with the following comments.

Suppose algebra $(\mathfrak{A}, \tau)$ is packed in $(\mathfrak{B}, \tau)$ so that (τ, τ^*) is an $\mathcal{E}$-pair in $\mathfrak{B}$. Now if a formula A is invalid on $\mathfrak{B}$, a substitution instance of it, A', takes a value unequal to $\mathbf{1}_{\mathfrak{B}}$ $(= \mathbf{1}_{\mathfrak{A}})$, when we assign elements $a_1, \ldots, a_k \in \mathfrak{A}$ and τ^* to the variables of A', i.e., $A'[a_1, \ldots, a_k, \tau^*] \neq \mathbf{1}_{\mathfrak{A}}$. For the second part of the corollary above, we need to refute A' on $\mathfrak{A}$, while dealing with a particular algebra $(\mathfrak{C}, \tau)$ which $(\mathfrak{A}, \tau)$ is also packed in, for the direct limit construction of [8], where $(\mathfrak{C}, \tau)$ is a resulting algebra obtained in the first step of a transfinite induction, allows to vary τ to obtain an embedding of any Heyting algebra into an enrichable one.

Theorem 3.1 shows that if we are able to shift down a refutation of A' from $(\mathfrak{A}_0, \tau)$ to $(\mathfrak{A}, \tau)$ then all first components in $\mathcal{S}$ generate the same variety as algebra $\mathfrak{A}$ does. Otherwise, as Theorem 3.2 witnesses, it would be difficult to find a needed algebra in $\mathcal{S}$ where the refutation can be conducted with embedding of this algebra into a fully enrichable one. Here the problem is that $(\mathcal{S}, \leq)$ is not necessarily a downward direct family. So there may exist infinitely many minimal algebras with respect to $\leq$, in each of which $(\mathfrak{A}, \tau)$ is packed.

References

[1] Leo Esakia, *The modalized Heyting calculus: a conservative modal extension of the intuitionistic logic*, J. Appl. Non-Classical Logics **16** (2006), no. 3-4, 349–366. MR 2300566 (2007m:03065)

[2] George Grätzer, *Universal algebra*, second ed., Springer-Verlag, New York, 1979. MR 538623 (80g:08001)

[3] A. V. Kuznetsov, *Proof-intuitionistic propositional calculus*, Izv. Akad. Nauk Moldav. SSR Ser. Fiz.-Tekhn. Mat. Nauk (1985), no. 1, 49–50, 79. MR 819845 (87d:03026)

[4] ______, *The proof-intuitionistic propositional calculus*, Dokl. Akad. Nauk SSSR **283** (1985), no. 1, 27–30. MR 796954 (86m:03021)

[5] A. V. Kuznetsov and A. Yu. Muravitsky, *Provability as modality*, Current Problems in Logic and Methodology of Science (Russian), "Naukova Dumka", Kiev, 1980, pp. 193–230.

[6] ______, *On superintuitionistic logics as fragments of proof logic extensions*, Studia Logica **45** (1986), no. 1, 77–99. MR 877303 (88d:03052)

[7] A. Yu. Muravitskiĭ, *Correspondence of proof-intuitionistic logic extensions to proof-logic extensions*, Dokl. Akad. Nauk SSSR **281** (1985), no. 4, 789–793. MR 785627 (87e:03036)

[8] ______, *Algebraic proof of the separation property for the proof-intuitionistic calculus*, Mat. Sb. (N.S.) **131(173)** (1986), no. 3, 403–412, 415. MR 881921 (88f:03009)

[9] ______, *Correspondence of proof-intuitionistic logic extensions to proof-logic extensions*, Trudy Inst. Mat. (Novosibirsk) **12** (1989), no. Mat. Logika i Algoritm. Probl., 104–120, 189. MR 1047657 (91i:03041)

Homogeneous orthocomplete effect algebras are covered by MV-algebras

JOSEF NIEDERLE and JAN PASEKA*

Department of Mathematics and Statistics
Faculty of Science
Masaryk University
Kotlářská 2
CZ-611 37 Brno
Czech Republic

e-mail: paseka@math.muni.cz

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Abstract

Generalizations of Boolean algebras as carriers of probability measures are (lattice) effect algebras. They are a common generalization of MV-algebras and orthomodular lattices ([1], [2], [3], [8]). In the present paper, we continue the study of homogeneous effect algebras started in [6]. This class of effect algebras includes orthoalgebras, lattice ordered effect algebras and effect algebras satisfying the Riesz decomposition property.

In [6] it was proved that every homogeneous effect algebra is a union of its blocks, which are defined as maximal sub-effect algebras satisfying the Riesz decomposition property. In [9] Tkadlec introduced the so-called property (W+) as a common generalization of orthocomplete and lattice effect algebras.

The aim of our paper is to show that every block of an Archimedean homogeneous effect algebra satisfying the property (W+) is lattice ordered. Therefore, any Archimedean homogeneous effect algebra satisfying the property (W+) is covered by MV-algebras. As a corollary, this yields that every block of a homogeneous orthocomplete effect algebra is lattice ordered.

As a by-product of our study we extend the results on sharp and meager elements of [7] into the realm of Archimedean homogeneous effect algebras satisfying the property (W+).

List of selected results and definitions

Definition 1. A partial algebra $(E; \oplus, 0, 1)$ is called an *effect algebra* if 0, 1 are two distinct elements, called the *zero* and the *unit* element, and $\oplus$ is a partially defined binary operation called the *orthosummation* on E which satisfy the following conditions for any $x, y, z \in E$:

(Ei) $x \oplus y = y \oplus x$ if $x \oplus y$ is defined,

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(Eii) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if one side is defined,

(Eiii) for every $x \in E$ there exists a unique $y \in E$ such that $x \oplus y = 1$ (we put $x' = y$),

(Eiv) if $1 \oplus x$ is defined then $x = 0$.

A subset $Q \subseteq E$ is called a *sub-effect algebra* of E if

(i) $1 \in Q$

(ii) if out of elements $x, y, z \in E$ with $x \oplus y = z$ two are in Q, then $x, y, z \in Q$.

$(E; \oplus, 0, 1)$ is called an *orthoalgebra* if $x \oplus x$ exists implies that $x = 0$.

An effect algebra E satisfies the *Riesz decomposition property* (or RDP) if, for all $u, v_1, v_2 \in E$ such that $u \leq v_1 \oplus v_2$, there are u_1, u_2 such that $u_1 \leq v_1, u_2 \leq v_2$ and $u = u_1 \oplus u_2$.

An effect algebra E is called *homogeneous* if, for all $u, v_1, v_2 \in E$ such that $u \leq v_1 \oplus v_2 \leq u'$, there are u_1, u_2 such that $u_1 \leq v_1$, $u_2 \leq v_2$ and $u = u_1 \oplus u_2$ (see [6]).

A subset B of E is called a *block* of E if B is a maximal sub-effect algebra of E with the Riesz decomposition property.

An element x of an effect algebra E is called

1. *sharp* if $x \wedge x' = 0$. The set $\mathrm{S}(E) = \{x \in E \mid x \wedge x' = 0\}$ is called a *set of all sharp elements* of E (see [5]).
2. *principal*, if $y \oplus z \leq x$ for every $y, z \in E$ such that $y, z \leq x$ and $y \oplus z$ exists.
3. *central*, if x and x' are principal and, for every $y \in E$ there are $y_1, y_2 \in E$ such that $y_1 \leq x, y_2 \leq x'$, and $y = y_1 \oplus y_2$ (see [4]). The *center* $\mathrm{C}(E)$ of E is the set of all central elements of E.

In what follows set (see [7])

$$\mathrm{M}(E) = \{x \in E \mid \text{ if } v \in \mathrm{S}(E) \text{ satisfies } v \leq x \text{ then } v = 0\}.$$

We also define

$$\mathrm{HM}(E) = \{x \in E \mid \text{ there is } y \in E \text{ such that } x \leq y \text{ and } x \leq y'\}$$

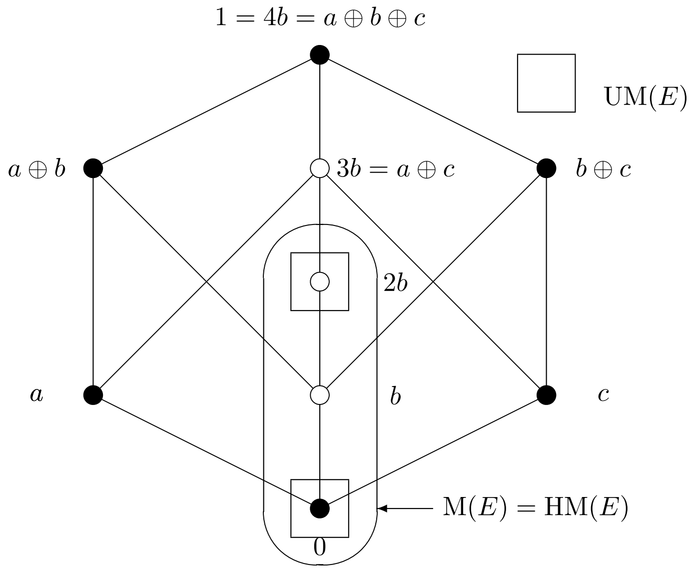
and

$$\mathrm{UM}(E) = \{x \in E \mid \text{ for every } y \in \mathrm{S}(E) \text{ such that } x \leq y \text{ it holds } x \leq y \ominus x\}.$$

An element $x \in \mathrm{HM}(E)$ is called *hypermeager*, an element $x \in \mathrm{UM}(E)$ is called *ultrameager*.

Lemma 2. *Let E be an effect algebra. Then $\mathrm{UM}(E) \subseteq \mathrm{HM}(E) \subseteq \mathrm{M}(E)$. Moreover, for all $x \in E$, $x \in \mathrm{HM}(E)$ iff $x \oplus x$ exists and, for all $y \in \mathrm{M}(E)$, $y \neq 0$ there is $h \in \mathrm{HM}(E)$, $h \neq 0$ such that $h \leq y$.*

Example 3. In the following non-homogeneous finite effect algebra, $\mathrm{M}(E) = \mathrm{HM}(E) \neq \mathrm{UM}(E)$. Sharp elements are denoted in black. One can easily check that E is a sub-effect algebra of the MV-effect algebra $[0,1] \times [0,1]$ such that $a \mapsto (\frac{3}{4}, 0), b \mapsto (\frac{1}{4}, \frac{1}{4}), c \mapsto (0, \frac{3}{4})$. Moreover, since $a \oplus c \notin \mathrm{S}(E)$ we obtain that $\mathrm{S}(E)$ is not a sub-effect algebra of E.



Definition 4. For an element x of an effect algebra E we write $\mathrm{ord}(x) = \infty$ if $nx = x \oplus x \oplus \cdots \oplus x$ (n-times) exists for every positive integer n and we write $\mathrm{ord}(x) = n_x$ if n_x is the greatest positive integer such that $n_x x$ exists in E. An effect algebra E is *Archimedean* if $\mathrm{ord}(x) < \infty$ for all $x \in E$.

We say that a finite system $F = (x_k)_{k=1}^n$ of not necessarily different elements of an effect algebra E is *orthogonal* if $x_1 \oplus x_2 \oplus \cdots \oplus x_n$ (written $\bigoplus_{k=1}^{n} x_k$ or $\bigoplus F$) exists in E. Here we define $x_1 \oplus x_2 \oplus \cdots \oplus x_n = (x_1 \oplus x_2 \oplus \cdots \oplus x_{n-1}) \oplus x_n$ supposing that $\bigoplus_{k=1}^{n-1} x_k$ is defined and $(\bigoplus_{k=1}^{n-1} x_k) \oplus x_n$ exists. We also define $\bigoplus \emptyset = 0$. An arbitrary system $G = (x_\kappa)_{\kappa \in H}$ of not necessarily different elements of E is called *orthogonal* if $\bigoplus K$ exists for every finite $K \subseteq G$. We say that for a orthogonal system $G = (x_\kappa)_{\kappa \in H}$ the element $\bigoplus G$ exists iff $\bigvee\{\bigoplus K \mid K \subseteq G \text{ is finite}\}$ exists in E and then we put $\bigoplus G = \bigvee\{\bigoplus K \mid K \subseteq G \text{ is finite}\}$. We say that $\bigoplus G$ is the *orthogonal sum* of G and G is *orthosummable.* (Here we write $G_1 \subseteq G$ iff there is $H_1 \subseteq H$ such that $G_1 = (x_\kappa)_{\kappa \in H_1}$). We denote $G^\oplus := \{\bigoplus K \mid K \subseteq G \text{ is finite}\}$.

E is called *orthocomplete* if every orthogonal system is orthosummable. *E fulfills the condition (W+)* [9] if for each orthogonal subset $A \subseteq E$ and each two upper bounds u, v of $A^\oplus$ there exists an upper bound w of $A^\oplus$ below u, v.

Every orthocomplete effect algebra is Archimedean.

Statement 5. [9, Theorem 2.2] *Lattice effect algebras and orthocomplete effect algebras fulfill the condition (W+).*

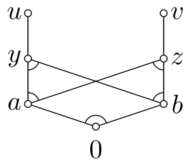
Proposition 6. *Let E be an Archimedean effect algebra fulfilling the condition (W+). Then every meager element of E is the orthosum of a system of hypermeager elements.*

Lemma 7. *Let E be an Archimedean effect algebra fulfilling the condition (W+), let $u, v \in E$, and let a, b be two maximal lower bounds of u, v. There exist elements y, z for which $y \leq u$, $z \leq v$, a, b are maximal lower bounds of y, z and y, z are minimal upper bounds of a, b.*

Lemma 8 (Shifting lemma)**.** *Let E be an Archimedean effect algebra fulfilling the condition (W+), let $u, v \in E$, and let a_1, b_1 be two maximal lower bounds of u, v. There exist elements y, z and two maximal lower bounds a, b of y, z for which $y \leq u$,*

$z \leq v$, $a \leq a_1$, $b \leq b_1$, $a \wedge b = 0$, a, b *are maximal lower bounds of* y, z *and* y, z *are minimal upper bounds of* a, b. *Furthemore,* $(y \ominus a) \wedge (z \ominus a) = 0$, $(y \ominus b) \wedge (z \ominus b) = 0$, $(y \ominus a) \wedge (y \ominus b) = 0$, $(z \ominus a) \wedge (z \ominus b) = 0$.

The Shifting lemma provides the following *minimax structure*.



Proposition 9. *Let E be an Archimedean homogeneous effect algebra fulfilling the condition (W+). Every two hypermeager elements u, v possess $u \wedge v$.*

Proposition 10. *Let E be an Archimedean homogeneous effect algebra fulfilling the condition (W+). For every orthogonal elements u, v, $u \wedge v$ and $u \vee_{[0, u \oplus v]} v$ exist and $[0, u \wedge v] \subseteq B$ for every block B containing u or v.*

Corollary 11. *Let E be an Archimedean homogeneous effect algebra fulfilling the condition (W+). For every element u, $u \wedge u'$ and $u \vee u'$ exist and $[0, u \wedge u'] \subseteq B$ for every block B containing u.*

Corollary 12. *Let E be an Archimedean homogeneous effect algebra fulfilling the condition (W+). For any block B and every elements $u, v \in B$ for which $u \wedge_B v = 0$, $u \wedge v = 0$.*

Theorem 13. *Let E be an Archimedean homogeneous effect algebra fulfilling the condition (W+). Then every block in E is a lattice and E can be covered by MV-algebras, i.e., E is a union of maximal sub-effect algebras of E with the Riesz decomposition property that are lattice ordered.*

Corollary 14. *Let E be an orthocomplete homogeneous effect algebra. Then E can be covered by MV-algebras.*

References

[1] C.C. Chang, Algebraic analysis of many-valued logics, Trans. Amer. Math. Soc. 88 (1958) 467–490.

[2] A. Dvurečenskij, S. Pulmannová: New Trends in Quantum Structures, Kluwer Acad. Publ., Dordrecht/Ister Science, Bratislava 2000.

[3] D.J. Foulis, M.K. Bennett, Effect algebras and unsharp quantum logics, Found. Phys. 24 (1994), 1325–1346.

[4] R. J. Greechie, D. J. Foulis, S. Pulmannová, The center of an effect algebra, Order 12 (1995), 91–106.

[5] S. P. Gudder, Sharply dominating effect algebras, Tatra Mt. Math. Publ. 15 (1998), 23–30.

[6] G. Jenča, Blocks of homogeneous effect algebras, Bulletin of the Australian Mathematical Society 64 (2001), 81–98.

[7] G. Jenča, Sharp and Meager Elements in Orthocomplete Homogeneous Effect Algebras, Order 27 (2010), 41-61.

[8] F. Kôpka, F. Chovanec, D-Posets, Math. Slovaca 44 (1994), 21–34.

[9] J. Tkadlec, Common generalizations of orthocomplete and lattice effect algebras, Inter. J. Theor. Phys., 49 (2010), 3279–3285.

Frobenius Algebras and Classical Proof Nets

François Lamarche and Novak Novaković

LORIA and INRIA Nancy – Grand Est

The semantics of proofs for classical logic is a very recent discipline; the construction of proofs semantics that are completely faithful to the natural symmetries of classical logic is even more recent. In this paper we present a theory of proof nets which is related to those in [LS05,Hyl04,FP05], but which differs from them in its ability to take account of *resources*, in the sense of linear logic. It also has the interesting property (like [Hyl04]) of being based on a topological foundation.

This work originated as an investigation in the denotational semantics of classical logic [LN09], furthering the work in [Lam07]. As it often happens here, it involved the construction of bialgebras, in this particular case in the category of posets and bimodules. The fact that these bialgebras were actually Frobenius algebras was noticed, but it took some time for the extreme interest of this property to sink in.

Definition 1 (Frobenius algebra). *Let* $(\mathbf{C}, \otimes, \mathbf{1})$ *be a symmetric monoidal category (SMC), and* A *an object of it. A* Frobenius algebra *is a sextuple* $(A, \Delta, \amalg, \nabla, \amalg)$ *where* $(A, \nabla, \amalg)$ *is a commutative monoid,* $(A, \Delta, \amalg)$ *a co-commutative comonoid, where the following diagram commutes:*

$$\begin{array}{ccccc}
A \otimes A & = & A \otimes A & = & A \otimes A \\
\downarrow{\scriptstyle \Delta\otimes \mathrm{Id}} & & \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle \mathrm{Id}\otimes\Delta} \\
A \otimes A \otimes A & & A & & A \otimes A \otimes A \\
\downarrow{\scriptstyle \mathrm{Id}\otimes\nabla} & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \nabla\otimes \mathrm{Id}} \\
A \otimes A & = & A \otimes A & = & A \otimes A
\end{array}$$

A Frobenius algebra is thin *if* $\amalg \circ \amalg$ *is the identity.*

The following is well-known.

Proposition 1. *The tensor of two Frobenius algebras is also a Frobenius algebra, where the monoid and comonoid operations are defined as usual in an SMC. It is thin if both factors are.*

Definition 2. *A Frobenius category* $\mathbf{C}$ *is a symmetrical monoidal category where every object* A *is equipped with a* thin *Frobenius algebra structure* $(A, \nabla_A, \amalg_A, \Delta_A, \amalg)$ *and such that the algebra on the tensor of two objects is the usual tensor algebra, as above.*

Frobenius algebras have gained a lot of attention after they were found to be closely related to 2-dimensional Topologica Quantum Field Theories (TQFTs). The main result was achieved by several people independently [Dij89,Koc04], and can be stated as follows. We present a slightly modified version of the standard result, which better fits our purposes and is an easy corollary of it.

Theorem 1. *The free Frobenius category* $\mathbf{F}$ *on one object generator is equivalent to the two following categories.*

1. *Take finite disjoint unions of* m *circles as an object* m. *A map* $m \to n$ *is a Riemann surface (with boundary) whose boundary is the disjoint sum* $m+n$ *(and would be orientable if the circles were extended to discs), such that every connected component has a nonempty boundary, where two surfaces are identified modulo homeomorphism. Composition of two maps* $m \to n, n \to p$ *is gluing, forgetting the boundaries in the middle, and dropping the components that do not touch the resulting boundary* $m + p$.
2. *Take finite sets* $[m] = \{0, 1, \ldots, m-1\}$ *as objects, seen as discrete topological spaces. A map* $[m] \to [n]$ *is a topological graph* G *(i.e. a CW-complex of dimension one), equipped with an injective function* $[m + n] \to G$ *such that every connected components of* G *is in the image of that function, with two graphs being identified if they are equivalent modulo homology. Composition is also gluing and dropping the components that are left out of the resulting set of endpoints.*

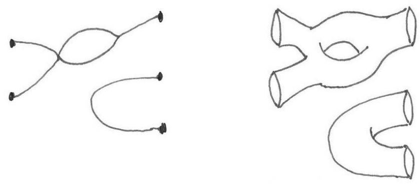


Fig 1. A map $2 \to 3$ shown in the two equivalent characterizations of the free Frobenius category. Objects are seen as distinguished end-points in the left or as circles to the right. One of the connected components has genus 1, the other 0. In both cases the map is determined by grouping of the atoms in a partition and an assignment of genera to the classes of the partition.

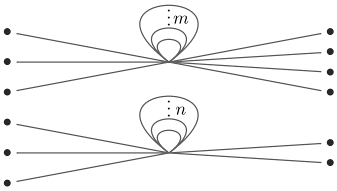
Since we are dealing with the universal algebra of categories, a free Frobenius category is defined only up to equivalence of categories, with the standard universal property associated to that situation. The two characterizations in Theorem 1 happen to be skeletal categories and are isomorphic. Our nonstandard notion of Frobenius category requires thinness; maps in the standard, non-thin free Frobenius category can contain several "floating" components that do not touch the border.

Since homology is much more technical than homotopy, we prefer to replace the second result above with:

2'. *Objects are sets of the form* $[m]$. *A map* $[m] \to [n]$ *is a topological graph* G *equipped with an injective function* $[m + n] \to G$ *that touches all connected components of* G, *where two such things are identified if they are homotopy equivalent in the co-slice* *category* $(m+n)/Top$*, where homotopies are defined to be constant on the base* $[m+n]$.

This allows for a treatment which is at the same time well-formalized and accessible to many more readers.

Theorem 2. *Every map in* $\mathbf{F}$ *can be represented by a graph* G *of the following form, where every connected component is a "star" whose central node has* n *loops attached to it, with* $n \geqslant 0$.



This prompts the following definition

Definition 3 (Linking). *We define a* linking *to be a triple*

$$P = (P, \mathcal{C}omp_P, \mathcal{G}en_P)$$

where

- *P is a finite set*
- *$\mathcal{C}omp_P$ is the set of classes of a* partition *of the set P. Its elements are called* components.
- *the function $\mathcal{G}en_P : \mathcal{C}omp_P \to \mathbb{N}$ (called* genus*) assigns a natural number to each component in $\mathcal{C}omp_P$*

Notice the abuse of notation, where a single letter P can be the full thing above or just its underlying set.

It should be obvious that a map $m \to n$ in $\mathbf{F}$ can be described as a linking on the set $m + n$. Naturally a formal definition of composition in terms of linkings is a bit trickier.

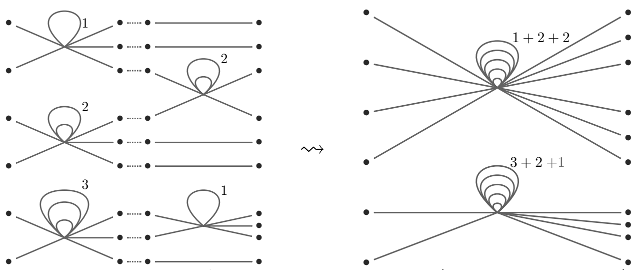


Fig. 2. Maps in a free Frobenius category (drawn horizontally) seen as topological graphs with object generators for nodes, and the bouquet of circles determining the genus. Composition of the maps amounts to glueing of graphs along nodes, and is determined by the homotopy type of the new graph as depicted.

Proposition 2. *The category* $\mathbf{F}$ *is compact-closed, the dual of an object being itself.*

This is easy to see, since given a map $m \to n$ stuff in m can be transfered to the right side by a purely formal manipulation, and vice-versa. More generally, *any* Frobenius category is compact-closed, but a proof of this requires some real algebra.

The relevance of the "Frobenius equations" for proof theory is due to the fact that they address the contraction-against-contraction case in cut elimination, seen for example in the proof to the right in Figure 3.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \overline{a}, a}\,Ax \quad \dfrac{}{\vdash \overline{a}, a}\,Ax}{\vdash \overline{a}, a, \overline{a}, a}\,Mix}{\vdash \overline{a}, \overline{a}, a}\,Contr \qquad \dfrac{\dfrac{\dfrac{}{\vdash \overline{a}, a}\,Ax \quad \dfrac{}{\vdash \overline{a}, a}\,Ax}{\vdash \overline{a}, a, \overline{a}, a}\,Mix}{\vdash \overline{a}, a, a}\,Contr}{\vdash \overline{a}, \overline{a}, a, a}\,Cut \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \overline{a}, a}\,Ax \quad \dfrac{}{\vdash \overline{a}, a}\,Ax}{\vdash \overline{a}, a, \overline{a}, a}\,Mix}{\vdash \overline{a}, a, \overline{a}}\,Contr \qquad \dfrac{}{\vdash \overline{a}, a}\,Ax}{\vdash \overline{a}, a, \overline{a}, \overline{a}, a}\,Mix}{\vdash \overline{a}, a, \overline{a}, a}\,Contr$$

Fig. 3. Two proofs identified by Frobenius equations

We introduce a standard language for classical propositional logic, with atoms $a, b, c, \ldots$, negatoms $\overline{a}, \overline{b}, \overline{c}, \ldots$ and conjunction $\wedge$, disjunction $\vee$. We call something which is either an atom ot a negatom a *literal*. Negation of a compound formula is defined by de Morgan duality. Sequents are defined as usual, and given a formula A or a sequent Γ we denote by $\mathcal{L}it(A)$, $\mathcal{L}it(\Gamma)$ their sets of occurences of literals.

Definition 4 (F-prenet). *We define an* F-prenet *to be a pair*

$$P \rhd \Gamma$$

consisting of a sequent Γ, and a linking $(P, \mathcal{C}omp_P, \mathcal{G}en_P)$ where the underlying set P is $\mathcal{L}it(\Gamma)$, and every class in $\mathcal{C}omp_P$ contains only atoms of the same type and their negation.

When we say that P "is" the set of literal occurences of Γ, we mean actually that P is an arbitrary set, equipped with a bijection with the actual literal occurences in Γ. The point is that this bijection never has to be made explicit in practice, while working directly with atom occurences would force ugly contortions.

Several deductive systems can be used with F-prenets. The first one is just the ordinary one-sided sequent calculus for classical logic, with the Mix rule (of linear logic) added. It is presented in full in [LS05], under the name CL. In general, a sequent calculus can be used to define a theory of proof nets is every n-ary introduction rule of the calculus

$$\frac{\vdash \Gamma_1 \quad \vdash \Gamma_2 \quad \cdots \quad \vdash \Gamma_n}{\vdash \Gamma}$$

can be transformed into a family of n morphisms $P_i \rhd \Gamma_i \to Q \rhd \Gamma$ in the following *syntactic category.*

Definition 5 (Syntactic Category). *Let $\mathcal{F}Synt$ have F-prenets for objects, where a map*

$$f : P \rhd \Gamma \to Q \rhd \Delta$$

is given by an ordinary function on the underlying set of literals

$$f : P \to Q \qquad \bigl(= \mathcal{L}it(\Gamma) \to \mathcal{L}it(\Delta)\bigr)$$

such that

1. *for every formula* A*, f maps $\mathcal{L}it(A)$ to a subset of $\mathcal{L}it(\Delta)$ which defines a subformula of a formula in Δ, while preserving the syntactic left-right order on literals.*

2. *for every $C \in \mathcal{C}omp_P$, one has that $f(C) \subseteq \mathcal{L}it(\Delta)$ is contained in a component $C' \in \mathcal{C}omp_Q$, with $\mathcal{G}en_P(C) \leq \mathcal{G}en_Q(C')$.*

The procedure to obtain an F-prenet $P \triangleright \Gamma$ from a proof of a sequent $\vdash \Gamma$ is absolutely straightforward. The cases that are worth mentioning specifically are Weakening and Contraction. Assuming we have constructed $P \triangleright \Gamma$ from a proof, then adding the formula A through weakening gives us a linking on the disjoint union $P \uplus \mathcal{L}it(\mathrm{A})$ where every added component is a singleton with associated genus 0. For contraction, if the two visible occurences of A in $P \triangleright \Gamma, \mathrm{A}, \mathrm{A}$ are contracted, we get an F-prenet $P(\mathrm{A} \curlyvee \mathrm{A}) \triangleright \Gamma, \mathrm{A}$ by connecting the ith literal of the first instance of A and the ith literal in the second instance to a single "terminal", where i ranges over the number of literals in A.

This $(- \curlyvee -)$ operation can be iterated, and can be applied to subformulas and subsequents as well as formulas. In what follows we use superscripts to disambiguate occurences when we feel it is useful.

Definition 6. *In the category $\mathcal{F}Synt$, we define the families of cospans Mix and $\wedge$ to be*

$$\begin{array}{ccc} P_l \triangleright \Gamma & & P_r \triangleright \Gamma \\ & \searrow^{Mix:l} \quad {}^{Mix:r}\swarrow & \\ & P_l \uplus P_r(\Gamma \curlyvee \Gamma) \triangleright \Gamma & \end{array}$$

and

$$\begin{array}{ccc} P_l \triangleright \Gamma, \mathrm{A}^1 \wedge \mathrm{B}^1, \mathrm{A}^2 & & P_r \triangleright \mathrm{B}^2, \mathrm{A}^3 \wedge \mathrm{B}^3, \Gamma \\ & \searrow^{\wedge:l} \quad {}^{\wedge:r}\swarrow & \\ & Q \triangleright \Gamma, \mathrm{A} \wedge \mathrm{B} & \end{array}$$

where Q is $P_l \uplus P_r\big(\Gamma \curlyvee \Gamma, (\mathrm{A}^1 \curlyvee \mathrm{A}^2) \curlyvee \mathrm{A}^3, (\mathrm{B}^1 \curlyvee \mathrm{B}^2) \curlyvee \mathrm{B}^3\big)$.

Definition 7. *An* anodyne map $P \triangleright \Gamma \xrightarrow{\ominus} Q \triangleright \Delta$ *is a syntactic map that can be decomposed*

$$P \triangleright \Gamma \xrightarrow{\sim} Q \triangleright \Delta_1 \xrightarrow{\vee} \cdots \xrightarrow{\vee} Q \triangleright \Delta_n = \Delta$$

as an isomorphism followed by a sequence of $\vee$-introduction maps (which do not affect the linking, only the sequent).

There is an important anodyne map, which corresponds to the removal of all outer disjuctions: We write

$$[P \triangleright \Gamma] \xrightarrow{\boxminus} P \triangleright \Gamma$$

to denote the anodyne map whose domain is the sequent where all outer disjunctions have been removed.

Definition 8 (Correct F-nets). *An F-prenet $P \triangleright \Gamma$ is a* CL-correct F-net*, (or simply an* F-net*) if it is at the root of a* correctness diagram *$\mathcal{T} \to \mathcal{F}Synt$, meaning a diagram for which:*

1. *$\mathcal{T}$ is a poset which is an inverted tree (i.e. the root is the top, the leaves are minimal), with $P \triangleright \Gamma$ at its root;*
2. *maps of the diagram $\mathcal{T}$ are either anodyne, or belong to a $\wedge$- or Mix-cospan;*
3. *the only branchings are $\wedge$- and Mix-cospans ;*
4. *every leaf of the tree is an F-prenet $Q \triangleright \Delta$ with $\mathcal{C}omp_Q = \{\{\mathrm{a}, \bar{\mathrm{a}}\}, \{x_1\}, \ldots, \{x_m\}\}$ and a map $\mathcal{G}en_Q$ which is 0 everywhere, i.e, an axiom with weakenings.*

This can be strengthened by forcing the anodyne maps always to be $\Box$-maps, and to have an alternation between these and maps from cospans. We show

Theorem 3 (Sequentialization). *Correct F-nets are precisely those F-prenets that come from* CL *without Cut.*

Given a linking P let $|P|$ be stand for the size of its underlying set, $|\mathcal{C}omp_P|$ for the number of components, and $|\mathcal{G}en_P|$ for the sum of all genera in P, i.e. $|\mathcal{G}en_P| = \sum_{C \in \mathcal{C}omp_P} \mathcal{G}en_P(C)$. The following observation is crucial to the proof:

Lemma 1 (Counting axiom links in an F-prenet). *If an F-prenet $P \triangleright \Gamma$ corresponds to a* CL *proof, then*

$$|Ax| = |P| - |\mathcal{C}omp_P| + |\mathcal{G}en_P|,$$

where $|Ax|$ is the number of axioms in the proof (corollary: any correctness diagram for this proof will have the same number of leaves).

This lemma, along with some additional analysis of proofs guarantees finiteness of the search space:

Theorem 4. *Given an F-prenet, its* CL*-correctness (*CL*-sequentializability) can be checked in finite time, i.e. the* CL*-correctness criterion yields a decision procedure for* CL*-correct F-nets.*

We have strong evidence that the procedure is NP-complete, actually.

When Cut comes into play, things change a bit. First of all, we define a *cut formula* to be $\mathrm{A} \diamondsuit \overline{\mathrm{A}}$, where $- \diamondsuit -$ is a new binary connective that is only allowed to appear as a root in a sequent.

Our original goal is to normalize these prenets with cuts by means of composition in **F** (remember it is compact-closed). This use of Frobenius algebras in classical logic is quite different from the one proposed by Hyland [Hyl04]. It more resembles the work in [LS05], where the equivalent to the category **F** there is obtained from an "interaction category" construction [Hyl04, Section 3] on sets and relations, where composition is defined by the means of a trace operator.

Cut elimination defined that way immediately causes problems. Look at the right part of Figure 4. For the resulting F-prenet to come from a proof we need the singleton component to come from a weakening, but this cannot happen according to our interpretation since its genus is > 0.

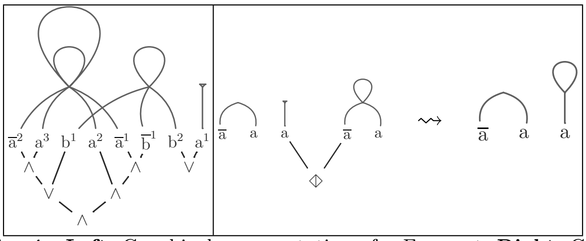


Fig. 4. Left: Graphical representation of a F-prenet. **Right:** Cut elimination performed on two CL-correct F-nets that results in a F-prenet not corresponding to a CL proof.

These issues can be dealt with by changing the deductive system and we define a new sound and complete calculus for classical logic, FL.

$$\frac{}{\vdash \overline{\mathrm{a}}, \mathrm{a}\ ;}\ Ax \qquad \frac{\vdash \Gamma\ ;\Delta}{\vdash \Gamma\ ;\Delta, \mathrm{a}, \mathrm{a}, \dots, \mathrm{a}, \overline{\mathrm{a}}, \overline{\mathrm{a}}, \dots, \overline{\mathrm{a}}}\ MulWeak$$

$$\frac{\vdash \Gamma, \mathrm{A}, \mathrm{B}\ ;\Delta}{\vdash \Gamma, \mathrm{A} \vee \mathrm{B}\ ;\Delta}\ \vee_l \qquad \frac{\vdash \Gamma, \mathrm{A}\ ;\Delta, \mathrm{B}}{\vdash \Gamma, \mathrm{A} \vee \mathrm{B}\ ;\Delta}\ \vee_c \qquad \frac{\vdash \Gamma\ ;\Delta, \mathrm{A}, \mathrm{B}}{\vdash \Gamma\ ;\Delta, \mathrm{A} \vee \mathrm{B}}\ \vee_r$$

$$\frac{\vdash \Gamma_1, \mathrm{A}\ ;\Delta_1 \quad \vdash \mathrm{B}, \Gamma_2\ ;\Delta_2}{\vdash \Gamma_1, \mathrm{A} \wedge \mathrm{B}, \Gamma_2\ ;\Delta_1, \Delta_2}\ \wedge_l \qquad \frac{\vdash \Gamma\ ;\Delta, \mathrm{A}, \mathrm{B}}{\vdash \Gamma\ ;\Delta, \mathrm{A} \wedge \mathrm{B}}\ \wedge_r$$

$$\frac{\vdash \Gamma_1, \mathrm{A}\ ;\Delta_1 \quad \vdash \Gamma_2\ ;\mathrm{B}, \Delta_2}{\vdash \Gamma_2\ ;\mathrm{A} \wedge \mathrm{B}, \Gamma_1, \Delta_1, \Delta_2}\ \wedge_c$$

$$\frac{\vdash \Gamma, \mathrm{A}, \mathrm{A}\ ;\Delta}{\vdash \Gamma, \mathrm{A}\ ;\Delta}\ Contr_l \qquad \frac{\vdash \Gamma\ ;\Delta, \mathrm{A}, \mathrm{A}}{\vdash \Gamma\ ;\Delta, \mathrm{A}}\ Contr_r$$

$$\frac{\vdash \Gamma, \mathrm{A}\ ;\Delta, \mathrm{A}}{\vdash \Gamma, \mathrm{A}\ ;\Delta}\ Contr_c$$

$$\frac{\vdash \Gamma\ ;\Delta_1 \quad \vdash \Delta\ ;\Delta_2}{\vdash \Gamma, \Delta\ ;\Delta_1, \Delta_2}\ Mix$$

$$\frac{\vdash \Gamma, \mathrm{A}\ ;\Delta_1 \quad \vdash \overline{\mathrm{A}}, \Delta\ ;\Delta_2}{\vdash \Gamma, \Delta\ ;\Delta_1, \Delta_2}\ Cut_l \qquad \frac{\vdash \Gamma\ ;\Delta, \mathrm{A}\overline{\mathrm{A}}}{\vdash \Gamma\ ;\Delta}\ Cut_r$$

$$\frac{\vdash \Gamma, \mathrm{A}\ ;\Delta_1 \quad \vdash \Delta\ ;\overline{\mathrm{A}}, \Delta_2}{\vdash \Delta\ ;\Gamma, \Delta_1, \Delta_2}\ Cut_c$$

Fig. 5. System FL.

The purpose of the stoup is to keep track the part that is sure to come from weakening, and also to allow the introduction of arbitrary linking configurations through weakening. This is because *MulWeak* is interpreted by adding to the linking a set $\{\mathrm{a}, \mathrm{a}, \dots, \mathrm{a}, \overline{\mathrm{a}}, \overline{\mathrm{a}}, \dots, \overline{\mathrm{a}}\}$, which contains a *single* component of genus zero. The definition of correctness for FL needing to accommodate the new connective for cut, we introduce another cospan in the syntactic category of F-prenets $\mathcal{F}Synt$. We also relax the definition of anodyne map to allow functions that are injective but not bijective, to take account of the new Weakening rule. With these modifications, Theorem 3 and Theorem 4 can be restated, with one marked difference: this time, for FL-correct net we have $|Ax| \leq |P| - |\mathcal{C}omp_P| + |\mathcal{G}en_P|$.

While problems like the counterexample above are solved, in general we still cannot eliminate the cuts on an FL-correct net and always get one which is also FL-correct. Thus we still do not have a category. This calls for a little more analysis. First notice that F-prenets do form a category themselves. It is easy to see that this category is equivalent to the free Frobenius category generated by the set of literal types (where an atom and its negation have the same "type"). And thus we can consider FL-correct (and CL-correct) nets to be a class of maps in that category, which is not closed under composition. But this large category (as usual objects are formulas and a map $\mathrm{A} \to \mathrm{B}$ is a $P \rhd \overline{\mathrm{A}}, \mathrm{B}$) has *two* order enrichments.

Definition 9. *Let $P \rhd \Gamma, Q \rhd \Gamma$ be two linkings over the same sequent. We write*

- *$P \leqslant Q$ if $\mathcal{C}omp_P = \mathcal{C}omp_Q$ and $\mathcal{G}en_P \leqslant \mathcal{G}en_Q$, i.e, the genus functions are ordered pointwise.*
- *$P \preceq Q$ if $\mathcal{C}omp_P$ is a finer partition than $\mathcal{C}omp_Q$ and the genus of every component in $\mathcal{C}omp_Q$ is greater than the sum of the genera of the components of $\mathcal{C}omp_P$ it contains.*

These order structures do define enrichments when they are considered as being defined on morphisms, as above. Both have their interests, but we don't have much space left. So we just state one of several corollarie of that analysis:

Theorem 5. *Let $P \rhd \Gamma$ be the result of eliminating the cuts on an FL-correct net. Then there exists an FL-correct linking $Q \geqslant P$.*

So we can obtain a category by cheating on our original goal and define a composition that "fattens" the one given by ordinary Frobenius categories, which we will describe in the full paper.

References

[Dij89] R H Dijkgraaf. A geometric approach to two dimensional conformal field theory. Technical report, Universiteit Utrecht, 1989.

[FP05] Carsten Führmann and David Pym. Order-enriched categorical models of the classical sequent calculus. *J. Pure App. Algebra*, 204(1):21–68, 2005.

[Hyl04] J. Martin E. Hyland. Abstract interpretation of proofs: Classical propositional calculus. In Jerzy Marcinkowski and Andrzej Tarlecki, editors, *Computer Science Logic, CSL 2004*, volume 3210 of *LNCS*, pages 6–21. Springer-Verlag, 2004.

[Koc04] Joachim Kock. *Frobenius Algebras and 2d Topological Quantum Field Theories*. Cambridge University Press, Cambridge, 2004.

[Lam07] F. Lamarche. Exploring the gap between linear and classical logic. *Theory and Applications of Categories*, 18(17):473–535, 2007.

[LN09] François Lamarche and Novak Novakovic. Two Denotational Interpretations of Proofs in Classical Logic. In Michel Parigot and Lutz Straßburger, editors, *Structures and Deduction 2009*, pages 117–133, Bordeaux France, 07 2009.

[LS05] F. Lamarche and L. Straßburger. Naming proofs in classical logic. In P. Urzyczyn, editor, *TLCA Proceedings*, volume 3461 of *LNCS*, pages 246–261. Springer, 2005.

On Reiterman Conversion

Jan Pavlík
Faculty of Mechanical Engineering, Brno University of Technology,
Brno, Czech Republic
pavlik@fme.vutbr.cz

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There are many ways how to define algebras with underlying objects on a general category. However, the usual concepts are not general enough. Categories of algebras for a monad involve existence of free objects, which may not occur in general, while f-algebraic categories (categories of algebras for an endofunctor) are not strong enough to describe varieties. There is a concept (see [1]) of varieties which avoid these restrictions but it needs colimits for the constructions. The concept of algebraic categories introduced in [5] involving proper classes of operation symbols is applicable on every category. The question of coherence with functor algebras was discussed and finally solved by Reiterman, but his result was not published in full generality.

The author's concept of l-algebraic categories ([3]) provide another option of approach. While l-algebraic categories are kind of Beck categories where an existence of some Kan extensions enables to define monadicity, its subfamily of polymeric categories is shown in [4] to be sufficient to cover many natural examples of algebraic structures. Moreover, if the category is cocomplete, each variety can be described as an ordinal limit of polymeric categories.

In this framework, we extend the Reiterman isomorphism onto all polymeric categories. This will provide an important step for the proof of algebraicity of all varieties which may be helpful to establish a counterpart for the equational logic such as injectivity logic by Adámek, Hébert and Sousa.

L-algebraic and Polymeric Categories

Given a base category $\mathcal{C}$, *l-algebraic categories* are $\mathcal{C}$-concrete limits of f-algebraic categories. Many natural examples of categories are l-algebraic (e.g. all monadic, f-algebraic categories, varieties etc.). Among them, a special role is played by *polymeric categories* (introduced in [4]), which we will focus on. One can show that every l-algebraic category is Beck, i.e. its forgetful U functor creates limits and U-absolute coequalizers.

Let (A, α) be an F-algebra. Given $n \in \omega$, a *n-polymer* of an algebra (A, α) is the morphism $\alpha^{(n)} : F^n(A) \to A$ in $\mathcal{C}$ defined recursively:

$$\alpha^{(0)} = \mathrm{id}_A, \alpha^{(n+1)} = \alpha \circ F\alpha^{(n)}.$$

Let $n \in \mathbb{N}$ and G be an endofunctor on $\mathcal{C}$. A natural transformation $\phi : G \to F^n$ is called *n-ary polymeric G-term* in category of F-algebras. A pair $(\phi, \psi)_p$ of polymeric G-terms of arities m,n, respectively, is called *polymeric identity.* Moreover, for an F-algebra (A, α), we define

$$(A, \alpha) \models (\phi, \psi)_p \quad \stackrel{def}{\Leftrightarrow} \quad \alpha^{(m)} \circ \phi_A = \alpha^{(n)} \circ \psi_A,$$

and we say that the F-algebra (A, α) *satisfies* the polymeric identity $(\phi, \psi)_p$.

For a class $\mathcal{I}$ of polymeric identities we define a *polymeric variety of F-algebras* as the class of all algebras satisfying all $(\phi, \psi)_p \in \mathcal{I}$. The corresponding subcategory of $\mathbf{Alg}\, F$ is denoted by $\mathbf{Alg}\,(F, \mathcal{I})$. Each category concretely isomorphic to a polymeric variety is called *polymeric.*

It is easy to show the relations: monadic $\Rightarrow$ polymeric $\Rightarrow$ l-algebraic.

Algebraic Categories

Given a category $\mathcal{C}$ and a class Ω of *operation symbols*, a *type* on $\mathcal{C}$ with the domain Ω is a mapping $t : \Omega \to (\mathrm{Ob}\mathcal{C})^2$. Given $\sigma \in \Omega$, $t(\sigma) = (t_0(\sigma), t_1(\sigma))$ is called an *arity-pair* for Ω.

An *algebra for a type t* is a pair (A, S) made up of a $\mathcal{C}$-object A and a mapping $S : \Omega \to \mathrm{Morph}\mathcal{S}et$ such that $S(\sigma) : \hom(t_0(\sigma), A) \to \hom(t_1(\sigma), A)$ for each $\sigma \in \Omega$. A morphism of t-algebras $f : (A, S) \to (B, T)$ is a morphism $f : A \to B$ such that, for every $\sigma \in \Omega$ and $m : t_0(\sigma) \to A$, $f \circ S(\sigma)(m) = T(\sigma)(f \circ m)$. The metacategory of t-algebras and their morphisms will be denoted by $t-\mathbf{alg}$.

The *t-terms* and their *arity-pairs* are defined recursively. Each $\sigma \in \Omega$ is a term of arity-pair $t(\sigma)$, there is a term $\overline{f}$ of arity-pair $(\mathrm{cod}(f), \mathrm{dom}(f))$ for every $\mathcal{C}$-morphism f and, for terms q, p of arity-pairs (Z, Y) and (Y, X), respectively, by $p \cdot q$ we denote another term of arity-pair (Z, X). The pairs of terms $\overline{f} \cdot \overline{g}$ and $\overline{g \circ f}$ are considered equal for every pair of composable morphisms g, f.

The class of all terms of type t will be denoted by $\mathcal{T}(t)$. By *(X, Y)-ary t-equation* we mean a pair of t-terms of the arity-pair (X, Y). Given a type $t : \Omega \to (\mathrm{Ob}\mathcal{C})^2$ then there is a universal $h : \Omega \to \mathcal{T}(t)$ such that for each algebra (A, S), the mapping $S : \Omega \to \mathrm{Morph}\mathcal{C}$ induces a partial-algebra homomorphism $\underline{S} : \mathcal{T}(t) \to \mathrm{Morph}\mathcal{C}$ such that $SS \circ h = S$. This mapping is called an *evaluation of terms* on (A, S). For each term p of arity-pair (X, Y), the evaluation on an algebra (A, S) defines a mapping $SS(p) : \hom(X, A) \to \hom(Y, A)$.

Given a t-equation (p, q), we say that (A, S) *satisfies* the (p, q) if $SS(p) = SS(q)$. Then we write $(A, S) \models (p, q)$.

For a class $\mathcal{I}$ of t-equations, the pair $(t, \mathcal{I})$ is called an *equational theory* over $\mathcal{C}$. The corresponding category will be denoted by $(t, \mathcal{I})$–**alg**. A category will be called *algebraic*, if it is isomorphic to $(t, \mathcal{I})-\mathbf{alg}$ for some equational theory $(t, \mathcal{I})$.

L-algebraic vs. Algebraic Categories

Author proved in [3] that every l-algebraic category with codensity monad is monadic and that if $\mathcal{C}$ has copowers, then every algebraic category over $\mathcal{C}$ is l-algebraic. Now we can show an algebraic category which is not l-algebraic.

Example 1 *Consider the category* $\mathcal{C} = \mathbf{2} + \mathbf{2}$ *consisting of objects* $0, 1, 0', 1'$ *and morphisms* $\iota : 0 \to 1$, $\iota' : 0' \to 1'$ *and identities. Let* $\mathcal{A} = \mathbf{1} + \mathbf{1}$ *and* $U : \mathcal{A} \to \mathcal{C}$ *be the inclusion of* $\{0, 0'\}$*. Then* $(\mathcal{A}, U)$ *is algebraic: the type is* $t : \{\rho, \sigma\} \to (\mathrm{Ob}\mathcal{C})^2$, $t(\rho) = (1, 0')$, $t(\sigma) = (1', 0)$ *and* $t-\mathbf{alg} \cong_{\mathcal{C}} \mathcal{A}$*. Moreover* $(\mathcal{A}, U)$ *is not l-algebraic since it has a codensity monad (the trivial monad) but* U *does not have an adjoint (1 does not have an universal arrow).*

Reiterman Conversion

We ask whether every l-algebraic category is algebraic. We are not able to answer this generally, but we can prove the algebraicity for all polymeric categories. First we focus on f-algebraic categories. J. Reiterman proved, but did not publish, the following result:

Theorem 2 (Reiterman theorem)
Every f-algebraic category is algebraic.

The paper of Kurz and Rosický [2] presents its proof for coalgebras on $\mathcal{S}et$. In general let $F : \mathcal{C} \to \mathcal{C}$ be a functor, Ω contain symbols σ_X of arity-pair (X, FX) for every object X in $\mathcal{C}$ and $\mathcal{I}$ be the closure of a class of equations

$$(\overline{Ff} \cdot \sigma_X, \sigma_Y \cdot \overline{f})$$

labeled by all morphisms $f : Y \to X$ in $\mathcal{C}$. We have a *Reiterman theory* $(t, \mathcal{I})$. Given $X \in \mathrm{Ob}\mathcal{C}$, F-algebra (A, α) and a morphism $h : X \to A$ we set

$$R_\alpha(\sigma_X)(h) = \alpha \circ Fh.$$

Now we have an assignment $R : \mathbf{Alg}\, F \to (t, \mathcal{I})-\mathbf{alg}$ (*Reiterman isomorphism*) given by $(A, \alpha) \mapsto (A, R_\alpha)$. It is straightforward to show that it is a functor and $(A, R_\alpha) \models (\overline{Ff} \cdot \sigma_X, \sigma_Y \cdot \overline{f})$ for every X. The inverse functor $S : (t, \mathcal{I})-\mathbf{alg} \to \mathbf{Alg}\, F$ is defined by $(A, H) \mapsto (A, H(\sigma_A)(\mathrm{id}_A))$. Since it is defined on algebras satisfying all equations from $\mathcal{I}$, it is easy to show that S is in fact the inverse functor for R.

We may use this isomorphism even to classify some other l-algebraic categories. If we modify the Reiterman theory for a *set of $\mathcal{C}$-endofunctors*, we may redefine the Reiterman isomorphism even for some non-homogenous l-algebraic categories such as products of f-algebraic categories. To prove the algebraicity for polymeric categories, let F be an endofunctor on $\mathcal{C}$.

Definition 3 *Given an object X in $\mathcal{C}$ and $k \in \omega$, then we define* k-polymeric t_F-term $\tau_X^{(k)}$ *by:* $\tau_X^{(0)} = \overline{\mathrm{id}_X}$ *and* $\tau_X^{(n+1)} = \sigma_{F^n X} \cdot \tau_X^{(n)}$ *for $n \in \omega$.*

Observe that the terms $\tau_X^{(k)}$ have arity pairs of $(X, F^k X)$. Let $(A, \alpha) \in \mathbf{Alg}\, F$ and $h : X \to A$ be a morphism. By induction we can show for every $k \in \omega$

$$\underline{R_\alpha}(\tau_X^{(k)})(h) = \alpha^{(k)} \circ F^k h.$$

Let $G : \mathcal{C} \to \mathcal{C}$ be a functor. For a natural transformation $\phi : G \to F^k$,

$$\underline{R_\alpha}(\overline{\phi_X} \cdot \tau_X^{(k)})(h) = \alpha^{(k)} \circ \phi_A \circ Gh.$$

Hence for an (m, n)-ary polymeric G-identity $(\phi, \psi)_p$,

$$(A, \alpha) \models (\phi, \psi)_p \Leftrightarrow (\forall X \in \mathrm{Ob}\mathcal{C})\ (A, R_\alpha) \models (\overline{\phi_X} \cdot \tau_X^{(m)}, \overline{\psi_X} \cdot \tau_X^{(n)}).$$

This, together with the Reiterman theorem, yields

Proposition 4 *Every polymeric category is algebraic.*

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References

[1] J. Adámek, H. Porst: *From Varieties of Algebras to Covarieties of Coalgebras*, Electronic Notes in Theoretical Computer Science (2001)

[2] A. Kurz, J. Rosický: *Modal Predicates and Coequations*, Electronic Notes in Theoretical Computer Science, Vol. 65, 156-175 (2002)

[3] J. Pavlík: *Kan Extensions in Context of Concreteness*, arXiv:1104.3542v1 [math.CT] (2011)

[4] J. Pavlík: *Varieties defined without colimits*, 7th Panhellenic Logic Symposium - proceedings (2009)

[5] J. Rosický: *On algebraic categories*, Colloquia Mathematica Societatis János Bolai(1977)

TWO RESULTS ON COMPACT CONGRUENCES

MIROSLAV PLOŠČICA

For a class $\mathcal{K}$ of algebras we denote $\operatorname{Con}\mathcal{K}$ the class of all lattices isomorphic to $\operatorname{Con}(A)$ (the congruence lattice of an algebra A) for some $A \in \mathcal{K}$. There are many papers investigating $\operatorname{Con}\mathcal{K}$ for various classes $\mathcal{K}$. However, the full description of $\operatorname{Con}\mathcal{K}$ has proved to be a very difficult (and probably intractable) problem, even for the most common classes of algebras, like groups or lattices. A recent evidence of this is the solution of the Congruence Lattice Problem (CLP) by F. Wehrung [6].

The lattice $\operatorname{Con}(A)$ is algebraic for every algebra A. The subset $\operatorname{Con}_c(A) \subseteq \operatorname{Con}(A)$ of all compact (finitely generated) congruences is closed under finite joins (thus forming a join-subsemilattice), but, in general, not closed under meets. This fact seems to be a major obstacle in the description of the class $\operatorname{Con}(\mathcal{K})$. In the few relevant cases when the class $\operatorname{Con}\mathcal{K}$ is well understood, the crucial assumptions are that

(i) class $\mathcal{K}$ is a congruence-distributive variety (equational class) of algebras;
(ii) for every $A \in \mathcal{K}$ the set $\operatorname{Con}_c(A)$ is closed under intersection.

There is also a considerable evidence that (ii) makes the study of congruence lattices easier even without the assumption (i). (See, for instance [5].)

Our first result is the characterization of locally finite congruence-distributive varieties satisfying (ii). Actually, we provide two such characterizations. One of them connects (ii) with a structure of subdirectly irreducible members of $\mathcal{K}$, another is formulated in the language of morphisms between congruence lattices.

Actually, this result is not completely new, as the equivalence of the first two conditions was proved in [1] using the concept of equationally definable principal meets. However, we provide a direct proof which does not refer to polynomials and, we believe, provides an insight helpful in describing $\operatorname{Con}\mathcal{K}$ for classes $\mathcal{K}$ satisfying (i) and (ii).

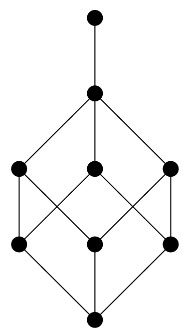
For every homomorphism $f: A \to B$ we define the mapping $\operatorname{Con}_c(f): \operatorname{Con}_c(A) \to \operatorname{Con}_c(B)$ by the rule that $\operatorname{Con}_c(f)(\alpha)$ is the congruence generated by the set $\{(f(x), f(y)) \mid (x, y) \in \alpha\}$, for every $\alpha \in \operatorname{Con}_c(A)$. The mapping $\operatorname{Con}_c(f)$ always preserves joins, but not necessarily meets. (Even if $\operatorname{Con}_c(A)$ is a lattice.)

Theorem 1. *Let $\mathcal{K}$ be a locally finite congruence-distributive variety. The following conditions are equivalent.*

(1) *For every $A \in \mathcal{K}$ the set $\operatorname{Con}_c(A)$ is closed under intersection.*
(2) *Every finite subalgebra of a subdirectly irreducible algebra in $\mathcal{K}$ is itself subdirectly irreducible.*
(3) *For every embedding $f: A \to B$ of algebras in $\mathcal{K}$ with A finite, the mapping $\operatorname{Con}_c(f)$ preserves meets.*

This theorem is a generalization of the result from [4], which says that if every subdirectly irreducible algebra in $\mathcal{K}$ is simple, then (1) is satisfied.

As an example, consider the varieties of distributive pseudocomplemented lattices. Let $\mathcal{B}_n$ be the variety generated by the pseudocomplemented lattice (the ordinal sum) $B_n = \mathbf{2}^n \oplus \mathbf{1}$. (See the picture of B_3 below.)



So, $\mathcal{B}_0$ is the variety of Boolean algebras, $\mathcal{B}_1$ is known as the variety of Stone algebras. The complete list of subdirectly irreducible algebras in $\mathcal{B}_n$ and their subalgebras (up to an isomorphism) is $\{B_0, B_1, \dots, B_n\}$. Hence the condition (2) of our theorem is satisfied. (The congruence lattices of distributive pseudocomplemented lattices have been investigated in [2] and elsewhere.)

There is also another way how to construct examples of varieties satisfying the conditions of the theorem. Take any finite, subdirectly irreducible algebra A, generating a congruence-distributive variety. Enhance the type of A by taking all elements of A as constants (nullary operations). Then the resulting algebra A^* generates a variety satisfying (2).

Using the above result, we are able to prove several general theorems about $\operatorname{Con}\mathcal{K}$ for some varieties $\mathcal{K}$.

Our second result is a solution of the following problem formulated at the problem session at Novi Sad Algebraic Conference 2003 ([3]).

Problem 2. *If an infinite algebra A belongs to a finitely generated congruence distributive variety, is it true that $|\operatorname{Con}_c A| = |A|$?*

Since every compact congruence is generated by a finite subset of A^2, the inequality $|\operatorname{Con}_c A| \leq |A|$ is clear. Without any assumptions the inverse inequality is false. (There are simple algebras of any cardinality.) However, the assumptions of the above Problem imply that A is a subdirect product of finite algebras and hence has infinitely many congruences. (Namely, the kernels of the natural projections.)

Our result is as follows.

Theorem 3. *Let A be an infinite subalgebra of the direct product $\Pi_{i\in I} A_i$. Suppose that there exists a natural number n such that $|A_i| \leq n$ for every $i \in I$. Then $|\operatorname{Con}_c A| = |A|$.*

This theorem answers Problem 2 affirmatively. Notice however, that the theorem is more general, for instance it does not assume the congruence distributivity.

The common finite bound on the cardinalities of A_i is necessary. The ring of p-adic integers is a subdirect product of finite rings, its cardinality is continuum, but it has only countably many congruences.

Our proof is based on the following combinatorial assertion.

Lemma 4. *Let n be a natural number, X an infinite set and $F \subseteq \{0,\dots,n-1\}^X$. Suppose that $D(x,y) = \{f \in F \mid f(x) \neq f(y)\}$ is nonempty for every $x, y \in X$. Then there are $|X|$ mutually different sets of the form $D(x,y)$.*

References

[1] W. J. Blok, D. Pigozzi, *A finite basis theorem for quasivarieties*, Algebra Universalis **22** (1986), 1–13.

[2] T. Katriňák, *Congruence lattices of distributive p-algebras*, Algebra Universalis **7** (1977), 265-271.

[3] *Problems posed at the Novi Sad Algebraic Conference '03*, Novi Sad J. Math. **34** (2004), 183-186.

[4] M. Ploščica, *Separation in distributive congruence lattices,* Algebra Universalis **49** (2003), 1–12.

[5] P. Růžička, *Lattices of two-sided ideals of locally matricial algebras and the Γ-invariant problem*, Israel J. Math. **142** (2004), 1–28.

[6] F. Wehrung, *A solution to Dilworth's congruence lattice problem*, Advances of Math. **216** (2007), 610-625.

Mathematical Institute, Slovak Academy of Sciences, Grešákova 6, 04001 Košice, Slovakia

E-mail address: miroslav.ploscica@upjs.sk

Priestley Duality for Bilattices*

Achim Jung
School of Computer Science
University of Birmingham
Edgbaston
B15 2TT Birmingham
United Kingdom
A.Jung@cs.bham.ac.uk

Umberto Rivieccio
Research Center for Integrated Science
Japan Advanced Institute
of Science and Technology
1-1 Asahidai, Nomi, Ishikawa 923-1292
Japan
rUmberto@jaist.ac.jp

Bilattices are algebraic structures introduced in 1988 by Matthew Ginsberg [7] as a uniform framework for inference in Artificial Intelligence. The main idea behind bilattices is to consider two different orders on truth values, one for the "degree of truth" and the other for the "degree of information" associated with a sentence. This approach turned out to be very flexible and useful in AI and computer science, as it allows to deal with partial as well as inconsistent data, also unifying several inference mechanisms used in default and non-monotonic reasoning (see especially the works of Ginsberg, O. Arieli & A. Avron and, for logic programming applications, of M. Fitting).

The bilattice formalism is also appealing from a purely mathematical point of view, as it constitutes a natural generalization of lattices (see for instance [9]) and was used to define natural (i.e., semantically motivated) examples of so-called non-protoalgebraic logics, a class of logics that has a particular interest within the theory of algebraization of deductive systems (see [2, 3]). An up-to-date review of the applications of this formalism and also of the motivation behind its study can be found in the dissertation [11].

In the present work we develop a Priestley-style duality theory for bilattices and some related algebras. This topic has already been investigated by Mobasher et al. [9] but only from an abstract category-theoretic point of view (see below for details). Here instead we are interested in a concrete study of the topological spaces that correspond to the these algebraic structures. We think that such study is interesting not only because it provides a deeper insight into the structure of bilattices, but also because it contains some ideas on how to develop a bitopological generalization of Priestley duality theory in the direction indicated in [8].

We recall the main definitions appearing in this framework. By a (bounded) *pre-bilattice* we mean an algebra $\langle B, \wedge, \vee, \otimes, \oplus, \mathtt{f}, \mathtt{t}, \bot, \top \rangle$ such that $\langle B, \leq_t, \wedge, \vee, \mathtt{f}, \mathtt{t} \rangle$ and $\langle B, \leq_k, \otimes, \oplus, \bot, \top \rangle$ are both bounded lattices. By a *bilattice* we mean an algebra $\langle B, \wedge, \vee, \otimes, \oplus, \neg, \mathtt{f}, \mathtt{t}, \bot, \top \rangle$ such that $\langle B, \wedge, \vee, \otimes, \oplus, \mathtt{f}, \mathtt{t}, \bot, \top \rangle$ is a pre-bilattice and the *negation* $\neg$ is a unary operation satisfying, for every $a, b \in B$,

(neg 1) if $a \leq_t b$, then $\neg b \leq_t \neg a$

(neg 2) if $a \leq_k b$, then $\neg a \leq_k \neg b$

(neg 3) $a = \neg\neg a$.

A (pre-)bilattice is *distributive* when all possible distributive laws concerning $\{\wedge, \vee, \otimes, \oplus\}$, i.e., all identities of the following form, hold: $x \circ (y \bullet z) = (x \circ y) \bullet (x \circ z)$ for every $\circ, \bullet \in \{\wedge, \vee, \otimes, \oplus\}$. A (pre-)bilattice is *interlaced* when all four lattice operations are monotone w.r.t. both lattice orders. It is easy to prove that distributive (pre-)bilattices form a proper subclass of the interlaced ones.

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We also consider some expansions of the standard bilattice language, for example *bilattices with conflation* [5], obtained by adding a kind of dual of the bilattice negation (an involutive unary operator that is monotone w.r.t. $\leq_t$ and antimonotone w.r.t. $\leq_k$), and *Brouwerian bilattices* [2].

A *Brouwerian bilattice* is an algebra $\langle B, \wedge, \vee, \otimes, \oplus, \supset, \neg \rangle$ such that the reduct $\langle B, \wedge, \vee, \otimes, \oplus, \neg \rangle$ is a bilattice and the following equations are satisfied:

(B1) $(x \supset x) \supset y = y$

(B2) $x \supset (y \supset z) = (x \wedge y) \supset z = (x \otimes y) \supset z$

(B4) $(x \vee y) \supset z = (x \supset z) \wedge (y \supset z) = (x \oplus y) \supset z$

(B4) $x \wedge ((x \supset y) \supset (x \otimes y)) = x$

(B5) $\neg(x \supset y) \supset z = (x \wedge \neg y) \supset z.$

Let us note that the bilattice reduct of any Brouwerian bilattice is distributive.

One of the key results for the study of bilattices is a representation theorem stating that any interlaced pre-bilattice is isomorphic to a certain product (similar to a direct product) of two lattices. An analogous theorem holds for bilattices: in this case we have that any interlaced bilattice is isomorphic to a product of two copies of the same lattice. These results were proved by Avron [1] for bounded interlaced (pre-)-bilattices, then generalized in [3] to the unbounded case. In [9] they are formulated in categorical terms, as follows: the category of bounded interlaced pre-bilattices is equivalent to the product of the category of bounded lattices (whose objects are pairs of bounded lattices) with itself, and the category of bounded interlaced bilattices is equivalent to the category of bounded lattices.

In [2] analogous representation theorems are proved for bilattices with conflation and Brouwerian bilattices, showing that every category of (pre-)bilattices on the left column of Table 2 is equivalent to the corresponding category of lattices on the right (all categories have as objects the corresponding algebras and as morphisms algebraic homomorphisms).

interlaced pre-bilattices	product of the category of lattices with itself
distributive pre-bilattices	product of the category of distributive lattices with itself
interlaced bilattices	lattices
distributive bilattices	distributive lattices
commutative interlaced bilattices with conflation	lattices with involution
commutative distributive bilattices with conflation	De Morgan lattices
Kleene bilattices with conflation	Kleene lattices
classical bilattices with conflation	Boolean lattices
Brouwerian bilattices	Brouwerian lattices

Table 1: Categorical equivalences

Using these results, it is immediate to conclude that, for instance, the category of bounded distributive bilattices is equivalent to the category of Priestley spaces. Given a bounded distributive bilattice, one constructs the lattice associated with it, then considers the Priestley space corresponding to this lattice. Conversely, for any Priestley space, one considers the distributive lattice associated with it and then constructs the corresponding bilattice. This is the approach taken in [9]. As mentioned above, we follow a different strategy, which has, in our opinion, the advantage of relating directly bilattices with Priestley spaces. Instead of focusing on the lattice factor(s) of (pre-)bilattice (as given by the above-mentioned representation theorems), we will focus on one of its lattice reducts. To do this, we use the fact that any bounded distributive pre-bilattice $\langle B, \wedge, \vee, \otimes, \oplus, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$ can be seen as a bounded distributive lattice $\langle B, \wedge, \vee, \mathsf{f}, \mathsf{t} \rangle$ having two extra constants $\bot, \top$ that are complement of each other, i.e., such that $\bot \wedge \top = \mathsf{f}$ and $\bot \vee \top = \mathsf{t}$. The two presentations are equivalent up to algebraic language (we express this saying that the two classes of algebras are *termwise equivalent*). The other two lattice operations can be recovered defining, for all $a, b \in B$,

$$a \otimes b := (a \wedge \bot) \vee (b \wedge \bot) \vee (a \wedge b)$$
$$a \oplus b := (a \wedge \top) \vee (b \wedge \top) \vee (a \wedge b)$$

This result, sometimes called the *90-degree lemma* (see [8]), can be extended even to interlaced (not necessarily distributive) pre-bilattices if we add some extra assumptions on $\bot$ and $\top$ (see [1, Theorem 5.3]). It is not difficult to see that it can also be extended to bilattices with negation, bilattices with conflation and Brouwerian bilattices. In these cases, instead of a bounded distributive lattices with two additional constants, we have a bounded distributive lattice with two additional constants plus operations for negation ($\neg$), conflation ($-$) and implication ($\supset$). Some of these algebras have been studied in the literature and corresponding duality theories have been developed. For instance, distributive lattices with an involutive negation are known as *De Morgan lattices* (see [6]; a duality theory for De Morgan lattices is developed in [4]), while De Morgan lattices with implication are known as *N4-lattices* (a duality theory for these structures is developed in [10]).

Our idea is to use the 90-degree lemma to extend the known duality results on these algebras to the study of spaces corresponding to bilattices and related structures, exploiting the following termwise equivalences:

bounded distributive pre-bilattices	bounded distributive lattices with extra constants $\bot, \top$
bounded distributive bilattices	De Morgan algebras with extra constants $\bot, \top$
bounded distributive bilattices with conflation	double De Morgan algebras with extra constants $\bot, \top$
bounded Brouwerian bilattices	bounded N-4 lattices with extra constants $\bot, \top$

Table 2: Termwise equivalences

In this way we obtain direct Priestley-style duality results for the above-mentioned classes of bilattices.

We have, for instance, that bounded distributive pre-bilattices correspond to *pre-bilattice spaces*, which we define as tuples $\langle X, \tau, \leq, X^1, X^2 \rangle$ such that $\langle X, \tau, \leq \rangle$ is a Priestley space and $X^1, X^2 \subseteq X$ are clopen up-sets satisfying that $X^1 \cap X^2 = \emptyset$ and $X^1 \cup X^2 = X$. We define a *pre-bilattice function* to be a function $f \colon X \to Y$ between two pre-bilattice spaces $\langle X, X^1, X^2, \tau, \leq \rangle$ and $\langle Y, Y^1, Y^2, \tau', \leq' \rangle$ that is monotone, continuous and such that $f(X^1) \subseteq Y^1$ and $f(X^2) \subseteq Y^2$. In this way we obtain a categorical equivalence between the category of bounded distributive pre-bilattices with algebraic $\{\wedge, \vee, \mathsf{f}, \mathsf{t}, \bot, \top\}$-homomorphisms as morphisms and the category of pre-bilattice spaces with pre-bilattice functions as morphisms.

A *De Morgan space* [4] is a tuple $\langle X, \tau, \subseteq, g \rangle$ such that $\langle X, \tau, \leq \rangle$ is a Priestley space and $g \colon X \to X$ is an order-reversing homeomorphism such that $g^2 = id_X$. Given two De Morgan spaces $\langle X, \tau, \leq \rangle$ and $\langle X', \tau', \leq' \rangle$, a *De Morgan function* is defined as an order-preserving continuous function $f \colon X \to X'$ such that $fg = g'f$. It is proved in [4] that the category of De Morgan algebras with algebraic homomorphisms as morphisms is equivalent to the category of De Morgan spaces with De Morgan functions as morphisms. We prove a similar result for bilattices defining a *bilattice space* to be a tuple $\langle X, \tau, \leq, g, X^1, X^2 \rangle$ such that

$\langle X,\tau,\leq,g\rangle$ is a De Morgan space, $\langle X,\tau,\leq,X^1,X^2\rangle$ is a pre-bilattice space and $g(X^1)=X^2$. A *bilattice function* is defined as a function that is both a pre-bilattice function and a De Morgan function. We prove then that the category of bounded distributive bilattices with algebraic $\{\wedge,\vee,\neg,\mathsf{f},\mathsf{t},\bot,\top\}$-homomorphisms as morphisms is equivalent to the category of bilattice spaces with bilattice functions as morphisms.

A similar result may be obtained for bounded distributive bilattices with conflation and the corresponding spaces, using the fact that that a bilattice with conflation can be defined as a structure $\langle B,\wedge,\vee,\otimes,\oplus,\neg,-\rangle$ such that both $\langle B,\wedge,\vee,\neg\rangle$ and $\langle B,\otimes,\oplus,-\rangle$ are De Morgan lattices.

A *Brouwerian bilattice space* $\langle X,\tau,\leq,g,X^1,X^2\rangle$ is a bilattice space such that X^1 with the induced topology is an Esakia space (i.e. a Priestley space where the down-set of any open set is open). A *Brouwerian bilattice function* is a function $f\colon X\to Y'$ between two Brouwerian bilattice spaces $\langle X,X^1,\tau,\leq,g\rangle$ and $\langle Y,Y^1,\tau',\leq',g'\rangle$ if f is a De Morgan function from $\langle X,\tau,\leq,g\rangle$ to $\langle Y,\tau',\leq',g'\rangle$ such that $f(X^1)\subseteq Y^1$ and $f\colon X^1\to Y$ is an Esakia function, i.e. for any open $O\in\tau'$,

$$f^{-1}((O\cap Y^1])\cap X^1=(f^{-1}(O\cap Y^1)]\cap X^1.$$

Since the $\{\wedge,\vee,\supset,\neg\}$-reduct of any Brouwerian bilattice is an N4-lattice, we can exploit the duality theory for N4-lattices developed in [10] to prove that the category of Brouwerian bilattices with algebraic $\{\wedge,\vee,\supset,\neg,\mathsf{f},\mathsf{t}\}$-homomorphisms as morphisms is equivalent to the category of Brouwerian bilattice spaces with Brouwerian bilattice functions as morphisms.

References

[1] A. Avron. The structure of interlaced bilattices. *Mathematical Structures in Computer Science*, 6(3):287–299, 1996.

[2] F. Bou, R. Jansana and U. Rivieccio. Varieties of interlaced bilattices. *Algebra Universalis*. To appear.

[3] F. Bou and U. Rivieccio. The logic of distributive bilattices. *Logic Journal of the I.G.P.L.*, 2010. To appear.

[4] W. H. Cornish and P. R. Fowler. Coproducts of De Morgan algebras. *Bull. Austral. Math. Soc.*, 16(1):1–13, 1977.

[5] M. Fitting. Kleene's three-valued logics and their children. *Fundamenta Informaticae*, 20(1-3):113–131, 1994. Special Anniversary Issue: 20th volume of Fundamenta Informaticae.

[6] J. M. Font. Belnap's four-valued logic and De Morgan lattices. *Logic Journal of the I.G.P.L.*, 5(3):413–440, 1997.

[7] M. L. Ginsberg. Multivalued logics: A uniform approach to inference in artificial intelligence. *Computational Intelligence*, 4:265–316, 1988.

[8] A. Jung and M. Andrew Moshier. On the bitopological nature of stone duality. Technical report, School Computer Science, The University of Birmingham, 2006. Technical Report CSR-06-13.

[9] B. Mobasher, D. Pigozzi, G. Slutzki, and G. Voutsadakis. A duality theory for bilattices. *Algebra Universalis*, 43(2-3):109–125, 2000.

[10] S. P. Odintsov. Priestley duality for paraconsistent Nelson's logic. *Studia Logica*, 96(1):65–93, 2010.

[11] U. Rivieccio. *An Algebraic Study of Bilattice-based Logics*. Ph. D. Dissertation, University of Barcelona, 2010.

AN EXTENSION OF STONE DUALITY TO FUZZY TOPOLOGIES AND MV-ALGEBRAS

CIRO RUSSO

Based on [3]

1. Introduction

In 1965, Zadeh [4] introduced the concept of *fuzzy subset* of a given set X by means of *membership* (or *characteristic*) *functions* defined on X and taking values in the real unit interval $[0,1]$, the value of such functions at a given point x representing the *degree of membership* of x to the fuzzy subset.

After Zadeh's paper, fuzzy sets and fuzzy logic have been deeply studied both from a strictly mathematical and foundational viewpoint (mainly in connection with many-valued logics, whose introduction came far before the one of fuzzy sets) and as a tool for applications to many areas, especially of Computer Science.

As a matter of fact, all the (propositional) many-valued logics which are standard complete with respect to some algebraic structure defined on $[0,1]$ are worthy candidates for being a proper logical setting for fuzzy sets. Nonetheless, if we look at the crisp and fuzzy powersets of a given set X as, respectively, $\{0,1\}^X$ and $[0,1]^X$, it is undoubtable that, among the various fuzzy logics and corresponding algebraic semantics, Łukasiewicz logic and MV-algebras are the ones that best succeed in both having a rich expressive power and preserving many properties of symmetry that are inborn qualities of Boolean algebras.

The introduction of several concepts of "fuzzy topology" came a few years after Zadeh's paper, and their study has been pursued for many years. Our aim is to use MV-algebras as a framework for fuzzy topology which, on the one hand, is sufficiently rich and complex and, on the other hand, reflects (up to a suitable reformulation) as many properties of classical topology as possible. For this reason we introduce the concept of *MV-topology*, a generalization of general topology whose main features can be summarized as follows.

- The Boolean algebra of the subsets of the universe is replaced by the MV-algebra of the fuzzy subsets.
- Classical topological spaces are examples of MV-topological spaces.
- The algebraic structure of the family of open (fuzzy) subsets has a quantale reduct $\langle \Omega, \bigvee, \oplus, \mathbf{0} \rangle$, which replaces the classical sup-lattice $\langle \Omega, \bigvee, \mathbf{0} \rangle$, and an idempotent semiring one $\langle \Omega, \wedge, \odot, \mathbf{1} \rangle$ in place of

the classical meet-semilattice $\langle \Omega, \wedge, \mathbf{1} \rangle$. Moreover, the lattice reduct $\langle \Omega, \bigvee, \wedge, \mathbf{0}, \mathbf{1} \rangle$ maintains the property of being a frame.

- The MV-algebraic negation * is, in the aforementioned classes of algebras, an isomorphism between the various structures of open subsets and the corresponding ones of closed subsets.
- A classical topology — called the *shadow topology* — is canonically associated to each MV-topology: it is obtained simply by restricting the family of open subsets to the crisp ones.

For notions and results on MV-algebras we refer the reader to [1].

2. Basic definitions

Throughout the paper, both crisp and fuzzy subsets of a given set will be identified with their membership functions and usually denoted by lower case latin or greek letters. In particular, for any set X, we shall use also $\mathbf{1}$ and $\mathbf{0}$ for denoting, respectively, X and $\varnothing$. In some cases, we shall use capital letters in order to emphasize that the subset we are dealing with is crisp.

Definition 2.1. Let X be a set, A the MV-algebra $[0,1]^X$ and $\Omega \subseteq A$. We say that $\langle X, \Omega \rangle$ is an *MV-topological space* if Ω is a subuniverse both of the quantale $\langle [0,1]^X, \bigvee, \oplus, \mathbf{0} \rangle$ and of the semiring $\langle [0,1]^X, \wedge, \odot, \mathbf{1} \rangle$. More explicitly, $\langle X, \Omega \rangle$ is an MV-topological space if

(i) $\mathbf{0}, \mathbf{1} \in \Omega$,

(ii) for any family $\{o_i\}_{i \in I}$ of elements of Ω, $\bigvee_{i \in I} o_i \in \Omega$,

and, for all $o_1, o_2 \in \Omega$ and $\bullet \in \{\odot, \oplus, \wedge\}$,

(iii) $o_1 \bullet o_2 \in \Omega$,

Ω is also called an *MV-topology* on X and the elements of Ω are the *open MV-subsets* of X. The set $\Xi = \{o^* \mid o \in \Omega\}$ is easily seen to be a subuniverse both of the quantale $\langle [0,1]^X, \bigwedge, \odot, \mathbf{1} \rangle$ and of the semiring $\langle [0,1]^X, \vee, \oplus, \mathbf{0} \rangle$.

Recalling that, for any MV-algebra A, the set $\mathrm{B}(A) = \{a \in A \mid a \oplus a = a\}$ forms a Boolean algebra, we can set the following

Definition 2.2. If $\langle X, \Omega \rangle$ is an MV-topology, then $\langle X, \mathrm{B}(\Omega) \rangle$ — where $\mathrm{B}(\Omega) := \Omega \cap \{0,1\}^X = \Omega \cap \mathrm{B}([0,1]^X)$ — is both an MV-topology and a topology in the classical sense. The topological space $\langle X, \mathrm{B}(\Omega) \rangle$ will be said the *shadow space* of $\langle X, \Omega \rangle$.

Let X and Y be sets. Any function $f : X \longrightarrow Y$ naturally defines a map $f^{\leftarrow\!\!\sim} : \alpha \in [0,1]^Y \longrightarrow \alpha \circ f \in [0,1]^X$.

Let $\langle X, \Omega_X \rangle$ and $\langle Y, \Omega_Y \rangle$ be two MV-topological spaces. A map $f : X \longrightarrow Y$ is said to be *continuous* if $f^{\leftarrow\!\!\sim}[\Omega_Y] \subseteq \Omega_X$. It is called an *MV-homeomorphism* if it is bijective and bi-continuous.

As in classical topology, we say that, given an MV-topological space $\mathcal{T} = \langle X, \Omega \rangle$, a subset Δ of $[0,1]^X$ is called a *base* for $\mathcal{T}$ if $\Delta \subseteq \Omega$ and every open set of $\mathcal{T}$ is a join of elements of Δ.

A *covering* of X is any subset Γ of $[0,1]^X$ such that $\bigvee \Gamma = \mathbf{1}$, while an *additive covering* ($\oplus$-covering, for short) is a finite subset $\{\alpha_1, \dots, \alpha_n\} \subseteq [0,1]^X$, $n \in \omega$, such that $\alpha_1 \oplus \cdots \oplus \alpha_n = \mathbf{1}$.

The presence of strong and weak conjunctions and disjunction, in the structure of open sets of an MV-topology, naturally suggests different fuzzy versions (weaker or stronger) of most of the classical topological concepts (separation axioms, compactness etc.). However, we shall limit our attention to the ones that serve the scope of this paper, namely *compactness* and *Hausdorff (or T_2) separation axiom.*

Definition 2.3. An MV-topological space $\mathbf{T} = \langle X, \Omega \rangle$ is said to be *compact* if any open covering of X contains an additive covering; it is called *strongly compact* if any open covering contains a finite covering.

$\mathbf{T}$ is called an *Hausdorff* (or *separated*) *space* if, for any $x \neq y \in X$, there exist $o_x, o_y \in \Omega$ such that

(i) $o_x(x) = 1 = o_y(y)$,
(ii) $o_x(y) = o_y(x) = 0$,
(iii) $o_x \odot o_y = \mathbf{0}$;

$\mathbf{T}$ is said to be *strongly Hausdorff* (or *strongly separated*) if there exist $o_x, o_y \in \Omega$ satisfying (i) and

(iv) $o_x \wedge o_y = \mathbf{0}$.

It is obvious that strong compactness implies compactness and, in the case of classical topologies, the two notions collapse to the usual one; for the same reason, the shadow spaces of both compact and strongly compact MV-spaces are compact. Analogously, it is self-evident that strong separation implies separation, they both coincide with the classical T_2 property on crisp topologies, and they both imply that the corresponding shadow space is Hausdorff in the classical sense.

3. The extension of Stone Duality

In this section we shall see that Stone duality can be extended to semisimple MV-algebras and compact separated MV-topological spaces having a base of clopens. Before the duality theorem, we recall that an algebra in a variety is called *semisimple* if it is a subdirect product of simple algebras. For any MV-algebra A, let $\operatorname{Max} A$ be the set of its maximal ideals; it is well-known that

$$A \text{ is semisimple} \quad \text{iff} \quad \bigcap \operatorname{Max} A = \{0\} \quad \text{iff} \quad A \hookrightarrow [0,1]^{\operatorname{Max} A}.$$

We shall denote by $\mathcal{MV}^{\mathrm{ss}}$ the full subcategory of $\mathcal{MV}$ whose objects are semisimple MV-algebras, by ${}^{\mathrm{MV}}\mathcal{T}\mathrm{op}$ the category of MV-topological spaces and MV-continuous functions and by ${}^{\mathrm{MV}}\mathcal{S}\mathrm{tone}$ its full subcategory whose objects are *Stone MV-spaces*, i.e., compact, separated MV-topological spaces having a base of clopen sets (*zero-dimensional*).

Theorem 3.1 (Duality theorem)**.** *The mappings*

$$
\begin{array}{lllllll}
\Phi: & \mathbf{T} & \in & {}^{\mathrm{MV}}\mathcal{T}\mathrm{op} & \longmapsto & \mathrm{Clop}\,\mathbf{T} & \in & \mathcal{MV}^{\mathrm{ss}} \\
\Psi: & A & \in & \mathcal{MV}^{\mathrm{ss}} & \longmapsto & \langle \mathrm{Max}\,A, \Omega_A\rangle & \in & {}^{\mathrm{MV}}\mathcal{T}\mathrm{op}
\end{array} \tag{1}
$$

define two contravariant functors. They form a duality between $\mathcal{MV}^{\mathrm{ss}}$ *and* ${}^{\mathrm{MV}}\mathcal{S}\mathrm{tone}$*.*

Moreover, the restriction of such a duality to Boolean algebras and crisp topologies coincide with the classical Stone duality.

Theorem 3.2. *For any Stone MV-space* $\mathbf{T}$*, its shadow space is a Stone space and its image under* Φ *is precisely the Boolean center of* $\Phi\mathbf{T}$*.*

Conversely, for any semisimple MV-algebra A*,* $\Psi\,\mathrm{B}(A)$ *coincide with the shadow topology of* ΨA*.*

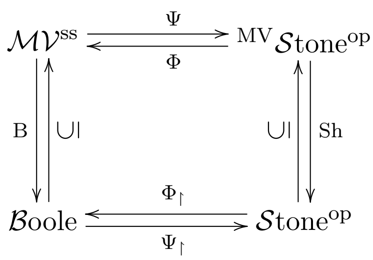
It is immediate to verify that

$$
\begin{array}{rlllll}
\mathrm{B}: & A & \in \mathcal{MV} & \longmapsto & \mathrm{B}(A) & \in \mathcal{B}\mathrm{oole} \\
\mathrm{Sh}: & \langle X, \Omega\rangle & \in {}^{\mathrm{MV}}\mathcal{T}\mathrm{op} & \longmapsto & \langle X, \mathrm{B}(\Omega)\rangle & \in \mathcal{T}\mathrm{op}
\end{array}
$$

are functors; they are, in fact, the left-inverses of the inclusion functors. Then Theorem 3.2 (together with last part of Theorem 3.1) can be reformulated as follows.

Corollary 3.3. $\Phi_{\restriction} \circ \mathrm{Sh} = \mathrm{B} \circ \Phi$ *and* $\Psi_{\restriction} \circ \mathrm{B} = \mathrm{Sh} \circ \Psi$*.*

Then we have the following commutative diagram of functors, where horizontal arrows are equivalences and vertical ones are inclusions of full subcategories and their respective left-inverses.



Corollary 3.4. *Stone MV-spaces which are strongly separated are dual to hyperarchimedean MV-algebras.*

References

[1] Cignoli R. L. O., D'Ottaviano I. M. L., Mundici D., *Algebraic Foundations of Many-valued Reasoning*, Trends in Logic, **7**, Kluwer Academic Publishers, Dordrecht, 2000.

[2] Hájek P., *Metamathematics of fuzzy logic*, Kluwer, 1998.

[3] Russo C., An extension of Stone duality to fuzzy topologies and MV-algebras, submitted.
`arXiv:1102.2000v2 [math.LO]`

[4] Zadeh L. A., Fuzzy sets, *Inform. and Control*, **8**, 338–353, 1965.

Dipartimento di Matematica, Università di Salerno, Italy, cirusso@unisa.it

THE CONTINUOUS WEAK BRUHAT ORDER

LUIGI SANTOCANALE

Abstract. Lattices of multipermutations $\mathsf{L}(v)$, $v \in \mathbb{N}^d$, generalize the weak Bruhat order on permutations.

For d_0 fixed, lattices $\mathsf{L}(v)$ with $v \in \mathbb{N}^{d_0}$ form a directed system, and can be glued into their colimit.

We give an explicit description of the Dedekind-MacNeille completion of this colimit, for each $d \geq 2$.

Lattices of Multipermutations

Lattices of multipermutations $\mathsf{L}(v)$ – see [1, 3] – generalize lattices of permutations (permutohedra, the weak Bruhat order on permutations) in a straightforward way. For a vector $v = (v_1, \ldots, v_d) \in \mathbb{N}^d$, elements of $\mathsf{L}(v)$ are words w on a linearly ordered alphabet $\{ a_1, \ldots, a_d \}$ such that the number of occurrences of the letter a_i in w equals v_i, for $i = 1, \ldots, d$. The order on words is the transitive closure of the covering relation $\prec$ defined by

$$w_0 a_i a_j w_1 \prec w_0 a_j a_i w_1 \,, \quad \text{whenever } i < j.$$

We can argue that $\mathsf{L}(v)$ is a lattice by embedding this poset as a principal ideal of the lattice of permutations $\mathsf{P}(k)$ with $k = \sum_{i=1}^d v_i$.

For $v, u \in \mathbb{N}^d$, we can transform a word $w \in \mathsf{L}(v)$ into a word $f_u(w) \in \mathsf{L}(v \cdot u)$: this is done by replacing each occurrence of the letter a_i by u_i occurrences of the same letter, for each $i = 1, \ldots, d$. The functions f_u give rise to a direct system in the category of lattices; let us denote by $\bigcup_{v \in \mathbb{N}^d} \mathsf{L}(v)$ its colimit. We deal with the following problem:

Problem. Provide an explicit characterization of the Dedekind-MacNeille completions of the lattices $\bigcup_{v \in \mathbb{N}^d} \mathsf{L}(v)$, for $d \geq 2$.

As we shall see in the next section, the problem has a rather intuitive solution if d, the dimension, equals 2. The geometrical flavor of this solution has stimulated us to look for a generalization to higher dimensions, with in mind that we could improve our understanding of the weak Bruhat order through geometry.

Among the lattices $\mathsf{L}(v)$, those of the form $\mathsf{L}(n, m)$ play a special role. Let us remark that they are distributive, while this is not the case if $v \in \mathbb{N}^d$ with $d \geq 3$. For $1 \leq i < j \leq d$ we have projection maps $\pi_{i,j} : \mathsf{L}(v) \to \mathsf{L}(v_i, v_j)$, obtained by erasing from a word all the letters distinct from a_i, a_j. The map $\pi = \langle \pi_{i,j} \mid 1 \leq i < j \leq d \rangle$, from $\mathsf{L}(v)$ to $\prod_{i<j} \mathsf{L}(v_i, v_j)$, is an order embedding; this means that we can identify $\mathsf{L}(v)$ with a sub-order of $\prod_{i<j} \mathsf{L}(v_i, v_j)$. For $u = \{ u_{i,j} \} \in \prod \mathsf{L}(v_i, v_j)$ we can ask whether u is in the image of $\mathsf{L}(v)$; this is answered as follows. For $1 \leq i < j \leq d$, $1 \leq x \leq v_i$ and $1 \leq y \leq v_j$, write $(j, y) <_u (i, x)$ if the y-th occurrence of a_j occurs before x-th occurrence of a_i in $u_{i,j}$; write $(i,x) <_u (j,y)$ otherwise. We say that u is *closed* if

$$(k,z) <_u (j,y) <_u (i,x) \text{ implies } (k,z) <_u (i,y)\,, \qquad i<j<k\,,$$

and that u is *open* if

$$(i,x) <_u (j,y) <_u (k,z) \text{ implies } (i,x) <_u (k,z)\,, \qquad i<j<k\,.$$

Lemma. *A tuple u lies in the image of π if and only if u is open and closed.*

The representation of $\mathsf{L}(v)$ as a subset of tuples is quite handy, for example it can be used to argue that $\mathsf{L}(v)$ is a lattice. As closed tuples are closed under meets and open tuples are stable under joins, we obtain a closure operator cl and an interior operator int on $\prod_{i<j}\mathsf{L}(v_i,v_j)$. As moreover the closure of an open tuple is open and the interior of a closed tuple is closed, we obtain formulas for computing meets and joins in $\mathsf{L}(v)$ given its representation as a subset of tuples:

$$u \wedge_{\mathsf{L}(v)} u' = \operatorname{int}(u \wedge_{\prod_{i<j}\mathsf{L}(v_i,v_j)} u')\,, \qquad u \vee_{\mathsf{L}(v)} u' = \operatorname{cl}(u \wedge_{\prod_{i<j}\mathsf{L}(v_i,v_j)} u')\,.$$

Meet-Continuous Functions on the Unit Interval

We denote by $\mathbb{I}$ the unit interval $[0,1]$. A monotone function $f:\mathbb{I}\to\mathbb{I}$ is *meet-continuous* if

$$f(\bigwedge X) = \bigwedge f(X)\,,$$

for every $X\subseteq\mathbb{I}$. We denote $\mathsf{L}(\mathbb{I}^2)$ the poset of meet-continuous functions with the pointwise ordering. $\mathsf{L}(\mathbb{I}^2)$ is complete, and it is a sub-lattice of the product lattice $\mathbb{I}^{\mathbb{I}}$ (notice however that the inclusion does not preserves the bottom element); it is therefore a distributive lattice.

Define a subdivision as a finite sequence $\{\,(x_i,y_i)\in\mathbb{I}^2 \mid i=0,\ldots,n\,\}$ such that the x_i,y_i are rational numbers, $0=x_0<x_1<\ldots<x_n=1$, and $y_0\leq y_1\leq\ldots\leq y_n=1$. The *rational step function* associated to the subdivision $S=\{\,(x_i,y_i)\mid i=0,\ldots,n\,\}$ is defined by the formula

$$f_S(x)=y_i \text{ if } x\in[x_i,x_{i+1}) \text{ for some } i<n, \text{ and } f_S(x)=1 \text{ if } x=1.$$

Rational step functions form a sublattice of $\mathsf{L}(\mathbb{I}^2)$ isomorphic to $\bigcup_{v\in\mathbb{N}^2}\mathsf{L}(v)$.

Proposition. *Every meet-continuous function is an infinite join and an infinite meet of rational step functions. Therefore $\mathsf{L}(\mathbb{I}^2)$ is the Dedekind-MacNeille completion of $\bigcup_{v\in\mathbb{N}^2}\mathsf{L}(v)$.*

Let us observe that elements of $\mathsf{L}(\mathbb{I}^2)$ are in bijection with other kind of geometric objects. By a path in $\mathbb{I}^2$ we mean the image of a bicontinuous (meet- and join-continuous) function $\pi:\mathbb{I}\to\mathbb{I}^2$. Paths can be characterized as dense chains of $\mathbb{I}^2$ that are sub-complete-lattices of $\mathbb{I}^2$. Elements of $\mathsf{L}(\mathbb{I}^2)$ are in bijection with paths. Of course, paths are also in bijection with join-continuous functions on the unit interval – let us denote by $\mathsf{L}_\vee(\mathbb{I}^2)$ the collection of those functions. Next, consider that:

(a) the function $\ell:\mathsf{L}(\mathbb{I}^2)\to\mathsf{L}_\vee(\mathbb{I}^2)$, mapping f to its left adjoint, is an order-reversing bijection,
(b) the function $(\,\cdot\,)^*:\mathsf{L}_\vee(\mathbb{I}^2)\to\mathsf{L}(\mathbb{I}^2)$, mapping f to f^* defined by $f^*(x)=\bigwedge_{x<x'}f(x')$, is an order-preserving bijection.

By composing these bijections, we obtain the following order-reversing operator on $\mathsf{L}(\mathbb{I}^2)$:

$$(\,\cdot\,)^{\perp} : \mathsf{L}(\mathbb{I}^2) \xrightarrow{\ \ell\ } \mathsf{L}_{\vee}(\mathbb{I}^2) \xrightarrow{(\,\cdot\,)^*} \mathsf{L}(\mathbb{I}^2)\,.$$

This operator, which actually is an involution, shall be needed next to define the weak Bruhat order in higher dimensions.

The Continuous Weak Bruhat Order

To construct something analogous to $\mathsf{L}(\mathbb{I}^2)$ in higher dimensions, we mimic the construction of $\mathsf{L}(v)$ as a suborder of $\prod_{i<j}\mathsf{L}(v_i,v_j)$. For $f = \{\, f_{i,j}\,\} \in \prod_{1\leq i<j\leq d}\mathsf{L}(\mathbb{I}^2)$, $1 \leq i < j \leq d$, and $x, y \in \mathbb{I}$, write:

$$(j,y) <_f (i,x) \text{ if } y < f_{i,j}(x)\,, \qquad (i,x) <_f (j,y) \text{ if } x < f^{\perp}_{i,j}(y)\,.$$

We remark that, contrary to the discrete case, $(j,y) \not<_f (i,x)$ does not imply $(j,y) <_f (i,x)$. We say that f is *closed* if

$$(k,z) <_f (j,y) <_f (i,x) \text{ implies } (k,z) <_f (i,x)\,, \qquad i<j<k\,,$$

and that f is *open* if

$$(i,x) <_f (j,y) <_f (k,z) \text{ implies } (i,x) <_f (k,z)\,, \qquad i<j<k\,.$$

Proposition. *The following statements hold:*

(i) closed tuples are stable under arbitrary intersections, and open tuples are stable under arbitrary unions,
(ii) the closure of an open tuple is open, and the interior of a closed tuple is closed,
(iii) the collection $\mathsf{L}(\mathbb{I}^d)$ of tuples that are both open and closed is a complete lattice where meets and joins are given by the formulas:

$$\bigwedge\nolimits_{\mathsf{L}(\mathbb{I}^d)} F = \mathrm{int}\big(\bigwedge\nolimits_{\prod_{i<j}\mathsf{L}(\mathbb{I}^2)} F\,\big)\,, \qquad \bigvee\nolimits_{\mathsf{L}(\mathbb{I}^d)} F = \mathrm{cl}\big(\bigvee\nolimits_{\prod \mathsf{L}(\mathbb{I}^2)} F\,\big)\,.$$

Let $i_{i,j} : \mathsf{L}(v_i,v_j) \to \mathsf{L}(\mathbb{I}^2)$ be the canonical inclusion. By taking the product of these maps, we obtain a map $i : \prod_{i<j}\mathsf{L}(v_i,v_j) \to \prod_{i<j}\mathsf{L}(\mathbb{I}^2)$ which has the property that $i(\mathrm{cl}(f)) = \mathrm{cl}(i(f))$; in particular i sends closed tuples to closed tuples; dually, i sends open tuples to open tuples. We obtain therefore families of monomorphisms

$$i_v : \mathsf{L}(v) \to \mathsf{L}(\mathbb{I}^d)$$

forming a cocone w.r.t. the directed diagram of the $\mathsf{L}(v)$. Therefore an entire copy of the colimit $\bigcup_{v\in\mathbb{N}^d}\mathsf{L}(v)$ resides inside $\mathsf{L}(\mathbb{I}^d)$.

Theorem. *The lattice $\mathsf{L}(\mathbb{I}^d)$ is the Dedekind-MacNeille completion of the lattice $\bigcup_{v\in\mathbb{N}^d}\mathsf{L}(v)$.*

Lattices and Discrete Geometry

A main motivation to develop this work came from discrete geometry. In this context, the Christoffel word [2] of type n, m, noted $C_{n,m}$, is the best lower approximation of straight line from $(0,0)$ to the point (n,m) by a discrete path on the plane (usually n and m are chosen to be coprime integers). Figure 1 illustrates this with the Christoffel word $C_{4,3} = xxyxyxy$.

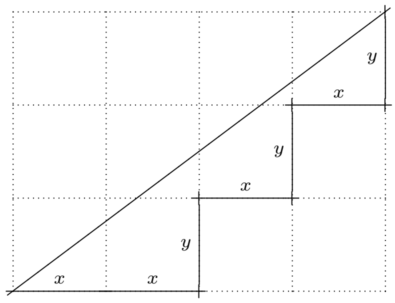


FIGURE 1. The Christoffel $C_{4,3}$

We can rephrase such a definition in a purely lattice-theoretic way, by saying that

$$C_{n,m} = \rho_{n,m}(\Delta)\,,$$

where $\rho_{n,m}$ is right adjoint to the inclusion $i_{n,m} : \mathsf{L}(n,m) \to \mathsf{L}(\mathbb{I}^2)$, and Δ is the identity.

Our work allows to propose a definition of Christoffel words in higher dimension, in a similar way. Indeed, every image of a bicontinuous map $\pi : \mathbb{I} \to \mathbb{I}^d$ gives rise to an open-closed tuple in $\mathsf{L}(\mathbb{I})$ – we do not know yet whether this is a bijection. By taking the diagonal as π, the corresponding tuple, noted Δ, is made up of identities. We can then define C_v, the Christoffel word associated to the vector $v \in \mathbb{N}^d$, by the following formula

$$C_v = \rho_v(\Delta)\,,$$

where now ρ_v is the right adjoint to the inclusion $i_v : \mathsf{L}(v) \to \mathsf{L}(\mathbb{I}^d)$.

REFERENCES

[1] M. K. Bennett and G. Birkhoff. Two families of Newman lattices. *Algebra Universalis*, 32(1):115–144, 1994.

[2] Jean Berstel, Aaron Lauve, Christophe Reutenauer, and Franco V. Saliola. *Combinatorics on words*, volume 27 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2009. Christoffel words and repetitions in words.

[3] Luigi Santocanale. On the join dependency relation in multinomial lattices. *Order*, 24(3):155–179, 2007.

LIF, CNRS UMR 6166, UNIVERSITÉ AIX-MARSEILLE I, 13453 MARSEILLE CEDEX 13, FRANCE
E-mail address: luigi.santocanale@lif.univ-mrs.fr
URL: http://www.lif.univ-mrs.fr/~lsantoca/

On Ultrafilter Extensions of Models

Denis I. Saveliev

We present a new area in model theory, concerning ultrafilter extensions of arbitrary models. Given a model $\mathfrak{A} = (X, F, \ldots, P, \ldots)$, we extend it, in a canonical way, to a model $\boldsymbol{\beta}\mathfrak{A} = (\boldsymbol{\beta}X, \tilde{F}, \ldots, \tilde{P}, \ldots)$, where the underlying set $\boldsymbol{\beta}X$ consists of all ultrafilters over X and carries the standard compact Hausdorff topology. The extension lifts homomorphisms between models: continuous extensions of homomorphisms of $\mathfrak{A}$ into $\mathfrak{B}$ are homomorphisms of $\boldsymbol{\beta}\mathfrak{A}$ into $\boldsymbol{\beta}\mathfrak{B}$. Moreover, if a model $\mathfrak{C}$ carries a compact Hausdorff topology which is compatible with its structure (like the topology and the structure of $\boldsymbol{\beta}\mathfrak{A}$), then continuous extensions of homomorphisms of $\mathfrak{A}$ into $\mathfrak{C}$ are homomorphisms of $\boldsymbol{\beta}\mathfrak{A}$ into $\mathfrak{C}$. These statements remain true for embeddings and other relationships between models. Thus the construction provides a right generalization of the Stone–Čech (or Wallman) compactification of a discrete space X to the case when X carries a first-order structure.

The construction, together with the mentioned basic results, appeared very recently in [1]. For algebras, ultrafilter extensions were independently discovered in [2]. A particular case, when algebras are semigroups, is known from 70s and used as a powerful tool in number theory, algebra, topological dynamics, and ergodic theory, see [3, 4]. There is also another construction of ultrafilter extensions, which came from modal logic, see [2, 5, 6, 7]. These extensions deal with relational structures and coincide with our extensions only for unary relations; we briefly discuss the connection.

Further, we present several new (yet unpublished) results from [8, 9]. We show that certain ultrafilters form submodels: given any infinite cardinal κ, the set of κ-complete ultrafilters over X forms a submodel of $\boldsymbol{\beta}\mathfrak{A}$, and under certain circumstances the set of κ-uniform ultrafilters over X forms a closed submodel of $\boldsymbol{\beta}\mathfrak{A}$. Although in simplest cases ultrafilter extensions can be elementary, in general they are highly complicated objects and their equational theories quite differ from the equational theories of extended models. We describe atomic formulas that are preserved under ultrafilter extensions. Finally, we mention some applications of these results in algebra.

References

[1] D. I. Saveliev. *Ultrafilter extensions of models.* Lecture Notes in AICS, Springer, 6521 (2011), 162–177.

[2] V. Goranko. *Filter and ultrafilter extensions of structures: universal-algebraic aspects.* Manuscript, 2007.

[3] J. Berglund, H. Junghenn, and P. Milnes. *Analysis on Semigroups.* Wiley, N.Y., 1989.

[4] N. Hindman, D. Strauss. *Algebra in the Stone–Čech Compactification.* W. de Gruyter, Berlin–N.Y., 1998.

[5] J. van Benthem. *Notes on modal definability.* Notre Dame J. Formal Logic, 30:1 (1988), 20–35.

[6] R. Goldblatt. *Varieties of complex algebras.* Annals of Pure and Applied Logic, 44 (1989), 173–242.

[7] Y. Venema. *Model definability, purely modal.* In: J. Gerbrandy et al. (eds). JFAK. Essays Dedicated to Johan vam Benthen on the Occasion on his 50th Birthday. Amsterdam, 1999.

[8] D. I. Saveliev. *On Hindman sets.* Manuscript, 2008.

[9] D. I. Saveliev. *On identities stable under ultrafilter extensions.* In progress.

Finite model property of pretransitive analogs of S5*

Andrey Kudinov† Ilya Shapirovsky†

We consider propositional normal unimodal *pretransitive* logics, i.e., logics with expressible 'transitive' modality. There is a long-standing open problem about the finite model property (fmp) and decidability of pretransitive logics, in particular – for the logics $\mathrm{K}_n^m = \mathrm{K} + \Box^m p \to \Box^n p$, $n > m > 1$.

A pretransitive logic L has the fmp or is decidable, only if these properties hold for the logic L.sym*, which is the extension of L with the symmetry axiom for 'transitive' modality: like S5 can be embedded into S4, L.sym* can be embedded into L.

We show that for all $n > m \geq 1$, the logics K_n^m.sym* have the fmp.

Pretransitive logics.

Definition 1 ([2])**.** A logic L is called *pretransitive* (according to [2] – *conically expressive*), if there exists a formula $\chi(p)$ with a single variable p such that for any Kripke model M with $\mathsf{M} \vDash \mathrm{L}$ and for any w in M we have:

$$\mathsf{M}, w \vDash \chi(p) \Leftrightarrow \forall u(wR^*u \Rightarrow \mathsf{M}, u \vDash p),$$

where R^* is the transitive closure of the acceptability relation on M.

To give a syntactic description of pretransitive logics, put $\Box^{\leq n}\varphi = \bigwedge_{i=0}^{n} \Box^i \varphi$, where $\Box^0\varphi = \varphi$, $\Box^{i+1}\varphi = \Box\Box^i\varphi$.

Lemma 2 (Shehtman, 2010)**.** L *is pretransitive iff* $\mathrm{L} \vdash \Box^{\leq m}p \to \Box^{\leq m+1}p$ *for some* $m \geq 1$.

By this lemma, for any pretransitive logic there exists the least m such that the formula $\Box^* p = \Box^{\leq m} p$ plays the role of $\chi(p)$ from Definition 1. Let $\Diamond^*\varphi = \neg\Box^*\neg\varphi$.

Consider the logics $\mathrm{K}_n^m = \mathrm{K} + \mathrm{A}_n^m$, where $\mathrm{A}_n^m = \Box^m p \to \Box^n p$, $n > m \geq 1$. For any m, n, A_n^m is a Sahlqvist formula, which corresponds to the property $R^n \subseteq R^m$; so all K_n^m are canonical, elementary and Kripke-complete pretransitive logics. If $m = 1, n = 2$, we obtain the well-known logic K4, which has the fmp. In fact, due to [1], all logics K_n^1 have the fmp. Logics with $m > 1$ were also considered (to our knowledge, K_3^2 appears already in the 1960s in papers by Segerberg and Sobociński); nevertheless, no results about the fmp or decidability for these logics are known yet.

Logics with the symmetry axiom for $\Box^*$. For a pretransitive logic L, put

$$\mathrm{L.sym}^* = \mathrm{L} + (p \to \Box^*\Diamond^* p).$$

(In [3], logics of this kind were considered in the particular case where $\mathrm{L} = \mathrm{K} + \Box^{\leq m}p \to \Box^{\leq m+1}p$.) It is well-known that for any formula φ, $\mathrm{S5} \vdash \varphi \Leftrightarrow \mathrm{S4} \vdash \Diamond\Box\varphi$ ([4]). The following is a generalization of this fact.

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†Institute for Information Transmission Problems, Moscow, Russia

Theorem 3. *If* L *is a pretransitive logic, then for any formula* φ *we have*

$$\mathrm{L.sym}^* \vdash \varphi \Leftrightarrow \mathrm{L} \vdash \Diamond^*\Box^*\varphi.$$

Before we prove this theorem, we formulate two simple corollaries of Lemma 2.

Proposition 4. *For a pretransitive* L *and a point generated* L*-frame* $\mathsf{F} = (W, R)$, $\mathsf{F} \vDash \mathrm{L.sym}^*$ *iff* R^* *is the universal relation on* W.

Proposition 5. *For a pretransitive* L *and a formula* φ, *let* φ^* *be the formula obtained from* φ *by replacing* $\Box$ *with* $\Box^*$ *and* $\Diamond$ *with* $\Diamond^*$. *Then for any* φ *we have:* $\mathrm{S4} \vdash \varphi \Rightarrow \mathrm{L} \vdash \varphi^*$, $\mathrm{S5} \vdash \varphi \Rightarrow \mathrm{L.sym}^* \vdash \varphi^*$.

Proof of Theorem 3. If $\mathrm{L} \vdash \Diamond^*\Box^*\varphi$, then $\mathrm{L.sym}^* \vdash \Diamond^*\Box^*\varphi$. $\mathrm{S5} \vdash (\Diamond\Box p \to p)$, so using the above proposition, we have $\mathrm{L} \vdash \varphi$.

To prove the converse direction, we proceed by induction on a derivation of φ.

Suppose $\varphi = p \to \Box^*\Diamond^* p$. Since $\mathrm{S4} \vdash \Diamond\Box(p \to \Diamond\Box p)$, by the above proposition $\mathrm{L} \vdash \Diamond^*\Box^*\varphi$.

Suppose $\mathrm{L.sym}^* \vdash \psi_1$, $\mathrm{L.sym}^* \vdash \psi_1 \to \varphi$. By the induction hypothesis, $\mathrm{L} \vdash \Diamond^*\Box^*\psi_1$, $\mathrm{L} \vdash \Diamond^*\Box^*(\psi_1 \to \varphi)$. Then $\mathrm{L} \vdash \Box^*\Diamond^*\Box^*\psi_1$, $\mathrm{L} \vdash \Box^*\Diamond^*\Box^*(\psi_1 \to \varphi)$ (using $\Box$-rule, one can easily show that $\Box^*$-rule is admissible in L). $\mathrm{S4} \vdash \Box\Diamond\Box p \wedge \Box\Diamond\Box(p \to q) \to \Diamond\Box q$, since this formula is valid in any finite S4-frames. So using Proposition 5, we have $\Diamond^*\Box^*\varphi$.

The case when φ is obtained by the substitution rule is trivial.

Suppose $\varphi = \Box\psi$, $\mathrm{L.sym}^* \vdash \psi$. It is easy to check (e.g., using the completeness of the logics $\mathrm{K}+\Box^{\leq m}p \to \Box^{\leq m+1}p$) that $\mathrm{L} \vdash \Diamond^*\Box^* p \to \Diamond^*\Box^*\Box p$. By the induction hypothesis, $\mathrm{L} \vdash \Diamond^*\Box^*\psi$, so $\mathrm{L} \vdash \Diamond^*\Box^*\varphi$. □

Corollary 6. *If* L *has the fmp, then* $\mathrm{L.sym}^*$ *also has the fmp.*

Proof. If a formula φ is $\mathrm{L.sym}^*$-consistent then $\Box^*\Diamond^*\varphi$ is satisfiable in a finite L-frame (W, R). It follows that φ is satisfiable in a maximal R^*-cluster, which is an $\mathrm{L.sym}^*$-frame. □

Thus, for a pretransitive L, any negative result about decidability or the fmp for $\mathrm{L.sym}^*$ transfers to L. At the same time, the authors do not know any examples of such $\mathrm{L.sym}^*$. Moreover, next we prove that $\mathrm{K}_n^m.\mathrm{sym}^*$ have the fmp for all $n > m \geq 1$.

Finite model property. By Sahlqvist's Theorem, all logics $\mathrm{K}_n^m.\mathrm{sym}^*$ are canonical and elementary. The class of all $\mathrm{K}_n^m.\mathrm{sym}^*$-frames can be easily characterized in terms of paths and cycles. By an *R-path* Σ in (W, R) we mean a finite sequence of at least two (not necessary distinct) points $(x_0, x_1, \ldots, x_l)$, such that $x_i R x_{i+1}$ for all $i < l$; we say that Σ *connects* x_0 and x_l. l is the *length* of Σ (notation: $[\Sigma]$). If $x_l = x_0$ then Σ is an *R-cycle.*

Proposition 7. *Suppose* $n > m \geq 1$, F *is a point generated frame which is not an irreflexive singleton. Then* $\mathsf{F} \vDash \mathrm{K}_n^m.\mathrm{sym}^*$ *iff any two points in* W *belong to an R-cycle, and for any* w, u, *if* w, u *are connected by an R-path with the length* n, *then* w, u *are connected by an R-path with the length* m.

Proposition 8. *For any* $s, r \geq 0$, $\mathrm{K}_n^m \vdash \Diamond^{m+(n-m)q+r}p \to \Diamond^{m+r}p$.

Proof. By an easy induction on q. □

Proposition 9. *All logics* $\mathrm{K}_n^m.\mathrm{sym}^*$ *are different.*

Proof. Let $L_1 = \mathrm{K}_n^m.\mathrm{sym}^*$ and $L_2 = \mathrm{K}_t^s.\mathrm{sym}^*$. First, we assume that $s < m$, then we consider the following frame

$$\mathsf{F} = (W, R),\ W = \{0, 1, \ldots, m\}\,,\ xRy \Leftrightarrow y = x \text{ or } y \equiv x + 1\ (mod\ m + 1).$$

It is easy to check that $\mathsf{F} \models L_1$ and $\mathsf{F} \not\models L_2$.

Now assume that $s = m$ and $t < n$. Put $k = n - m$,

$$\mathsf{F}' = (W', R'),\ W' = \{0, 1, \ldots, k-1\}\,,\ xR'y \Leftrightarrow y \equiv x + 1\ (mod\ k).$$

It is also easy to see that $\mathsf{F}' \models L_1$ and $\mathsf{F}' \not\models L_2$. □

Theorem 10. *The logics* $\mathrm{K}_n^m.\mathrm{sym}^*$ *have the fmp for all* $n > m \geq 1$.

If $m = 1$, the statement of the theorem immediately follows from [1] and Corollary 6. Also, for the case $m = n+1$, this theorem can be easily proved by the straightforward filtration argument (the same reasoning works if we consider $\mathrm{K} + \Box^{\leq m}p \to \Box p^{\leq m+1}$ instead of K_{m+1}^m, [3]). Nevertheless, the standard filtration argument does not work for the arbitrary case: to preserve validity of A_n^m, we have to construct a countermodel in a more subtle way. First, we need the following slightly modified version of filtration.

Definition 11. Let $\mathsf{M} = (W, R, \theta)$ be a model, φ be a formula, $\sim$ be an equivalence relation on W. For $u, v \in W$, we define

$$u \sim^\varphi v \text{ iff } u \sim x \text{ and } M, u \vDash \psi \Leftrightarrow M, v \vDash \psi \text{ for every subformula } \psi \text{ of } \varphi.$$

Let $\overline{W} = W/\sim^\varphi$, $\overline{u}\overline{R}\overline{v} \Leftrightarrow \exists u' \in \overline{u}\, \exists v' \in \overline{v} (u'Rv')$, $\overline{\theta}(p) = \{\overline{u} \mid u \in \theta(p)\}$ for all variables of φ (and put $\overline{\theta}(p) = \varnothing$ for other variables). The model $(\overline{W}, \overline{R}, \overline{\theta})$ is called *the (minimal)* $\sim$*-filtration of* M *through* φ.

Note that in the case when $\sim$ is the universal relation, the $\sim$-filtration is the standard *minimal filtration.* Clearly, $\sim$-filtrations preserve truth of subformulas of φ. Also, if $W/\sim$ is finite, then $W/\sim^\varphi$ is finite too.

Proposition 12. *Let* $(\overline{W}, \overline{R}, \overline{\theta})$ *be a* $\sim$*-filtration of* (W, R, θ).

- *For any* $l > 0$, $xR^l y$ *implies* $\overline{x}\,\overline{R}^l\,\overline{y}$.
- *If* R^* *is universal on* W, *then* $\overline{R}^*$ *is universal on* $\overline{W}$.

The proof of the above proposition is straightforward. The main difficulty in the proof of the theorem is to find an appropriate equivalence relation to make sure that A_n^m is valid in the resulting frame.

For a set of integers I, let $gcd(I)$ denotes its greatest common devisor.

Proof of Theorem 10. Let $\mathrm{L} = \mathrm{K}_n^m.\mathrm{sym}^*$, $k = m - n$. Consider an infinite rooted L-frame $\mathsf{F} = (W, R)$, and suppose that $\mathsf{M} = (W, R, \theta), x \vDash \varphi$. We construct a finite L-frame $\overline{\mathsf{F}} = (\overline{W}, \overline{R})$ where φ is satisfiable.

For a positive integer d, consider the relation $\sim_d$ on W: $u \sim_d w$ iff there exists an R-path Γ from u to w such that d divides $[\Gamma]$.

Claim 1. If d divides the length of any R-cycle in F, then $\sim_d$ is an equivalence relation and $W/{\sim_d}$ is finite.

Clearly, $\sim_d$ is transitive. $\sim_d$ is reflexive, since for any $w \in W$ there exists an R-path from w to w. If $u \sim_d w$, then d divides $[\Gamma^\uparrow]$ for some R-path $\Gamma^\uparrow$ from u to w. Let $\Gamma^\downarrow$ be an R-path from w to u. Then d divides $[\Gamma^\uparrow] + [\Gamma^\downarrow]$, so d divides $[\Gamma^\downarrow]$, and $w \sim_d u$.

To show that $W/{\sim_d}$ is finite, take points $w_1 R w_2 R \ldots R w_d$ (we can choose these points because F is serial). If $u \in W$, then some Γ connects w_d and u. Then $w_{d-r} \sim_d u$, where r is the remainder of the division $[\Gamma]$ by d.

To illustrate the following construction, first we consider the simplest case when k is a prime number or $k = 1$. In this case, we have two possibilities:
(a) there exists an R-cycle Γ_0 such that $gcd([\Gamma_0], k) = 1$;
(b) k divides the length of any R-cycle in F.

Suppose (a). Let us show that $wR^l u$ for any $l \geq m$, $w, u \in W$. Let v be the starting point of Γ_0, Γ_1 be an R-path from w to v, and Γ_2 be an R-path from v to u. For some $r < k$ we have $l + [\Gamma_1] + [\Gamma_2] \equiv r \ (mod\ k)$. Consider the path $\Gamma = \Gamma_1 \Gamma_0^{l+k-r} \Gamma_2$ (that is, Γ goes along Γ_1 then $l + k - r$ times along Γ_0 and then along Γ_2). Thus Γ connects w and u, and $[\Gamma] = l + qk$ for some $q > 0$. By Proposition 8, $wR^l u$.

Let $(\overline{\mathsf{F}}, \overline{\theta})$ be the minimal filtration of M through φ. By Proposition 12, between any two point in $\overline{W}$ there exists an $\overline{R}$-path with the length m, so $\overline{\mathsf{F}} \vDash \mathrm{L}$.

Suppose (b). In this case, $\sim_k$ is an equivalence relation on W. Let $(\overline{W}, \overline{R}, \overline{\theta})$ be the $\sim_k$-filtration of M through φ. Let us show that $(\overline{W}, \overline{R}) \vDash \mathrm{A}_n^m$. Suppose that $\overline{x}\overline{R}^n\overline{y}$. It means that we have for some $x_0, x_0', \ldots, x_n, x_n'$: $x_0 = x$, $x_n' = y$, and $x_i \sim_d x_i'$ & $x_i' R x_{i+1}$ for all $i < n$. Now, since $x_i \sim_d x_i'$ implies $x_i R^{q_i k} x_i'$ (for some q_i), there is an R-path Γ from x to y with $[\Gamma] = n + qk$, $q = \sum q_i$. Thus, $xR^{m+(q+1)k}y$, and $xR^m y$ (Proposition 8), and so $\overline{x}\overline{R}^m\overline{y}$ (Proposition 12). Hence $\overline{\mathsf{F}} \vDash \mathrm{L}$.

Now we extend the above construction for arbitrary k. In this case, we need a combination of reasonings from (a) and (b).

Let $D = \{gcd([\Gamma], k) \mid \Gamma \text{ is an } R\text{-cycle in } W\}$, and let d be the greatest common devisor of D. Let us assume that $D = \{d_1, \ldots, d_s\}$.

Claim 2. There exists positive integers $a_1, \ldots, a_s$ and R-cycles $\Gamma_1, \ldots, \Gamma_s$ such that

$$a_1[\Gamma_1] + \cdots + a_s[\Gamma_s] \equiv d \ (mod\ k).$$

To prove this claim, note that for every d_i there exists an R-cycle Γ_i and a positive integer l_i, such that

$$[\Gamma_i] = l_i d_i \text{ and } l_i \equiv 1 \ (mod\ k).$$

By the Euclidean algorithm, we have $\sum_{i=1}^s b_i d_i = d$ for some integers b_i, therefore $\sum_{i=1}^s a_i d_i \equiv d \ (mod\ k)$ for some $a_i > 0$. Since $l_i \equiv 1 \ (mod\ d)$, $\sum_{i=1}^s a_i l_i d_i \equiv d \ (mod\ k)$, which proves the claim.

By Claim 1, $\sim_d$ is an equivalence on W. Let $(\overline{W}, \overline{R}, \overline{\theta})$ be the $\sim_d$-filtration of M through φ. Similarly to the case (b), we obtain that if $\overline{u}\overline{R}^n\overline{w}$, then $u \in R^{n+dr}w$ for some $r \geq 0$. By Proposition 8, we may assume that $r < k$.

Let v_i denote the starting point of Γ_i, $\Delta_i^{\uparrow}$ be an R-path from w to v_i, $\Delta_i^{\downarrow}$ – from v_i to w. Let $\Sigma_i = \Delta_i^{\uparrow\,k-1} \Delta_i^{\downarrow\,k-1} \Delta_i^{\uparrow} \Gamma_i^{(k-r)a_i} \Delta_i^{\downarrow}$. So Σ_i is an R-path from w to w and $[\Sigma_i] \equiv (k - r)a_i[\Gamma_i] \ (mod\ k)$. Let $\Gamma = \Sigma_0 \Sigma_1 \ldots \Sigma_s$, where Σ_0 is an R-path from u to w with the length $n + dr$. By Claim 2, $[\Gamma] \equiv m \ (mod\ k)$. Thus, $uR^m w$, $\overline{u}\overline{R}^m\overline{w}$ and $\overline{\mathsf{F}} \vDash \mathrm{A}_n^m$. □

References

[1] D. Gabbay. A general filtration method for modal logics. *Journal of Philosophical Logic*, 1(1):29–34, 1972.

[2] D. Gabbay, V. Shehtman, and D. Skvortsov. *Quantification in Nonclassical Logic*. Elsevier, 2009.

[3] R. Jansana. Some logics related to von Wright's logic of place. *Notre Dame Journal of Formal Logic*, 35(1):88–98, 1994.

[4] K. Matsumoto. Reduction theorem in Lewis's sentential calculi. *Mathematica Japonicae*, 3:133–135, 1955.

Hybrid products of modal logics

Valentin Shehtman
Institute for Information Transmission Problems, Moscow, Russia

1. In this talk we consider normal polymodal propositional logics. Standard n-modal formulas are built from a countable set PV of propositional variables using the classical connectives $\to, \bot$ and the unary modal connectives $\Box_1, \dots, \Box_n$. *Closed formulas* do not contain propositional variables. An *n-modal logic* is a set of n-modal formulas containing the minimal logic $\mathbf{K}_n$ and closed under the standard rules (Modus Ponens, Necessitation, and Substitution). $\mathbf{\Lambda} + S$ denotes the smallest modal logic containing the modal logic $\mathbf{\Lambda}$ and a set of modal formulas S.

An *n-frame* has n accessibility relations: $F = (W, R_1, ..., R_n)$. The logic $\mathbf{L}(F)$ is the set of all n-modal formulas valid in F; for an n-modal logic $\mathbf{\Lambda}$, $\mathbf{V}(\mathbf{\Lambda})$ denotes the class of all n-frames validating $\mathbf{\Lambda}$ (*$\mathbf{\Lambda}$-frames*).

Recall that the *product* of an n-frame $F = (W, R_1, ..., R_n)$ and an m-frame $G = (V, S_1, ..., S_m)$ is the $(n+m)$-frame

$$F \times G = (W \times V, R_{11}, \dots, R_{n1}, S_{12}, \dots, S_{m2}),$$

where

$$(x, y)R_{i1}(x', y') \Leftrightarrow xR_ix' \;\&\; y = y';\;\; (x, y)S_{j2}(x', y') \Leftrightarrow x = x' \;\&\; yS_jy'.$$

For n-modal logic L_1 and an m-modal logic L_2, with $\mathbf{V}(L_1), \mathbf{V}(L_2) \neq \varnothing$, the *product* is the $(n+m)$-modal logic

$$L_1 \times L_2 := \mathbf{L}(\{F_1 \times F_2 \;\mid F_1 \vDash L_1,\; F_2 \vDash L_2\}.$$

The *commutative join* of an n-modal logic L_1 and an m-modal logic L_2 is defined as follows:

$$[L_1, L_2] := L_1 * L_2 + \{Com_{ij} \mid 1 \le i \le n,\; 1 \le j \le m\} + \{CR_{ij} \mid 1 \le i \le n,\; 1 \le j \le m\},$$

where $L_1 * L_2$ is the fusion,

$$Com_{ij} := (\Box_i \Box_{n+j} p \leftrightarrow \Box_{n+j} \Box_i p), \quad CR_{ij} := (\Diamond_i \Box_{n+j} p \to \Box_{n+j} \Diamond_i p).$$

The logics L_1, L_2 are called *product matching* if $L_1 \times L_2 = [L_1, L_2]$.

Definition 1 *A modal formula A is* Horn *if the class of its frames $\mathbf{V}(A)$ is axiomatized by a first-order Horn sentence. A modal logic is* Horn axiomatizable *if it can be axiomatized by a set of modal formulas that are Horn or closed.*

The proof of the following result can be found in :

Theorem 2 *(cf. [3], [4]) Every two Horn axiomatizable modal logics are product matching.*

2. The *hybrid modal language* $H(E)_{m,n}$ contains the same symbols as the standard $(n+m)$-modal language (we use the symbols $\blacksquare_i$ rather than $\Box_{n+i}$) and also the countable set of *nominals* (propositional constants) Nom and the universal modality $[\forall]$ (which may be regarded as $\blacksquare_{m+1}$).

These hybrid modal formulas are interpreted in hybrid Kripke models (F, θ), where $F = (W_1, R_1, \ldots, R_n) \times (W_2, W_2 \times W_2, S_1, ...S_m)$ and θ is a function defined on $Nom \cup PV$ such that $\theta(q) \subseteq W_1 \times W_2$ for $q \in PV$ and $\theta(c) \in W_2$ for $c \in Nom$. The truth definition $(M, x \vDash A)$ for $M = (F, \theta)$ and $A \in H(E)_{m,n}$ is quite standard, with

- $M, (x, y) \vDash q$ iff $(x, y) \in \theta(q)$; $M, (x, y) \vDash c$ iff $y = \theta(c)$;
- $M, (x, y) \vDash [\forall]A$ iff $\forall z \in W_2\ M, (x, z) \vDash A$;
- $M, (x, y) \vDash \Box_i A$ iff $\forall z \in R_i(x)\ M, (z, y) \vDash A$;
- $M, (x, y) \vDash \blacksquare_j A$ iff $\forall z \in S_j(y)\ M, (x, z) \vDash A$.

The *validity* in F is defined as the truth in all hybrid models over F. The hybrid modal logic $\mathbf{L^H}(F)$ of F is defined as the set of all $H(E)_{m,n}$-formulas valid in F.

Definition 3 *Let L_1, L_2 be (usual) propositional modal logics. Their* hybrid product *is*

$$(L_1 \times L_2)^H := \bigcap \{\mathbf{L^H}(F_1 \times F_2) \mid F_1 \vDash L_1,\ F_2 \vDash L_2\}.$$

It is obvious that $\mathbf{L^H}(F)$ is conservative over $\mathbf{L}^(F)$, so the hybrid product is conservative over the usual product.

Definition 4 *Let L_1, L_2 be (usual) propositional modal logics. Their* hybrid commutative join *is the hybrid modal logic (in $H(E)_{m,n}$) obtained by extending $[L_1, L_2]$ with the following axioms (where $q \in PV$, $c \in Nom$)*

(1) **S5**-*axioms for* $[\forall]$, *(2)* $[\forall]q \to \blacksquare_j q$, *(3)* $[\exists]c$,

(4) $[\exists](c \wedge q) \to [\forall](c \to q)$, *(5)* $c \to \Box_i c$, *(6)* $\Diamond_i c \to c$

and the rules

$$R1\ \frac{A}{[\forall]A}\ ; \qquad (R2)\ \frac{[\exists](c \wedge A)}{A}\ \textit{(if } c \textit{ does not occur in } A\textit{);}$$

$$R3\ \frac{[\exists](c \wedge \blacklozenge_j d) \to [\exists](d \wedge A)}{[\exists](c \wedge \blacksquare_j A)}\ \textit{(if } c \neq d \textit{ and } d \textit{ does not occur in } A\textit{).}$$

Recall that a modal logic $\mathbf{\Lambda}$ is called *r-persistent* if the validity of $\mathbf{\Lambda}$ is transferred from any refined general frame to the underlying Kripke frame. In particular, every logic $\mathbf{\Lambda} \supseteq \mathbf{K4}$, with universally first-order definable $\mathbf{V}(\mathbf{\Lambda})$, is r-persistent [2]; every tabular logic is r-persistent [5].

Theorem 5 *If the logics L_1, L_2 are r-persistent, then they are* hybrid product matching*:*

$$(L_1 \times L_2)^H = [L_1, L_2]^H$$

Note that L_1, L_2 may be hybrid product matching, while L_1, L_2 are not product matching. In these cases $[L_1, L_2]^H$ is not conservative over $[L_1, L_2]$.

Remark 6 One of the anonymous referees has pointed out that Theorem 5 follows from a stronger result by Katsuhiko Sano [6]. Sano considers a more general type of hybrid product using both horizontal and vertical nominals. Instead of the universal modalities he uses @-operators, which does not matter for the axiomatization result, but may be essential for the decision problem (see below).

3. The translation of $H(E)_{m,n}$-formulas into predicate modal formulas is defined by induction:

- $p_i^\sharp(y) := P_i(y)$, $\bot^\sharp(y) := \bot$, $(A \to B)^\sharp(y) := A^\sharp(y) \to B^\sharp(y)$,
- $(\Box_i A)^\sharp(y) := \Box_i A^\sharp(y)$, $([\forall]A)^\sharp(y) := \forall y A^\sharp(y)$,
- $(\blacksquare_j A)^\sharp(y) := \forall z(S_j(y,z) \to A^\sharp(z))$
- $c^\sharp(y) := (y = c)$.

Now suppose L_1 is *conically expressive* [5], i.e., in L_1 there is a derived modal operator $\Box^*$ corresponding to the reflexive transitive closure of $R_1 \cup \dots \cup R_n$. The following modal first-order formulas are called the *rigidity axioms*:

$$Rig_k^* := \Box^* \forall y \forall z((S_k(y,z) \to \Box_i S_k(y,z)) \wedge (\neg S_k(y,z) \to \Box_i \neg S_k(y,z)),$$

where $1 \leq k \leq m$.

Let L_1 be an n-modal logic, $CK(L_1)$ the class of all predicate Kripke frames with constant domains based on L_1-frames. Let $\mathbf{L}(CK(L_1))$ be the corresponding modal predicate logic, i.e., the set of all n-modal predicate formulas valid in these frames.

Let L_2 be an n-modal logic such that the class $\mathbf{V}(L_2)$ is first-order definable by a sentence φ (involving binary predicate letters $S_1, \dots, S_m$).

Proposition 7 *Under the above assumptions, for any $H(E)_{m,n}$-formula A,*

$$(L_1 \times L_2)^H \vdash A \text{ iff } \mathbf{L}(CK(L_1)) \vdash \Box^* \varphi \wedge \bigwedge_{k=1}^{m} Rig_k^* \to \forall y A^\sharp(y).$$

So hybrid products can be regarded as fragments of modal predicate logics with constant domains.

4. The general problem about decidability of hybrid products from Theorem 5 is open. In some cases they are undecidable, because the standard products are undecidable; this happens e.g. in the case of $\mathbf{S4.3} \times \mathbf{S4.3}$ [3]. In other cases (like $(\mathbf{S5} \times \mathbf{S5})^H$) the answer is probably positive. But in the interesting case of the minimal hybrid product $(\mathbf{K} \times \mathbf{K})^H$ we have obtained only a partial result.

Definition 8 *A* @-formula *is an $H(E)_{m,n}$-formula built from propositional variables and nominals using classical connectives, the modalities $\blacksquare_i$, $\Box_j$ and $@_c$ for each nominal c, where $@_c A := [\exists](c \wedge A)$ ($[\forall]$ is not allowed).*

Theorem 9 *$(\mathbf{K}_n \times \mathbf{K}_n)^H$ has the finite model property for @-formulas.*

The proof is by the "filtration through bisimulation" method from [7].

Hence by the finite axiomatizability (Theorem 5), we obtain

Corollary 10 *The @-fragment of $(\mathbf{K}_n \times \mathbf{K}_n)^H$ is decidable.*

However the problem about decidability of Sano's minimal product $[\mathbf{K}^+_{H(@)}, \mathbf{K}^+_{H(@)}]$ remains open.

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References

[1] B. Ten Cate. *Model theory for extended modal languages.* ILLC Dissertation Series DS-2005-01. Amsterdam, 2005.

[2] A.V. Chagrov, M.V. Zakharyaschev. *Modal logic.* Oxford University Press, 1996.

[3] D. M. Gabbay, A. Kurucz, F. Wolter, M. Zakharyaschev. *Many-dimensional modal logics: theory and applications.* Elsevier, 2003.

[4] D. Gabbay, V. Shehtman. Products of modal logics, part 1. *Logic Journal of the IGPL*, 6: pp. 73-146, 1998.

[5] D. Gabbay, V. Shehtman, D. Skvortsov. *Quantification in Nonclassical Logic, Volume 1.* Elsevier, 2009.

[6] K. Sano. Axiomatizing hybrid products: How can we reason many-dimensionally in hybrid logic? *J. Applied Logic*, v.8, 2010, pp. 459-474.

[7] V. Shehtman. Filtration via bisimulation. In: *Advances in Modal Logic*, Volume 5. Renate Schmidt et al., eds. King's College Publications, 2005, pp. 289-308.

Morphisms of quantum triads

Radek Šlesinger

The concept of quantum triad was introduced by D. Kruml in [3], which provides a base for this submission, as a means to reconstruct a quantale acting on a couple of modules equipped with a "bilinear form" into another quantale. In the model case, quantales were built from a pair of quantale modules, yet the triad construction can be performed for some other algebraic structures as well, including non-ordered ones. We will look at morphisms appearing in triads in more detail to obtain additional information on their behaviour.

Just recall that a *sup-lattice* stands for a complete join-semilattice, and sup-lattice homomorphisms are mappings that preserve arbitrary joins. A *(unital) quantale* is a sup-lattice endowed with an associative binary operation (with unit) which distributes over arbitrary joins on both sides. Quantale homomorphisms are supposed to preserve both joins and multiplication. Finally, a *right (left) module* over a quantale Q is a sup-lattice L equipped with a right (left) action of the quantale that distributes over joins on both the module and the quantale side and satisfies the associativity LQQ (QQL). A module over a unital quantale is called unital if the quantale unit acts as identity. Module homomorphisms are sup-lattice homomorphisms preserving the action of the quantale. For quantales Q and R, a (Q, R)-*bimodule* stands for a sup-lattice L which is both a left Q-module and a right R-module, and the associativity QLR holds. For a right Q-module R and a left Q-module L, a map $f\colon L \times R \to Q$ is a bimorphism if $f(l, -)$ and $f(-, r)$ are module homomorphisms. A sup-lattice $R \otimes_Q L$ can be constructed by analogy to the tensor product of a right and a left module over a ring. For more information on quantales and quantale modules, please refer to [4, 6, 7].

Definition. *Let T be a quantale, L be a left T-module, and R be a right T-module. The triple (L, T, R) is called a* triad *if there exists a (T, T)-bimorphism $L \times R \to T$ making respective combinations with quantale actions (TLR, LRT) associative. Since the bimorphism usually does not have to be distinguished, we shall write simply lr.*

Quantum triads can be viewed as a generalization of sup-lattice 2-forms introduced in [5]. The 2-forms represent the case where T is fixed to be the 2-element frame.

Definition. *A quantale Q is called a* solution *of the triad if L is a (T, Q)-bimodule, R is a (Q, T)-bimodule, and there is a bimorphism $R \times L \to Q$ (again written just rl) associating with admissible actions (QRL, RLQ, RTL, LQR, LRL, RLR).*

For every triad there are two special solutions, called simply 0-solution and 1-solution here, with multiplication and actions on L, R given as follows:

- $Q_0 = R \otimes_T L$
  - $(r_1 \otimes l_1) \cdot (r_2 \otimes l_2) = r_1(l_1 r_2) \otimes l_2$
  - $l'(r \otimes l) = (l'r)l$
  - $(r \otimes l)r' = r(lr')$
- $Q_1 = \{(\alpha, \beta) \in \mathrm{End}(L) \times \mathrm{End}(R) \mid \alpha(l)r = l\beta(r) \text{ for any } l \in L, r \in R\}$
  - $(\alpha_1, \beta_1) \cdot (\alpha_2, \beta_2) = (\alpha_2 \circ \alpha_1, \beta_1 \circ \beta_2)$
  - $l'(\alpha, \beta) = \alpha(l')$
  - $(\alpha, \beta)r' = \beta(r')$

These two solutions form a so-called couple [1]. In general, a couple consists of two quantales Q_0 and Q_1 and a coupling map (which is a quantale homomorphism) $\phi\colon Q_0 \to Q_1$ satisfying that Q_0 is a (Q_1, Q_1)-bimodule with $\phi(q)r = qr = q\phi(r)$ for any $q, r \in Q_0$. In the case of triads, the coupling homomorphism is $\phi : (r \otimes l) \mapsto ((- \cdot r)l, r(l \cdot -))$. All solutions Q of (L, T, R) then correspond exactly to factorizations of the couple $Q_0 \to Q_1$, i.e. quantale homomorphisms $\phi_0\colon Q_0 \to Q$ and $\phi_1\colon Q \to Q_1$ satisfying:

- $\phi_1 \circ \phi_0 = \phi$,
- $\phi_0(\phi_1(k)q) = k\phi_0(q)$ and $\phi_o(q\phi_1(k)) = \phi_0(q)k$ (so ϕ_0 becomes a coupling map under scalar restriction over ϕ_1).

Definition. *Let (L, T, R) and $(\overline{L}, T, \overline{R})$ be triads over the same quantale T, and $\varphi_L\colon L \to \overline{L}$ and $\varphi_R\colon R \to \overline{R}$ be module homomorphisms satisfying $lr = \varphi_L(l)\varphi_R(r)$ for any $l \in L$, $r \in R$. The pair (φ_L, φ_R) is then called a* morphism of triads.

When related to the setting of sup-lattice 2-forms, triad morphisms generalize orthomorphisms of 2-forms.

One of possible directions in the study of quantum triads is to apply existing results from the theory of quantales and quantale modules to get new information on the triad structure and to investigate how the solutions change along with the initial structures. For instance, one can start with the following easy result.

A right Q-module M is called flat if the functor $M \otimes_Q -$ which maps left Q-modules to sup-lattices preserves injective homomorphisms.

Proposition. *Let* $(\varphi_L, \varphi_R)\colon (L, T, R) \to (\overline{L}, T, \overline{R})$ *be a triad morphism. It induces a sup-lattice homomorphism* $\varphi_R \otimes_T \varphi_L \colon R \otimes_T L \to \overline{R} \otimes_T \overline{L}$ *between the 0-solutions. Following from the definition of triad morphism,* $\varphi_R \otimes_T \varphi_L$ *is also a quantale homomorphism, which simply implies that*

1. *if both* φ_L *and* φ_R *are surjections,* $\overline{Q_0}$ *is a quantale quotient of* Q_0*,*
2. *if* φ_L *and* φ_R *are injections and* R *and* $\overline{L}$ *(or* L *and* $\overline{R}$*) are flat,* Q_0 *is isomorphic to a subquantale of* $\overline{Q_0}$*.*

Note that, for quantale modules, flatness is equivalent to projectivity (this can be shown in the same way as a specific case in [2]). Characterization of projective quantale modules is available for modules generated by a suitable set of elements [8].

References

[1] J. M. Egger and D. Kruml, *Girard couples of quantales*, Applied Categorical Structures, 18 (2010), pp. 123–133.

[2] A. Joyal and M. Tierney, *An extension of the Galois theory of Grothendieck*, American Mathematical Society, Providence, Rhodes Island, USA, 1984.

[3] D. Kruml, *Quantum triads: an algebraic approach.* http://arxiv.org/abs/0801.0504.

[4] D. Kruml and J. Paseka, *Algebraic and categorical aspects of quantales*, in Handbook of Algebra, vol. 5, North-Holland, 2008, pp. 323–362.

[5] P. Resende, *Sup-lattice 2-forms and quantales*, Journal of Algebra, 276 (2004), pp. 143–167.

[6] K. I. Rosenthal, *Quantales and their applications*, Pitman Research Notes in Mathematics, Longman Scientific & Technical, New York, 1990.

[7] S. A. Solovyov, *On the category Q-Mod*, Algebra universalis, 58 (2008), pp. 35–58.

[8] R. Šlesinger, *Decomposition and projectivity of quantale modules*, Acta Universitatis Matthiae Belii, Series Mathematics, 16 (2010), pp. 81–89.

Department of Mathematics and Statistics, Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic
E-mail address: xslesing@math.muni.cz

Topological categories versus categorically-algebraic topology*

Sergejs Solovjovs[1,2]

[1] Department of Mathematics, University of Latvia
Zellu iela 8, LV-1002 Riga, Latvia
sergejs.solovjovs@lu.lv
[2] Institute of Mathematics and Computer Science, University of Latvia
Raina bulvaris 29, LV-1459 Riga, Latvia

Abstract. This paper shows that a concrete category is fibre-small and topological iff it is concretely isomorphic to a subcategory of some category $\mathbf{Top}(T)$ of categorically-algebraic topological structures that is definable by topological co-axioms in $\mathbf{Top}(T)$.

1 Introduction

Motivated by the considerable diversification in the approaches to lattice-valued topology, we introduced in [6] a common framework for the majority of them, aimed to provide the means of interaction between different topological settings. The new approach is based in category theory and universal algebra, and, therefore, is called *categorically-algebraic* (*catalg*) *topology*. It originates in topological theories of J. Adámek, H. Herrlich, G. E. Strecker [1] (originally, due to O. Wyler [10]) and S. E. Rodabaugh [5]. While the latter approach fits easily into the catalg framework, up to now, there has been no explicit elaboration of relationships between catalg topology and topological theories of J. Adámek *et al.* These theories are briefly touched in [5], which claims to resolve completely the relationships between them and those of [5]. Since the claimed resolution is neither complete nor error-free, we decided to consider a more general question on relations between *universal topology* [1] (the term is due to H. Herrlich [3]) and catalg topology. As the main result of the study, it appears that a concrete category is fibre-small and topological iff it is isomorphic to a full subcategory of some category $\mathbf{Top}(T)$ of catalg topological structures that is definable by topological co-axioms in $\mathbf{Top}(T)$ (in the sense of [1, 3]).

Since catalg topology is implicitly fuzzy topology, the achievement can be interpreted as a general fuzzification procedure for the objects of fibre-small topological categories (in the sense of, e.g., *fuzzy theories* of E. G. Manes [4]).

2 Preliminaries

This section provides the theoretical background for the results of the paper.

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2.1 Catalg topology

Definition 1. *1. A* catalg backward powerset theory *(*cabp-theory*) in a category* $\mathbf{X}$ *is a functor* $\mathbf{X} \xrightarrow{P} \mathbf{A}^{op}$ *to the dual category of some variety* $\mathbf{A}$.

2. Let $\mathbf{X}$ *be a category and let* $\mathcal{T} = (P, (\|-\|, \mathbf{B}))$ *comprise a cabp-theory* $\mathbf{X} \xrightarrow{P} \mathbf{A}^{op}$ *and a reduct* $(\|-\|, \mathbf{B})$ *of* $\mathbf{A}$. *A* catalg topological theory *(*cat-theory*)* in $\mathbf{X}$ induced by $\mathcal{T}$ *is the functor* $\mathbf{X} \xrightarrow{T} \mathbf{B}^{op} = \mathbf{X} \xrightarrow{P} \mathbf{A}^{op} \xrightarrow{\|-\|^{op}} \mathbf{B}^{op}$.

3. Let T *be a cat-theory in a category* $\mathbf{X}$. $\mathbf{Top}(T)$ *is the concrete category over* $\mathbf{X}$*, whose objects (*catalg topological spaces *or* T-spaces*) are pairs* (X, τ)*, where* X *is an* $\mathbf{X}$*-object and* τ *is a subalgebra of* TX *(*catalg topology *or* T-topology *on* X*), and whose morphisms (*catalg continuous *or* T-continuous $\mathbf{X}$-morphisms*)* $(X, \tau) \xrightarrow{f} (Y, \sigma)$ *are* $\mathbf{X}$*-morphisms* $X \xrightarrow{f} Y$ *such that* $(Tf)^{op}(\alpha) \in \tau$ *for every* $\alpha \in \sigma$ *(*catalg continuity *or* T-continuity*).*

The following is one of the main results of the emerging theory (see also [8]).

Theorem 1. *Every category* $\mathbf{Top}(T)$ *is fibre-small and topological over* $\mathbf{X}$.

2.2 Universal topology

Definition 2. *1. A* topological theory *in a category* $\mathbf{X}$ *is a functor* $\mathbf{X} \xrightarrow{\mathfrak{T}} \mathbf{CSLat}(\bigvee)$*, where* $\mathbf{CSLat}(\bigvee)$ *is the variety of* $\bigvee$*-semilattices.*

2. Let $\mathfrak{T}$ *be a topological theory in a category* $\mathbf{X}$. $\mathfrak{Top}(\mathfrak{T})$ *is the concrete category over* $\mathbf{X}$*, whose objects (*$\mathfrak{T}$-models*) are pairs* (X, t)*, where* X *is an* $\mathbf{X}$*-object and* $t \in \mathfrak{T}X$*, and whose morphisms (*$\mathfrak{T}$-morphisms*)* $(X, t) \xrightarrow{f} (Y, s)$ *are* $\mathbf{X}$*-morphisms* $X \xrightarrow{f} Y$ *such that* $\mathfrak{T}f(t) \leqslant s$.

One of the main results in the theory is given by the so-called *fibre-functor*.

Theorem 2. *For every fibre-small topological category* $(\mathbf{M}, |-|)$ *over* $\mathbf{X}$*, there is a topological theory* $\mathfrak{T}$ *such that* $(\mathbf{M}, |-|)$ *is concretely isomorphic to* $\mathfrak{Top}(\mathfrak{T})$.

3 Catalg topology versus universal topology

This section clarifies the relationships between catalg and universal topologies.

3.1 From universal topology to catalg topology

Following the idea of [5, Lemma 3.30(1)], we obtain the next result.

Lemma 1. *1. There exists a functor* $\mathbf{CSLat}(\bigvee) \xrightarrow{(-)^{\vdash}} \mathbf{CSLat}(\bigvee)^{op}$ *defined by* $(A_1 \xrightarrow{\varphi} A_2)^{\vdash} = A_1^d \xrightarrow{(\varphi^{\vdash})^{op}} A_2^d$*, where* A_i^d *has the dual partial order of* A_i *and* $\varphi^{\vdash}$ *is the upper adjoint of* φ *in the sense of partially ordered sets [2].*

2. *Every topological theory* $\mathbf{X} \xrightarrow{\mathfrak{T}} \mathbf{CSLat}(\bigvee)$ *provides the cat-theory* $\mathbf{X} \xrightarrow{T_{\mathfrak{T}}} \mathbf{CSLat}(\bigvee)^{op} = \mathbf{X} \xrightarrow{\mathfrak{T}} \mathbf{CSLat}(\bigvee) \xrightarrow{(-)^{\vdash}} \mathbf{CSLat}(\bigvee)^{op}$.

Lemma 1 helps to clarify the relationships between the categories $\mathfrak{Top}(\mathfrak{T})$ and $\mathbf{Top}(T_{\mathfrak{T}})$ (the first item, apart from fullness, is due to [5, Theorem 3.31(1)]).

Theorem 3. *1. There is a full concrete embedding* $\mathfrak{Top}(\mathfrak{T}) \xhookrightarrow{F} \mathbf{Top}(T_{\mathfrak{T}})$ *defined by* $F((X,t) \xrightarrow{f} (Y,s)) = (X, \downarrow^d t) \xrightarrow{f} (Y, \downarrow^d s)$, *where* $\downarrow^d (-)$ *is the lower set in the dual partial order.*

2. *There exists a concrete functor* $\mathbf{Top}(T_{\mathfrak{T}}) \xrightarrow{G} \mathfrak{Top}(\mathfrak{T})$ *defined by* $G((X,\tau) \xrightarrow{f} (Y,\sigma)) = (X, \bigwedge \tau) \xrightarrow{f} (Y, \bigwedge \sigma)$.
3. *F is a left-adjoint-right-inverse to G, the respective co-universal arrows given by* $FG(X,\tau) \xrightarrow{1_X} (X,\tau)$.
4. $\mathfrak{Top}(\mathfrak{T})$ *is concretely isomorphic to a full concretely coreflective subcategory of* $\mathbf{Top}(T_{\mathfrak{T}})$.

The last item of Theorem 3 induces the main result of this paper.

Theorem 4. *For a concrete category* $(\mathbf{M}, |-|)$, *the following are equivalent:*

1. $(\mathbf{M}, |-|)$ *is fibre-small and topological;*
2. $(\mathbf{M}, |-|)$ *is concretely isomorphic to a subcategory of some category* $\mathbf{Top}(T)$ *that is definable by topological co-axioms in* $\mathbf{Top}(T)$.

Proof. Use Theorem 2, Theorem 3(4) and the following two facts: (1) every full concretely coreflective subcategory of $\mathbf{Top}(T)$ is finally closed in $\mathbf{Top}(T)$ (and, therefore, is topological); (2) a full subcategory of $\mathbf{Top}(T)$ is concretely coreflective in $\mathbf{Top}(T)$ iff it is definable by topological co-axioms in $\mathbf{Top}(T)$. □

3.2 From catalg topology to universal topology

Lemma 2. *1. Given a variety* $\mathbf{A}$, *there exists a functor* $\mathbf{A}^{op} \xrightarrow{(-)^{\leftarrow\varphi}} \mathbf{CSLat}(\bigvee)$ *defined by* $(A_1 \xrightarrow{\varphi} A_2)^{\leftarrow\varphi} = (Sub(A_1))^d \xrightarrow{(\varphi^{op})^{\leftarrow}} (Sub(A_2))^d$, *where* $Sub(A_i)$ *is the* $\bigwedge$*-semilattice of subalgebras of* A_i.

2. *Every cat-theory* $\mathbf{X} \xrightarrow{T} \mathbf{A}^{op}$ *gives the topological theory* $\mathbf{X} \xrightarrow{\mathfrak{T}_T} \mathbf{CSLat}(\bigvee) = \mathbf{X} \xrightarrow{T} \mathbf{A}^{op} \xrightarrow{(-)^{\leftarrow\varphi}} \mathbf{CSLat}(\bigvee)$.

Notice that S. E. Rodabaugh [5, Lemma 3.30(2)] considered the variety **SQuant** of semi-quantales and used the non-dual order on their sub(semi-quantales), mistakenly claiming that their joins coincide with the unions. The flaw made him to consider the category $\mathfrak{Top}_{\mathrm{alt}}(\mathfrak{T})$ instead of $\mathfrak{Top}(\mathfrak{T})$, in which the morphism condition is changed from "$\mathfrak{T}f(t) \leqslant s$" to "$\mathfrak{T}f(t) \geqslant s$" (Definition 2).

Lemma 2 provides the second important result of this paper.

Theorem 5. *The categories* $\mathbf{Top}(T)$ *and* $\mathfrak{Top}(\mathfrak{T}_T)$ *are equal.*

Since every category of the form $\mathfrak{Top}(\mathfrak{T})$ is topological [1], Theorem 5 provides another proof of Theorem 1.

4 Conclusion

This paper clarified the relationships between two different approaches to topology. Following the discussion of [5, Remark 3.32(6)], one could ask which of them is more general. In case of semi-quantales, S. E. Rodabaugh was confronted with a mixed situation, having constructed just the embedding of Theorem 3(1) and an erroneous version of Theorem 5, for the particular case of semi-quantales and a modification (due to the error) of the categories $\mathfrak{Top}(\mathfrak{T})$. The variety-based case of this paper suggests that universal topology is more general than catalg topology, in the sense that every category of the form $\mathbf{Top}(T)$ can be reconstructed completely through a suitable category of the form $\mathfrak{Top}(\mathfrak{T})$, whereas the converse way requires the application of some topological co-axioms, whose ultimate description in each case can be problematic. On the other hand, in concrete applications, the catalg setting is more suitable, since it provides the underlying algebraic structures of the topological structures in question, whereas universal topology contains the information on their ground category only.

Recently [7], we showed that given a cat-theory T, the category $\mathbf{Top}(T)$ can be fully embedded into the category $\mathbf{TopSys}(T)$ of *catalg topological systems* (in the sense of S. Vickers [9]), which is (essentially) algebraic over its ground category, under reasonable requirements on the theory T. In view of the fact, the main result of this paper (Theorem 4) says that every fibre-small topological category can be fully embedded into a (potentially) algebraic category, which ultimately determines the degree of algebraicity of the topological category in question. As a consequence, we provide another answer to the problem, posed by S. E. Rodabaugh [5] on the extent to which (lattice-valued) topology is algebraic.

References

[1] J. Adámek, H. Herrlich, and G. E. Strecker, *Abstract and Concrete Categories: The Joy of Cats*, Dover Publications, 2009.

[2] M. Erné, J. Koslowski, A. Melton, and G. E. Strecker, *A primer on Galois connections*, Ann. N. Y. Acad. Sci. **704** (1993), 103–125.

[3] H. Herrlich, *Universal Topology*, Categorical topology, Proc. Int. Conf., Toledo/Ohio 1983 (H. L. Bentley, H. Herllich, M. Rajagopalan, and H. Wolff, eds.), Sigma Ser. Pure Math., vol. 5, Heldermann Verlag, 1984, pp. 223–281.

[4] E. G. Manes, *A class of fuzzy theories*, J. Math. Anal. Appl. **85** (1982), 409–451.

[5] S. E. Rodabaugh, *Relationship of Algebraic Theories to Powerset Theories and Fuzzy Topological Theories for Lattice-Valued Mathematics*, Int. J. Math. Math. Sci. **2007** (2007), 1–71.

[6] S. Solovjovs, *Categorically-algebraic topology*, Abstracts of the International Conference on Algebras and Lattices, Charles University, Prague, 2010, pp. 20–22.

[7] S. Solovyov, *Categorical foundations of variety-based topology and topological systems*, submitted to Fuzzy Sets Syst. (Special Issue: Linz 2009).

[8] S. Solovyov, *Categorically-algebraic dualities*, Acta Univ. M. Belii, Ser. Math. **17** (2010), 57–100.

[9] S. Vickers, *Topology via Logic*, Cambridge University Press, 1989.

[10] O. Wyler, *TOP categories and categorical topology*, General Topology Appl. **1** (1971), 17–28.

The Modal Logic of the Bi-topological Rational Plane*

Levan Uridia

Universidad Rey Juan Carlos, Spain
C. Tulipan, s/n E 28933 Mostoles, (Madrid)

Abstract. We introduce the multi-modal logic $\mathbf{KD4} \oplus \mathbf{KD4}$. As a main result we prove that in derived set interpretation of modalities $\mathbf{KD4} \oplus \mathbf{KD4}$ is the modal logic of bi-topological rational plane $Q \times Q$ with horizontal and vertical topologies.

1 Introduction

Topological semantics for modal logic starts with McKinsey and Tarski [9] and their celebrated completeness result that **S4** is a complete and sound modal logic of a real line. The semantics they use in this result is called C-semantics because the diamond modality is interpreted as a closure operator of a given topological space. Nevertheless much more attention has been attracted to the C-semantics [2],[1], [5] the derived set interpretation of the diamond modality, so called d-semantics is proven to be more expressive and in some situations gives more elegant characterization of properties of topological spaces [4], [3]. This direction was extensively studied by Esakia [6],[7] who introduced **wK4**, the modal logic of all topological spaces, with the desired (derivative operator) interpretation of the $\Diamond$-modality. **K4** is a logic over **wK4** and is characterized in this semantics by the class of all T_D-spaces whereas adding axiom **D** amounts to requiring that the space is dense-in-itself [3].

In the beginning of 80s Shehtman proved the analogous results to [9]. In his PhD thesis he presented modal logics of the rational line Q and the real line R in d-semantics. Different from C-semantics, d-semantics can distinguish these two topological spaces and therefore the logics are not the same. It turned out that the modal logic of rational line is **KD4**. Quite recently this completeness result was reproved [8]. The new proof is much simpler and uses a new technique of obtaining rational line from ω-branching tree. In this paper we will make use of the technique from [8] and generalise the result to the fusion of **KD4** with itself i.e., we have two modalities, where each modality satisfies $\mathbf{K}, \mathbf{4}$ and **D** axioms. Following standard notations we call this logic $\mathbf{KD4} \oplus \mathbf{KD4}$. As a main result, we prove that $\mathbf{KD4} \oplus \mathbf{KD4}$ is sound and complete with respect to the

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bi-topological space $(Q \times Q, \tau_v, \tau_h)$ where τ_h, τ_v are respectively horizontal and vertical topologies on the rational plane.

The paper is organised in the following way. In section 2 we introduce the modal logic **KD4** $\oplus$ **KD4**, recall the Kripke semantics and present the completeness proof which is a direct generalisation of a standard proof for **KD4**. In section 3 we show that **KD4** $\oplus$ **KD4** is the modal logic of (ω, ω)-branching tree $T_{\omega,\omega}$. This result is an easy generalisation of the result from [8]. In section 4 we present the topological d-semantics and show the main result of the paper that in this semantic the modal logic of $Q \times Q$ with horizontal and vertical topologies is **KD4** $\oplus$ **KD4**.

2 Kripke Semantics

The normal modal logic **KD4** $\oplus$ **KD4** is defined in a modal language with infinite set $Prop$ of propositional letters and connectives $\wedge, \neg, \Box_1, \Box_2$. The set of formulas $Form$ is constructed in a standard way: $Prop \subseteq Form$, If $\alpha, \beta \in Form$ then $\neg\alpha, \alpha \wedge \beta, \Box_1\alpha, \Box_2\alpha, \in Form$. We will use standard abbreviations for disjunction and implication, $\alpha \vee \beta \equiv \neg(\neg\alpha \wedge \neg\beta)$ and $\alpha \to \beta \equiv \neg\alpha \vee \beta$.

• The axioms are all classical tautologies, each box satisfies all **KD4** axioms i.e., for each $i \in \{1, 2\}$ we have:

(K) $\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$,

(D) $\Box_i p \to \Diamond_i p$,

(4) $\Box_i p \to \Box_i \Box_i p$,

• The rules of inference are: Modus-Ponens, Substitution, Necessitation.

The Kripke semantics for the modal logic **KD4** $\oplus$ **KD4** is provided by the transitive and serial, bi-relational Kripke frames.

Definition 1 *The relation $R \subseteq W \times W$ is:*
serial if - $(\forall x)(\exists y)(xRy)$,
transitive if - $(\forall x, y, z)(xRy \wedge yRz \Rightarrow xRz)$.

The triple (W, R_1, R_2), with W an arbitrary set and $R_i \subseteq W \times W$ where $i \in \{1, 2\}$, is a *transitive and serial bi-relational Kripke frame* if both R_1 and R_2 are transitive and serial relations. A quadruple (W, R_1, R_2, V) is a transitive and serial bi-relational Kripke model if (W, R_1, R_2) is a transitive and serial bi-relational Kripke frame and $V : Prop \to P(W)$ is a valuation function.

Definition 2 *For a given transitive and serial bi-relational Kripke model $\mathcal{M} = (W, R_1, R_2, V)$ the satisfaction of a formula at a point $w \in W$ is defined inductively as follows:*
$w \Vdash p$ *iff* $w \in V(p)$,
$w \Vdash \alpha \wedge \beta$ *iff* $w \Vdash \alpha$ *and* $w \Vdash \beta$,
$w \Vdash \neg\alpha$ *iff* $w \nVdash \alpha$,
$w \Vdash \Box_i\phi$ *iff* $(\forall v)(wR_iv \Rightarrow v \Vdash \phi)$,

The validity of a formula in a model, frame or class of frames is defined in a standard way.

Proposition 3 *(Completeness) Modal logic* $\mathbf{KD4} \oplus \mathbf{KD4}$ *is sound and complete with respect to the class of all finite, transitive and serial bi-relational Kripke frames.*

3 $T_{\omega,\omega}$

We borrow the ideas from [8] and [1] and introduce $T_{\omega,\omega}$, infinitely branching irreflexive tree. Let $N = \omega - \{0\}$ where τ denotes the least infinite ordinal. Let $\alpha \in N \cup \{\omega\}$. The infinite (α, α)-ary tree is defined as follows. The set of worlds is denoted by $T_{\alpha,\alpha}$ and each world is called a node. The set of nodes of $T_{\alpha,\alpha}$ satisfies the defining properties:

$\forall t \in T_{\alpha,\alpha}$, t is parent to exactly $4 \times \alpha$ many nodes in $T_{\alpha,\alpha}$,

$\exists r \in T_{\alpha,\alpha}$, r is the root of $T_{\alpha,\alpha}$ and it is unique,

$\forall t \in T_{\alpha,\alpha}$, if t is not the root, then t has a unique parent.

The main result for this section states that infinitely branching tree captures the fusion logic.

Theorem 4 *The fusion* $\mathbf{KD4} \oplus \mathbf{KD4}$ *is sound and complete w.r.t.* $T_{\omega,\omega}$.

4 Topological Semantics

The derived set topological semantics for $\mathbf{KD4} \oplus \mathbf{KD4}$ is provided by the class of all bi-topological spaces. As a main result we prove the soundness and completeness of the logic $\mathbf{KD4} \oplus \mathbf{KD4}$ with respect to the rational plane $Q \times Q$ with horisontal and vertical topologies. We start with the basic definitions.

Definition 5 *A pair* (X, τ) *is called a topological space if* X *is a set and* τ *is a collection of subsets of* X *with the following properties: 1)* $X, \varnothing \in \tau$*, 2)* $A, B \in \tau$ *implies* $A \cap B \in \tau$*, 3)* $A_i \in \tau$ *implies* $\bigcup A_i \in \tau$.

Now we define the set of all colimits of a given topological space. An operator which assigns to a set, the set of all its colimits is called colimit operator. Colimit operator is central for defining satisfaction relation in a derived set semantics.

Definition 6 *Given a topological space* (X, τ) *and a set* $A \subseteq X$ *we will say that* $x \in X$ *is a colimit point of* A *if there exists an open neighborhood* U_x *of* x *such that* $U_x - \{x\} \subseteq A$*. The set of all colimit points of* A *will be denoted by* $col(A)$ *and will be called colimit set of* A.

Colimit set serves for giving semantics of box modality, consequently semantics for diamond is provided by the dual of colimit set, which is called derived set. The derived set of A is denoted by $der(A)$. So we have $col(A) = X - der(X - A)$. Below we give the definition of the satisfaction of modal formulas in a derived set topological semantics.

Definition 7 *The satisfaction of a modal formula on a bi-topological model* $(M) = (W, \tau_1, \tau_2, V)$ *at a point* $w \in W$ *is defined in the following way:*
$w \Vdash p$ *iff* $w \in V(p)$,
$w \Vdash \alpha \wedge \beta$ *iff* $w \Vdash \alpha$ *and* $w \Vdash \beta$,
$w \Vdash \neg\alpha$ *iff* $w \nVdash \alpha$,
$w \Vdash \Box_i \phi$ *iff* $w \in col_i(V(\phi))$, *where* col_i *is a colimit operator of* τ_i, $i \in \{1, 2\}$.

Now let us define the desired bi-topological space $(Q \times Q, \tau_h, \tau_v)$. The underline set is $Q \times Q$ i.e., consists of all pares of rational numbers. As for the topologies we have $U \in \tau_h$ iff for every element $(x, y) \in U$ there exists an open interval $I \subseteq Q$ such that $I \times \{y\} \in U$. Analogously $V \in \tau_v$ iff for every element $(x, y) \in V$ there exists an open interval $I \subseteq Q$ such that $\{x\} \times I \in U$. The following lemma serves as a main intermediate step towards a proof of the main result.

Lemma 1. *For every formula* α *in the fusion language* **KD4** $\oplus$ **KD4** *the following holds: If* $T_{\omega,\omega} \nVdash \alpha$ *then* $(Q \times Q, \tau_h, \tau_v) \nVdash \alpha$.

Now we are ready to prove the main result of the paper.

Theorem 8 **KD4** $\oplus$ **KD4** *is sound and complete with respect to* $(Q \times Q, \tau_h, \tau_v)$.

References

1. Benthem, J. V., G. Bezhanishvili, B. T. Cate and D. Sarenac, *Modal logics for products of topologies*, Studia Logica **84** (2004).
2. Benthem, J. V. and D. Sarenac, *The geometry of knowledge*, Aspects of universal logic **17 of Travaux Log.** (2004), pp. 1–31.
3. Bezhanishvil, G., L. Esakia and D. Gabelaia, *Some results on modal axiomatization and definability for topological spaces*, Studia Logica **81** (2005), pp. 325–355.
4. Bezhanishvili, G., L. Esakia and D. Gabelaia, *The modal logic of stone spaces: diamond as derivative.*, The Review of Symbolic Logic **3**, pp. 26–40.
5. Bezhanishvili, G. and M. Gehrke, *Completeness of s4 with respect to the real line: revisited*, Annals of Pure and Applied Logic (2005), pp. 287–301.
6. Esakia, L., *Weak transitivity - restitution*, Logical Studies **8** (2001), pp. 244–255.
7. Esakia, L., *Intuitionistic logic and modality via topology*, Annals of Pure Applied Logic, 127(1-3) **127** (2004), pp. 155–170.
8. Lucero-Bryan, J. G., "Modal logics of some subspaces of the real numbers: Diamond as derivative," Ph.D. thesis, New Mexico State University (2010).
9. McKinsey, J. and A. Tarski, *The algebra of topology*, Annals of Mathematics **45** (1944), pp. 141–191.

On the transport of finiteness structures

Lionel Vaux
Institut de Mathématiques de Luminy, CNRS UMR 6206, Marseille, France
lionel.vaux@univmed.fr

Abstract

Finiteness spaces were introduced by Ehrhard as a model of linear logic, which relied on a finitess property of the standard relational interpretation and allowed to reformulate Girard's quantitative semantics in a simple, linear algebraic setting.

We review recent results obtained in a joint work with Christine Tasson, providing a very simple and generic construction of finiteness spaces: basically, one can *transport* a finiteness structure along any relation mapping finite sets to finite sets. Moreover, this construction is functorial under mild hypotheses, satisfied by the interpretations of all the positive connectives of linear logic.

Recalling that the definition of finiteness spaces follows a standard orthogonality technique, fitting in the categorical framework established by Hyland and Schalk, the question of the possible generalization of transport to a wider setting is quite natural. We argue that the features of transport do not stand on the same level as the orthogonality category construction; rather, they provide a simpler and more direct characterization of the obtained structure, in a webbed setting.

1 Finiteness spaces and finitary relations

Sets and relations. We write $\mathfrak{P}(A)$ for the powerset of A, $\mathfrak{P}_{\mathrm{f}}(A)$ for the set of all finite subsets of A and $!A$ for the set of all finite multisets of elements of A.

Let A and B be sets and f be a relation from A to B: $f \subseteq A \times B$. We then write ${}^t f$ for the transpose relation $\{(\beta,\alpha) \in B \times A;\ (\alpha,\beta) \in f\}$. For all subset $a \subseteq A$, we write $f \cdot a$ for the *direct image* of a by f: $f \cdot a = \{\beta \in B;\ \exists \alpha \in a,\ (\alpha,\beta) \in f\}$. If $\alpha \in A$, we will also write $f \cdot \alpha$ for $f \cdot \{\alpha\}$. We say that a relation f is *quasi-functional* if $f \cdot \alpha$ is finite for all α. If $b \subseteq B$, we define the *division* of b by f as $f \setminus b = \{\alpha \in A;\ f \cdot \alpha \subseteq b\}$. Notice that in general $f \cdot (f \setminus b)$ may be a strict subset of b, and $f \setminus (f \cdot a)$ may be a strict superset of a.

We write Rel for the category of sets and relations. It is a very simple model of linear logic: multiplicatives are given by the compact closed structure associated with cartesian products of sets (linear negation is then the transposition of relations, which is also a dagger); additives are modelled by disjoint union of sets, which gives a biproduct; the exponential modality is that of finite multisets.

Let T and U be two endofunctors of Rel, and let f be the data of a relation f^A (which we may also write f) from TA to UA for all set A: we say f is a *lax natural transformation* from T to U if, for all relation g from A to B, $f^B \circ (Tg) \subseteq (Ug) \circ f^A$. As an example, consider the finite multiset functor !, and for all A, let σ^A be the only relation from $!A$ to A such that for all $\overline{\alpha} \in !A$, $\sigma^A \cdot \overline{\alpha}$ is the support set of $\overline{\alpha}$. This defines a quasi-functional lax natural transformation from ! to the identity functor: notice that in that case, the inclusion $\sigma \circ !g \subseteq g \circ \sigma$ may be strict.

Finiteness spaces. We briefly recall the basic definition of finiteness spaces as given by Ehrhard [Ehr05]. Let A and B be sets, we write $A \perp_{\mathrm{f}} B$ if $A \cap B$ is finite. If $\mathfrak{A} \subseteq \mathfrak{P}(A)$, we define the *predual* of $\mathfrak{A}$ on A as $\mathfrak{A}^\perp = \{a' \subseteq A;\ \forall a \in \mathfrak{A},\ a \perp_{\mathrm{f}} a'\}$. A *finiteness structure* on A is a set $\mathfrak{A}$ of subsets of A such that $\mathfrak{A}^{\perp\perp} = \mathfrak{A}$. A *finiteness space* is then a pair $\mathcal{A} = (|\mathcal{A}|, \mathfrak{F}(\mathcal{A}))$ where $|\mathcal{A}|$ is the underlying set, called the *web* of $\mathcal{A}$, and $\mathfrak{F}(\mathcal{A})$ is a finiteness structure on $|\mathcal{A}|$.

We write $\mathcal{A}^{\perp}$ for the *dual* finiteness space: $\left|\mathcal{A}^{\perp}\right| = |\mathcal{A}|$ and $\mathfrak{F}\left(\mathcal{A}^{\perp}\right) = \mathfrak{F}(\mathcal{A})^{\perp}$. The elements of $\mathfrak{F}(\mathcal{A})$ are called the *finitary subsets* of $\mathcal{A}$. Standard arguments on closure operators defined by orthogonality apply and in particular $\mathfrak{A}^{\perp} = \mathfrak{A}^{\perp\perp\perp}$, for all $\mathfrak{A} \subseteq \mathfrak{P}(A)$; hence finiteness structures are exactly preduals. More specific to the orthogonality $\perp_{\mathrm{f}}$, for all finiteness structure $\mathfrak{A}$ on A, we obtain:

(1) $\mathfrak{A}$ is downwards closed for inclusion, i.e. $a \subseteq a' \in \mathfrak{A}$ implies $a \in \mathfrak{A}$;

(2) $\mathfrak{P}_{\mathrm{f}}(A) \subseteq \mathfrak{A}$ and $\mathfrak{A}$ is closed under finite unions, i.e. $a, a' \in \mathfrak{A}$ implies $a \cup a' \in \mathfrak{A}$.

The first property is similar to the one for coherence spaces. The second one is distinctive of finiteness spaces, and is a non-uniformity property: union of finitary subsets models some form of computational non-determinism, which is crucial to interpret the differential λ-calculus [ER03].

Finitary relations. Let $\mathcal{A}$ and $\mathcal{B}$ be two finiteness spaces: we say a relation f from $|\mathcal{A}|$ to $|\mathcal{B}|$ is *finitary* from $\mathcal{A}$ to $\mathcal{B}$ if: for all $a \in \mathfrak{F}(\mathcal{A})$, $f \cdot a \in \mathfrak{F}(\mathcal{B})$, and for all $b' \in \mathfrak{F}\left(\mathcal{B}^{\perp}\right)$, ${}^{t}f \cdot b' \in \mathfrak{F}\left(\mathcal{A}^{\perp}\right)$. The identity relation is finitary from $\mathcal{A}$ to itself, and finitary relations compose: this defines the category $\underline{\mathrm{Fin}}$ whose objects are finiteness spaces and morphisms are finitary relations.

Finitary relations form a finiteness structure: remark that $f \subseteq |\mathcal{A}| \times |\mathcal{B}|$ is finitary iff $f \in \left\{a \times b';\ a \in \mathfrak{F}(\mathcal{A}) \text{ and } b' \in \mathfrak{F}\left(\mathcal{B}^{\perp}\right)\right\}^{\perp}$. This reflects the $*$-autonomous structure of $\underline{\mathrm{Fin}}$, with tensor product given by $|\mathcal{A} \otimes \mathcal{B}| = |\mathcal{A}| \times |\mathcal{B}|$ and $\mathfrak{F}(\mathcal{A} \otimes \mathcal{B}) = \{a \times b;\ a \in \mathfrak{F}(\mathcal{A}) \text{ and } b \in \mathfrak{F}(\mathcal{B})\}^{\perp\perp}$, and $*$-functor given by duality on finiteness spaces and transposition on finitary relations: $f \in \underline{\mathrm{Fin}}(\mathcal{A}, \mathcal{B}) \mapsto {}^{t}f \in \underline{\mathrm{Fin}}(\mathcal{B}^{\perp}, \mathcal{A}^{\perp})$.

2 Transport

Transport of finiteness structures In the following, we present the basic results obtained in recent work with Tasson [TV11]. The starting point is the following lemma, which allows to generate a finiteness structure on a set A, by transporting that of a finiteness space $\mathcal{B}$ along any relation f from A to $|\mathcal{B}|$, provided f maps finite subsets of A to finitary subsets of $\mathcal{B}$.

Lemma 2.1 (Transport)**.** *Let A be a set, $\mathcal{B}$ a finiteness space and f a relation from A to $|\mathcal{B}|$ such that $f \cdot \alpha \in \mathfrak{F}(\mathcal{B})$ for all $\alpha \in A$. Then $\mathfrak{F} = \{a \subseteq A;\ f \cdot a \in \mathfrak{F}(\mathcal{B})\}$ is a finiteness structure on A and, more precisely, $\mathfrak{F} = \{f \setminus b;\ b \in \mathfrak{F}(\mathcal{B})\}^{\perp\perp}$.*

The proof of this *transport lemma* [TV11, Lemma 3.4] is very similar to that of the characterization of the exponential modality, given in Ehrhard's paper [Ehr05, Lemma 4]. Actually, we obtain this characterization as a straightforward application of transport, through the support relation, which is quasi-functional: if $\mathcal{A}$ is a finiteness space, then for all $\overline{\alpha} \in\, !\,|\mathcal{A}|$, $\sigma \cdot \overline{\alpha} \in \mathfrak{P}_{\mathrm{f}}(A) \subseteq \mathfrak{F}(\mathcal{A})$; moreover, if $a \subseteq |\mathcal{A}|$, then $\sigma \setminus a = !a$. We thus obtain a finiteness space $!\mathcal{A}$ such that $|!\mathcal{A}| = !\,|\mathcal{A}|$ and $\mathfrak{F}(!\mathcal{A}) = \{\overline{a} \subseteq !|\mathcal{A}|;\ \sigma \cdot \overline{a} \in \mathfrak{F}(\mathcal{A})\} = \{!a;\ a \in \mathfrak{F}(\mathcal{A})\}^{\perp\perp}$.

Corollary 2.2. *Let A be a set, $(\mathcal{B}_i)_{i \in I}$ a family of finiteness spaces and $(f_i)_{i \in I}$ a family of relations such that, for all $\alpha \in A$ and all $i \in I$, $f_i \cdot \alpha \in \mathfrak{F}(\mathcal{B}_i)$. Then $\mathfrak{F} = \{a \subseteq A;\ \forall i \in I,\ f_i \cdot a \in \mathfrak{F}(\mathcal{B}_i)\}$ is a finiteness structure on A and, more precisely, $\mathfrak{F} = \left\{\bigcap_{i \in I}(f_i \setminus b_i);\ \forall i \in I,\ b_i \in \mathfrak{F}(\mathcal{B}_i)\right\}^{\perp\perp}$.*

Again, we obtain the following characterization of the tensor product, by applying this generalized transport lemma: denoting π_1 and π_2 the two obvious projection relations we obtain $\mathfrak{F}(\mathcal{A} \otimes \mathcal{B}) = \{c \subseteq |\mathcal{A}| \times |\mathcal{B}|\,;\ \pi_1 \cdot c \in \mathfrak{F}(\mathcal{A}) \text{ and } \pi_2 \cdot c \in \mathfrak{F}(\mathcal{B})\}$. Similarly, the direct product of an arbitrary family of finiteness spaces is given by $\left|\&_{i\in I}\mathcal{A}_i\right| = \biguplus_{i\in I}|\mathcal{A}_i| = \bigcup_{i\in I}\{i\}\times|\mathcal{A}|_i$ and $\mathfrak{F}\left(\&_{i\in I}\mathcal{A}_i\right) = \left\{\biguplus_{i\in I}a_i;\ \forall i\in I,\ a_i\in\mathfrak{F}(\mathcal{A}_i)\right\}$: this is otained by transport through the restrictions $\rho_i = \{((i,\alpha),\alpha);\ \alpha\in|\mathcal{A}_i|\}$. It turns out that the transport lemma is very versatile: for any sensible notion of datatype (lists, trees, graphs, *etc.*), it allows to form a finiteness spaces of such objects, with finiteness given by that of the elements (or nodes), possibly with an additional finiteness condition on the shape (e.g., bounded length).

Transport functors. We say an endofunctor $\mathcal{T}$ of Fin *has a web* if there exists an endofunctor T of Rel, such that $|\mathcal{T}\mathcal{A}| = T|\mathcal{A}|$ for all finiteness space $\mathcal{A}$, and $\mathcal{T}f = Tf$ for all $f\in \underline{\mathrm{Fin}}(\mathcal{A},\mathcal{B})\subseteq \underline{\mathrm{Rel}}(|\mathcal{A}|,|\mathcal{B}|)$. We then say T is the web of $\mathcal{T}$ and write $T = |\mathcal{T}|$. Notice that in that case, if $f\subseteq A\times B$, Tf must be finitary from $\mathcal{T}(A,\mathfrak{A})$ to $\mathcal{T}(B,\mathfrak{B})$ for all finiteness structures $\mathfrak{A}$ and $\mathfrak{B}$ making f finitary from $(A,\mathfrak{A})$ to $(B,\mathfrak{B})$. We show that, under mild hypotheses, the transport lemma allows to define such functors.

Let T be a functor in Rel. We call *ownership relation* on T the data of a quasi-functional lax natural transformation ϵ from T to the identity functor. Given such an ownership relation, we can transport the finiteness structure of any space $\mathcal{A}$ to the web $T|\mathcal{A}|$: indeed, $\epsilon^{|\mathcal{A}|}$ then satisfies the condition of Lemma 2.1 because it is quasi-functional and finite subsets are always finitary. In such a situation, we write $T_\epsilon\mathcal{A}$ for the finiteness space $(T|\mathcal{A}|, \{\widetilde{a}\subseteq T|\mathcal{A}|;\ \epsilon\cdot\widetilde{a}\in\mathfrak{F}(\mathcal{A})\})$. If $f\in\underline{\mathrm{Rel}}(|\mathcal{A}|,|\mathcal{B}|)$, we also write $T_\epsilon f = Tf$: then T_ϵ defines a functor on Fin (with web T) iff Tf is finitary from $T_\epsilon\mathcal{A}$ to $T_\epsilon\mathcal{B}$ as soon as f is finitary from $\mathcal{A}$ to $\mathcal{B}$. In that case, we say T_ϵ is the *transport functor* deduced from the *transport situation* (T,ϵ).

We now provide sufficient conditions for a transport situation to give rise to a transport functor. A *shape relation* on (T,ϵ) is the data of a fixed set M of *shapes* and a quasi-functional lax natural transformation μ from T to the constant functor which sends every set to M and every relation to the identity, subject to the following additional condition: for all $\widetilde{a}\subseteq TA$, if $\mu\cdot\widetilde{a}$ and $\epsilon\cdot\widetilde{a}$ are both finite, then $\widetilde{a}$ is finite.

Lemma 2.3. *Let (T,ϵ) be a transport situation. If T is symmetric (i.e. ${}^t(Tf) = T\,{}^tf$ for all f) and there exists a shape relation on (T,ϵ), then T_ϵ is an endofunctor in* Fin.

The symmetry of T is essential in the proof, since it allows ϵ and μ to interact with ${}^tTf$ as well as with Tf (the definition of finitary relations is related with both directions). Moreover, the existence of a shape relation is also crucial, since some transport situations on symmetric functors do not preserve finitary relations. This is in particular the case of a would-be infinitary tensor: although we can apply the transport lemma to define $\bigotimes_{i\in I}\mathcal{A}_i$ for all family $(\mathcal{A}_i)_{i\in I}$ of finiteness spaces (consider the projections $(\pi_i)_{i\in I}$), the tensor of finitary relations is not necessarily finitary. It is however important to note that the shape relation plays no rôle in the definition of T_ϵ: its existence is a mere side condition ensuring functoriality.

A direct consequence is the functoriality of the exponential !: the shape of a finite multiset is its size. Lemma 2.3 is easily generalized to functors of arbitrary arity, such as the direct product of finiteness spaces, given by disjoint union of webs and finitary subsets: the shape of an element $(j,\alpha)\in\biguplus_{i\in I}|\mathcal{A}_i|$ is the index j. The functoriality of binary tensor product also follows, this time with no need of an additional shape relation: the binary cartesian product of finite sets is always finite.

The properties of transport functors are further studied in [TV11]: we show that, under additional hypotheses, transport functors are Scott-continuous, which allows to take fixed points of such; this is put to use by giving an account of recursive algebraic datatypes in Fin.

3 On possible generalizations of transport

The orthogonality category of finiteness spaces. The category $\underline{\mathrm{Fin}}$ is the tight orthogonality category associated with $\perp_{\mathrm{f}}$ on $\underline{\mathrm{Rel}}$, following the theory of Hyland and Schalk [HS03]. The transport lemma can be used to establish the self-stability of $\perp_{\mathrm{f}}$ easily. More generally, it provides simple and concrete characterizations of the abstract structure generated by double-glueing. In fact, the very merit of transport lies precisely in making the bidual closure typical of the orthogonality construction almost trivial, since it simply amounts to the downwards closure for inclusion.

This difference in approach shows in the formulation of transport. Key ingredients seem to rely strongly on the fact that we consider a webbed model (interpretations of proofs are particular subsets of their types), and in particular on the order enrichment of the category given by inclusion of relations. We can only remark that the condition "f sends finite subsets to finitary subsets" can be rephrased as f being negative from $(A, \mathfrak{P}(A), \mathfrak{P}_{\mathrm{f}}(A))$ to $(|\mathcal{B}|, \mathfrak{F}(\mathcal{B}), \mathfrak{F}(\mathcal{B}^{\perp}))$. The possible generalization of transport to a wider setting is nonetheless an appealing perspective. As a first step, we turn our attention to other models of linear logic related with the relational model.

Transport in other webbed models. Recall that a coherence space $\mathcal{A}$ is the data of a set $|\mathcal{A}|$ and a reflexive binary relation $\frown_{\mathcal{A}}$ on $|\mathcal{A}|$ (its *coherence*). Equivalently, $\mathcal{A}$ can be characterized by the set $\mathfrak{C}(\mathcal{A}) \subseteq \mathfrak{P}(|\mathcal{A}|)$ of its *cliques*, i.e. sets of pairwise coherent elements. A relation $f \subseteq |\mathcal{A}| \times |\mathcal{B}|$ is said to be linear if, for all $a \in \mathfrak{C}(\mathcal{A})$, $f \cdot a \in \mathfrak{C}(\mathcal{B})$ and for all $b' \in \mathfrak{C}(\mathcal{B})^{\perp}$, ${}^{t}f \cdot b' \in \mathfrak{C}(\mathcal{A})^{\perp}$, where $\mathfrak{C}(\mathcal{A})^{\perp}$ denotes the dual for the *partial orthogonality*: $a \perp_{\mathrm{p}} a'$ iff $a \cap a'$ has at most one element.

The transport lemma is easily adapted to coherence spaces: just replace "finiteness structure" with "clique", and observe that, if $f \cdot \alpha$ is always a clique, then $f \cdot a$ is a clique iff $f \cdot \{\alpha, \alpha'\}$ is a clique for all $\alpha, \alpha' \in a$, which defines a new coherence. The technique we used to establish the functoriality of transport, however, does not apply directly: if (T, ϵ) is a transport situation and $f \in \mathfrak{C}(\mathcal{A} \multimap \mathcal{B})$, then Tf sends cliques to cliques by lax-naturality of ϵ, but establishing the reverse direction (inverse images of anticliques are anticliques) will require to tweak the notion of shape relation to accomodate coherence rather than finiteness. This is the subject of ongoing work.

Transport does not seem to be meaningful for the webbed model obtained from the *total orthogonality*: $a \perp_{\mathrm{t}} a'$ iff $a \cap a'$ is a singleton. This defines Loader's totality spaces: intuitively, total subsets represent maximal cliques. This maximality property is not compatible with the fact that, by construction, the structures obtained by transport are downwards closed for inclusion.

References

[Ehr05] Thomas Ehrhard. Finiteness spaces. *Mathematical Structures in Computer Science*, 15(4):615–646, 2005.

[ER03] Thomas Ehrhard and Laurent Regnier. The differential lambda-calculus. *Theoretical Computer Science*, 309:1–41, 2003.

[HS03] Martin Hyland and Andrea Schalk. Glueing and orthogonality for models of linear logic. *Theor. Comput. Sci.*, 294(1/2):183–231, 2003.

[TV11] Christine Tasson and Lionel Vaux. Transport of finiteness structures and applications. *Mathematical Structures in Computer Science*, 2011. To appear.

A Goal-oriented Graph Calculus for Relations*

Paulo A. S. VELOSO
COPPE-UFRJ: Systems; BRAZIL

Sheila R. M. VELOSO
UERJ: Engin., Systems; BRAZIL

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1 Introduction

We introduce a goal-oriented graphical calculus for relational inclusions.[1]
Graphs and diagrams provide convenient visualization in many areas and their heuristic appeal is evident. Venn diagrams, for instance, may be very helpful in visualizing connections between sets. They are not, however, accepted as proofs: one has to embellish the connections discovered in terms of standard methods of reasoning. This is not the case with our graph calculus: there is no need to compile the steps into standard reasoning. Graph manipulations, as they have precise syntax and semantics, are proof methods.
Formulas are traditionally written down on a single line [5]. A basic idea behind graph calculi is a two-dimensional representation. Using drawings for relations is a very natural idea: represent the fact that *a* is related to *b* via relation r by an arrow $a \xrightarrow{\mathrm{r}} b$. Then, some operations on relations correspond to simple manipulations on arrows (e.g. transposal amounts to arrow reversal, intersection to parallel arcs and relative product to consecutive arcs); so one can reason about relations by manipulating their representations. This is the basic idea of graph methods for reasoning about relations [4, 5, 7, 8, 9, 10]. Some relational operations (like complementation) are not so easy to handle. (It can be introduced by definition, if one can reason from hypotheses [8], or it may be handled via arcs labeled by boxes [10].)
Here, we introduce a sound and complete goal-oriented graph calculus for relational inclusions. It is conceptually simpler and easier to use than the usual ones. Also, it can handle hypotheses, as well.

2 Basic Ideas

We now examine the basic ideas behind our graph calculus for relational inclusions.
We wish to establish inclusions between relational terms. Relational terms are expression like r, $\overline{\mathrm{s}}$, $\mathrm{r} \sqcap \overline{\mathrm{s}}$ and $\mathrm{r}^{\smile};\overline{(\mathrm{r};\mathrm{s})}$. The relational terms are generated from relation names by relational constants and operations. We employ the RelMiCs notation [1]. A relation name corresponds to an arbitrary binary relation. The constants $\perp$, $\top$, $\mathbb{I}$ and $\mathbb{D}$ denote respectively the empty relation, the universal relation, the identity relation and the diversity relation. Unary operations $^{-}$ and $^{\smile}$ stand for complementation $^{\sim}$ and transposition $^{\mathsf{T}}$. Operations $\sqcap$ and $\sqcup$ stand for Boolean intersection $\cap$ and union $\cup$, respectively, while operations ; and † stand for relative product | and sum $\dagger$, respectively.
A graph is a finite set of alternative slices. A slice consists of finite sets of nodes and labeled arcs together with 2 distinguished nodes. We have two kinds of rules: reduction rules (to obtain reduced graphs) and an expansion rule (creating alternative instances). To prove an inclusion $\mathrm{P} \sqsubseteq \mathrm{Q}$ we start with the slice corresponding to $\mathrm{P} \sqcap \overline{\mathrm{Q}}$ and apply the rules so that one obtains a graph whose slices are inconsistent.
We now examine some examples illustrating our graph methods (cf. Sections 3 and 4).

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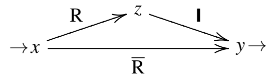
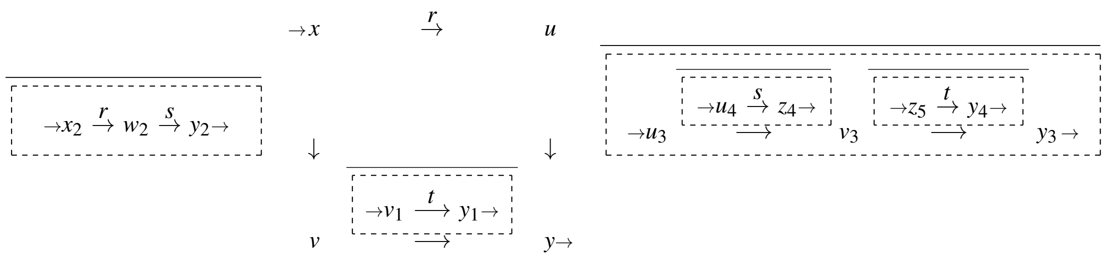
Given a relational term R, to establish $R^{\smile\smile} \sqsubseteq R$, we show $R^{\smile\smile} \sqcap \overline{R} \sqsubseteq \mathbb{L}$. We form the slice with 2 parallel arcs $\to x \xrightarrow{R^{\smile\smile}} y \to$ and $\to x \xrightarrow{\overline{R}} y \to$. We can reverse the arrow of a $^{\smile}$-arc, thus converting arc $\to x \xrightarrow{R^{\smile\smile}} y \to$ to $\to x \xleftarrow{R^{\smile}} y \to$ and then to $\to x \xrightarrow{R} y \to$. So, we obtain a slice S with 2 parallel arcs $\to x \xrightarrow{R} y \to$ and $\to x \xrightarrow{\overline{R}} y \to$, which is inconsistent.

To establish $R;\mathbb{I} \sqsubseteq R$, we show $(R;\mathbb{I}) \sqcap \overline{R} \sqsubseteq \mathbb{L}$; form the slice with 2 parallel arcs $\to x \xrightarrow{R;\mathbb{I}} y \to$ and $\to x \xrightarrow{\overline{R}} y \to$. We can enlarge a ;-arc by a new node, thus converting arc $\to x \xrightarrow{R;\mathbb{I}} y \to$ to consecutive arcs $\to x \xrightarrow{R} z \xrightarrow{\mathbb{I}} y \to$. So, we obtain slice T with 3 arcs $\to x \xrightarrow{R} z$, $z \xrightarrow{\mathbb{I}} y \to$ and $\to x \xrightarrow{\overline{R}} y \to$, namely 

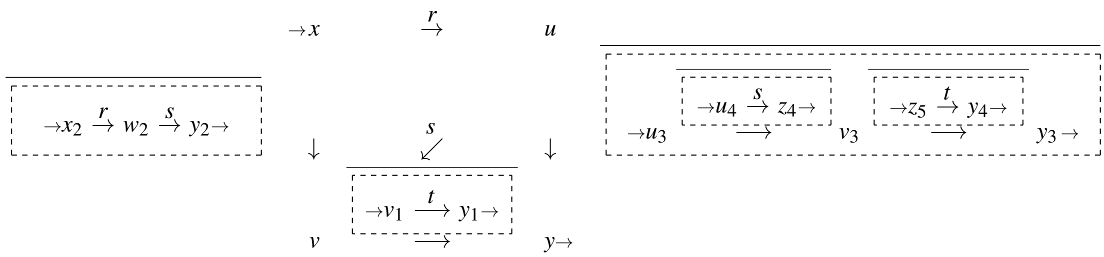
. We can eliminate the $\mathbb{I}$-arc by renaming z to y, thus obtaining the above slice S.

Now, to establish $R \sqsubseteq R;\mathbb{T}$, we show $R \sqcap \overline{R;\mathbb{T}} \sqsubseteq \mathbb{L}$. We form the slice with 2 parallel arcs $\to x \xrightarrow{R} y \to$ and $\to x \xrightarrow{\overline{R;\mathbb{T}}} y \to$. We can replace label $R;\mathbb{T}$ by the 2-arc slice L: $\to x' \xrightarrow{R} z' \xrightarrow{\mathbb{T}} y' \to$, which can be converted to the 2-arc slice L′: $\to x' \xrightarrow{R} z' \quad y' \to$ We thus reduce arc $\to x \xrightarrow{\overline{R;\mathbb{T}}} y \to$ to arc $\to x \xrightarrow{\overline{L'}} y \to$, and slice $DS(R \setminus R;\mathbb{T})$ to a slice S′ with 2 parallel arcs $\to x \xrightarrow{R} y \to$ and $\to x \xrightarrow{\overline{L'}} y \to$. Now, consider the node mapping given by $x' \mapsto x$ and $y', z' \mapsto y$, it maps arc $x' \xrightarrow{R} z'$ of L′ to arc $x \xrightarrow{R} y$ of S′; so slice S′ is inconsistent.

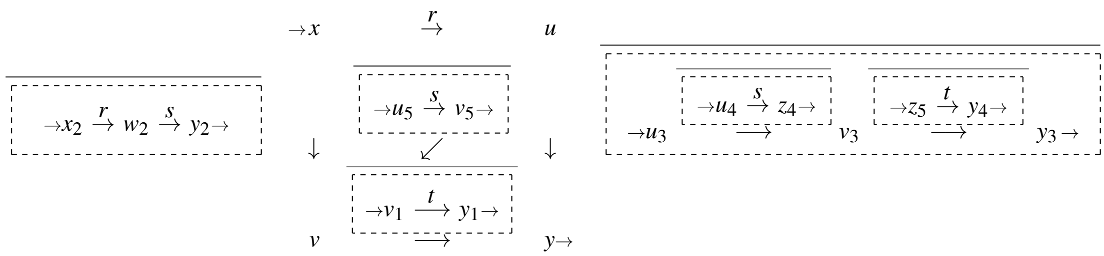
To establish $r;(s \dagger t) \sqsubseteq (r;s) \dagger t$, as before, we form the slice $DS(r;(s \dagger t) \setminus (r;s) \dagger t)$ with 2 parallel arcs $\to x \xrightarrow{r;(s \dagger t)} y \to$ and $\to x \xrightarrow{\overline{(r;s) \dagger t}} y \to$. We can convert it to a slice S′ (with complemented slices as arc labels):



This slice S′ is not yet inconsistent. But, we can expand it to a graph G consisting of 2 alternative slices S_+ and S_-, where slice S_+ is as follows



and slice S_- is as follows



Now, both slices S_+ and S_- represent inconsistent situations. Therefore, we have established the inclusion $\mathsf{DS}(r;(s\dagger t) \setminus (r;s)\dagger t) \sqsubseteq \bot$, whence also $r;(s\dagger t) \sqsubseteq (r;s)\dagger t$.

3 Graph Language

We now introduce our graph language. We will consider two fixed denumerably infinite sets: set $\mathbb{R}n$ of *relation names* $(r,s,t,\dots)$ and set $\mathbb{N}d$ of (individual) *nodes* (in alphabetical order: $x,y,z,\dots$).
We first introduce the syntax of our graph language (by mutual recursion). Labels are obtained from relation names, slices and graphs (see below), by relational operations. An *arc* is a triple $u\,\mathsf{L}\,v$, where u,v are nodes and L is a label. A *sketch* consists of sets and of arcs. A *draft* is a sketch with finite sets of nodes and of arcs. A *slice* S consists of a draft $\underline{\mathsf{S}}$ (its *underlying draft*) together with a pair of nodes $x_\mathsf{S}, y_\mathsf{S}$ (its *input* and *output* nodes). A graph is a finite set of alternative slices. An inclusion is a pair of labels, noted $\mathsf{L} \sqsubseteq \mathsf{K}$. The *difference slice* of a pair of labels L and K is the 2-arc slice $\mathsf{DS}(\mathsf{L} \setminus \mathsf{K}) := \langle\{x,y\},\{x\mathsf{L}y, x\overline{\mathsf{K}}y\} : x,y\rangle$.
We use models for semantics. A *model* is a structure $\mathfrak{M} = \langle M,(r^{\mathfrak{M}})_{r\in\mathbb{R}n}\rangle$, consisting of a set M and a binary relation $r^{\mathfrak{M}} \subseteq M^2$ (where $M^2 := M \times M$) for each relation name $r \in \mathbb{R}n$. We introduce the semantics of our graph language (by mutual recursion). Given a model $\mathfrak{M}$, the *relation of a label* is the relation $[\mathsf{L}]_{\mathfrak{M}} \subseteq M^2$ obtained by extending those of the relation names ($[r]_{\mathfrak{M}} := r^{\mathfrak{M}}$) by means of the concrete versions of the operations: e.g. $[\bot]_{\mathfrak{M}} := \emptyset$, $[\top]_{\mathfrak{M}} := M^2$, $[\mathsf{I}]_{\mathfrak{M}} := I_M$, $[\mathsf{L}^{\smile}]_{\mathfrak{M}} := [\mathsf{L}]_{\mathfrak{M}}{}^{\mathsf{T}}$, $[\mathsf{L}\sqcap\mathsf{K}]_{\mathfrak{M}} := [\mathsf{L}]_{\mathfrak{M}} \cap [\mathsf{K}]_{\mathfrak{M}}$, $[\mathsf{L};\mathsf{K}]_{\mathfrak{M}} := [\mathsf{L}]_{\mathfrak{M}} \mid [\mathsf{K}]_{\mathfrak{M}}$; for a slice or a graph, $[\mathsf{S}]_{\mathfrak{M}} := [\![\mathsf{S}]\!]_{\mathfrak{M}}$ and $[\mathsf{G}]_{\mathfrak{M}} := [\![G]\!]_{\mathfrak{M}}$. Given an assignment g into M, g *satisfies* arc $u\,\mathsf{L}\,v$ (in $\mathfrak{M}$) iff $(u^{\mathrm{g}}, v^{\mathrm{g}}) \in [\mathsf{L}]_{\mathfrak{M}}$ and g *satisfies* a sketch (in $\mathfrak{M}$) iff it satisfies all its arcs. The *extension* of a slice $\mathsf{S} = \langle\underline{\mathsf{S}} : x_\mathsf{S}, y_\mathsf{S}\rangle$ is the binary relation $[\![\mathsf{S}]\!]_{\mathfrak{M}}$ consisting of the pair of values of x_S and y_S for the assignments satisfying $\underline{\mathsf{S}}$. The *extension* of a graph is given by $[\![\mathsf{G}]\!]_{\mathfrak{M}} := \bigcup_{\mathsf{S}\in\mathsf{G}} [\![\mathsf{S}]\!]_{\mathfrak{M}}$.
An inclusion $\mathsf{L} \sqsubseteq \mathsf{K}$ *holds* in model $\mathfrak{M}$ iff $[\mathsf{L}]_{\mathfrak{M}} \subseteq [\mathsf{K}]_{\mathfrak{M}}$. Call an inclusion *valid* iff it holds in every model. Call labels L and K *equivalent* ($\mathsf{L} \equiv \mathsf{K}$) iff both inclusions $\mathsf{L} \sqsubseteq \mathsf{K}$ and $\mathsf{K} \sqsubseteq \mathsf{L}$ are valid. Call a label *null* when it is equivalent to $\bot$. The empty graph $\{\ \}$ (with no slices) is null.
Our reduction is guaranteed by the fact that an inclusion $\mathsf{L} \sqsubseteq \mathsf{K}$ holds in a model $\mathfrak{M}$ iff the difference slice $\mathsf{DS}(\mathsf{L} \setminus \mathsf{K})$ has empty extension in $\mathfrak{M}$.
Given sketches Σ' and Σ'', a *morphism* $\theta : \Sigma' \dashrightarrow \Sigma''$ is a node renaming function that preserves arcs, mapping each arc uLv of Σ' to an arc $u^\theta L v^\theta$ of Σ''.
Morphisms transfer satisfying assignments by composition: given a morphism $\theta : \Sigma' \dashrightarrow \Sigma''$ and a model $\mathfrak{M}$, for every assignment g satisfying Σ'' (in $\mathfrak{M}$), the composite assignment $\mathrm{g}\cdot\theta$ satisfies Σ' (in $\mathfrak{M}$).
A sketch Σ is *zero* iff, for some slice $\mathsf{T} = \langle\underline{\mathsf{T}} : x_\mathsf{T}, y_\mathsf{T}\rangle$, there exists a morphism $\theta : \underline{\mathsf{T}} \dashrightarrow \Sigma$, such that $x_\mathsf{T}{}^\theta\, \overline{\mathsf{T}}\, y_\mathsf{T}{}^\theta$ is an arc of Σ. Clearly, no assignment can satisfy a zero sketch. A slice is *zero* iff its underlying draft is zero. A *zero graph* is a graph consisting of zero slices.
The category of sketches and morphisms has co-limits. In particular, the pushout of drafts gives a draft. Consider a slice T. Given a draft $\mathsf{D} = \langle N,A\rangle$ and nodes $(u,v) \in N^2$, the *glued draft* $\mathsf{D}\frac{u}{v}\mathsf{T}$ is the pushout of drafts D and $\underline{\mathsf{T}}$ over the arcless draft $\langle\{x,y\},\emptyset\rangle$ and natural morphisms. Given a slice S, we obtain the *glued slice* $\mathsf{S}\frac{u}{v}\mathsf{T}$ by transferring the input and output nodes of S to the glued draft $\underline{\mathsf{S}}\frac{u}{v}\mathsf{T}$. Also, we *glue* a graph naturally by gluing its slices: $\mathsf{S}\frac{u}{v}\mathsf{H} := \{\mathsf{S}\frac{u}{v}\mathsf{T} \,/\, \mathsf{T} \in \mathsf{H}\}$.
We define reduced labels, arcs, sketches, slices and graphs by mutual recursion. A label L is *reduced* iff it is a relation name or $\overline{\mathsf{T}}$, where T is a reduced slice (see below). An arc $u\,\mathsf{L}\,v$ is *reduced* iff its label L is reduced. A sketch is *reduced* iff all its arcs are reduced arcs. A slice S is *reduced* iff its underlying draft $\underline{\mathsf{S}}$ is a reduced sketch. A graph is *reduced* iff all its slices are reduced slices.

4 Graph Calculus

We now introduce the rules of our graph calculus: reduction and expansion.
The reduction rules are will be of two kinds: operational and structural.
The operational rules come from labels that are equivalent to graphs. For the constants, $\bot$ is equivalent to the null graph $\{\ \}$, $\top$ and I are equivalent to graphs with a single arcless slice, namely $\to x\ \ y\to$ and $\to x \to$; also,

$\mathsf{D} \equiv \{\to x \xrightarrow{\overline{\to x \to}} y\to\}$. For the operations, $\mathrm{L}^{\smile} \equiv \{\to x \xleftarrow{\mathrm{L}} y\to\}$, $\mathrm{L}\sqcap\mathrm{K}$ is equivalent to the graph whose single slice consists of the 2 parallel arcs $\to x \xrightarrow{\mathrm{L}} y\to$ and $\to x \xrightarrow{\mathrm{K}} y\to$, $\mathrm{L}\sqcup\mathrm{K} \equiv \overline{\{\to x \xrightarrow{\mathrm{L}} y\to, \to x \xrightarrow{\mathrm{K}} y\to\}}$, $\mathrm{L};\mathrm{K} \equiv \{\to x \xrightarrow{\mathrm{L}} z \xrightarrow{\mathrm{K}} y\to\}$ and $\mathrm{L}\dagger\mathrm{K} \equiv \{\to x \xrightarrow{\mathrm{M}} y\to\}$, where $\mathrm{M} := \to x \xrightarrow{\overline{\mathrm{L}}} z \xrightarrow{\overline{\mathrm{K}}} y\to$. We have no such rule for complementation, but we do have $\overline{\overline{\mathrm{L}}} \equiv \mathrm{L}$ By applying the operational rules in any context, we can eliminate all relational operations but complement. We can however obtain graphs and their complements as labels. (For instance $r;(s\sqcup t) \rhd^{(;)} \{\to x \xrightarrow{r} z \xrightarrow{s\sqcup t} y\to\} \rhd^{(\sqcup)} \left\{ \to x \xrightarrow{r} z \xrightarrow{\overline{\{\to x \xrightarrow{s} y\to, \to x \xrightarrow{t} y\to\}}} y\to \right\} = \mathsf{G}'$.)
The structural rules address such cases. We can replace a graph-label by glued slices, as (using '+' for addition of an arc) $\{\mathsf{S} + u\,\mathsf{H}\,v\} \equiv \mathsf{S}\frac{u}{v}\mathsf{H}$. (So, $\mathsf{G}' \rhd \{\to x \xrightarrow{r} z \xrightarrow{s} y\to, \to x \xrightarrow{r} z \xrightarrow{t} y\to\}$.) Also, we can replace a label that is a complemented graph by a slice, since $\overline{\mathsf{G}} \equiv \{\{\to x \xrightarrow{\overline{\mathsf{S}}} y\to / \mathsf{S} \in \mathsf{G}\}\}$. For a small slice, we can replace the complemented slice by a graph, moving complement inside. An *I-O arc* of S is an arc $u\,\mathrm{L}\,v$ with $\{u,v\} \subseteq \{x_{\mathsf{S}}, y_{\mathsf{S}}\}$. The *transformed* of I-O arc $\mathsf{a} = u\,\mathrm{L}\,v$ is the arc $\mathsf{a}^{\mathtt{tr}}$ obtained by replacing x_{S} by x, y_{S} by y and label L by $\overline{\mathrm{L}}$. Now, the *graph of slice* S is the graph $\mathsf{Gr}(\mathsf{S})$ with a single-arc slice $\langle\{x,y\},\{\mathsf{a}^{\mathtt{tr}}\} : x,y\rangle$, for each I-O arc a of S. Call a slice S *small* iff $N = \{x_{\mathsf{S}}, y_{\mathsf{S}}\}$. For a small slice S, $\overline{\{\mathsf{S}\}} \equiv \mathsf{Gr}(\mathsf{S})$. Finally, we can replace a label $\overline{r}$, with $r \in \mathbb{N}\mathsf{d}$, by $\overline{\to x \xrightarrow{r} y\to}$ (as $\overline{\mathrm{L}} \equiv \overline{\to x \xrightarrow{\mathrm{L}} y\to}$).
Thus, we can convert every label to an equivalent reduced graph. To establish that a reduced graph is null, we try to obtain a zero graph by applying the expansion rule. The expansion rule replaces a single-slice graph $\{\mathsf{S}\}$ by a two-slice graph $\{\mathsf{S}\frac{u}{v}\mathsf{T}, \mathsf{S} + u\,\overline{\mathsf{T}}\,v\}$, where u and v are nodes of S.
Soundness in not difficult to see. For completeness, consider a reduced graph G such that H is not zero, whenever $\mathsf{G} \vdash^{(\mathtt{Exp})} \mathsf{H}$. We can then obtain a family $\mathcal{R}$ of non-zero reduced slices (whose underlying drafts are connected by morphisms), which is saturated by applications of the expansion rule. This family can be used to obtain a co-limit sketch, which gives rise to a canonical counter model $\mathfrak{C}$: $[\![\mathsf{G}]\!]_{\mathfrak{C}} \neq \emptyset$.

References

[1] BRINK, C., KAHL, W. & SCHMIDT, G. (eds.) *Relational Methods in Computer Science*. Wien: Springer-Verlag, 1997.

[2] BROWN, C. & HUTTON, G. Categories, allegories and circuit design. In *Proc. LICS 94*: 372–381. Paris: IEEE, 1994.

[3] BROWN, C. & JEFFREY, A. Allegories of circuits. In *Proc. Logical Foundations of Computer Science*: 56–68, St. Petersburg, 1994.

[4] CURTIS, S. & LOWE, G. A graphical calculus. In *Mathematics of Program Construction* (LNCS **947**): 214–231; Berlin: Springer, 1995.

[5] CURTIS, S. & LOWE, G. Proofs with graphs. *Sci. Comput. Programm.* **26**:197–216, 1996.

[6] FREITAS, R., VELOSO, P. A. S., VELOSO, S. R. M. & VIANA, P. Reasoning with graphs. *Electronic Notes in Theoretical Computer Science* **165**: 201–212, 2006.

[7] FREITAS, R., VELOSO, P. A. S., VELOSO, S. R. M. & VIANA, P. On positive relational calculi. *Logic J. IGPL* **15**: 577–601, 2007.

[8] FREITAS, R., VELOSO, P. A. S., VELOSO, S. R. M. & VIANA, P. On a graph calculus for algebras of relations. In Hodges, W. & de Queiroz, R. (eds.) *WoLLIC 08* (LNAI **5110**): 298 - 312, Heiderberg: Springer-Verlag, 2008.

[9] FREITAS, R., VELOSO, P. A. S., VELOSO, S. R. M. & VIANA, P. On graph reasoning. *Information and Computation* **207**: 1000–1014, 2009.

[10] FREITAS, R., VELOSO, P. A. S., VELOSO, S. R. M. & VIANA, P. A calculus for graphs with complement. In Goel, A. K., Jamnik, M. & de Narayanan, N. H. (eds.) *Diagrammatic Representation and Inference* (LNAI **6170**): 84 –98, Berlin: Springer-Verlag, 2010.

[11] MACLANE, S. *Categories for the Working Mathematician*. Berlin: Springer-Verlag, 1971.

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