Intuitionistic modalities in topology and algebra

Mamuka Jibladze

Razmadze Mathematical Institute, Tbilisi

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Classical modal systems and topology

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The system **S4** is normal and satisfies

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These were considered by Rasiowa and Sikorski under the name of topological Boolean algebras; Blok used the term interior algebra which is mostly used nowadays along with **S4**-algebra.

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- $\delta\varnothing = \varnothing$,
- $\delta(A \cup B) = \delta A \cup \delta B$,
- $\delta\delta A \subseteq A \cup \delta A$.

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This τ satisfies the dual identities

- $\tau X = X$,
- $\tau(A \cap B) = \tau A \cap \tau B$,
- $A \cap \tau A \subseteq \tau\tau A$.

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If one wants to have a modal system with topological semantics interpreting $\Diamond$ as δ, and $\Box$ as τ, one thus arrives at a normal system with an additional axiom

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This is **wK4**, or weak **K4**.

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For the algebraic semantics one has derivative algebras $(B, \boldsymbol{\delta})$ — Boolean algebras with an operator $\boldsymbol{\delta}$ satisfying

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Or, one might define them in terms of the dual coderivative operator $\boldsymbol{\tau} = \neg\,\boldsymbol{\delta}\,\neg$ with axioms

- $\boldsymbol{\tau}\, 1 = 1$,
- $\boldsymbol{\tau}(b \wedge b') = \boldsymbol{\tau}\, b \wedge \boldsymbol{\tau}\, b'$,
- $b \wedge \boldsymbol{\tau}\, b \leqslant \boldsymbol{\tau}\,\boldsymbol{\tau}\, b$.

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HC

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also in general $\mathbf{c}$ does not leave any manageable “trace” on H.

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And now, $\boldsymbol{\tau}$ restricts to H in a nontrivial way:

since $\boldsymbol{\tau}$ is obviously monotone, $h \leqslant \boldsymbol{\tau}\, h$ implies $\boldsymbol{\tau}\, h \leqslant \boldsymbol{\tau}\,\boldsymbol{\tau}\, h$, i. e. $\boldsymbol{\tau}\,|_H \subseteq H$.

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Thus, each coderivative algebra $(B, \boldsymbol{\tau})$ gives rise to a Heyting algebra $H = \{h \in B \mid h \leqslant \boldsymbol{\tau}\, h\}$ equipped with an operator $\boldsymbol{\tau} = \boldsymbol{\tau}\,|_H : H \to H$.

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Then obviously

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Now if $(B, \boldsymbol{\tau}) = (\mathscr{P}(X), \tau)$ for a topological space X, the corresponding operator $\tau = \tau|_{\mathscr{O}(X)} : \mathscr{O}(X) \to \mathscr{O}(X)$ can be defined entirely in terms of $\mathscr{O}(X)$ as follows:

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In particular, one has

$$\tau(U) \leqslant V \cup \left(V \xrightarrow[\mathscr{O}(X)]{} U \right)$$

for any $U, V \in \mathscr{O}(X)$.

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It is given by adding to the axioms of **HC** the axioms

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However in *intuitionistic* modal systems such things happen.

E. g. for those familiar with nuclei — a nucleus is an inflationary multiplicative idempotent operator.

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Algebraic models of **mHC** are thus of the form $(H, \boldsymbol{\tau})$ where H is a Heyting algebra and $\boldsymbol{\tau} : H \to H$ satisfies

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Back to the classics
K4.Grz

Moreover one has

Theorem. *For every* **mHC**-*algebra* $(H, \boldsymbol{\tau})$ *there exists a coderivative algebra* $(\mathrm{B}(H), \boldsymbol{\tau})$ *such that* $H = \{h \in \mathrm{B}(H) \mid h \leqslant \boldsymbol{\tau}\, h\}$ *and* $\boldsymbol{\tau}\,|_H = \boldsymbol{\tau}$*.*

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Here $\mathrm{B}(H)$ is the free Boolean extension of H, so that every element of $\mathrm{B}(H)$ is a finite meet of elements of the form $\neg h' \vee h$ for some $h, h' \in H$. One defines

$$\boldsymbol{\tau}(\neg h' \vee h) := h' \underset{H}{\longrightarrow} h$$

and then extends to the whole $\mathrm{B}(H)$ by multiplicativity (correctness must be ensured).

Back to the classics
K4.Grz

It turns out that actually the coderivative algebras of the above form $(\mathrm{B}(H), \boldsymbol{\tau})$ land in a proper subvariety: they all are **K4**-algebras, i. e. satisfy $\boldsymbol{\tau}\, b \leqslant \boldsymbol{\tau}\, \boldsymbol{\tau}\, b$; moreover they satisfy the identity

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His $\#$ is defined on propositional variables p by $\#p := p \wedge \Box p$, commutes with $\wedge$, $\vee$, $\bot$, $\Box$ and moreover

$$\#(\varphi \to \psi) := (\#\varphi \to \#\psi) \wedge \Box(\#\varphi \to \#\psi).$$

Back to the classics

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Theorem. **mHC** $\vdash \varphi$ *iff* **K4.Grz** $\vdash \#\varphi$. *Moreover, the lattice of all extensions of* **mHC** *is isomorphic to the lattice of all normal extensions of* **K4.Grz**.

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This result may be viewed as an analog/generalization of the Kuznetsov-Muravitsky theorem.

Canonical choices of the modality

KM

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From the point of view of topological/algebraic semantics, this system is interesting in that in its models, the coderivative operator is in fact uniquely determined.

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Whereas if $(H, \boldsymbol{\tau})$ happens to be a model of **KM**, i. e. $\boldsymbol{\tau}\, h \to h$ is equal to h for all $h \in H$, then in addition to $\boldsymbol{\tau}\, h \leqslant h' \vee (h' \to h)$, also $\boldsymbol{\tau}\, h$ itself is of the form $h' \vee (h' \to h)$ for some h' (in fact for $h' = \boldsymbol{\tau}\, h$).

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Thus $\boldsymbol{\tau}\, h$ is the smallest element of the set $\{h' \vee (h' \to h) \mid h' \in H\}$, and this property makes it uniquely determined.

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Thus $\boldsymbol{\tau}\,h$ is the smallest element of the set $\{h' \vee (h' \to h) \mid h' \in H\}$, and this property makes it uniquely determined.

An algebra of the form $(\mathscr{O}(X), \tau)$ is a model of **KM** iff the space X is scattered (every nonempty subspace has an isolated point).

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Another way to ensure the preferred choice of $\boldsymbol{\tau}$ is to enrich the syntax.

Note that whenever H is a complete Heyting algebra, it comes with the “correct” $\boldsymbol{\tau}$, viz.

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One way to do this syntactically is to enrich the language with propositional quantifiers.

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The calculus **QHC** is obtained from **HC** by adding operators $\forall_p$, one for each propositional variable p.

Canonical choices of the modality

QHC

The calculus **QHC** is obtained from **HC** by adding operators $\forall_p$, one for each propositional variable p.

In **QHC**, one has axioms $(\forall_p\varphi) \to \varphi|_{\psi\leftarrow p}$ and inference rules

$$\frac{\psi \to \varphi}{\psi \to \forall_p\varphi}$$

whenever p does not occur freely in ψ (here $\varphi|_{\psi\leftarrow p}$ is the result of substituting ψ for p everywhere in φ).

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whenever p does not occur freely in ψ (here $\varphi|_{\psi\leftarrow p}$ is the result of substituting ψ for p everywhere in φ).

It is then easy to see that the modality $\Box$ given by

$$\Box\varphi := \forall_p(p \vee (p \to \varphi)),$$

for any p which does not occur freely in φ, satisfies all axioms of **mHC**.

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Q^+HC

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The corresponding system **Q^+HC** is given by adding to **QHC** the Casari schema

$$\forall_p((\varphi \to \forall_p \varphi) \to \forall_p \varphi) \to \forall_p \varphi.$$

Canonical choices of the modality

Kripke-Joyal semantics

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In particular, on the subobject classifier Ω one has the corresponding operator $\tau : \Omega \to \Omega$ given, in the Kripke-Joyal semantics, by

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Thus τ classifies the Higgs subobject $\{\mu \in \Omega \mid \{\top\} \cup {\downarrow}\mu = \Omega\}$ of Ω.

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Scattered toposes

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There are non-Boolean scattered toposes — e. g. the sheaves on any scattered space form a scattered topos.

Canonical choices of the modality

Scattered toposes

Theorem. *For an elementary topos $\mathscr{E}$, the following are equivalent:*

(i) $\mathscr{E}$ *is scattered, i. e.* $(\tau p \to p) \to p$ *holds in* $\mathscr{E}$*;*

(ii) *The Casari schema* $\forall_p((\varphi \to \forall_p\varphi) \to \forall_p\varphi) \to \forall_p\varphi$ *holds in* $\mathscr{E}$*;*

(iii) $(\forall_x\neg\neg\varphi(x)) \to \neg\neg\forall_x\varphi(x)$ *holds in every closed subtopos of* $\mathscr{E}$*.*

Temporal intuitionistic logic

tHC

The temporal Heyting Calculus **tHC** results from adding to **mHC** one more modal operator $\Diamond$, with additional axioms

- $p \to \Box\Diamond p$;
- $\Diamond\Box p \to p$;
- $\Diamond(p \vee q) \to (\Diamond p \vee \Diamond q)$;
- $\Diamond\bot \to \bot$

and an additional rule

$$\frac{p \to q}{\Diamond p \to \Diamond q}.$$

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If such an adjoint exists on a coderivative algebra $(\mathscr{P}(X), \boldsymbol{\tau})$ corresponding to a space X, then one can show that the topology on X is the Alexandroff topology for some preorder on X (namely, for the specialization preorder).

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For $(\mathscr{O}(X), \boldsymbol{\tau})$ existence of the adjoint is less stringent. There are “almost Alexandroff” spaces with this property which are not Alexandroff.

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$$\boldsymbol{\tau}_H(h) = \bigwedge \left\{ h' \vee (h' \to h) \mid h' \in H \right\}$$

has a left adjoint $\boldsymbol{\tau}_{H^\circ}$, where H° is H with the order reversed.

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has a left adjoint $\boldsymbol{\tau}_{H^\circ}$, where H° is H with the order reversed. There are still more general spaces with this property. The (probably) simplest Heyting algebra which is not bi-Heyting is given by $\top = a_0 > a_1 > a_2 > a_3 > \ldots$ and $b_0 > b_1 > b_2 > b_3 > \ldots > \bot$, with $a_n > b_n$ for each n. It is the algebra of open sets of a space, so has a canonical coderivative operator $\boldsymbol{\tau}$. The left adjoint $\rotatebox[origin=c]{180}{$\boldsymbol{\tau}$}$ to $\boldsymbol{\tau}$ is given by $\rotatebox[origin=c]{180}{$\boldsymbol{\tau}$}\, a_n = \rotatebox[origin=c]{180}{$\boldsymbol{\tau}$}\, b_n = b_{n+1}$.