The Equivalence of Game and Denotational Semantics for the Probabilistic μ-Calculus

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Outline

- Introduction to the standard modal μ-calculus
  - Labeled Transition Systems
  - Syntax, Denotational Semantics
  - Examples
  - Game Semantics
- **Probabilistic** modal μ-calculus
  - Probabilistic Labeled Transition Systems
  - Syntax, Denotational Semantics
  - Examples
  - Game Semantics
- Sketch of the Proof Technique

Labeled Transition Systems

A LTS is a pair $\langle P, \{\xrightarrow{a}\}_{a\in L}\rangle$ where

- P is a countable set of states,
- L is a countable set of labels, or *atomic* actions,
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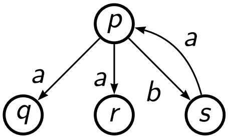

Modal μ-Calculus

The modal μ-calculus extends Hennessy-Milner Logic with least and greatest fixed points:

$$F ::= F \vee F \mid F \wedge G \mid \langle a \rangle F \mid [a] F \mid X \mid \mu X.F \mid \nu X.F$$

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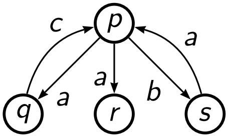
One can define a *negation* operator $\sim$by induction as follows:

- $\sim(F \vee G) = \sim F \wedge \sim G$
- $\sim(F \wedge G) = \sim F \vee \sim G$
- $\sim(\langle a \rangle F) = [a] \sim F$
- $\sim([a] F) = \langle a \rangle \sim F$
- $\sim(\mu X.F) = \nu X. \sim F[\sim X/X]$
- $\sim(\nu X.F) = \mu X. \sim F[\sim X/X]$
- $\sim\sim X = X$

Fact: $[\![\sim F]\!](p) = \neg\big([\![F]\!](p)\big)$

Examples



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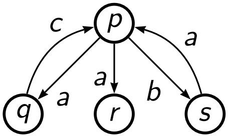

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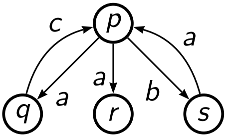

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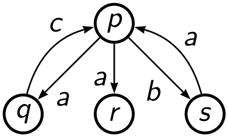

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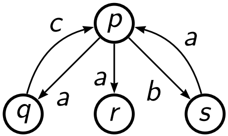

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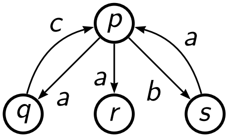

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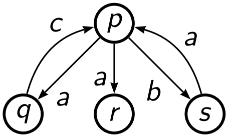

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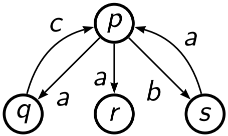

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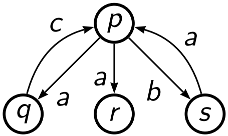

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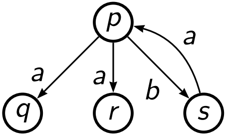
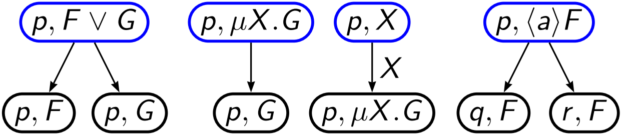
2 Player Game Semantics

The modal μ-calculus has a complementary game semantics (Emerson and Jutla 1991, Stirling 1996)



A game is an infinite directed graph (V, E). The states $v \in V$ of the game are pairs $\langle p, G \rangle$. E is defined using the structure of G.



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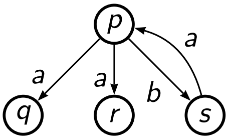

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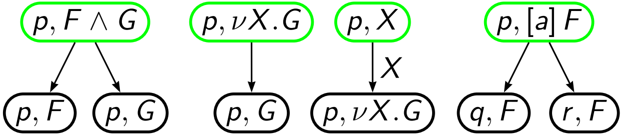

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Determinacy of Gale-Stewart Games [Martin 1975]:
Either P_1 has a winning strategy or P_2 has a winning strategy.

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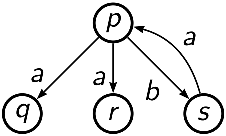
The game semantics of the formula F is the map $(\!| F |\!) : P \rightarrow \{\bot, \top\}$ defined as

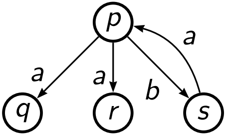
$$(\!| F |\!)(p) = \top \text{ if } P_1 \text{ has a winning strategy in } \langle p, F \rangle$$

$$(\!| F |\!)(p) = \bot \text{ if } P_2 \text{ has a winning strategy in } \langle p, F \rangle$$

example



$$[\![\nu X.(\langle a\rangle tt \wedge \langle b\rangle\langle a\rangle X)]\!]\,(p) = ?$$



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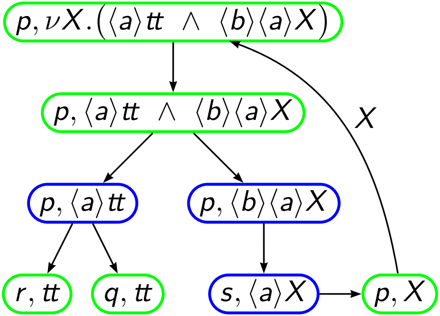
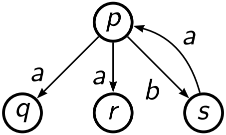


example



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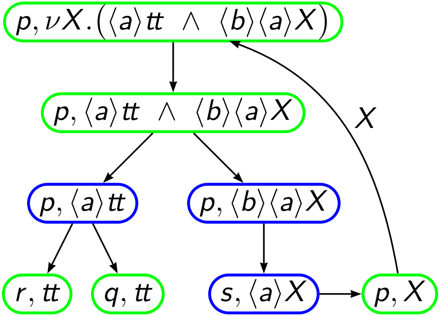

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Probabilistic LTS

A PLTS is a pair $\langle P, \{\xrightarrow{a}\}_{a \in L}\rangle$ where

- P is a countable set of states,
- L is a countable set of labels, or *atomic* actions,
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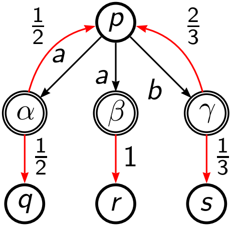

Probabilstic Modal μ-calculus

The Probabilistic modal μ-Calculus, was introduced in

- Huth and Kwiatkowska 1997
- McIver and Morgan 2003
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as a logic for expressing properties of PLTS:

It has the same syntax of standard μ-calculus:

$$F ::= F \vee F \mid F \wedge G \mid \langle a \rangle F \mid [a] F \mid X \mid \mu X.F \mid \nu X.F$$

The semantics of a formula is: $[\![F]\!]_\rho : P \to [0,1] \cong \mathcal{D}\{\top, \bot\}$

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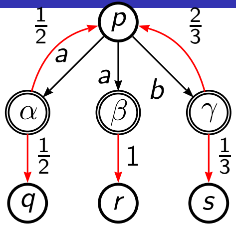
where $[\![F]\!]_\rho (\alpha) = \sum_{p \in supp(\alpha)} \alpha(p) \cdot [\![F]\!]_\rho (p)$

Examples



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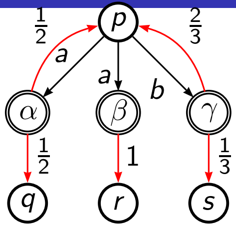
Examples



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$$\llbracket \langle a \rangle tt \rrbracket (p) = 1$$
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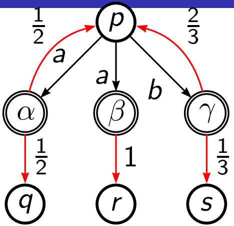
Examples



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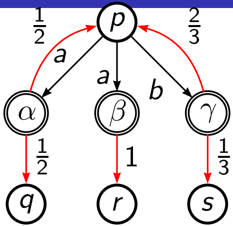
Examples



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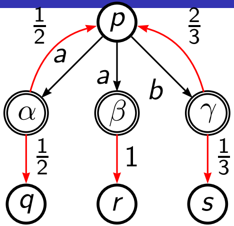

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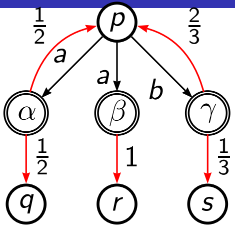
Examples



$$\llbracket \langle a \rangle F \rrbracket_\rho (p) = \bigsqcup_{p \xrightarrow{a} \alpha} \llbracket F \rrbracket_\rho ({\color{red}\alpha})$$

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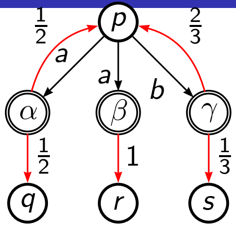
Examples



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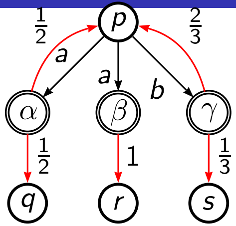
$\llbracket [b] [b] [b] ff \rrbracket (p) = \frac{5}{9}$

Examples



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$$\llbracket \mu X.\, [b]\, X \rrbracket (p) = 1$$

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Remark 2: at early stages [Huth and Kwiatkowska 1997] of the development of this logic, different semantics were proposed:

$$[\![F \wedge G]\!]_\rho = [\![F]\!]_\rho \sqcap [\![G]\!]_\rho$$

$$[\![F \wedge G]\!]_\rho = [\![F]\!]_\rho \cdot [\![G]\!]_\rho$$

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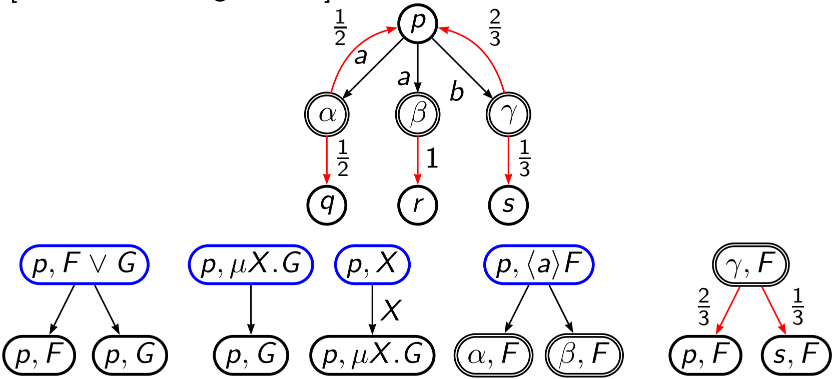
A game semantics for the probabilistic μ-calculus was proposed in [McIver and Morgan 2003].



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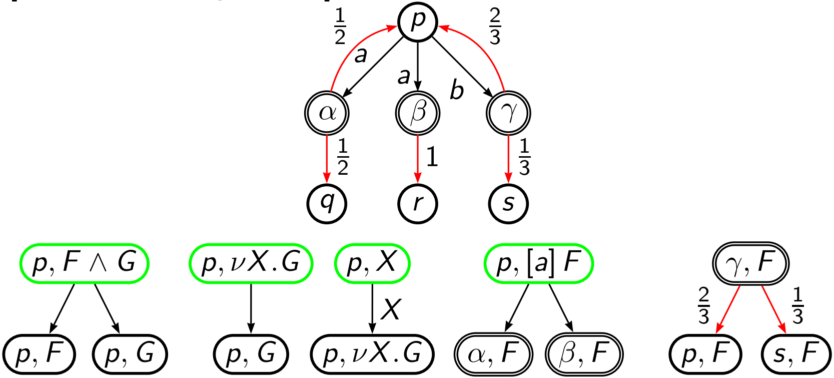

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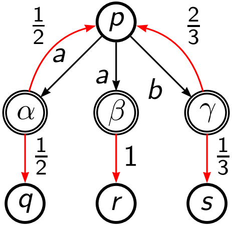

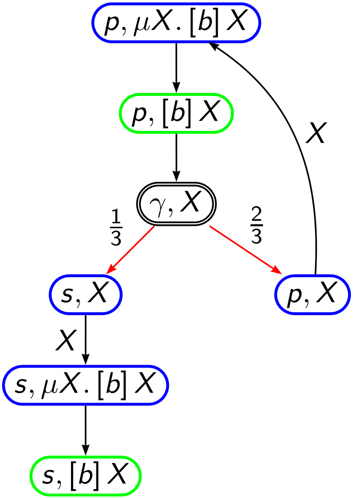

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Idea: When the two Players play accordingly with $\langle \sigma_1, \sigma_2 \rangle$ Player 1 wins with probability $\mathbb{V}^{v}_{\sigma_1,\sigma_2}$

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Determinacy of Blackwell Games [Martin 1998, Maitra and Sudderth 1998]: $1 = 2$

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$$\mathcal{V}(v) \stackrel{\text{def}}{=} \bigsqcup_{\sigma_1} \bigsqcap_{\sigma_2} \mathbb{V}^{v}_{\sigma_1,\sigma_2} = \bigsqcap_{\sigma_2} \bigsqcup_{\sigma_1} \mathbb{V}^{v}_{\sigma_1,\sigma_2}$$

is called the *value* of the game at v.

Fact 1: No *optimal* strategies! only ϵ-optimal strategies.

Fact 2: unbounded amount memory is needed, in general!

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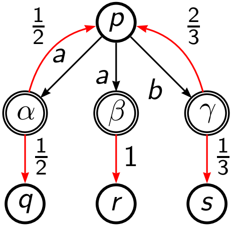
The proof uses a technique recently introduced in [Fischer, Gradel and Kaiser 2009]

example



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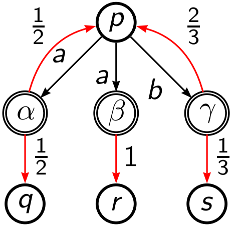

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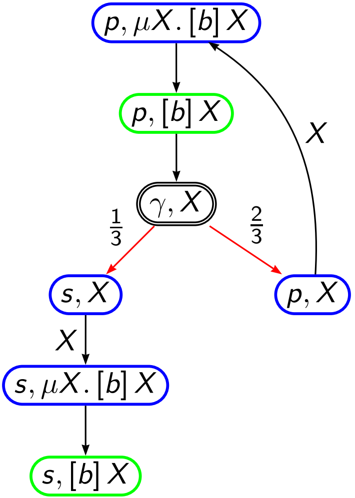

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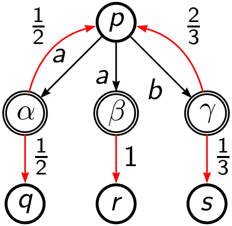

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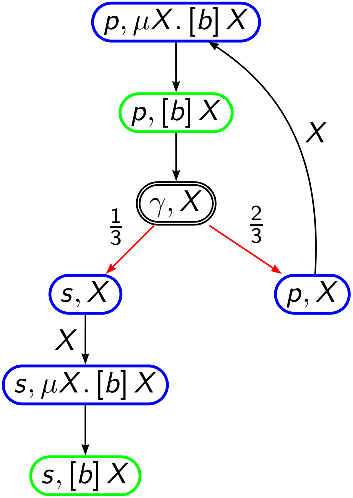

Proof Technique

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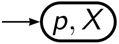

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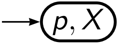

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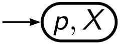

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Intuition: Player 2 wins in $(\!| \mu X.F |\!)_{\rho}$ at least as in $(\!| F |\!)_{\rho^{\gamma}}$, i.e. Player 1 loses $(\!| \mu X.F |\!)_{\rho}$ at least as in $(\!| F |\!)_{\rho^{\gamma}}$.

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