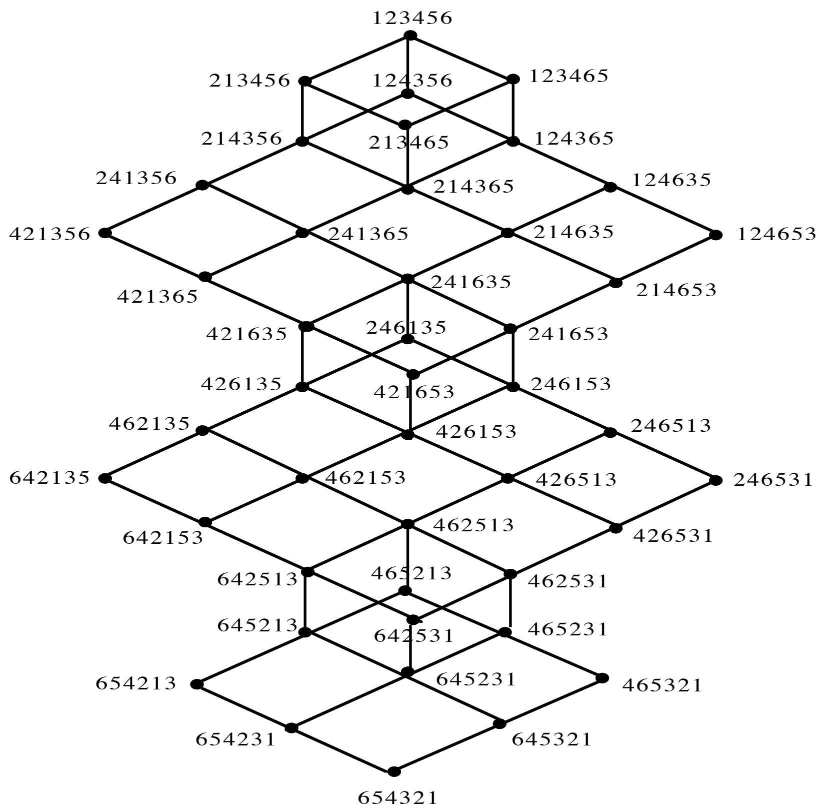
CONDORCET DOMAINS and DISTRIBUTIVE LATTICES

Bernard Monjardet

CES (CERMSEM) Université Paris I Panthéon Sorbonne, Maison des Sciences Économiques, 106-112 bd de l'Hopital 75647 Paris Cédex 13, FRANCE, and CAMS, EHESS, (e-mail monjarde@univ-paris1.fr)



CONDORCET DOMAINS and DISTRIBUTIVE LATTICES

SUMMARY

- Condorcet domains
  - Definition
  - Characterization (Ward, Sen...)
  - Examples
  - Maximum size ?
- Distributive lattices
  - Birkhoff's duality
- CH-Condorcet domains

(Black, Guilbaud, Blin, Romero, Frey, Abello, Arrow and Raynaud, Chameni-Nembua, Craven, Fishburn, Galambos and Reiner ...)

  - Definition (closure operator)
  - Examples
  - Main results

3 types of CH-Condorcet domains

- Maximal chain (Blin)
- Single-peaked (Black)
- Alternating-scheme (Fishburn)

Maximum size
Conjectures

CONDORCET DOMAINS

A *CONDORCET DOMAIN* is a set of linear orders where the majority rule works well:
the STRICT MAJORITY RELATION IS always
A (not necessarily linear) ORDER
(equivalently, it has never cycles)

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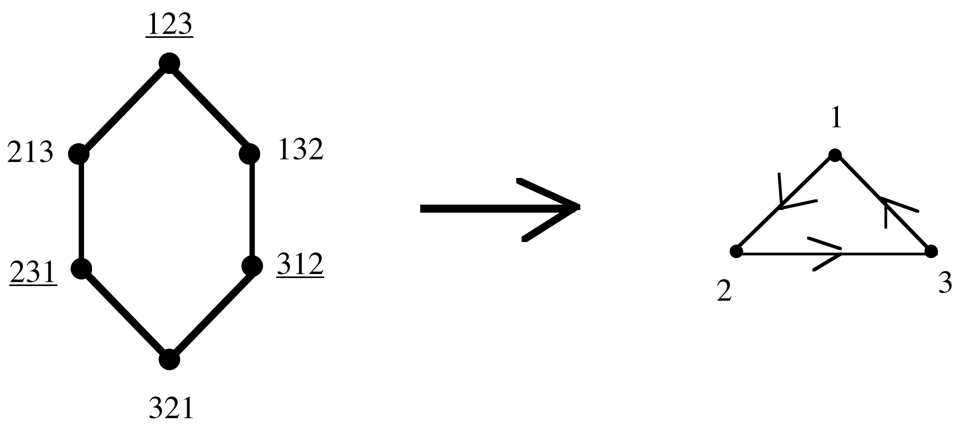
A COUNTER-EXAMPLE



123
231
321

FORMALLY…

$A = \{1,2\ldots n\}$ (alternatives, candidates, decisions,…)

$L = x_1 < x_2 < \ldots . x_n$ linear order on A (permutation $x_1 x_2 \ldots . x_n$; rank of $x_i = i$)

$\mathcal{D} \subseteq \mathcal{L}_n = \{n!$ linear orders on $A\}$

$\pi \in \mathcal{D}^V$ *profile* of v "voters"

$yL_q x$ for voter q if (s)he prefers x to y

$$\mathbf{yR_{MAJ}(\pi)x \text{ if } |\{q \in V : yL_q x\}| > v/2}$$

A set $\mathcal{D}$ of linear orders is a *Condorcet domain* if

$\forall\, v \geq 1, \forall\, \pi \in \mathcal{D}^V$, $R_{MAJ}(\pi)$ has no cycles

Terminology : transitive simple majority domains*,* consistent sets*,* majority-consistent sets*,* **acyclic sets,** "domaines Condorcéens"

CONDORCET DOMAINS CHARACTERIZATIONS

Ward, Sen,…..

$\mathcal{D} \subset \mathcal{L}_n$ is a Condorcet domain

$\Leftrightarrow$

$\mathcal{D}$ does not contain *3-cyclic sets*

$\Leftrightarrow$

$\mathcal{D}$ is *value-restricted*

3-cyclic set (latin square):

$x_1x_2x_3$

$x_2x_3x_1$

$x_3x_1x_2$

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3-cyclic set (latin square):

$x_1x_2x_3$
$x_2x_3x_1$
$x_3x_1x_2$

$\mathcal{D} \subset \mathcal{L}_n$ is ***value-restricted*** if, for every subset {i,j,k} of A, **there exists an alternative** which either has **never rank 1 or never rank 2 or never rank 3** in the set $\mathcal{D}_{/\{i,j,k\}}$ (of the restrictions of the orders of $\mathcal{D}$ to the set {i,j,k})

NEVER CONDITIONS

For $i<j<k$, $h \in \{i,j,k\}$ and $r \in \{1,2,3\}$,

$\mathcal{D}$ satisfies the *Never Condition* $hN_{\{i,j,k\}}r$

if h has never rank r in the set $\mathcal{D}_{/\{i,j,k\}}$

$\mathcal{D} \subset \mathcal{L}_n$ is a Condorcet domain

$\Leftrightarrow$

for every $i<j<k$, there exists $h \in \{i,j,k\}$ and $r \in \{1,2,3\}$: $hN_{\{i,j,k\}}r$

$\mathcal{D}$ satisfies the *Never Condition hNr*

if for every $i<j<k$, and for h and r fixed $hN_{\{i,j,k\}}r$

NEVER CONDITIONS

$\mathcal{D}$ satisfies the *Never Condition hNr*
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$\mathcal{B}(4)$ = {4321, 4312, 4132, 4123, 1432, 1423, 1243, 1234}
satisfies jN1 (i<j<k)

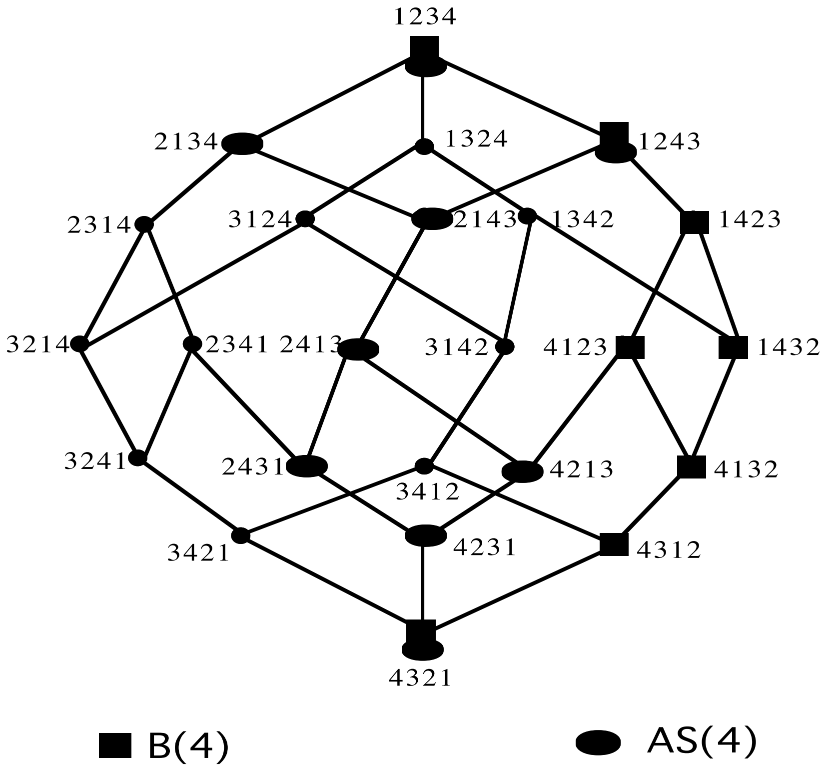
$\mathcal{B}(4)$	123	124	134	234
1234	123	124	134	234
1243	123	124	143	243
1423	123	142	143	423
1432	132	142	143	432
4123	123	412	413	423
4132	132	412	413	432
4312	312	412	431	432
4321	321	421	431	432
	2 NEVER 1	2 NEVER 1	3 NEVER 1	3 NEVER 1

$\mathcal{C}(4)$ = {4321, 4231, 2431, 2341, 2314, 2134,1234} satisfies

C(4)	123	124	134	234
1234	123	124	134	234
2134	213	214	134	234
2314	231	214	314	234
2341	231	241	341	234
2431	231	241	431	243
4231	231	421	431	423
4321	321	421	431	432
	2 NEVER 3	2 NEVER 3	3 NEVER 3	3 NEVER 1

$\mathcal{AS}(4)$ = {4321,42312431,4213,2413,2143,2134,1243,1234} satisfies ......

THE PERMUTOEDRE LATTICE



$$\mathcal{L}_4$$

$$x_1 x_2 \ldots \mathbf{x_i x_{i+1}} \ldots x_n \prec x_1 x_2 \ldots \mathbf{x_{i+1} x_i} \ldots x_n$$

HOW LARGE CAN BE CONDORCET DOMAINS ?

A Condorcet domain $\mathcal{D}$ is *maximal*, if for any linear order L not in $\mathcal{D}$, $\mathcal{D} \cup \{L\}$ is no more a Condorcet domain.

A Condorcet domain $\mathcal{D} \subset \mathcal{L}_n$ is *maximum* if it has the maximum cardinality among all Condorcet domains in $\mathcal{L}_n$.

PROBLEM

What is the size of a maximum Condorcet domain ?

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CONJECTURE (Johnson 1978, Craven 1992)

$$\text{maximum size} = 2^{n-1}$$

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CONJECTURE (Johnson 1978, Craven 1992)

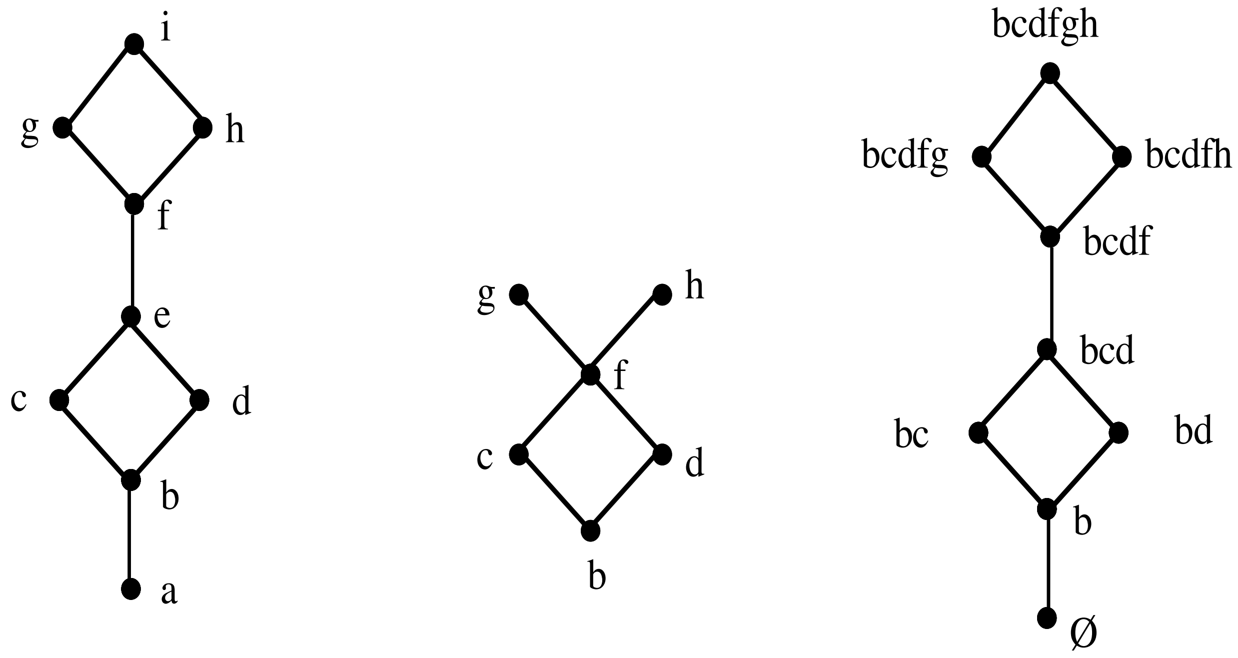
$$\text{maximum size} = \mathbf{2^{n-1}}$$

DISPROVED : Kim and Roush 1980 !
for n = 4 ! !

The three previous examples are maximal Condorcet domains and $\mathcal{AS}(4)$ is maximum of size 9.

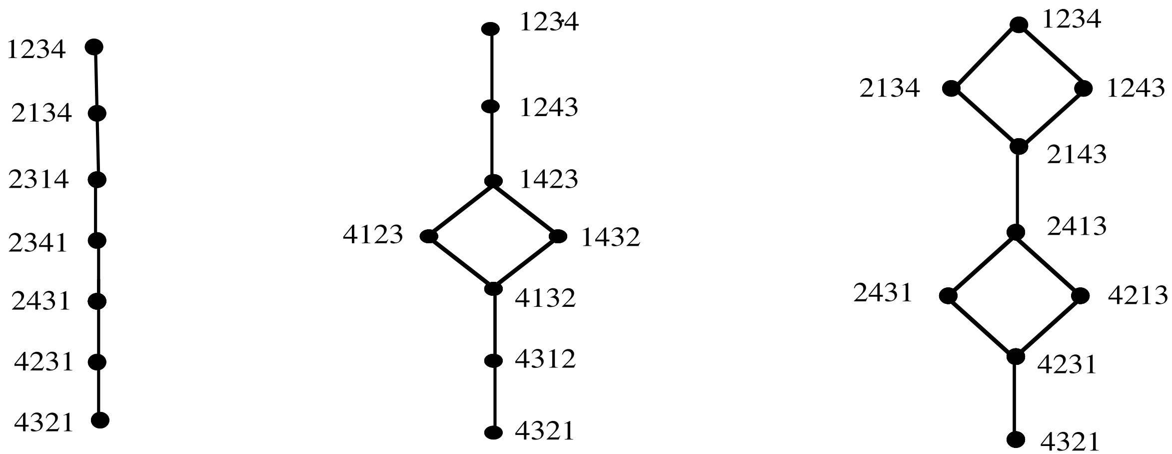
DISTRIBUTIVE LATTICES
Birkhoff's representation theorem

A distributive lattice L is isomorphic to the lattice of ideals of the poset J_L of its join-irreducible elements



L $\quad$ J_L $\quad$ $\mathcal{I}(J_L)$

The THREE EXAMPLES of CONDORCET DOMAINS in $\mathcal{L}_4$



$\mathcal{C}(4)$ $\mathcal{B}(4)$ $\mathcal{AS}(4)$

t($\mathcal{C}$(4)) = {123,124,134,234,213,214,231,314,241, 341,243,431,423,421,432,321}

t($\mathcal{B}$(4)) = {123,124,134,234,143,243,142,423,132, 432,412,413,431,312,432,321}

t($\mathcal{AS}$(4)) = {123,124,134,234,213,214,143,243,241,413,231,431, 421,423,432,321}

16 ordered triples Why ?

THREE OBSERVATIONS

A CONDORCET DOMAIN of $\mathcal{L}_n$ contains at most

4n(n-1)(n-2)/6 ordered triples

(16 for n = 4)

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A CONDORCET DOMAIN of $\mathcal{L}_n$ contains at most **4n(n-1)(n-2)/6** ordered triples (16 for n = 4)

Any MAXIMAL CHAIN of $\mathcal{L}_n$ is a CONDORCET DOMAIN (Blin, 1972) containing

$$n(n-1)(n-2)/6+(n-2)[n(n-1)/2] =$$

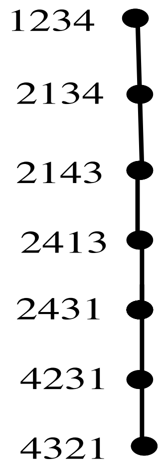
4n(n-1)(n-2)/6 ordered triples

THREE OBSERVATIONS

A CONDORCET DOMAIN of $\mathcal{L}_n$ contains at most **4n(n-1)(n-2)/6** ordered triples (16 for n = 4)

Any MAXIMAL CHAIN of $\mathcal{L}_n$ is a CONDORCET DOMAIN (Blin, 1972) containing n(n-1)(n-2)/6+(n-2)[n(n-1)/2] = **4n(n-1)(n-2)/6** ordered triples

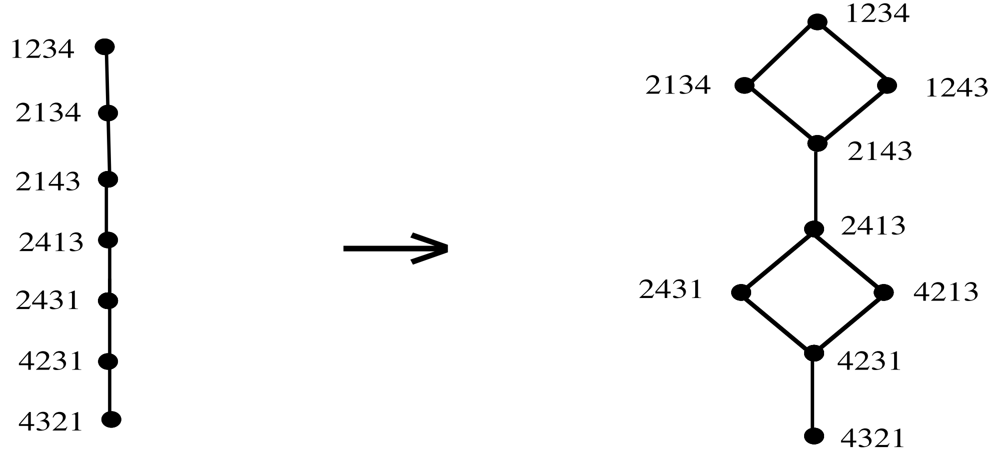
A MAXIMAL CHAIN of $\mathcal{L}_n$ is not generally a MAXIMAL CONDORCET DOMAIN



One can add 1243 and 4213

CH-CONDORCET DOMAINS

The closure operator

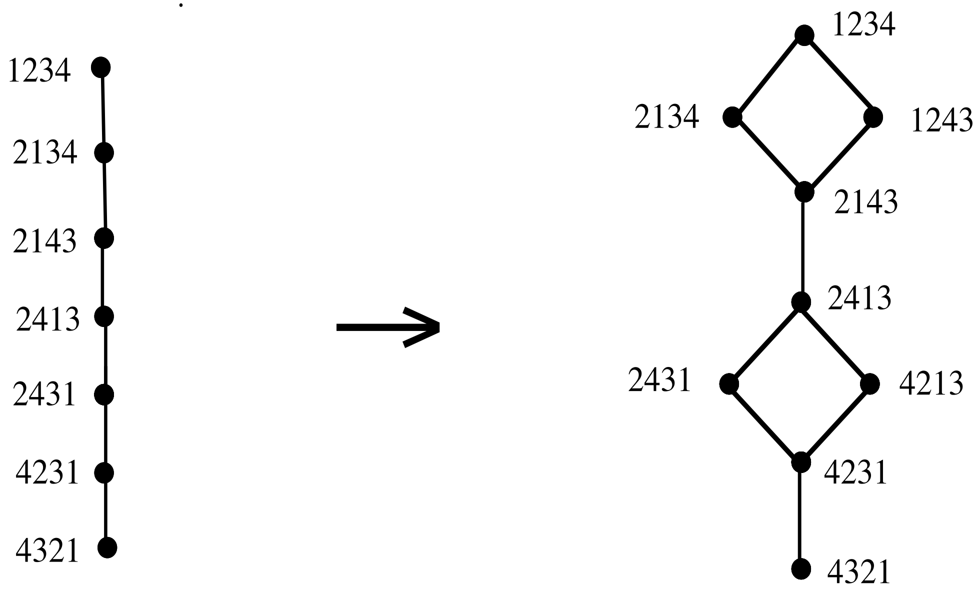


$$\mathcal{E} \subset \mathcal{L}_n \rightarrow \quad \mathcal{E} \cup \{L \in \mathcal{L}_n : t(L) \subset t(\mathcal{E})\}$$

(Closure operator defined by Kim and Roush, 1980)

To ADD to a set $\mathcal{E}$ of linear orders all the linear orders not increasing the set of ordered triples contained in $\mathcal{E}$

CH-CONDORCET DOMAINS



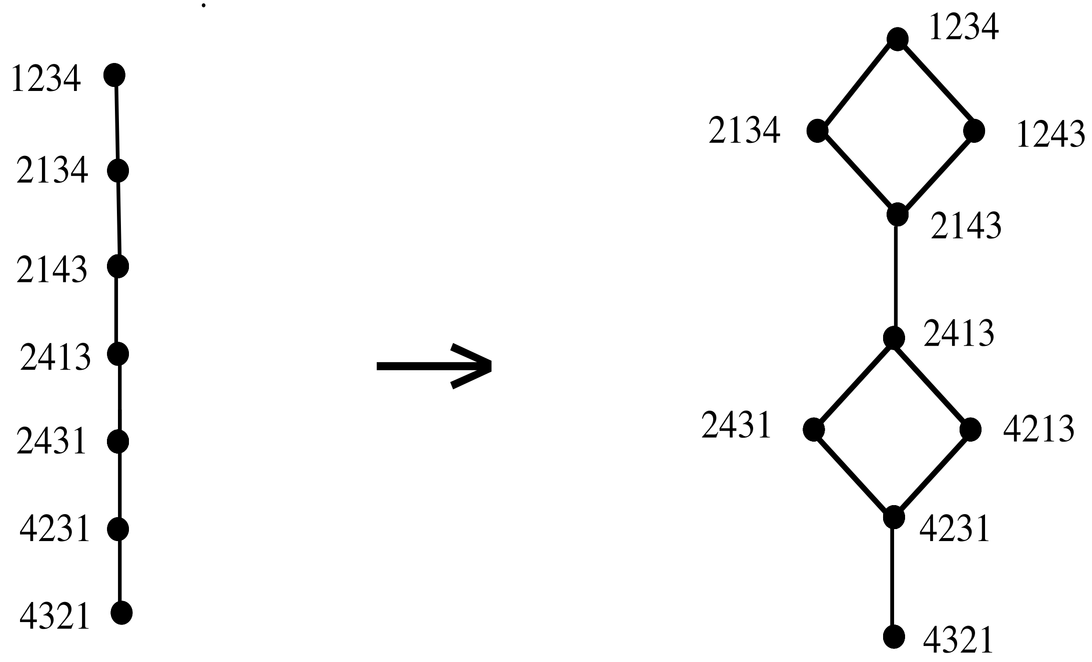
A CH-CONDORCET DOMAIN IS
the closure $\mathcal{D}$ of a maximal chain C of $\mathcal{L}_n$, and so is a
A MAXIMAL CONDORCET DOMAIN

Abello, 1984, 1985

CH-Condorcet domains of size $3.2^{n-2}-4$

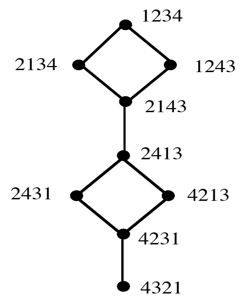
An EXAMPLE of CH-CONDORCET

DOMAIN: $\mathcal{AS}(4)$



$\mathcal{AS}(4)$ is a distributive lattice, maximal covering distributive sublattice of $\mathcal{L}_n$

An EXAMPLE of CH-CONDORCET DOMAIN: $\mathcal{AS}(4)$



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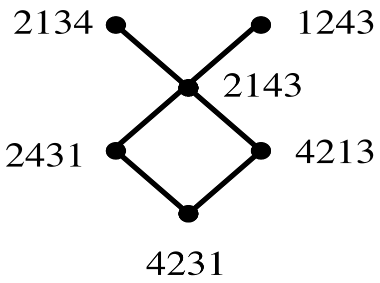
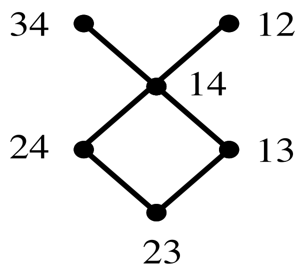


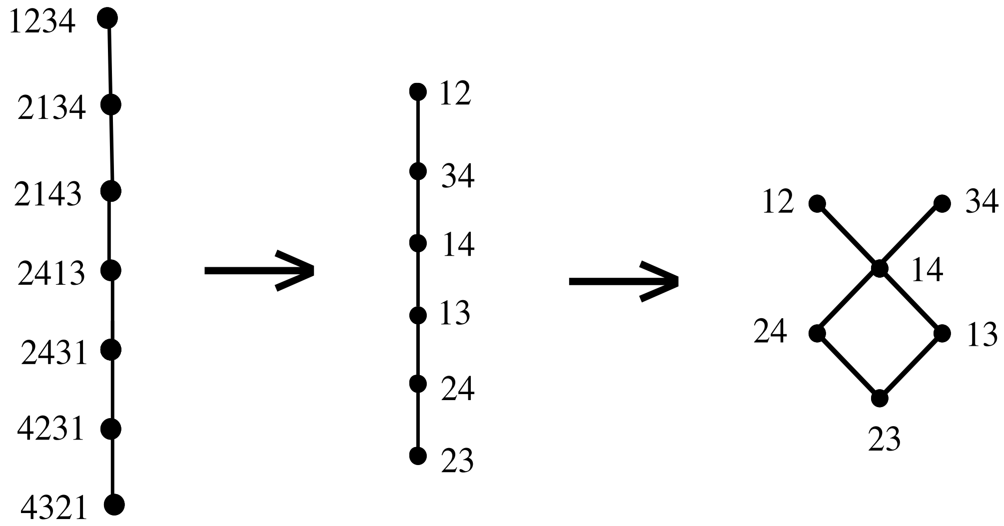
$J_{\mathcal{AS}(4)}$



$P_{\mathcal{AS}(4)}$

$P_{\mathcal{AS}(4)}$ is defined on the set $P^2(4)$ of ordered pairs of {1,2,3,4}. It induces $\mathcal{AS}(4)$

$P_{\mathcal{AS}(4)}$ can be obtained from any maximal chain of $\mathcal{AS}(4)$:



- $4321 \prec 4231 \prec 2431 \prec 2413 \prec 2143 \prec 2134 \prec 1234$
- associate the linear order : $23 \prec 24 \prec 13 \prec 14 \prec 34 \prec 12$
- and ………

N.B. This construction allows to get a maximal Condorcet domain (which is a distributive lattice) from any maximal chain of $\mathcal{L}_n$

NEVER CONDITIONS for $\mathcal{AS}(4)$

$\mathcal{AS}(4)$	123	124	134	234
1234	123	124	134	234
2134	213	214	134	234
1243	123	124	143	243
2143	213	214	143	243
2413	213	241	413	243
2431	231	241	431	243
4213	213	421	413	423
4231	231	421	431	423
4321	321	421	431	432
	2 NEVER 3	2 NEVER 3	3 NEVER 1	3 NEVER 1

3 NEVER 1 in {134} and {234}
2 NEVER 3 in {123} and {124}

NEVER CONDITIONS for $\mathcal{AS}(4)$

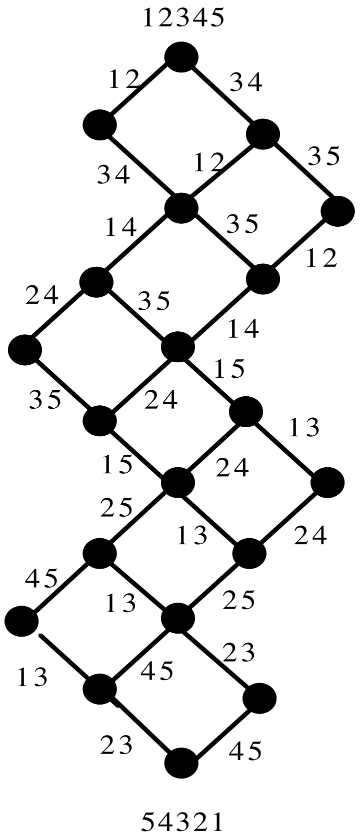
3 NEVER 1 in {134} and {234}
2 NEVER 3 in {123} and {124}

GENERALIZATION:

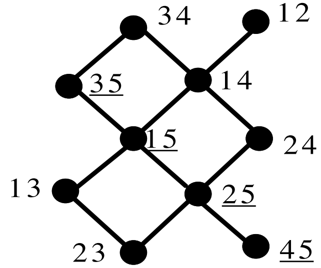
FISHBURN'S ALTERNATING SCHEME (1997) giving $\mathcal{AS}$(n)

$$\forall\ i < j < k \text{ and } j\ \underline{\text{odd}},\ jN1 \text{ in } L_{/\{i,j,k\}}$$
$$\forall\ i < j < k \text{ and } j\ \underline{\text{even}},\ jN3 \text{ in } L_{/\{i,j,k\}}$$

$\mathcal{AS}(5)$

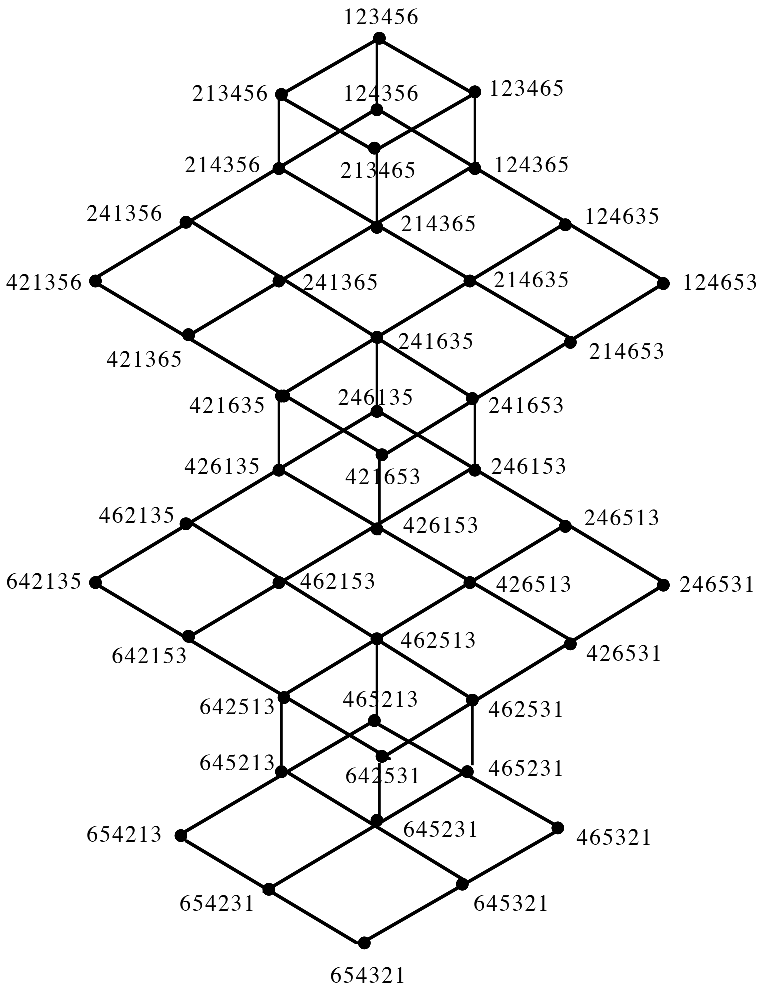


$\mathcal{AS}(5)$



$P_{\mathcal{AS}(5)}$

$\mathcal{AS}(6)$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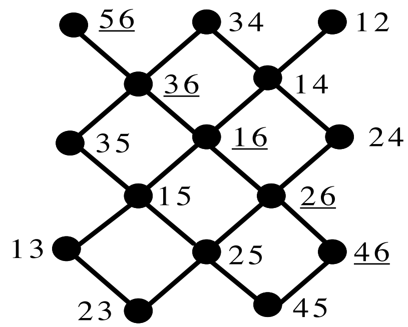


$P_{\mathcal{AS}(6)}$

N.B. For the « little history »

I get this Condorcet domain when I was director of Chameni-Nembua's Thesis (1989),

where he proved that any covering distributive sublattice of $\mathcal{L}_n$ is a Condorcet domain

(generalizing Guilbaud's 1952 observation on Black's domains and Frey's 1971 results)

$$45 > 44 = 3.2^{6-2}-4 \quad !$$

A FUNDAMENTAL OBSERVATION

Maximal chain of $\mathcal{AS}(4)$:

$4321 \prec 4231 \prec 2431 \prec 2413 \prec 2143 \prec 2134 \prec 1234$

Associated linear order on $P^2(4)$:

$\lambda = 23 \prec 24 \prec 13 \prec 14 \prec 34 \prec 12$

The restriction of the order of λ to the set {(ij), (ik), (jk)} of the ordered pairs of an ordered triple ijk is

- either the **lexicographic** order: $\{13<14<34\}, \{23<24<34\}$

- or the **dual lexicographic** order: $\{23<13<12\}, \{24<14<12\}$

In fact, a linear order λ on $P^2(n)$ is induced by a maximal chain of $\mathcal{L}_n$ iff for every ordered triple ijk, the three ordered pairs ij, ik and jk are ordered by λ either lexicographically or dually lexicographically .

CH-CONDORCET DOMAINS MAIN RESULTS

Let C be a maximal chain of the lattice $\mathcal{L}_n$

1 The closure $\mathcal{D} = \mathcal{D}(C)$ of C is

- a maximal Condorcet domain,
- a maximal covering distributive sublattice of $\mathcal{L}_n$.

One goes from a maximal chain of $\mathcal{D}$ to another one by a sequence of «quadrangular transformations» of the linear orders in the chains:

let $L = x_1 \ldots x_k x_{k+1} \ldots x_i x_{i+1} \ldots x_n$ be a linear order such that x_k, x_{k+1}, x_i and x_{i+1} are four different alternatives ; then L is transformed into $L' = x_1 \ldots x_{k+1} x_k \ldots x_{i+1} x_i \ldots x_n$.

CH-CONDORCET DOMAINS MAIN RESULTS

Let C be a maximal chain of the lattice $\mathcal{L}_n$

1 The closure $\mathcal{D} = \mathcal{D}(C)$ of C is

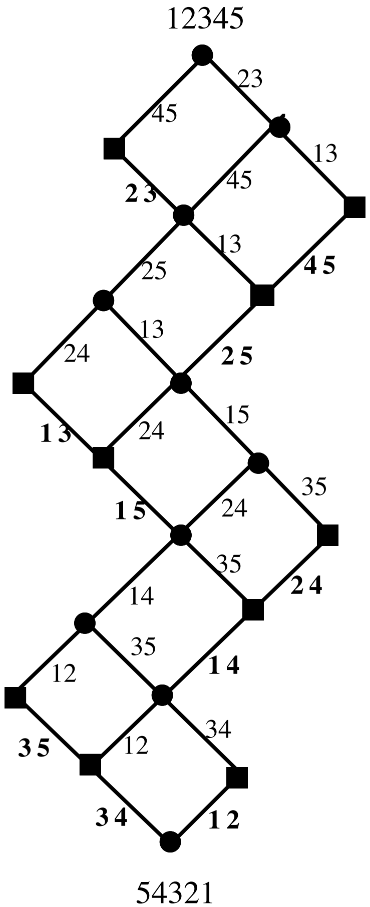
- a maximal Condorcet domain,
- a maximal covering distributive sublattice of $\mathcal{L}_n$.

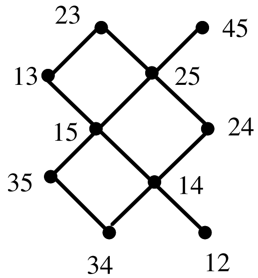
2 The poset $J_{\mathcal{D}}$ of the join-irreducible elements of the distributive lattice $\mathcal{D}$ is isomorphic to a poset $P_{\mathcal{D}}$ defined on the set of all ordered pairs $(i<j)$.

Any order in $\mathcal{D}$ corresponds to an ideal of this poset obtained by applying to $L_0 = n<\ldots 2<1$ all the transpositions of the ordered pairs belonging to this ideal.

The poset $P_{\mathcal{D}}$ can be obtained from any maximal chain of $\mathcal{D}$

EXAMPLE $\mathcal{AS}(5)$





$\mathcal{AS}(5)$

$P_{\mathcal{AS}(5)}$

CH-CONDORCET DOMAINS MAIN RESULTS

Let C be a maximal chain of the lattice $\mathcal{L}_n$ and λ the associated linear order on $P^2(n)$

1 The closure $\mathcal{D} = \mathcal{D}(C)$ of C is

- a maximal Condorcet domain,
- a maximal covering distributive sublattice of $\mathcal{L}_n$.

2 The poset $J_{\mathcal{D}}$ of the join-irreducible elements of the distributive lattice $\mathcal{D}$ is isomorphic to a poset $P_{\mathcal{D}}$ defined on the set of all ordered pairs (i<j).

3 $\mathcal{D}$ is the set of all linear orders satisfying the following Never Conditions:

jN1, $\forall$ i<j<k with ijk $\in LEX_3\lambda$

jN3, $\forall$ i<j<k with ijk $\in ALEX_3\lambda$.

where $LEX_3\lambda$ (resp. $ALEX_3\lambda$) is the set of ordered triples ijk where the three ordered pairs ij, ik and jk are lexicographically (resp. dually lexicographically) ordered by λ.

ALGORITHM CONSTRUCTING

$P_{\mathcal{D}}$ from a MAXIMAL CHAIN of $\mathcal{D}$

L_0 (= n...21) $\prec$ L_1.....$L_k$$\prec$ L_{k+1}....$L_{n(n-1)/2}$(= 12...n)

iterative construction of $P_{\mathcal{D}}$:

λ associated linear order on $P^2(n)$

$$\lambda = (i,j)_1 \prec (i,j)_2 \; .... \prec (i,j)_{n(n-1)/2}, \text{ where}$$

$$L_{k+1} = L_k \backslash (j,i)_k + \{(i,j)_k\} \text{ (and } i < j).$$

<u>First step :</u>

$P_{\mathcal{D}} = \{(i,j)_1\}$

Second step

$P_{\mathcal{D}} =$

-$(i,j)_1 + (i,j)_2$ if there is no the same element in the two ordered pairs $(i,j)_1$ and $(i,j)_2$;

- if not, one has for instance $(i,j)_1 = (x,y)$ and $(i,j)_2 = (y,z)$ and in this case $P_{\mathcal{D}}$ contains (x,y), (y,z) and the ordered pair (x,z) obtained by transitive closure of the two others.

Iterating this procedure one obtains finally the partial order $P_{\mathcal{D}}$ on the ordered pairs.

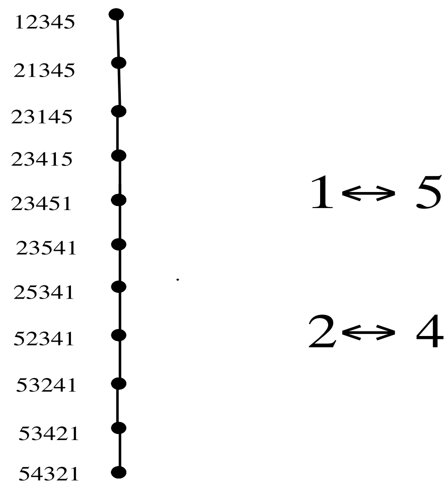
3 TYPES of CH-CONDORCET DOMAINS

- **Minimal CH-Condorcet domains**

- **CH-Condorcet domains given by Fishburn's alternating scheme**

- **CH-Condorcet domains given by Black's single-peaked orders**

Minimal CH-Condorcet domains



This maximal chain is obtained from I2345 by the sequence of transpositions exchanging successively the ranks of 1 and 5, then the ranks of 2 and 4 :

The set of following Never Conditions defines a maximal CH-Condorcet domain which is a maximal chain of $\mathcal{L}_n$:

jN1 $\forall$ i<j<k with

k $\in$ {n,n-1,....(n+t)/2} where t = 4 (respectively,3) for n even (respectively, n odd) and i > n+1-k.

jN3 $\forall$ i<j<k with i $\in$ {1,2...... $\lfloor$(n-1)/2$\rfloor$} and k < n+2-i.

CH-Condorcet domains $\mathcal{AS}$(n)

given by Fishburn's alternating scheme

For n odd, the covering pairs $(i,j)\prec(k,l)$

$(1\leq i<j\leq n)$ of the poset $P_{\mathcal{AS}(n)}$ are given by :

$\forall\ 2 < j,\ (1,j)\prec(2,j)$

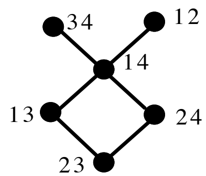
$\forall\ i < n-1,\ (i,n-1)\prec(i,n)$

For i even $< j-2$, $(i,j)\prec(i+2,j)$

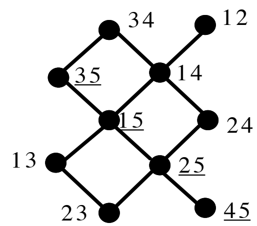
For i odd > 2, $(i,j)\prec(i-2,j)$

For j even $< n-2$, $(i,j)\prec(i,j+2)$

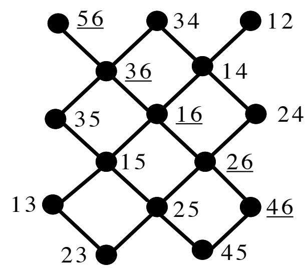
For j odd $> i+2$, $(i,j)\prec(i,j-2)$



$P_{\mathcal{AS}(4)}$



$P_{\mathcal{AS}(5)}$



$P_{\mathcal{AS}(6)}$

CH-Condorcet domains $\mathcal{B}$(n) (Black's single-peaked orders)

The set A is linearly ordered as 1<2…<p…<n by a "reference" order.

Let p be the preferred alternative of a linear order L

L is *single-peaked* (w.r.t. <) if

$$i<j<p \Rightarrow iLj(Lp), \text{ and}$$
$$p<i<j \Rightarrow jLi(Lp).$$

FACT

A linear order L is single-peaked (w.r.t. <)

$$\Leftrightarrow$$

for every ordered triple i<j<k, L satisfies the never condition jN1.

N.B On n alternatives, there are 2^{n-1} single-peaked linear orders (w.r.t. <) (Kreweras, 1962).Arrow-Black domain: for every 3-subset {i,j,k} of A, there exists $h \in \{i,j,k\}$ such that $hN_{\{i,j,k\}}1$

CH-Condorcet domains $\mathcal{B}(n)$ (Black's single-peaked orders)

The poset $P_{\mathcal{B}(n)}$ is a **lattice** of which the covering relation is given by:

$(i,j) \prec (k,h)$ $(1 \leq i < j \leq n)$ if $i = k$ and $h = j+1$,

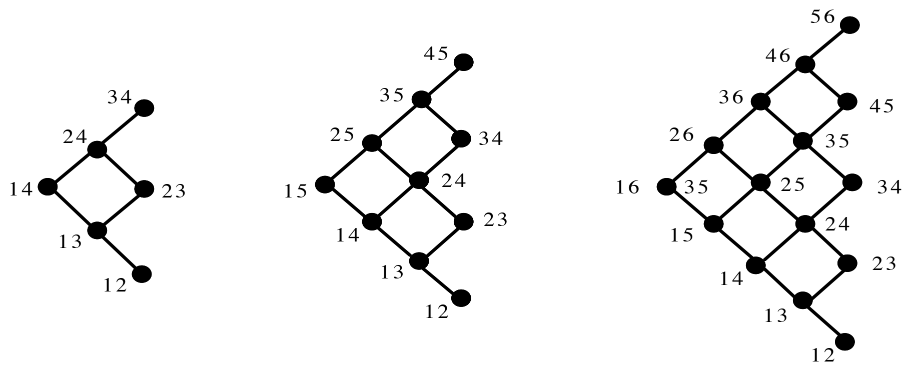
or if $k = i+1$ and $j = h$.

The join and meet operations of this lattice are:

$(i,j) \vee (k,h) = (\max(i,k), \max(j,h))$ and

$(i,j) \wedge (k,h) = (\min(i,k), \min(j,h))$.

A maximal chain of $\mathcal{B}(n)$ is: $12 \prec .... \prec 1n \prec 23 \prec .... \prec 2n \prec 34 \prec ..... \prec 3n \prec ..... \prec 1n$.



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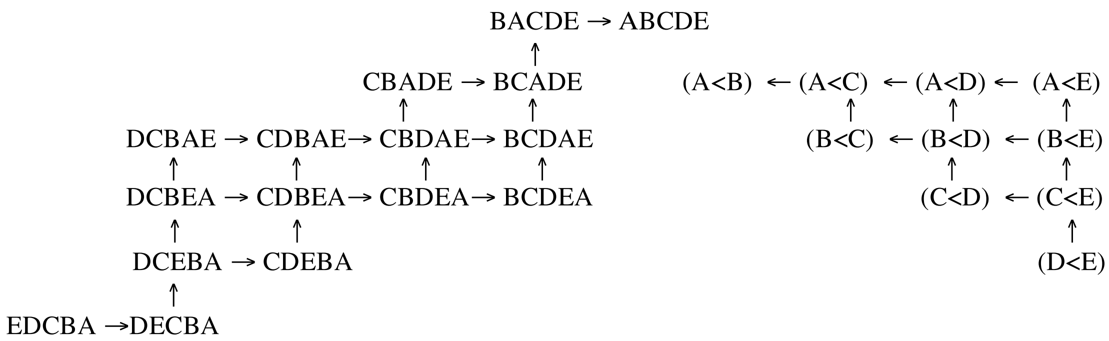

Guilbaud 1952

MAXIMUM SIZE

$$f(n) = \mathrm{MAX}\{|\mathcal{D}|,\ \mathcal{D} \text{ Condorcet domain} \subset \mathcal{L}_n\}$$

A Condorcet domain $\mathcal{D} \subset \mathcal{L}_n$ is *connected*, if there always exists a *path* in the permutoèdre graph $\mathcal{L}_n$ between two linear orders in $\mathcal{D}$.

$$g(n) = \mathrm{MAX}\{|\mathcal{D}|,\ \mathcal{D} \text{ connected Condorcet domain of maximum diameter} \subset \mathcal{L}_n\}$$

g(n) = MAXIMUM SIZE of a CONNECTED ACYCLIC SET
f(n) = MAXIMUM SIZE of a MAXIMAL ACYCLIC SET

	A	B	C	D	E	F	G	H
n	2^{n-1}	$2^{n-1}+2^{n-3}-1$	$3.2^{n-2}-4$	$\mathcal{AS}(n)$	**g(n)**	C(n)	RS(n)	**f(n)**
3	4	4	2	4	**4**	5	4	**4**
4	8	9	8	9	**9**	14	8	**9**
5	16	19	20	20	**20**	42	16	**20**
6	32	39	44	45	**45**	132	36	**45**
7	64	79	92	100	**100**	429	81	?
8	128	159	188	222	?	1430	180	?
9	256	319	380	488	?	4862	400	?
10	512	639	764	1069	?	16796	900	?
11	1024	1279	1532	2324	?	58786	2025	?
12	2048	2559	3068	5034	?	208012	4500	?
13	4096	5119	6140	10840	?	742900	10000	?
14	8192	10239	12284	23266	?	2674440	22200	?
15	16384	20479	24572	49704	?	9694845	49284	?
16	32768	40959	49148	105884	?	35357670	108336	?
17	65536	81919	98300	224720	?		238144	?
18	131072	163840	196604	475773	?		521672	?
19	262144	826680	393216	1004212	?		1142761	?
20	524288	671359	805628	2115186	?		2484356	?

EXACT VALUES

E: $n \leq 4$ folklore, $n = 5,6$ Fishburn 1997, 2002

H: $n \leq 4$ folklore, $n = 5,6$ Fishburn 1997, 2002

LOWER BOUNDS

A: Craven's conjecture, 1992 (!)

B: Kim and Roush,1980

C: Abello and Johnson 1984 (N.B. $3.2^{n-2}-4 = 2^{n-1}+2^{n-2}-4$)

D: Fishburn 1997 (Alternating scheme, $n \leq 6$ BM 1989)

G: Fishburn 1997 (Replacement scheme $f(n+m) \geq f(n).f(m+1)$)

For all large n, $(2.17)^n < f(n)$ (Fishburn 1997)

UPPER BOUNDS

F: $g(n) < C(n) =$ Catalan number $2n!/n!(n+1)!$ (Abello 1991)

For all n, $f(n) < c^n$ for some $c > 0$ (Raz 2000

Fishburn's REPLACEMENT SCHEME

0 1 2....... m-1 m	m+1.................m+p
1 0 2..........m-1 m	....
......0.....................	....
....................0.......	....
.............0.............	....
m m-1...............10	m+1.................m+p
$\mathcal{D}$(m+1)	$\mathcal{D}$(p)

FOR EVERY ORDER IN (m+1) REPLACES 0

BY each of the ORDERS IN $\mathcal{D}$(p)

The domain of linear orders obtained on {0,1,2…m, m+1…m+p} is a Condorcet domain. Hence

$$\boxed{f(m+p) \geq f(p)f(m+1)}$$

$f(16) \geq 108.336 > 105.884 = |\mathcal{AS}(16)|$

$f(n) > (2.17)^n$, for all large n

CONJECTURES

Conjecture 1 (Fishburn 1996, 1997)

$$f(n+m) \leq f(n+1)f(m+1) \text{ for all } n, m \geq 1$$

The proof of this conjecture would imply

$$(2.17)^n < f(n) < (2.591)^{n-2} \text{ for all } n \geq 12$$

since Fishburn (1997) proved the lower bound and the implication for the upper bound (2002).

Then if true it would give a much better upper bound that the bound 4^{n-1} conjectured by Abello (1991). In the same paper Abello conjectures $g(n) \leq 3^{n-1}$ for which the conjectured upper bound $(2.591)^{n-2}$ would still be much better.

Let $|\mathcal{AS}(n)|$ be the size of the acyclic domain given by the alternating scheme.

Conjecture 2_ (Galambos and Reiner 2006)

$$g(n) = |\mathcal{AS}(n)|$$

This conjecture is true for $n \leq 6$ since in this case $f(n) = |\mathcal{AS}(n)|$ and Galambos and Reiner checked it for $n = 7$.

ANOTHER CONJECTURE….

There always exists a maximal covering distributive sublattice of $\mathcal{L}_n$ of size 1+ n(n-1)/2 (it is a maximal chain of $\mathcal{L}_n$) and of maximum size g(n).

For n = 4, there exist maximal covering distributive sublattices of $\mathcal{L}_4$ of size 7, 8 and 9.

Conjecture (2006)

For any integer i in the interval [1+ n(n-1)/2, g(n)] there exists a maximal covering distributive sublattice of $\mathcal{L}_n$ of size i, for instance for n = 5, for i in [11,20]

TOO BAD !

Conjecture (2006)

For any integer i in the interval [1+ n(n-1)/2, g(n)] there exists a maximal covering distributive sublattice of $\mathcal{L}_n$ of size i, for instance for n = 5, for i in [11,20]

FALSE ! even for n = 5

SIZE of the MCDS	Number of types
20	1
19	2
17	4
16	6
15	4
14	3
12	9
11	2

There does not exist MCDS of $\mathcal{L}_5$ of sizes 13 and 18

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PETITE ANNONCE

TOUT ce que VOUS AVEZ TOUJOURS VOULU SAVOIR sur les ENSEMBLES ORDONNES (finis) dans :

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