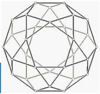
Finite Coxeter lattices and lattices of finite closure systems: some (lower) bounded lattices

Nathalie Caspard

LACL, University Paris 12 Val-de-Marne, France
and CAMS, EHESS, Paris, France

ICFCA'07, Clermont-Ferrand



Sketch of the talk

1. (Lower) bounded lattices and the doubling operation
2. Finite Coxeter lattices
   - Coxeter lattices
   - The class $\mathcal{HH}$ of lattices
   - All lattices of $\mathcal{HH}$ are bounded
   - Finite Coxeter lattices are in $\mathcal{HH}$
3. The lattice of finite closure systems

Outline

1. (Lower) bounded lattices and the doubling operation
2. Finite Coxeter lattices
   - Coxeter lattices
   - The class $\mathcal{HH}$ of lattices
   - All lattices of $\mathcal{HH}$ are bounded
   - Finite Coxeter lattices are in $\mathcal{HH}$
3. The lattice of finite closure systems

(Lower) bounded lattices

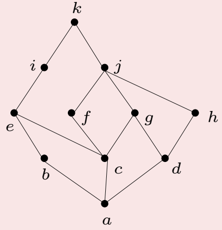
Definition (McKenzie [10], 1972)

A homomorphism $\alpha : L \to L'$ is called *lower bounded* if the inverse image of each element of L' is either empty or has a minimum.

A lattice is *lower bounded* if it is the lower bounded homomorphic image of a free lattice.

An *upper bounded* lattice is defined dually and a lattice is *bounded* if it is lower and upper bounded.

The doubling construction, Day [6], 1970



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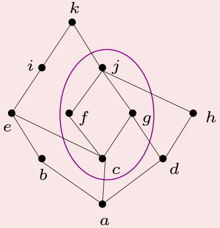

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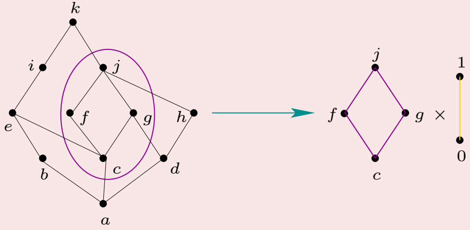

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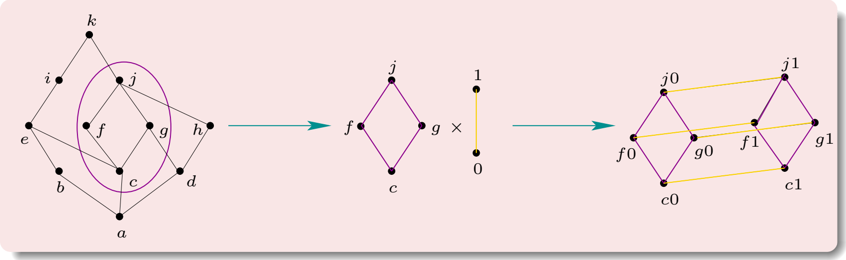

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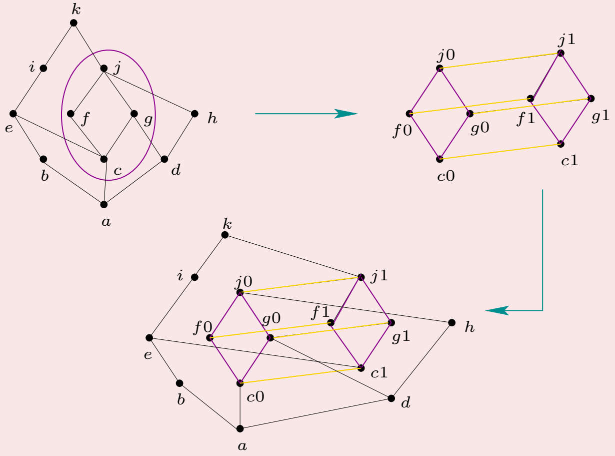
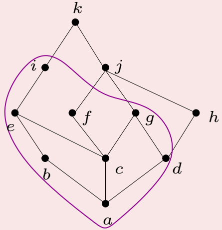


Generalisation to lower pseudo-intervals



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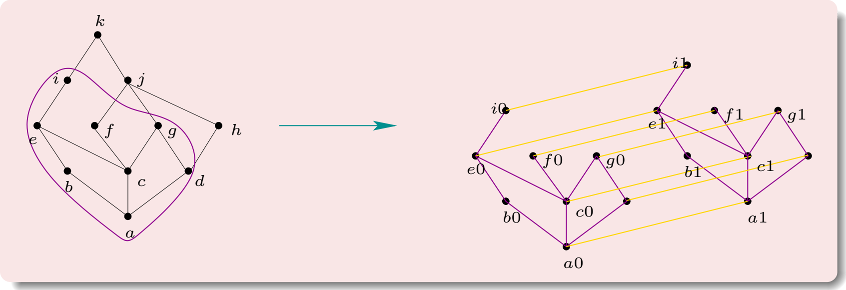

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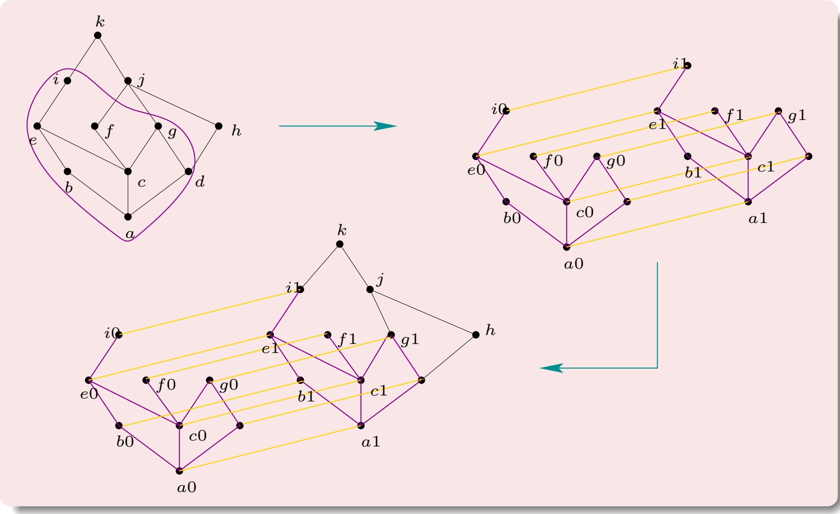
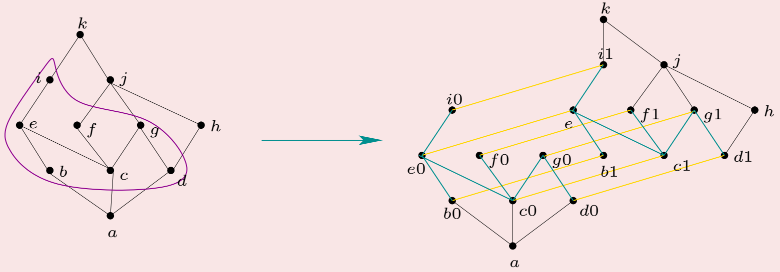


Generalisation to convex sets



Characterisation of bounded lattices

Theorem (Day [7], 1979)

Let L be a lattice. The following are equivalent :

- *L is bounded,*
- *it can be constructed starting from $\underline{2}$ by a finite sequence of interval doublings.*

Characterisation of lower bounded lattices

Theorem (Day [7], 1979)

Let L be a lattice. The following are equivalent :

- *L is lower bounded,*
- *it can be constructed starting from $\underline{2}$ by a finite sequence of lower pseudo-intervals.*

Characterisation of upper bounded lattices

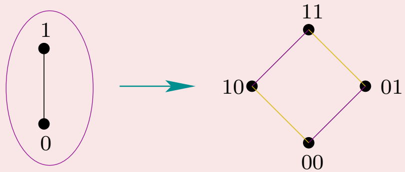
Theorem (Day [7], 1979)

Let L be a lattice. The following are equivalent :

- *L is upper bounded,*
- *it can be constructed starting from $\underline{2}$ by a finite sequence of upper pseudo-intervals.*

An example of bounded lattice

An example of bounded lattice



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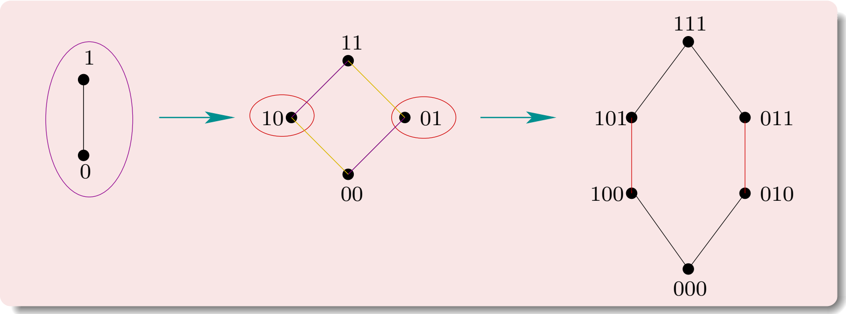
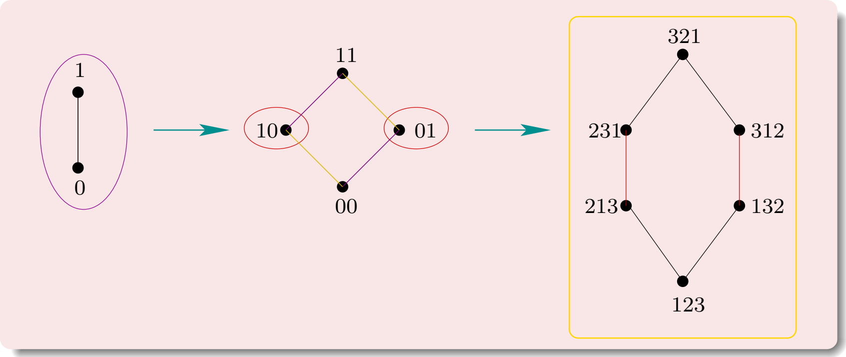
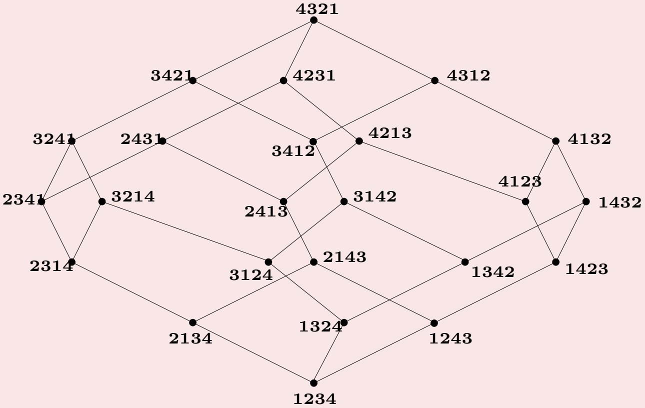


Perm(3) is bounded



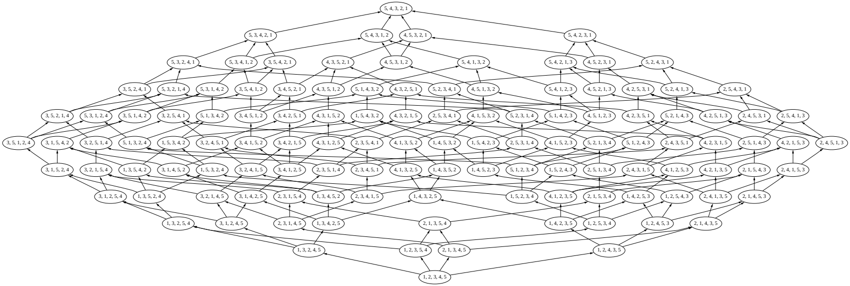
Permutohedron on 4 elements :

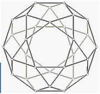
Permutohedron on 4 elements : bounded too



Permutohedron on 5 elements :

Permutohedron on 5 elements : bounded again





In fact...

Permutohedron is bounded

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And in fact...

All finite Coxeter lattices are bounded

Outline

What is a Coxeter group ?

Definition

A group W is a *Coxeter group* if W has a set of generators $S \subset W$, subject only to relations of the form

$$(ss')^{m(s,s')} = e$$

where $m(s,s) = 1$ for any s in S (all generators have order 2), and $m(s,s') = m(s',s) \geq 2$ for $s \neq s'$ in S.

List of all finite irreducible Coxeter groups

1. The four infinite families :
   - A_n (symmetric groups),
   - B_n,
   - D_n,
   - and I_n (dihedral groups).

List of all finite irreducible Coxeter groups

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   - A_n (symmetric groups),
   - B_n,
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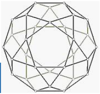
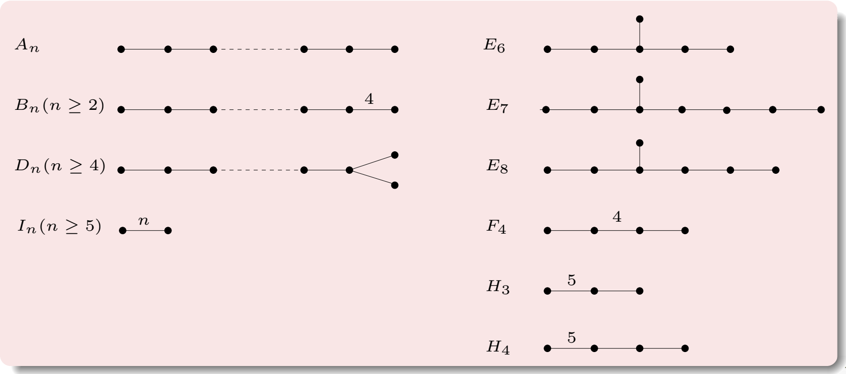
   - and I_n (dihedral groups).
2. and the six isolated groups : E_6, E_7, E_8, F_4, H_3 and H_4.



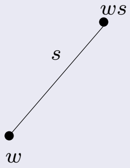
Coxeter lattices

Coxeter graph of finite irreducible Coxeter groups



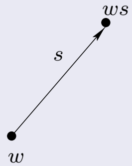
The lattice structure of Coxeter groups

Cayley graph of a group



The lattice structure of Coxeter groups

Cayley graph of a group ordered by the (right) weak order



If $\ell(w) < \ell(ws)$.

The lattice structure of Coxeter groups

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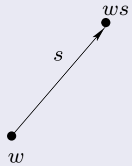

If $\ell(w) < \ell(ws)$.

Theorem (Björner, 1984)

The weak order on any finite Coxeter group is a (autodual) lattice.

Finite Coxeter lattices are bounded

Sketch of the proof :

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1. Defining a new class of lattices : $\mathcal{HH}$,

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1. Defining a new class of lattices : $\mathcal{HH}$,
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3. Showing that finite Coxeter lattices are in $\mathcal{HH}$.

Finite Coxeter lattices are bounded

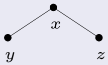
Sketch of the proof :

1. Defining a new class of lattices : $\mathcal{HH}$,

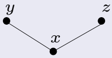
Hat, antihat and 2-facet

Definition

- a *Hat* $(y, x, z)^{\wedge}$:



- an *antiHat* $(y, x, z)^{\vee}$:



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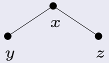

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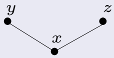

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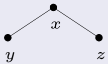

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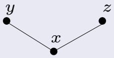
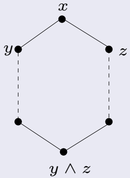


- a *2-facet* $F^{y,x,z}$:



Definition of a 2-facet labelling

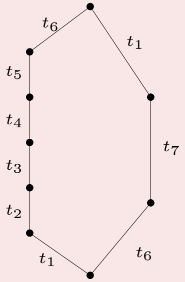


Fig.: Example of a 2-facet labelling

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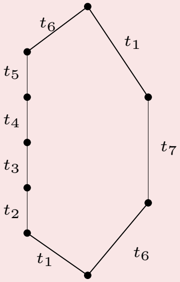

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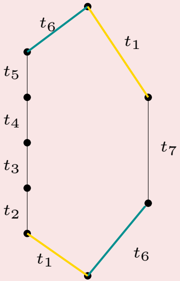

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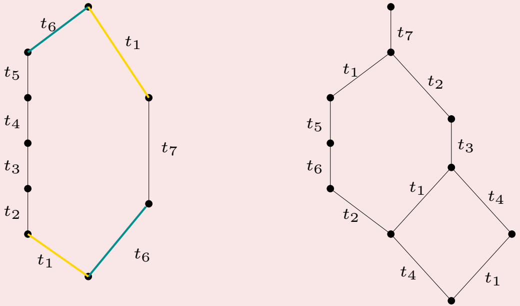


Fig.: Another example of a 2-facet labelling

Definition of a 2-facet labelling

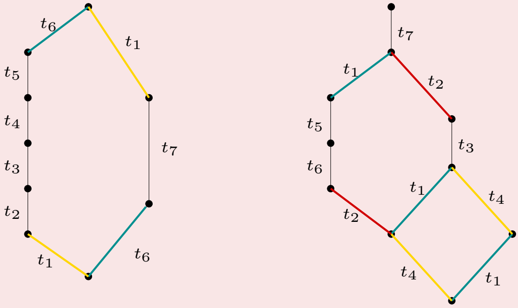
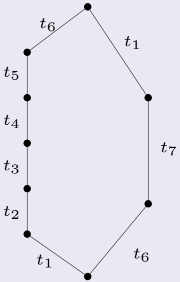


Fig.: Another example of a 2-facet labelling

2-FACET RANK FUNCTION ON A 2-FACET LABELLING

Definition



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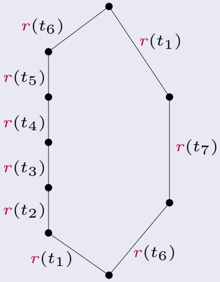

This is a function r from $T = \{t_1, ..., t_i, ...t_p\}$ to $\mathbb{R}$

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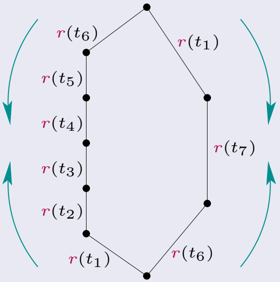

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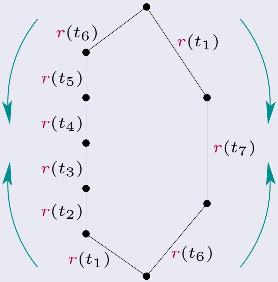
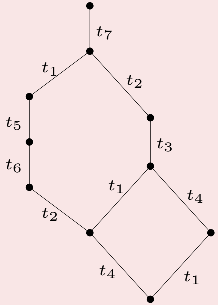


This is a function r from $T = \{t_1, ..., t_i, ...t_p\}$ to $\mathbb{R}$ such that :

So : $r(t_1) < r(t_2) < r(t_3)$
and $r(t_6) < r(t_5) < r(t_4)$
and $r(t_1), r(t_6) < r(t_7)$

2-facet rank function on a 2-facet labelling



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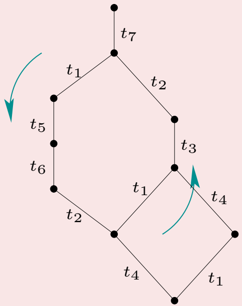
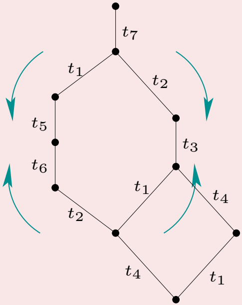


Here $r(t_1) < r(t_5), r(t_3)$

2-FACET RANK FUNCTION ON A 2-FACET LABELLING



Here $r(t_1) < r(t_5), r(t_3)$ and $r(t_2) < r(t_6), r(t_3)$

On semidistributivity

Definition

A lattice is *semidistributive* if, for all $x, y, z \in L$:

- $x \wedge y = x \wedge z$ implies $x \wedge y = x \wedge (y \vee z)$
- $x \vee y = x \vee z$ implies $x \vee y = x \vee (y \wedge z)$

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Proposition (Day, Nation, Tschantz [8], 1989)

Bounded lattices are semidistributive.

The class $\mathcal{HH}$ of lattices

Definition

A finite lattice L is in the class $\mathcal{HH}$ if it satisfies :

The class $\mathcal{HH}$ of lattices

Definition

A finite lattice L is in the class $\mathcal{HH}$ if it satisfies :

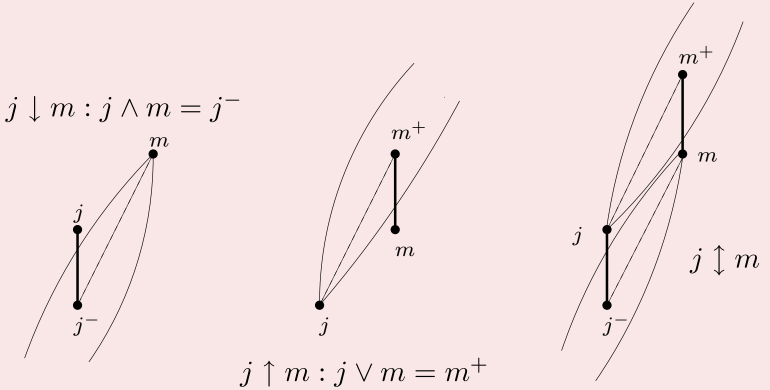
1. L is semidistributive,
2. to every hat $(y, x, z)^{\wedge}$ of L is associated an anti-hat $(y', y \wedge z, z')_{\vee}$ of L such that $[y \wedge z, x]$ is a 2-facet,
3. to every anti-hat $(y, x, z)_{\vee}$ of L is associated a hat $(y', y \vee z, z')^{\wedge}$ of L such that $[x, y \vee z]$ is a 2-facet,
4. there exists a 2-facet labelling T on the (covering) edges of L and a 2-facet rank function r on T.

First part of the theorem

All lattices of $\mathcal{HH}$ are bounded

How do we prove this ?

Recalling arrow relations...



Characterising semidistributivity with arrow relations

Proposition (Day [7], 1979)

A lattice L is semidistributive if and only if the relation $\updownarrow$ on $J \times M$ induces a bijection between J and M.

Characterising semidistributivity with arrow relations

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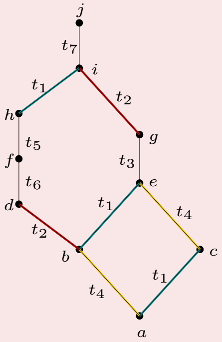
Notation

In any semidistributive lattice L, we can denote by (j, m_j) – or by (j_m, m) – the elements of $J_L \times M_L$ which are bijective for the relation $\updownarrow$.

Relations on the edges of the lattices of $\mathcal{HH}$



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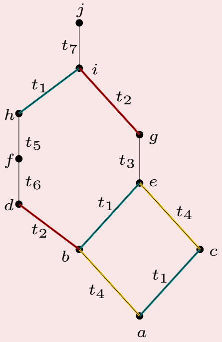
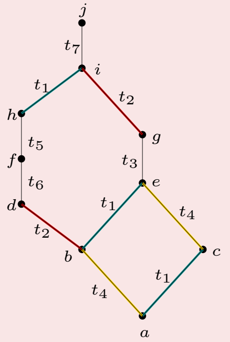


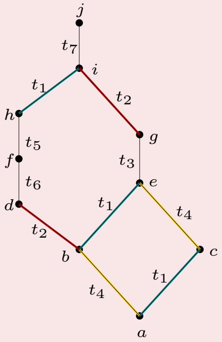
We write : $bd \prec_{t_2} gi$

RELATIONS ON THE EDGES OF THE LATTICES OF $\mathcal{HH}$



We write : $bd \prec_{t_2} gi$
and $ab \prec_{t_4} ce$

Relations on the edges of the lattices of $\mathcal{HH}$



We write : $bd \prec_{t_2} gi$
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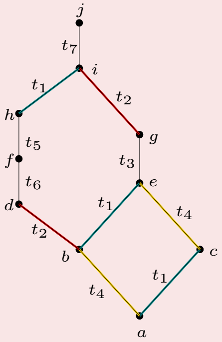
Relations on the edges of the lattices of $\mathcal{HH}$



We write : $bd \prec_{t_2} gi$
and $ab \prec_{t_4} ce$
and $ac \prec_{t_1} be \prec_{t_1} hi$
and so : $ac \leq_{t_1} hi$.

Using the $\leq_t$ relations

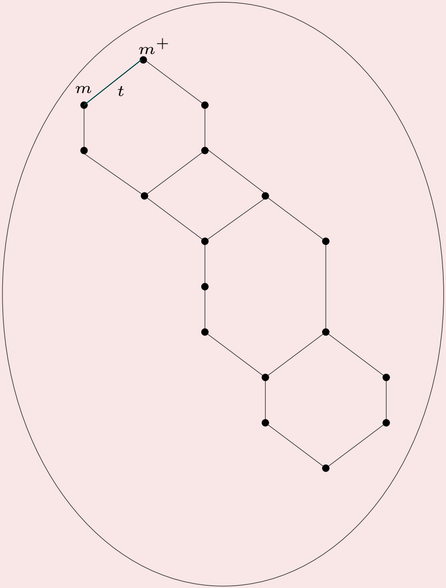
Theorem

Let m be meet-irreducible in $L \in \mathcal{HH}$ and let (m, m^+) be labelled by t.

The set $E_m = \{(x, y) : (x, y) \leq_t (m, m^+)\}$ is not empty and has a least element (u, v).

Moreover v is a join-irreducible, $v^- = u$ and $v \updownarrow m$.

All lattices of $\mathcal{HH}$ are bounded



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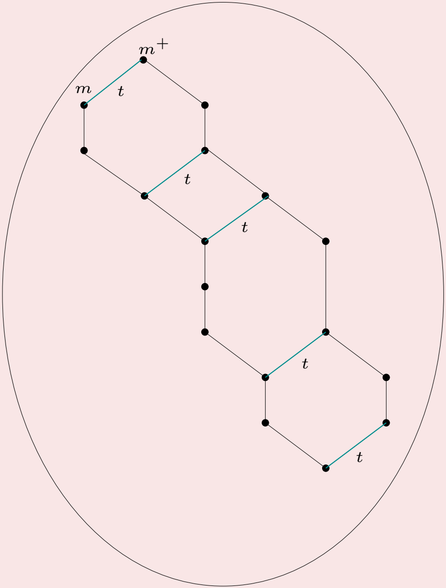
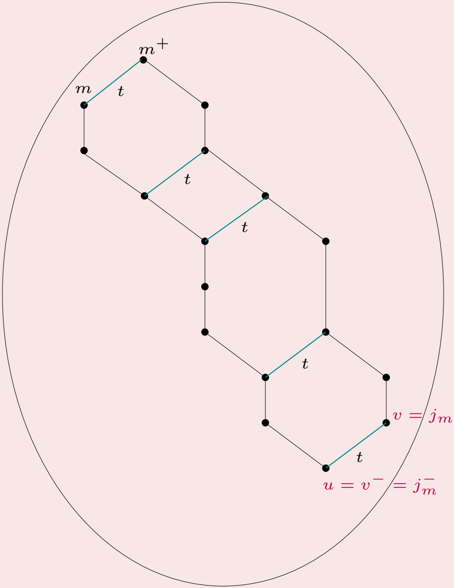


All lattices of $\mathcal{HH}$ are bounded



Lemma

Let $L \in \mathcal{HH}$ and T a 2-facet labelling of L. There exists a label $t \in T$ such that for any hat $(y, x, z)^\wedge$ whose arc (y, x) or (z, x) is labelled by t, $F^{(y,x,z)}$ is a diamond.

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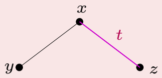

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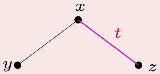

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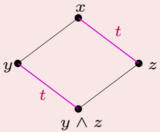

"Disconstructing" an interval to construct a second lemma

Definition

Let L be a lattice and $I \subseteq L$ an interval of L. We say that I is *contractible* (in L) if L can be obtained from a lattice L_0 by the doubling of an interval $I_0 \subseteq L_0$ (with $I = I_0 \times \underline{2}$).

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Lemma

Let $L \in \mathcal{HH}$, $j \in J_L$ and t the label of the arcs (j^-, j) and (m_j, m_j^+).

Assume all 2-facets contained in $[j^-, m_j^+]$ and which have one edge labelled by t are isomorphic with diamonds.

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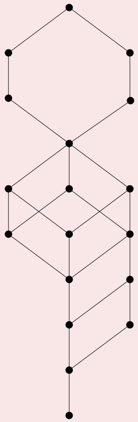
Then the interval $I_{j,m_j} = [j^-, m_j^+]$ is contractible.

Illustration of the lemma



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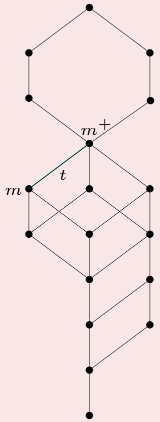

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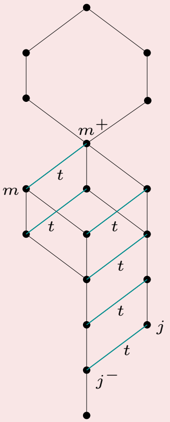

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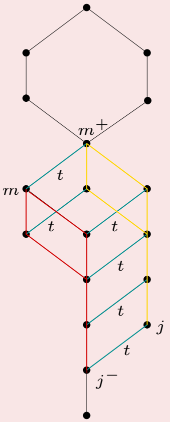

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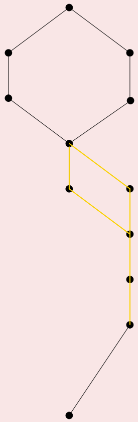

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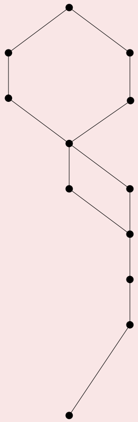

At last...

Theorem

The class $\mathcal{HH}$ of lattices is closed for the contraction of a contractible interval w.r.t. a label whose 2-facet rank function is maximal.

Hence the result : lattices of $\mathcal{HH}$ are bounded !

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Not all bounded lattices are in $\mathcal{HH}$

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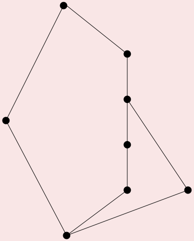
A bounded lattice that does not belong to $\mathcal{HH}$



WHY ? ? ?

Second part of the theorem

Finite Coxeter lattices are in $\mathcal{HH}$

HOW DO WE PROVE THIS ?

A STRONG RESULT

Proposition (L.C.D.P.-B., 1994)

Finite Coxeter lattices are semidistributive.

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Proposition (DUQUENNE AND CHERFOUH, 1994)

Permutohedron is semidistributive.

Reflections as elements and edge labels

Definition

$$T_W = \{t \in W : \exists s \in S, \exists w \in W \text{ such that } t = wsw^{-1}\}$$

is the set of the *reflections* of the Coxeter group W.

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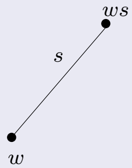

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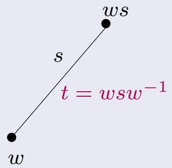
Two labellings of the edges : the g-labelling and the r-labelling



Properties of the reflections

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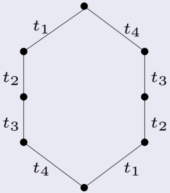

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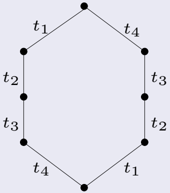

Corollary

The r-labelling on the edges of any finite Coxeter lattice is a 2-facet labelling.

Properties of the length function

Theorem (L.C.d.P.-B.)

The length function ℓ on every Coxeter lattice L_W is a 2-facet rank function when defined on the r-labelling of the edges of L_W.

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Theorem (L.C.d.P.-B.)

The length function ℓ on every Coxeter lattice L_W is a 2-facet rank function when defined on the r-labelling of the edges of L_W.

So :

Theorem

Every Coxeter lattice is in the class $\mathcal{HH}$ and therefore is bounded.

Two additional results

Theorem

Let L_W be a Coxeter lattice and W_H a parabolic subgroup of W. There exists a series of interval contractions that leads from L_W to the lattice L_{W_H} of its parabolic subgroup W_H.

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Proposition

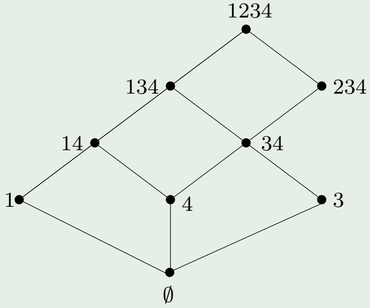
There exists a particular interval doubling series from a given Coxeter lattice generated by n generators to the Coxeter lattice of the same family, generated by $n+1$ generators.

Outline

Definition

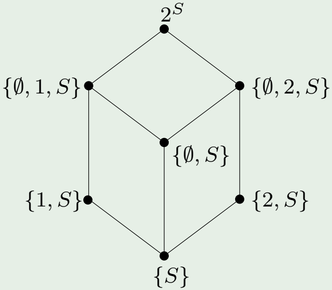
A *closure system* $\mathcal{C}$ on S : a subset of 2^S which contains S and is closed under set intersection.

Example ($S = \{1, 2, 3, 4\}$)



The lattice $(\mathbb{M}_n, \subseteq)$ of closure systems on a finite set S

Example (n=2)



Structures cryptomorphic with :

- closure operators,
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Theorem

The lattice $(\mathbb{M}_n, \subseteq)$ of closure systems is lower bounded.

HOW DO WE PROVE THIS ?

Two dependence relations on the join-irreducibles of $\mathbb{M}_n$

- The *dependence relation* δ (**Monjardet** [11], 1990),
- The *strong dependence relation* δ_d (**Day** [7], 1979).

Definition

1. $j\delta j'$ if $j = j'$ or if $\exists x \in L$ with $j < j' \vee x$, $j \not\leq x$ and $j' \not\leq x$.
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In particular, we have $\delta_d \subseteq \delta$.

Characterising δ and δ_d with the arrow relations

Proposition

1. $j \delta j' \iff \exists m \in M : \; j \uparrow m \; and \; j' \not\leq m.$
2. $j \delta_d j' \iff \exists m \in M : \; j \uparrow m \; and \; j' \downarrow m.$

Some results

Proposition

In any lattice L, the following are equivalent :

1. L *is atomistic,*
2. $\forall\ j \in J,\ \forall\ m \in M,\ j \not\leq m$ *implies* $j \downarrow m$,
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Moreover :

Proposition (Day [7], 1979)

A lattice L is lower bounded if and only if δ_d has no circuit.

THE JOIN-IRREDUCIBLES OF $\mathbb{M}_n$

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Proposition

The lattice $\mathbb{M}_n$ is atomistic and $\delta_d = \delta$.

What about the meet-irreducibles of $\mathbb{M}_n$?

Proposition

Let $\mathcal{C}$ be a closure system of $\mathbb{M}_n$. The following holds :

$$\mathcal{C} \in M_{\mathbb{M}_n} \iff \mathcal{C} = \mathcal{C}_{A,i} = \{X \subseteq S : \; A \not\subseteq X \text{ or } i \in X\}.$$

Finally...

Proposition

Let $\mathcal{C}_A$ and $\mathcal{C}_B$ be two join-irreducible elements of $\mathbb{M}_n$.

$$\mathcal{C}_A \delta \mathcal{C}_B \iff A \subseteq B \subset S$$

Finally...

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The lattice $\mathbb{M}_n$ of closure systems is lower bounded.

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Theorem

The lattice $\mathbb{M}_n$ of closure systems is lower bounded.

It is not bounded since it is not semidistributive.

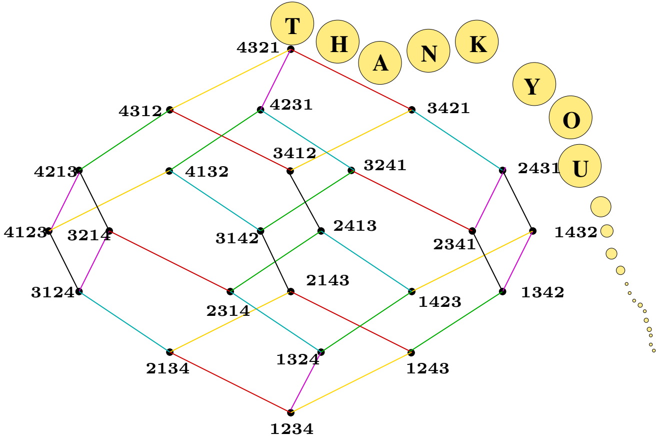
My colleagues (and friends !)

Fig.: C. le Conte de Poly-Barbut, CAMS, EHESS, Paris.

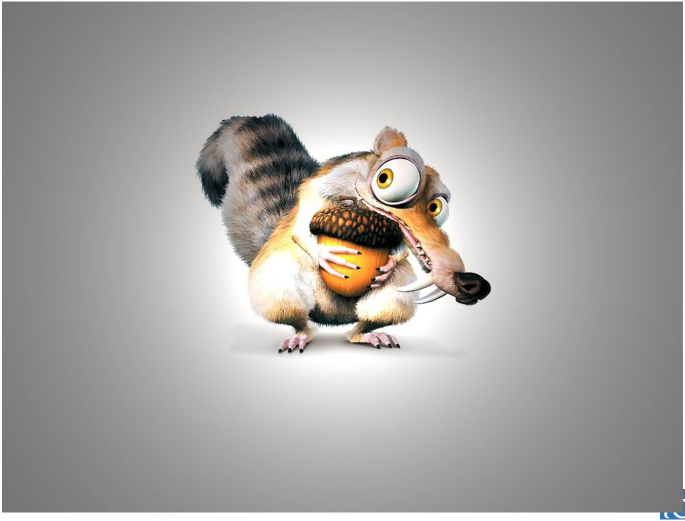
My colleagues (and friends!)

Fig.: B. Leclerc and B. Monjardet, CAMS, EHESS, Paris.

The final word.



No questions.. ?



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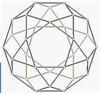
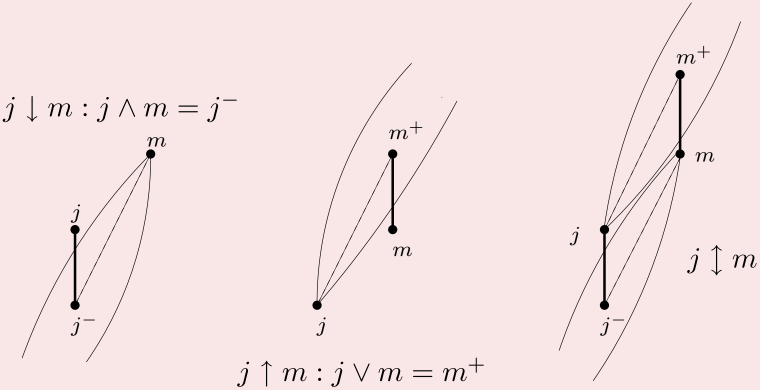
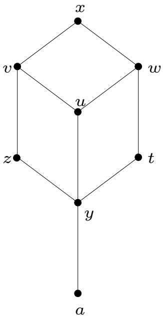


Recalling arrow relations...



... And the A-context of a lattice



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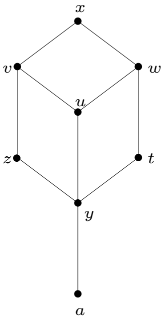

	a	z	t	v	w
y	$\updownarrow$	$\times$	$\times$	$\times$	$\times$
z		$\times$	$\downarrow$	$\times$	$\updownarrow$
t		$\downarrow$	$\times$	$\updownarrow$	$\times$
u		$\updownarrow$	$\updownarrow$	$\times$	$\times$

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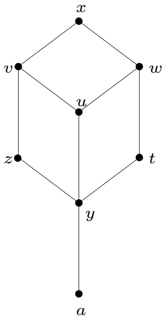

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y	$\updownarrow$	$\times$	$\times$	$\times$	$\times$
z		$\times$	$\downarrow$	$\times$	$\updownarrow$
t		$\downarrow$	$\times$	$\updownarrow$	$\times$
u		$\updownarrow$	$\updownarrow$	$\times$	$\times$

Any lattice has $(|J|! \times |M|!)$ *tableaux* to describe its A-context.

ON SEMIDISTRIBUTIVITY

Definition

A lattice is *meet-semidistributive* if, for all $x, y, z \in L$, $x \wedge y = x \wedge z$ implies $x \wedge y = x \wedge (y \vee z)$.

Join-semidistributive lattices are defined dually and a lattice is *semidistributive* if it is meet- and join-semidistributive.

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Proposition (Day, Nation, Tschantz [8], 1989)

Bounded lattices are semidistributive.

Results

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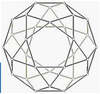

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Hence :

Given any total order on $J_{Perm(n)}$, there exists a unique total order on $M_{Perm(n)}$ – say L^*_M – such that $T = (A_L, L_J, L^*_M)$ has all $\updownarrow$ on the principal diagonal.

A simple idea from a strong result

Definition

Let L be a semidistributive lattice. A tableau $T = (A_L, L_J, L_M)$ of the A-context of L is a *B-tableau* if the following hold :

1. the $|J|$ arrows $\updownarrow$ of T are on the principal diagonal of T,
2. All arrows $\uparrow_.$ are below this diagonal and all arrows $\downarrow_.$ are above.

Proposition (Geyer [9], 1994)

A lattice is bounded if and only if its A-context admits a B-tableau.

A B-tableau of Perm(4)

$J \setminus M$	3421	4231	3241	2431	4312	4213	3214	2413	4132	3142	1432
1243	$\updownarrow$	$\times$	$\downarrow$	$\times$	$\times$	$\times$	$\downarrow$	$\times$	$\times$	$\downarrow$	$\times$
1324	$\times$	$\updownarrow$	$\times$	$\downarrow$	$\times$	$\downarrow$	$\times$	$\downarrow$	$\times$	$\times$	$\times$
1342	$\times$	$\uparrow$	$\updownarrow$		$\times$		$\downarrow$		$\times$	$\times$	$\times$
1423	$\uparrow$	$\times$		$\updownarrow$	$\times$	$\times$		$\downarrow$	$\times$		$\times$
2134	$\times$	$\times$	$\times$	$\times$	$\updownarrow$	$\times$	$\times$	$\times$	$\downarrow$	$\downarrow$	$\downarrow$
2314	$\times$	$\times$	$\times$	$\times$	$\uparrow$	$\updownarrow$	$\times$	$\downarrow$			
2341	$\times$	$\times$	$\times$	$\times$	$\uparrow$	$\uparrow$	$\updownarrow$				
2413	$\uparrow$	$\times$		$\times$	$\uparrow$	$\times$		$\updownarrow$			
3124	$\times$	$\uparrow$	$\times$		$\times$		$\times$		$\updownarrow$	$\times$	$\downarrow$
3412	$\times$	$\uparrow$	$\uparrow$		$\times$				$\uparrow$	$\updownarrow$	
4123	$\uparrow$	$\times$		$\uparrow$	$\times$	$\times$			$\times$		$\updownarrow$

A B-tableau of Perm(4)

$J \setminus M$	3421	4231	3241	2431	4312	4213	3214	2413	4132	3142	1432
1243	$\updownarrow$	$\times$	$\downarrow$	$\times$	$\times$	$\times$	$\downarrow$	$\times$	$\times$	$\downarrow$	$\times$
1324	$\times$	$\updownarrow$	$\times$	$\downarrow$	$\times$	$\downarrow$	$\times$	$\downarrow$	$\times$	$\times$	$\times$
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2314	$\times$	$\times$	$\times$	$\times$	$\uparrow$	$\updownarrow$	$\times$	$\downarrow$			
2341	$\times$	$\times$	$\times$	$\times$	$\uparrow$	$\uparrow$	$\updownarrow$				
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3124	$\times$	$\uparrow$	$\times$		$\times$		$\times$		$\updownarrow$	$\times$	$\downarrow$
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4123	$\uparrow$	$\times$		$\uparrow$	$\times$	$\times$			$\times$		$\updownarrow$

Here : L_J is equal to $Lex(J)$.

In fact :

Theorem

The tableau $T = (A_{Perm(n)}, Lex_J, L^*_M)$ of the A-context of the lattice $Perm(n)$ is a B-tableau.

Recalls

Definition

- $A(\alpha)$: the set of *agreements* of α,
- $D(\alpha)$: the set of *disagreements* of α.

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EXPRESSION OF THE ELEMENTS OF $J_{Perm(n)}$

Result

$\alpha \in J_{Perm(n)}$ *if and only if there exists a unique ordered pair vu of* adjacent *elements in* α *such that* $u < v$.

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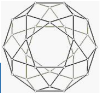

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So, in other words :

$$\alpha \in J_{Perm(n)} \iff \alpha = A|\overline{A} = Bv|u\overline{B}$$

with $u < v$ *and* $A = Bv$ *and* $\overline{A} = u\overline{B}$ *the two maximal linear suborders of* α *compatible with* $0_{Perm(n)} = 1 \ldots i \ldots n$.

Expression of the elements of $M_{Perm(n)}$

Result

$\alpha \in M_{Perm(n)}$ if and only if there exists a unique ordered pair lp of adjacent elements in α such that $l < p$.

Example

$4213, 1432 \in M_{Perm(4)}$ but $1342, 4231 \notin M_{Perm(4)}$.

Expression of the elements of $M_{Perm(n)}$

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$\alpha \in M_{Perm(n)}$ *if and only if there exists a unique ordered pair* lp *of* adjacent *elements in* α *such that* $l < p$*.*

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$$\alpha \in M_{Perm(n)} \iff \alpha = C|\overline{C} = Dl|p\overline{D}$$

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CHARACTERISING THE A-CONTEXT OF PERM(n)

Lemma

Let $\gamma = Bv|u\overline{B} \in J_{Perm(n)}$ *and* $\mu = Cl|p\overline{C} \in M_{Perm(n)}$.

Characterising the A-context of Perm(n)

Lemma

Let $\gamma = Bv|u\overline{B} \in J_{Perm(n)}$ *and* $\mu = Cl|p\overline{C} \in M_{Perm(n)}$.

1. $\gamma \leq \mu \iff D(\gamma) \subseteq D(\mu) \iff A(\mu) \subseteq A(\gamma)$.
2. $\gamma \uparrow \mu \iff pl \in D(\gamma)$ *and* $D(\gamma) \subseteq D(\mu^+)$.
3. $\gamma \downarrow \mu \iff uv \in A(\mu)$ *and* $A(\mu) \subseteq A(\gamma^-)$.
4. $\gamma \updownarrow \mu \iff pl \in D(\gamma)$, $uv \in A(\mu)$, $D(\gamma) \subseteq D(\mu^+)$ *and* $A(\mu) \subseteq A(\gamma^-)$.

Characterising the bijection between J and M induced by $\updownarrow$

Proposition

1. *Let $\gamma = Bu|v\overline{B}$ be a join-irreducible and μ a meet-irreducible of Perm(n).*

$$\gamma \updownarrow \mu \iff \mu = Cu|v\overline{C} \quad \text{with} \left\{ \begin{array}{l} C = \left(\{x \in B : u < x\} \cup \{x \in \overline{B} : v < x\}, > \right) \\ \overline{C} = \left(\{x \in B : x < u\} \cup \{x \in \overline{B} : x < v\}, > \right) \end{array} \right.$$

2. *Let $\mu = Cl|p\overline{C}$ be a meet-irreducible and γ a join-irreducible of Perm(n).*

$$\gamma \updownarrow \mu \iff \gamma = Bp|l\overline{B} \quad \text{with} \left\{ \begin{array}{l} B = \left(\{x \in C : x < p\} \cup \{x \in \overline{C} : x < l\}, \ < \right) \\ \overline{B} = \left(\{x \in C : p < x\} \cup \{x \in \overline{C} : l < x\}, \ < \right) \end{array} \right.$$

An additional result

Theorem

Let L_J be a linear order on $J_{Perm(n)}$ and L^*_M the "associated" linear order on $M_{Perm(n)}$. The following are equivalent :

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Theorem

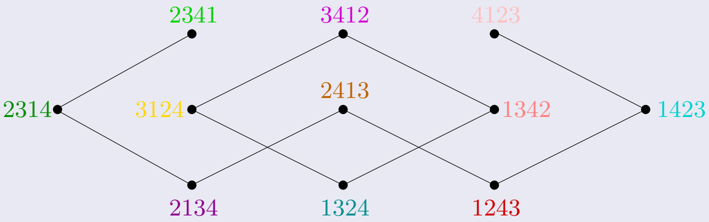
Let L_J be a linear order on $J_{Perm(n)}$ and L^*_M the "associated" linear order on $M_{Perm(n)}$. The following are equivalent :

1. $T = (A_{Perm(n)}, L_J, L^*_M)$ *is a B-tableau of* $Perm(n)$,
2. L_J *is a linear extension of* $(J, \leq_{Perm(n)})$ *and* L^*_M *a linear extension of* $(M, \geq_{Perm(n)})$.

Not all tableaux of Perm(n) are B-tableaux

NOT ALL TABLEAUX OF $\text{PERM}(n)$ ARE B-TABLEAUX

Proof :



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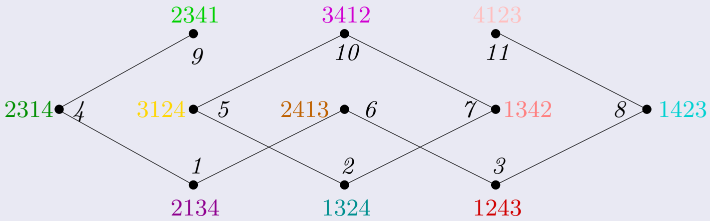

Fig.: A linear extension L_J of $(J, \leq_{Perm(4)})$ for which L^*_M on M is not a linear extension of $(M, \geq_{Perm(4)})$.

Not all tableaux of Perm(n) are B-tableaux

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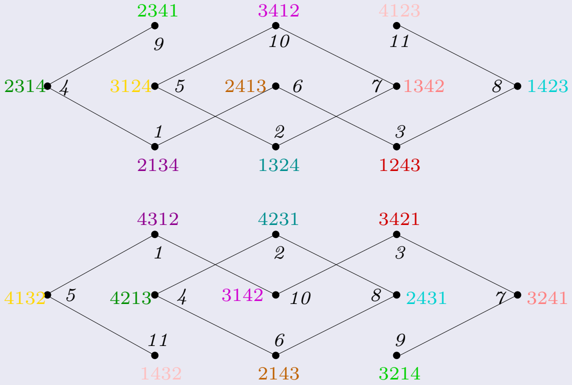
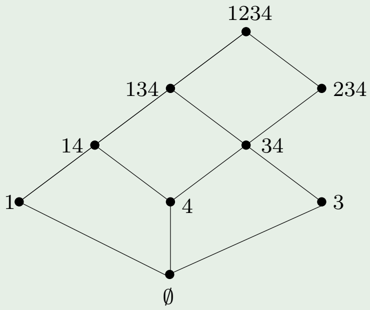


Definition

A *closure system* $\mathcal{C}$ on S : a subset of 2^S which contains S and is closed under set intersection.

Example ($S = \{1, 2, 3, 4\}$)



Proposition

The set of all the lattices that can be obtained from $L \in \mathcal{HH}$ by a series of interval contractions is a distributive lattice when ordered by the following natural order relation : $L < L'$ if L can be obtained from L' by a series of interval contractions.