Critical points between varieties of algebras

P. Gillibert

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  - A ring is a $\{+, -, \cdot, 0\}$-algebra.
  - If R is a ring. An R-module is a $\{+, -, 0\} \cup \{\lambda_a \mid a \in R\}$-algebra.

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- An $\{*, {}^{-1}, 1\}$-algebra is a group if it satisfies the following identities :

$$x * (y * z) = (x * y) * z,$$
$$1 * x = x * 1 = x,$$
$$x * x^{-1} = x^{-1} * x = 1.$$

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- A variety is *finitely generated* if it is generated by a finite class of finite algebras.

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- Those identities define the variety of lattices.

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- Denote by $\operatorname{Con} A$ the poset of all congruences of A, under inclusion. It is an algebraic lattice.

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- Every congruence is a (infinite) join of finitely generated congruences.

$$\theta = \bigvee (\Theta_A(x, y) \mid (x, y) \in \theta).$$

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- If it is the case, we say that A is *congruence-distributive*.

The $\mathrm{Con_c}$ functor

- For $f\colon A \to B$. We put :

$$\begin{aligned}\mathrm{Con_c}\, f\colon\ \mathrm{Con_c}\, A &\to \mathrm{Con_c}\, B\\ \alpha &\mapsto \Theta_B(\{(f(x), f(y)) \mid (x,y) \in \alpha\})\end{aligned}$$

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- $\mathrm{Con_c}$ is a functor from any variety of algebras to the variety of semilattices.

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- We have a good description of $\mathrm{Con_c}\, \mathcal{V}$ for very few varieties of algebras.
- We denote by $\mathcal{D}$ the variety of all distributive lattices, then $\mathrm{Con_c}\, \mathcal{D}$ is the class of all generalized Boolean algebras.

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- There exists a distributive semilattice of cardinality $\aleph_2$ that does not belong to $\mathrm{Con_c}\,\mathcal{L}$ (Růžička 2008).

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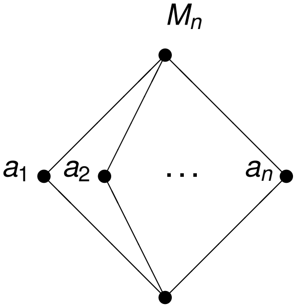

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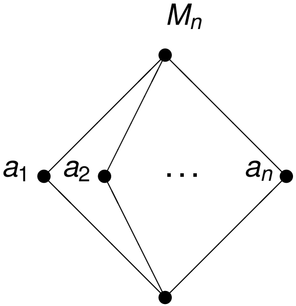

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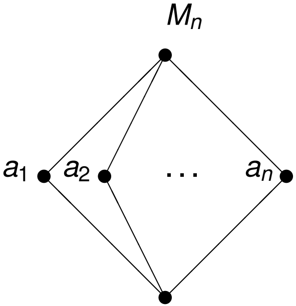

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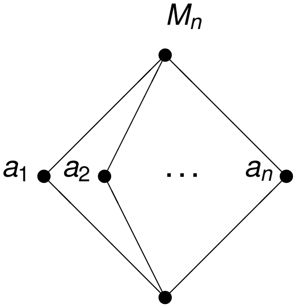

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- We have $M_3 \in \mathrm{Con_c}\,\mathcal{G}$ and $M_3 \notin \mathrm{Con_c}\,\mathcal{L}$, hence :

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Then $\mathrm{crit}(\mathcal{V};\mathcal{W}) = \infty$.

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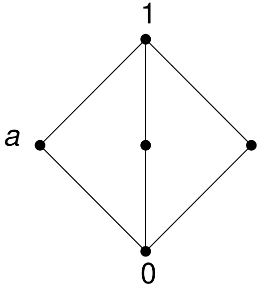

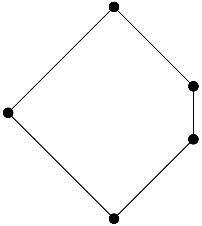

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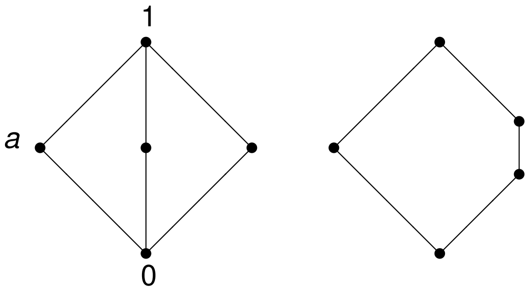

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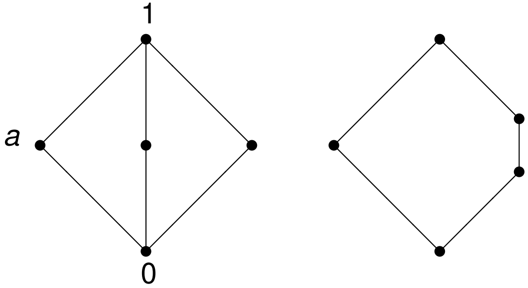

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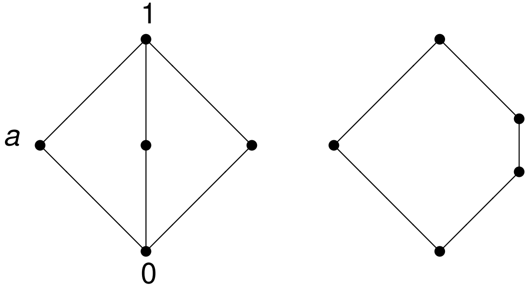

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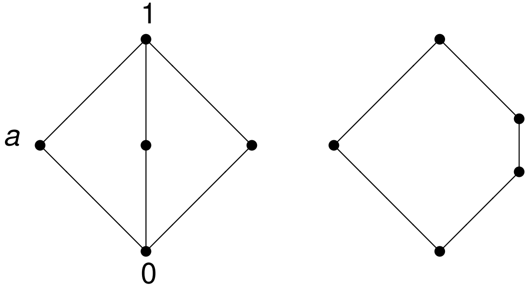

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Moreover if $\mathcal{V}$ is a finitely generated congruence-distributive variety, then the theorem is also true for $\lambda = \aleph_0$.

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Theorem (G. 2008)

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$$\operatorname{crit}(\mathcal{V};\mathcal{W}) \leq \aleph_{n-1}$$

Two varieties with critical point $\aleph_1$

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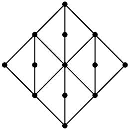
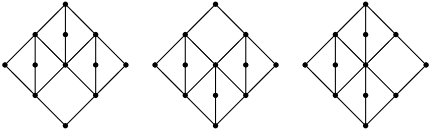
$\mathcal{V} = \textbf{Var}\Big($  $\Big)$, $\quad \mathcal{W} = \textbf{Var}\Big($  $\Big)$

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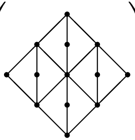
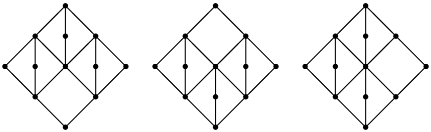
$\mathcal{V} = \textbf{Var}\Big($  $\Big)$, $\quad \mathcal{W} = \textbf{Var}\Big($  $\Big)$

- Assume that $\vec{A}$ is a ω-indexed diagram of finite lattices in $\mathcal{V}$, then $\mathrm{Con_c} \circ \vec{A}$ has a lifting in $\mathcal{W}$.
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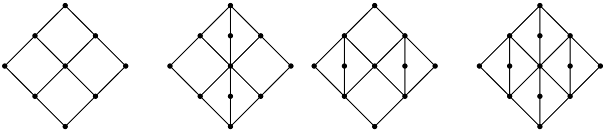
Two varieties with critical point $\aleph_1$

FIG.: A_0, A_1, A_2, $A_3 = T_1$



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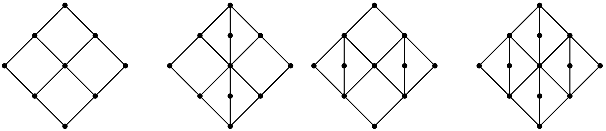
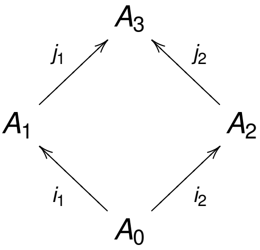
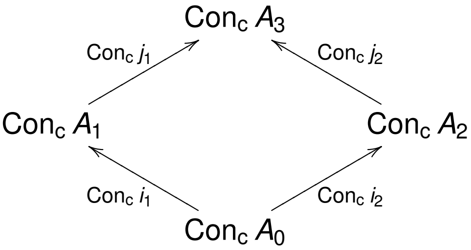
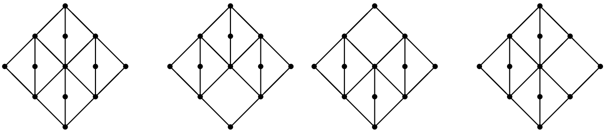






Two varieties with critical point $\aleph_1$

FIG.: T_1, T_2, T_3, and T_4



Theorem (G. 2008)

Let $\mathcal{V} = \mathbf{Var}(T_1)$ and $\mathcal{W} = \mathbf{Var}(T_2, T_3, T_4)$, where T_1, T_2, T_3, and T_4 are the lattices above, then :

$$\operatorname{crit}(\mathcal{V}; \mathcal{W}) = \aleph_1$$

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Corollary (G. 2008)

Let $\mathcal{V}$ be locally finie variety of algebras, let $\mathcal{W}$ be a finitely generated congruence-distributive variety of algebras. Then either $\operatorname{crit}(\mathcal{V};\mathcal{W}) < \aleph_\omega$ *or* $\operatorname{crit}(\mathcal{V};\mathcal{W}) = \infty$.

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The result can be generalized assuming only that $\mathcal{W}$ is finitely generated congruence-modular varieties of algebras (G., Wehrung).

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- Then we add many “decorations”, we obtain a diagram $\vec{A}$ in $\mathbf{Var}(L)$.
- Let $\mathcal{V}$ be a variety of lattices, $\mathrm{Con_c} \circ \vec{A}$ has a lifting in $\mathcal{V}$ if and only if L belongs to $\mathcal{V}$ or $\mathcal{V}^{\mathrm{d}}$.

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Theorem (G. 2010)

Let $\mathcal{V}$ and $\mathcal{W}$ be finitely generated variety of lattices. Let L be a finite lattice in $\mathcal{V}$. If $L \notin \mathcal{W} \cup \mathcal{W}^{\mathrm{d}}$, then $\mathrm{crit}(\mathcal{V};\mathcal{W}) \leq \aleph_2$.

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- This solve the two, above mentione, conjecture of J. Tůma et F. Wehrung in the finitely generated case.

That is all

Thank you for your attention.
Any questions ?