Some properties of the lattice of finite Moore families

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Outline

1. Introduction and definitions

2. Lower bounded lattices and the doubling operation

Definition

Moore family

A *Moore family* $\mathcal{M}$ on S : any subset of 2^S which is $\cap$-stable and contains S.

- $M_1, M_2 \in \mathcal{M} \Longrightarrow M_1 \cap M_2 \in \mathcal{M}$.
- $S \in \mathcal{M}$.

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- Closure system
- Intersection ring (of sets)
- Protopology
- Intersection semilattice,
- ...

Numbers of finite closure systems on S

Known up to $|S|=7$.

1. 2,
2. 7,
3. 61,
4. 2480,
5. 1.385.552 (see Higuchi),
6. 75.973.751.474 (Habib & Nourine, Discrete Maths, 2005)
7. 14.087.648.235.707.352.472 (Colomb, Irlande & Raynaud, LNCS, 2010)

Some cryptomorphic notions

Folklore : closure operators on S (Moore/Birkhoff)

Any map ϕ on 2^S which is :

- isotone ($A \subseteq B \Longrightarrow \phi(A) \subseteq \phi(B)$),
- extensive ($A \subseteq \phi(A)$),
- idempotente ($\phi^2(A) = \phi(A)$).

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- for any $\phi, \mathcal{M}_\phi$ = fixed points of ϕ,
- for any $\mathcal{M}, \phi_\mathcal{M}$ is such that $\phi_\mathcal{M}(X) = \bigcap\{M \in \mathcal{M} : X \subseteq M\}$

Some cryptomorphic notions

Other kinds of families

Example : Families $\mathcal{O}$ on S containing the emptyset and the union of any subset of $\mathcal{O}$.

Sperner Villages on S (Demetrovics & Hua, 1991)

A Sperner village on S : set $\mathcal{V}$ of Sperner families on S satisfying some particular properties.
(A *Sperner family* $\mathcal{F}$ on S is such that two distinct elements of $\mathcal{F}$ are incomparable for set inclusion.)

Some cryptomorphic notions

Complete implicational systems Σ on S (Armstrong, 1974)

Any binary relation Σ on 2^S satisfying the following three properties :

- $A \longrightarrow B$ and $B \longrightarrow C \in \Sigma$ imply $A \longrightarrow C \in \Sigma$,
- $A \supseteq B$ implies $A \longrightarrow B \in \Sigma$,
- $A \longrightarrow B$ and $C \longrightarrow D \in \Sigma$ imply $(A \cup C) \longrightarrow (B \cup D) \in \Sigma$.

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- For a given Σ, $\phi_\Sigma : \phi_\Sigma(A) = \bigcup\{x \in S : A \longrightarrow x \in \Sigma\}$
- For a given closure operator ϕ, $\Sigma_\phi = \{X \longrightarrow Y : Y \subseteq \phi(X)\}$.

Some cryptomorphic notions

Congruences on $(2^S, \cup)$

Any equivalence relation θ on 2^S such that, for all $A, B, C \subseteq S$, $A\theta B$ implies $(A \cup C)\theta(B \cup C)$.

- For a given closure operator ϕ, θ_ϕ is such that $A\theta_\phi B$ iff $\phi(A) = \phi(B)$.
- Given a congruence θ on 2^S, ϕ_θ is such that $\phi_\theta(A) = \bigcup\{B \subseteq S : B\theta A\}$

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Set representations of finite lattices

- Any Moore family can be ordered as a lattice,
- Any lattice is the lattice of the fixed points of a closure operator.

Set representations of finite lattices

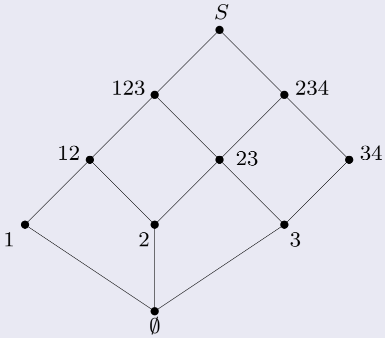

Fig.: A Moore family on $S = \{1, 2, 3, 4\}$

Examples of particular Moore families

Topologies

Any Moore family which contains $\emptyset$ and is $\cup$-stable.

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Convex geometries

Any Moore family containing $\emptyset$ and such that : "for every closed set M different from S there exists $x \notin M$ such that $M + \{x\}$ is a closed set".

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——> Set representations of *meet-distributive lattices*,

——> Families of fixed points of *anti-exchange closures*.

——> *path-independent choice functions* in microeconomics.

The lattice $(\mathbb{M}_n, \subseteq)$ of Moore families on a finite set S

Lattice structure

Ordered with set inclusion, $\mathbb{M}_n$ is a lattice since :

- it is an $\cap$-semilattice
- with a maximum (2^S)

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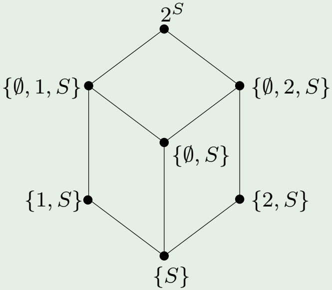
Lattice operations :

- $\mathcal{M} \wedge \mathcal{M}' = \mathcal{M} \cap \mathcal{M}'$,
- $\mathcal{M} \vee \mathcal{M}' = \{M \cap M' : M \in \mathcal{M} \text{ and } M' \in \mathcal{M}'\}$.

The lattice $(\mathbb{M}_n, \subseteq)$ of Moore families on a finite set S

Example ($|S|$=2)



Covering relation of the lattice $(\mathbb{M}_n, \subseteq)$

Characterization

The following are equivalent :

- $\mathcal{M} \prec \mathcal{M}'$,
- $\mathcal{M}' = \mathcal{M} + \{Q\}$, with Q an $\cap$-irreducible element of $\mathcal{M}'$.

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NB. "Q" for "Quasi-closed set" (of $\mathcal{M}$).

Atoms and join-irreducibles of $(\mathbb{M}_n, \subseteq)$

Clearly :

The following are equivalent :

- $\mathcal{M}$ is an atom of $\mathbb{M}_n$,
- there exists $A \subset S$ such that $\mathcal{M} = \mathcal{M}_A = \{A, S\}$.

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Now :

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So :

Theorem

The lattice $(\mathbb{M}_n, \subseteq)$ is atomistic.

Meet-irreducibles of $(\mathbb{M}_n, \subseteq)$

Implicational Moore family

For all distinct $A, B \subseteq S$,

$$\mathcal{M}_{A,B} = \{X \subseteq S : A \not\subseteq X \text{ or } B \subseteq X\}$$

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Note : $\mathcal{M}_{A,i} \prec \mathcal{M} \iff \mathcal{M} = \mathcal{M}_{A,i} + \{A\}$.

A strong constructive property

Theorem (N.C., 1998)

The lattice $(\mathbb{M}_n, \subseteq)$ is lower bounded.

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Lower bounded lattices

Definition (McKenzie, 1972)

A homomorphism $\alpha : L \to L'$ is called *lower bounded* if the inverse image of each element of L' is either empty or has a minimum.

A lattice is *lower bounded* if it is the lower bounded homomorphic image of a free lattice.

Upper bounded lattices

Definition (McKenzie, 1972)

A homomorphism $\alpha : L \to L'$ is called *upper bounded* if the inverse image of each element of L' is either empty or has a maximum.

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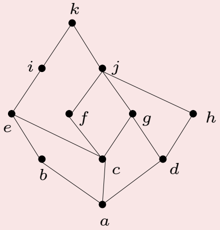
Bounded lattices

Definition (McKenzie, 1972)

A lattice is *bounded* if it is lower and upper bounded.

The interval doubling construction (Day, 1970)



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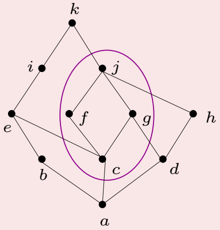

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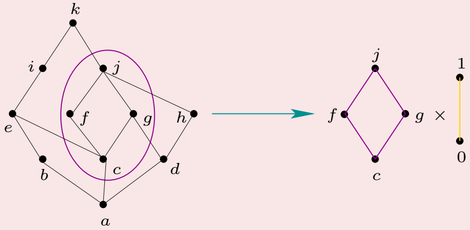

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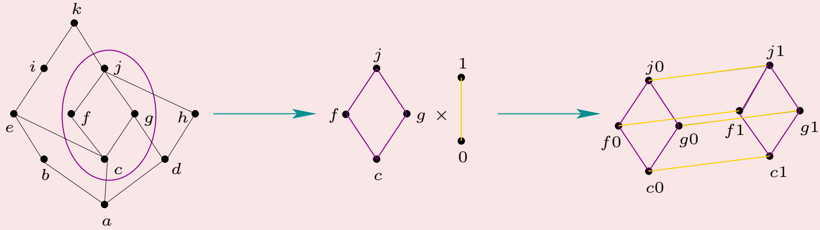

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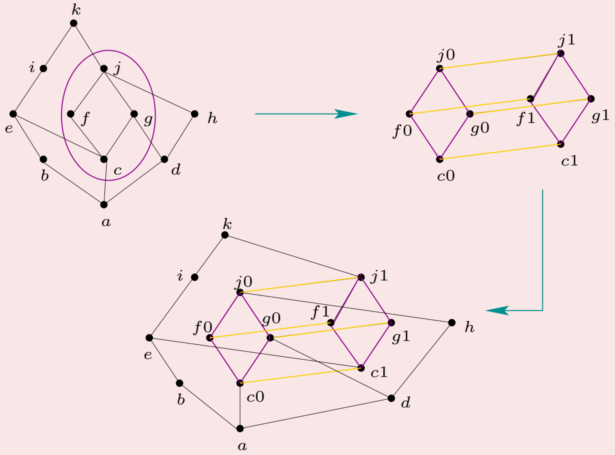


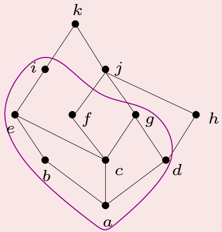
Characterization of bounded lattices

Theorem (Day, 1979)

Let L be a lattice. The following are equivalent :

- *L is bounded,*
- *it can be constructed starting from $\underline{2}$ by a finite sequence of interval doublings.*

Generalization to lower pseudo-interval doublings



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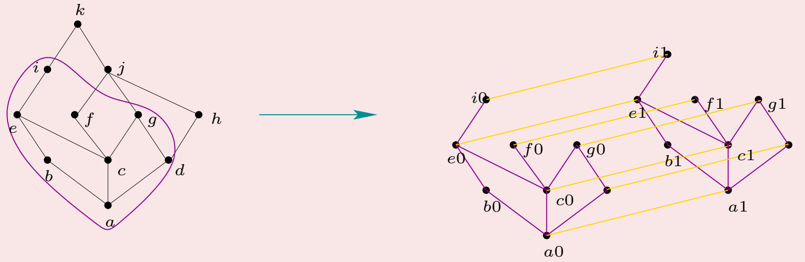

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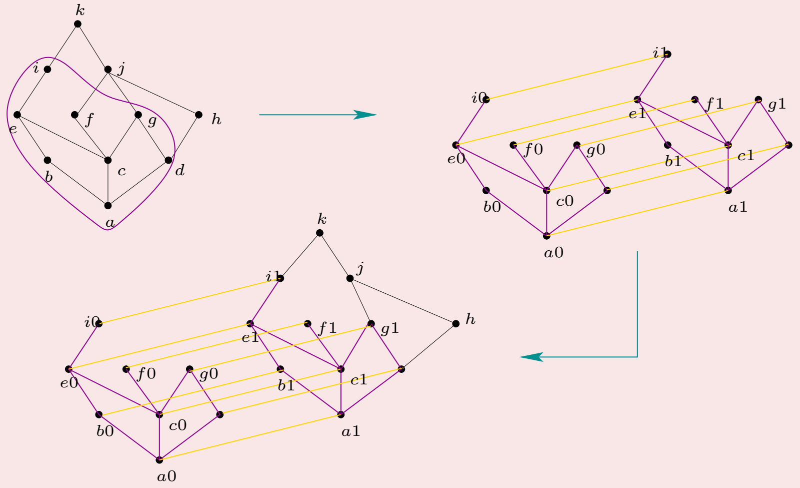

Characterization of lower bounded lattices

Theorem (Day, 1979)

Let L be a lattice. The following are equivalent :

- *L is lower bounded,*
- *it can be constructed starting from $\underline{2}$ by a finite sequence of lower pseudo-intervals.*

Another characterization of lower bounded lattices

Theorem (Day, 1979)

Let L be a lattice. The following are equivalent :

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What is the definition of δ_d ?

Two dependence relations on the join-irreducibles of a lattice

The *strong dependence relation* δ_d (Day, 1979)

$$j\delta_d j' \text{ if } j = j' \text{ or if } \exists x \in L \text{ with } j < j' \vee x \text{ and } j \not\leq j'^{-} \vee x.$$

- Inspired from a relation owed to Pudlak and Tuma.
- Provides a characterization of lower bounded lattices.
- Used by Freese, Jezek & Nation in the study of free lattices.

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The *dependence relation* δ (Monjardet, 1990)

$j\delta j'$ if $j = j'$ or if $\exists x \in L$ with $j < j' \vee x$, $j \not\leq x$ and $j' \not\leq x$.

For the study of consensus problems in lattices.

On the dependence relations

Clearly : $\delta_d \subseteq \delta$.

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Moreover :

Lemma (N.C, Monjardet, 1998)

The following two conditions are equivalent :

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Lemma (N.C, Monjardet, 1998)

The following two conditions are equivalent :

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So :

Corollary

$\delta_d = \delta$ in $\mathbb{M}_n$.

Finally the result :

Recalling :

- In $(\mathbb{M}_n, \subseteq)$, the dependence relations δ_d and δ are equal,
- A lattice L is lower bounded $\iff$ δ_d is cycle-free in L,

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Proposition (N.C., 1998)

In $\mathbb{M}_n$: $\mathcal{M}_A \delta \mathcal{M}_B$ if and only if $A \subseteq B \subset S$.

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Hence the result.

It is not bounded since it is not semidistributive.

Induced properties of the lattice $(\mathbb{M}_n, \subseteq)$

Corollary

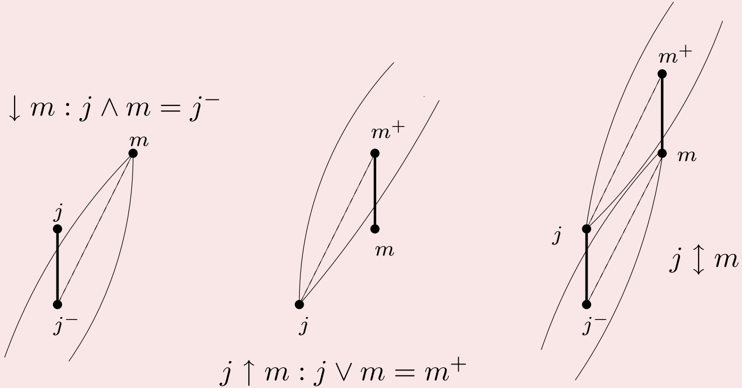
1. *Lower Bounded,*
2. *Join SemiDistributive*
   $(x \vee y = x \vee y \Longrightarrow x \vee y = x \vee (y \wedge z))$,
3. *Join PseudoComplemented* ($\forall x, \{t : t \vee x = 1\}$ *has a minimum*),
4. *Atomistic,*
5. *Meet Distributive* (= *JSD* + *LSM*),
6. *Lower SemiModular* $(x \prec x \vee y \Longrightarrow x \wedge y \prec y)$,
7. *Ranked* (*and* $r(\mathcal{M}) = |\mathcal{M}| - 1$).

$(1) \Longrightarrow (2) \Longrightarrow (3)$.

$(1)+(4) \Longrightarrow (5)$ and moreover $(5)=(2)+(6)$.

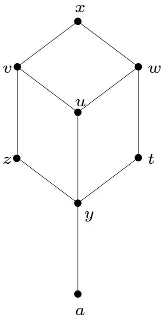
Annex : recalling about the arrow relations...

$$j \downarrow m : j \wedge m = j^-$$



$$j \uparrow m : j \vee m = m^+$$

… And the A-table of a lattice



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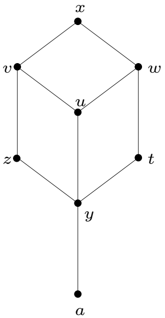

	a	z	t	v	w
y	$\updownarrow$	$\times$	$\times$	$\times$	$\times$
z		$\times$	$\downarrow$	$\times$	$\updownarrow$
t		$\downarrow$	$\times$	$\updownarrow$	$\times$
u		$\updownarrow$	$\updownarrow$	$\times$	$\times$

Examples of classes of lattices characterized by means of the arrow relations and/or the dependence relations

Proposition

- *Boolean,*
- *Semi-distributive,*
- *Distributive,*
- *Lower (resp. upper) bounded,*
- *Meet-(resp. join-)distributive,*
- *Atomistic,*
- *Coatomistic.*

Quasi-closed sets and critical sets of a Moore family

Quasi-closed set and critical set

- A subset $Q \subset S$ is a *quasi-closed set* of a Moore family $\mathcal{M}$ if $Q \notin \mathcal{M}$ and $\mathcal{M} + \{Q\} \in \mathbb{M}_n$.
- Q is called a *F-quasi-closed set* if $\phi(Q) = F$.

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Canonical basis of a Moore family (Guigues-Duquenne, 1986)

Let $\mathcal{M}$ be a Moore family on S and ϕ its associated closure. The set $\{\mathcal{M}_{C,\phi(C)} : C \text{ is a critical set of } \mathcal{M}\}$ is called the *canonical basis* of $\mathcal{M}$.

Characterization result : an extension to a 1987 result of Burosch, Demetrovics and Katona

Theorem (N.C., 1998)

Let $\{(C_i, F_i)\}_{i=1}^{m}$ be a set of m ordered pairs of subsets of S. There exists a Moore family $\mathcal{M}$ on S such that the C_i's are all critical sets of $\mathcal{M}$ and the F_i's are all respective $\phi_{\mathcal{M}}(C_i)$ if and only if the following hold :

- $\forall i \leq m, C_i \subset F_i \subseteq S$,
- $\forall i, j \leq m, (C_i \subset C_j$ implies $F_i \subset C_j)$,
- $\forall i, j \leq m, (C_i \subseteq F_j$ implies $F_i \subseteq F_j)$.

N. Caspard and B. Monjardet, The lattice of Moore families and closure operators on a finite set : a survey, *Electronic Notes in Discrete Mathematics*, **2** (1999).

N. Caspard et B. Monjardet, The lattice of closure systems, closure operators and implicational systems on a finite set : a survey, *Discrete Applied Mathematics*, 127(2), 241–269 (2003).

A. Day, A simple solution to the word problem for lattices, *Canad. Math. Bull.* **13**, 253–254 (1970).

A. Day, characterisations of finite lattices that are bounded-homomorphic images or sublattices of free lattices, *Canadian J. Math.* **31**, 69–78 (1979).

A. Day, J.B. Nation and S. Tschantz, Doubling Convex Sets in Lattices and a Generalized Semidistributivity Condition, *Order* **6**, 175–180 (1989).

W. Geyer, The generalized doubling construction and formal concept analysis, *Algebra Universalis* **32**, 341–367 (1994).

R. McKenzie, Equational bases and non-modular lattice varieties, *Trans. Amer. Math. Soc* **174**, 1–43 (1972).

B. Monjardet, Arrowian characterizations of latticial federation consensus functions, *Mathematical Social Sciences* **20**(1), 51-71 (1990).

Characterizing δ and δ_d with the arrow relations

Proposition

1. $j \delta j' \iff \exists m \in M : \; j \uparrow m \; and \; j' \not\leq m.$
2. $j \delta_d j' \iff \exists m \in M : \; j \uparrow m \; and \; j' \downarrow m.$

In particular : $\delta_d \subseteq \delta$.

Some results

Proposition

In any lattice L, the following are equivalent :

1. *L is atomistic,*
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