Existence varieties of regular rings and complemented modular lattices

Christian Herrmann[1] Marina Semenova[2]

[1]TU Darmstadt, Germany
[2]Institute of Mathematics SB RAS, Russia

Luminy, November 2010

Lattices are considered in signature $\{\vee, \wedge, 0\}$

Lattices are considered in signature $\{\vee, \wedge, 0\}$

$$\Sigma_l = \text{axioms of modular lattices} +$$
$$+\ \forall xy\, \exists z\ [x \wedge y \wedge z = 0]\&[(x \wedge y) \vee z = x].$$

Lattices are considered in signature $\{\vee, \wedge, 0\}$

$$\begin{aligned}\Sigma_l = {} & \text{axioms of modular lattices} + \\ & + \ \forall xy\, \exists z\ [x \wedge y \wedge z = 0] \& [(x \wedge y) \vee z = x].\end{aligned}$$

If $L \in \mathbf{Mod}\,\Sigma_l$, then L is a sectionally complemented modular lattice (a SCML for short);

Lattices are considered in signature $\{\vee, \wedge, 0\}$

$$\begin{aligned}\Sigma_I &= \text{axioms of modular lattices} + \\ &+ \ \forall xy\, \exists z\ [x \wedge y \wedge z = 0] \& [(x \wedge y) \vee z = x].\end{aligned}$$

If $L \in \mathbf{Mod}\, \Sigma_I$, then L is a sectionally complemented modular lattice (a SCML for short);
if $L \in \mathbf{Mod}\, \Sigma_I$ has 1, then it is a complemented modular lattice (a CML for short);

Λ-algebras are considered in signature $\{\Lambda, +, -, \cdot, 0\}$;

Λ-algebras are considered in signature $\{\Lambda, +, -, \cdot, 0\}$;
Rings are considered in signature $\{+, -, \cdot, 0\}$;

Λ-algebras are considered in signature $\{\Lambda, +, -, \cdot, 0\}$;
Rings are considered in signature $\{+, -, \cdot, 0\}$;

$$\Sigma_\Lambda = \text{axioms of } \Lambda\text{-algebras} \; + \; \forall x \, \exists y \; [xyx = x].$$

Λ-algebras are considered in signature $\{\Lambda, +, -, \cdot, 0\}$;
Rings are considered in signature $\{+, -, \cdot, 0\}$;

$$\Sigma_\Lambda = \text{axioms of } \Lambda\text{-algebras} \ + \ \forall x \, \exists y \ [xyx = x].$$

If $A \in \mathbf{Mod}\, \Sigma_\Lambda$, then A is a (von Neumann) regular algebra.

Let R be a regular ring.

Let R be a regular ring.
$\mathbb{L}(R)$ is the lattice of principal right ideals of R.

Let R be a regular ring.
$\mathbb{L}(R)$ is the lattice of principal right ideals of R.
$\mathbb{L}(R)$ is a SCML.

Let R be a regular ring.
$\mathbb{L}(R)$ is the lattice of principal right ideals of R.
$\mathbb{L}(R)$ is a SCML.
If R is Artinian, then $\mathbb{L}(R)$ is a finite height CML.

(X, φ) is a combinatorial geometry, if it has the exchange property:

$$a \in \varphi(Y \cup \{b\}) \;\rightarrow\; b \in \varphi(Y \cup \{a\})$$

for any $a, b \in X$ and any $Y \subseteq X$.

(X, φ) is a combinatorial geometry, if it has the exchange property:

$$a \in \varphi(Y \cup \{b\}) \;\rightarrow\; b \in \varphi(Y \cup \{a\})$$

for any $a, b \in X$ and any $Y \subseteq X$.

Closure lattices of combinatorial geometries are often modular.

(X, φ) is a combinatorial geometry, if it has the exchange property:

$$a \in \varphi(Y \cup \{b\}) \ \rightarrow \ b \in \varphi(Y \cup \{a\})$$

for any $a, b \in X$ and any $Y \subseteq X$.

Closure lattices of combinatorial geometries are often modular.

Let $V_{\mathbb{D}}$ be a vector space over a division ring $\mathbb{D}$.

(X, φ) is a combinatorial geometry, if it has the exchange property:

$$a \in \varphi(Y \cup \{b\}) \;\rightarrow\; b \in \varphi(Y \cup \{a\})$$

for any $a, b \in X$ and any $Y \subseteq X$.

Closure lattices of combinatorial geometries are often modular.

Let $V_{\mathbb{D}}$ be a vector space over a division ring $\mathbb{D}$.
$\mathrm{Sub}(V_{\mathbb{D}})$ is the subspace lattice.

(X, φ) is a combinatorial geometry, if it has the exchange property:

$$a \in \varphi(Y \cup \{b\}) \;\rightarrow\; b \in \varphi(Y \cup \{a\})$$

for any $a, b \in X$ and any $Y \subseteq X$.

Closure lattices of combinatorial geometries are often modular.

Let $V_{\mathbb{D}}$ be a vector space over a division ring $\mathbb{D}$.
$\mathrm{Sub}(V_{\mathbb{D}})$ is the subspace lattice.
$\mathrm{Sub}(V_{\mathbb{D}}) \cong \mathbb{L}(\mathrm{End}(V_{\mathbb{D}}))$, $\quad \mathrm{End}(V_{\mathbb{D}})$ is a regular ring.

$\mathrm{Sub}(V_{\mathbb{D}})$ is a subdirectly irreducible Arguesian SCL:

$\mathrm{Sub}(V_{\mathbb{D}})$ is a subdirectly irreducible Arguesian SCL:

$$\forall\, x_0\, x_1\, x_2\, y_0\, y_1\, y_2 \quad \bigwedge_{i<3}(x_i \vee y_i) \leq (x_0 \wedge (x_1 \vee c)) \;\vee\; (y_0 \wedge (y_1 \vee c)),$$

where

$$c_i = (x_j \vee x_k) \wedge (y_j \vee y_k), \quad \{i, j, k\} = \{0, 1, 2\},$$
$$c = (c_0 \vee c_1) \wedge c_2.$$

$\mathrm{Sub}(V_{\mathbb{D}})$ is a subdirectly irreducible Arguesian SCL:

$$\forall\, x_0\, x_1\, x_2\, y_0\, y_1\, y_2 \quad \bigwedge_{i<3}(x_i \vee y_i) \leq \big(x_0 \wedge (x_1 \vee c)\big) \ \vee \ \big(y_0 \wedge (y_1 \vee c)\big),$$

where

$$c_i = (x_j \vee x_k) \wedge (y_j \vee y_k), \quad \{i,j,k\} = \{0,1,2\},$$
$$c = (c_0 \vee c_1) \wedge c_2.$$

If $\dim V_{\mathbb{D}} < \omega$, then $\mathrm{Sub}(V_{\mathbb{D}})$ is simple finite height.

A partial converse is true:

A partial converse is true:

Theorem (von Neumann, 1939; Jónsson, 1960)

Let L be a simple Arguesian CL of finite height $n \geqslant 3$.

A partial converse is true:

Theorem (von Neumann, 1939; Jónsson, 1960)

Let L be a simple Arguesian CL of finite height $n \geqslant 3$. Then there is a division ring $\mathbb{D}$ such that $L \cong \mathrm{Sub}(\mathbb{D}^n_{\mathbb{D}})$.

Problem (Dilworth)

Is the class of lattices embeddable into CML-s (SCML-s) a variety?

Problem (Dilworth)

Is the class of lattices embeddable into CML-s (SCML-s) a variety?

Due to the Maltsev theorem, this class is a quasivariety.

Problem (Dilworth)

Is the class of lattices embeddable into CML-s (SCML-s) a variety?

Due to the Maltsev theorem, this class is a quasivariety.

Theorem (Jónsson, 1960)

The following are equivalent for a SCML:

Problem (Dilworth)

Is the class of lattices embeddable into CML-s (SCML-s) a variety?

Due to the Maltsev theorem, this class is a quasivariety.

Theorem (Jónsson, 1960)

The following are equivalent for a SCML:

1. $L \in \mathbf{S}(\mathrm{Cl}(X, \varphi))$ *for a projective geometry* (X, φ);

Problem (Dilworth)

Is the class of lattices embeddable into CML-s (SCML-s) a variety?

Due to the Maltsev theorem, this class is a quasivariety.

Theorem (Jónsson, 1960)

The following are equivalent for a SCML:

1. $L \in \mathbf{S}(\mathrm{Cl}(X, \varphi))$ *for a projective geometry* (X, φ);
2. $L \in \mathbf{S}(\mathrm{Sub}(A))$ *for an Abelian group* A;

Problem (Dilworth)

Is the class of lattices embeddable into CML-s (SCML-s) a variety?

Due to the Maltsev theorem, this class is a quasivariety.

Theorem (Jónsson, 1960)

The following are equivalent for a SCML:

1. $L \in \mathbf{S}(\mathrm{Cl}(X, \varphi))$ *for a projective geometry* (X, φ)*;*
2. $L \in \mathbf{S}(\mathrm{Sub}(A))$ *for an Abelian group* A*;*
3. $L \in \mathbf{S}(\prod_{i \in I} \mathrm{Sub}(V_i))$, V_i *is a vector space for all* $i \in I$*;*

Problem (Dilworth)

Is the class of lattices embeddable into CML-s (SCML-s) a variety?

Due to the Maltsev theorem, this class is a quasivariety.

Theorem (Jónsson, 1960)

The following are equivalent for a SCML:

1. $L \in \mathbf{S}(\mathrm{Cl}(X, \varphi))$ *for a projective geometry* (X, φ)*;*
2. $L \in \mathbf{S}(\mathrm{Sub}(A))$ *for an Abelian group* A*;*
3. $L \in \mathbf{S}(\prod_{i \in I} \mathrm{Sub}(V_i))$, V_i *is a vector space for all* $i \in I$*;*
4. L *is Arguesian.*

Let $\mathcal{K} \subseteq \mathbf{Mod}\,\Sigma$.

Let $\mathcal{K} \subseteq \mathbf{Mod}\,\Sigma$.
$\mathbf{S}_{\exists}(\mathcal{K}) = \mathbf{Mod}\,\Sigma \cap \mathbf{S}(\mathcal{K})$

Let $\mathcal{K} \subseteq \mathbf{Mod}\,\Sigma$.
$\mathbf{S}_\exists(\mathcal{K}) = \mathbf{Mod}\,\Sigma \cap \mathbf{S}(\mathcal{K})$

Definition

$\mathcal{K} \subseteq \mathbf{Mod}\,\Sigma$ is an $\exists$-variety, if it is closed under $\mathbf{H}$, $\mathbf{S}_\exists$, and $\mathbf{P}$.

Let $\mathcal{K} \subseteq \mathbf{Mod}\,\Sigma$.
$\mathbf{S}_{\exists}(\mathcal{K}) = \mathbf{Mod}\,\Sigma \cap \mathbf{S}(\mathcal{K})$

Definition

$\mathcal{K} \subseteq \mathbf{Mod}\,\Sigma$ is an $\exists$-variety, if it is closed under $\mathbf{H}$, $\mathbf{S}_{\exists}$, and $\mathbf{P}$.

Theorem

Let $\mathcal{K} \subseteq \mathbf{Mod}\,\Sigma$.

Let $\mathcal{K} \subseteq \mathbf{Mod}\,\Sigma$.
$\mathbf{S}_\exists(\mathcal{K}) = \mathbf{Mod}\,\Sigma \cap \mathbf{S}(\mathcal{K})$

Definition

$\mathcal{K} \subseteq \mathbf{Mod}\,\Sigma$ is an $\exists$-variety, if it is closed under $\mathbf{H}$, $\mathbf{S}_\exists$, and $\mathbf{P}$.

Theorem

Let $\mathcal{K} \subseteq \mathbf{Mod}\,\Sigma$.

1. $\mathbf{V}_\exists(\mathcal{K}) = \mathbf{HS}_\exists\mathbf{P}(\mathcal{K})$ *is the smallest* $\exists$*-variety containing* $\mathcal{K}$*;*

Let $\mathcal{K} \subseteq \mathbf{Mod}\,\Sigma$.
$\mathbf{S}_\exists(\mathcal{K}) = \mathbf{Mod}\,\Sigma \cap \mathbf{S}(\mathcal{K})$

Definition

$\mathcal{K} \subseteq \mathbf{Mod}\,\Sigma$ is an $\exists$-variety, if it is closed under $\mathbf{H}$, $\mathbf{S}_\exists$, and $\mathbf{P}$.

Theorem

Let $\mathcal{K} \subseteq \mathbf{Mod}\,\Sigma$.

1. $\mathbf{V}_\exists(\mathcal{K}) = \mathbf{HS}_\exists\mathbf{P}(\mathcal{K})$ *is the smallest* $\exists$*-variety containing* $\mathcal{K}$*; moreover,* $\mathbf{TV}_\exists(\mathcal{K}) = \mathbf{VT}(\mathcal{K})$.

Let $\mathcal{K} \subseteq \mathbf{Mod}\,\Sigma$.
$\mathbf{S}_\exists(\mathcal{K}) = \mathbf{Mod}\,\Sigma \cap \mathbf{S}(\mathcal{K})$

Definition

$\mathcal{K} \subseteq \mathbf{Mod}\,\Sigma$ is an $\exists$-variety, if it is closed under $\mathbf{H}$, $\mathbf{S}_\exists$, and $\mathbf{P}$.

Theorem

Let $\mathcal{K} \subseteq \mathbf{Mod}\,\Sigma$.

1. $\mathbf{V}_\exists(\mathcal{K}) = \mathbf{HS}_\exists\mathbf{P}(\mathcal{K})$ *is the smallest* $\exists$*-variety containing* $\mathcal{K}$*; moreover,* $\mathbf{TV}_\exists(\mathcal{K}) = \mathbf{VT}(\mathcal{K})$.
2. *The reduct of any free algebra from* $\mathbf{VT}(\mathcal{K})$ *belongs to* $\mathbf{P}_{s\exists}(\mathcal{K})$.

Let $\mathcal{K} \subseteq \mathbf{Mod}\,\Sigma$.
$\mathbf{S}_\exists(\mathcal{K}) = \mathbf{Mod}\,\Sigma \cap \mathbf{S}(\mathcal{K})$

Definition

$\mathcal{K} \subseteq \mathbf{Mod}\,\Sigma$ is an $\exists$-variety, if it is closed under $\mathbf{H}$, $\mathbf{S}_\exists$, and $\mathbf{P}$.

Theorem

Let $\mathcal{K} \subseteq \mathbf{Mod}\,\Sigma$.

1. $\mathbf{V}_\exists(\mathcal{K}) = \mathbf{HS}_\exists\mathbf{P}(\mathcal{K})$ *is the smallest $\exists$-variety containing $\mathcal{K}$; moreover,* $\mathbf{TV}_\exists(\mathcal{K}) = \mathbf{VT}(\mathcal{K})$.
2. *The reduct of any free algebra from* $\mathbf{VT}(\mathcal{K})$ *belongs to* $\mathbf{P}_{s\exists}(\mathcal{K})$.
3. *Any SI algebra from* $\mathbf{V}_\exists(\mathcal{K})$ *belongs to* $\mathbf{HS}_\exists\mathbf{P}_u(\mathcal{K})$.

Let $\mathcal{K} \subseteq \mathbf{Mod}\,\Sigma$.
$\mathbf{S}_\exists(\mathcal{K}) = \mathbf{Mod}\,\Sigma \cap \mathbf{S}(\mathcal{K})$

Definition

$\mathcal{K} \subseteq \mathbf{Mod}\,\Sigma$ is an ∃-variety, if it is closed under $\mathbf{H}$, $\mathbf{S}_\exists$, and $\mathbf{P}$.

Theorem

Let $\mathcal{K} \subseteq \mathbf{Mod}\,\Sigma$.

1. $\mathbf{V}_\exists(\mathcal{K}) = \mathbf{HS}_\exists\mathbf{P}(\mathcal{K})$ *is the smallest ∃-variety containing* $\mathcal{K}$*; moreover,* $\mathbf{TV}_\exists(\mathcal{K}) = \mathbf{VT}(\mathcal{K})$.
2. *The reduct of any free algebra from* $\mathbf{VT}(\mathcal{K})$ *belongs to* $\mathbf{P}_{s\exists}(\mathcal{K})$.
3. *Any SI algebra from* $\mathbf{V}_\exists(\mathcal{K})$ *belongs to* $\mathbf{HS}_\exists\mathbf{P}_u(\mathcal{K})$.
4. *Any ∃-variety is generated by its finitely generated SI-s.*

Free algebras exist in $\exists$-varieties.

Free algebras exist in $\exists$-varieties.
Any $\exists$-variety can be defined by positive sentences as well as by Horn sentences.

Free algebras exist in $\exists$-varieties.
Any $\exists$-variety can be defined by positive sentences as well as by Horn sentences.

Problem

Can an $\exists$-variety be defined by positive Horn sentences?

Regular rings

Regular rings

Theorem

$\mathbf{V}_\exists(\mathbb{F}^{n\times n} \mid n_0 < n < \omega,\ \mathbb{F}$ *is a quotient field of* $\Lambda)$ *is the* $\exists$*-variety of regular* Λ*-algebras.*

Regular rings

Theorem

$\mathbf{V}_\exists(\mathbb{F}^{n\times n} \mid n_0 < n < \omega,\ \mathbb{F}$ *is a quotient field of* $\Lambda)$ *is the* $\exists$*-variety of regular* Λ*-algebras.*

Corollary

1. $\mathbf{V}_\exists(\mathbb{F}_p^{n\times n} \mid n_0 < n < \omega,\ p$ *is prime*$)$ *is the* $\exists$*-variety of regular rings.*

Regular rings

Theorem

$\mathbf{V}_\exists(\mathbb{F}^{n\times n} \mid n_0 < n < \omega,\ \mathbb{F}$ is a quotient field of $\Lambda)$ is the $\exists$-variety of regular Λ-algebras.

Corollary

1. *$\mathbf{V}_\exists(\mathbb{F}_p^{n\times n} \mid n_0 < n < \omega,\ p$ is prime$)$ is the $\exists$-variety of regular rings.*
2. *Free regular rings are residually finite.*

Regular rings

Theorem

$\mathbf{V}_{\exists}(\mathbb{F}^{n\times n} \mid n_0 < n < \omega,\ \mathbb{F}$ *is a quotient field of* $\Lambda)$ *is the* $\exists$*-variety of regular* Λ*-algebras.*

Corollary

1. $\mathbf{V}_{\exists}(\mathbb{F}_p^{n\times n} \mid n_0 < n < \omega,\ p$ *is prime*$)$ *is the* $\exists$*-variety of regular rings.*
2. *Free regular rings are residually finite.*
3. *The equational theory of regular rings with quasi-inversion as a fundamental operation is decidable.*

Theorem

Let R be a SI non-Artinian regular Λ-algebra.

Theorem

Let R be a SI non-Artinian regular Λ-algebra. There is a field F: $\mathsf{V}_{\exists}(R) = \mathsf{V}_{\exists}(F^{n \times n} \mid n_0 < n < \omega)$.

Theorem

Let R be a SI non-Artinian regular Λ-algebra. There is a field F: $\mathsf{V}_\exists(R) = \mathsf{V}_\exists(F^{n\times n} \mid n_0 < n < \omega)$.

Corollary

Any $\exists$-variety of regular Λ-algebras is generated by its simple Artinian members.

Theorem

Let R be a SI non-Artinian regular Λ-algebra. There is a field F: $\mathsf{V}_\exists(R) = \mathsf{V}_\exists(F^{n\times n} \mid n_0 < n < \omega)$.

Corollary

Any $\exists$-variety of regular Λ-algebras is generated by its simple Artinian members.

Corollary

Free regular Λ-algebras are residually Artinian.

Theorem

Let R be a SI non-Artinian regular Λ-algebra. There is a field F: $\mathsf{V}_\exists(R) = \mathsf{V}_\exists(F^{n\times n} \mid n_0 < n < \omega)$.

Corollary

Any $\exists$-variety of regular Λ-algebras is generated by its simple Artinian members.

Corollary

Free regular Λ-algebras are residually Artinian.

Goodearl, Menal, and Moncasi (1993) proved the latter statement for algebras with unit.

For an $\exists$-variety $\mathcal{V}$ of regular rings,
$C(\mathcal{V})$ is the class of simple Artinian members of $\mathcal{V}$.

For an $\exists$-variety $\mathcal{V}$ of regular rings,
$C(\mathcal{V})$ is the class of simple Artinian members of $\mathcal{V}$.

By the Wedderburn-Artin theorem, $C(\mathcal{V})$ consists of matrix rings over division rings.

For a class $\mathcal{C}$ of simple Artinian regular rings and for $n > 0$:

$$D \in \mathbf{D}_n(\mathcal{C}) \quad \text{if and only if} \quad D^{n \times n} \in \mathcal{C}\,.$$

For a class $\mathcal{C}$ of simple Artinian regular rings and for $n > 0$:

$$D \in \mathbf{D}_n(\mathcal{C}) \quad \text{if and only if} \quad D^{n\times n} \in \mathcal{C}\,.$$

Definition

$\mathcal{C}$ is closed, if the following holds:

For a class $\mathcal{C}$ of simple Artinian regular rings and for $n > 0$:

$$D \in \mathbf{D}_n(\mathcal{C}) \quad \text{if and only if} \quad D^{n \times n} \in \mathcal{C}\,.$$

Definition

$\mathcal{C}$ is closed, if the following holds:

1. $\mathbf{D}_n(\mathcal{C})$ is a universal class of division rings for all $n > 0$;

For a class $\mathcal{C}$ of simple Artinian regular rings and for $n > 0$:

$$D \in \mathbf{D}_n(\mathcal{C}) \quad \text{if and only if} \quad D^{n\times n} \in \mathcal{C}\,.$$

Definition

$\mathcal{C}$ is closed, if the following holds:

1. $\mathbf{D}_n(\mathcal{C})$ is a universal class of division rings for all $n > 0$;
2. $\mathbf{D}_n(\mathcal{C}) \subseteq \mathbf{D}_m(\mathcal{C})$ for all $n \geqslant m > 0$;

For a class $\mathcal{C}$ of simple Artinian regular rings and for $n > 0$:

$$D \in \mathbf{D}_n(\mathcal{C}) \quad \text{if and only if} \quad D^{n\times n} \in \mathcal{C}\,.$$

Definition

$\mathcal{C}$ is closed, if the following holds:

1. $\mathbf{D}_n(\mathcal{C})$ is a universal class of division rings for all $n > 0$;
2. $\mathbf{D}_n(\mathcal{C}) \subseteq \mathbf{D}_m(\mathcal{C})$ for all $n \geqslant m > 0$;
3. if $n = mk > 0$, $F \in \mathbf{D}_n(\mathcal{C})$, and $D \in \mathrm{S}(F^{k\times k})$ is a division ring, then $D \in \mathbf{D}_m(\mathcal{C})$;

For a class $\mathcal{C}$ of simple Artinian regular rings and for $n > 0$:

$$D \in \mathbf{D}_n(\mathcal{C}) \quad \text{if and only if} \quad D^{n\times n} \in \mathcal{C}\,.$$

Definition

$\mathcal{C}$ is closed, if the following holds:

1. $\mathbf{D}_n(\mathcal{C})$ is a universal class of division rings for all $n > 0$;
2. $\mathbf{D}_n(\mathcal{C}) \subseteq \mathbf{D}_m(\mathcal{C})$ for all $n \geqslant m > 0$;
3. if $n = mk > 0$, $F \in \mathbf{D}_n(\mathcal{C})$, and $D \in \mathrm{S}(F^{k\times k})$ is a division ring, then $D \in \mathbf{D}_m(\mathcal{C})$;
4. p is a prime;
   if for any $n > 0$, there is $D \in \mathbf{D}_n(\mathcal{C})$ with char $D = p$, then $F \in \bigcap_{n>0} \mathbf{D}_n(\mathcal{C})$ for any F with char $F = p$;

For a class $\mathcal{C}$ of simple Artinian regular rings and for $n > 0$:

$$D \in \mathbf{D}_n(\mathcal{C}) \quad \text{if and only if} \quad D^{n\times n} \in \mathcal{C}\,.$$

Definition

$\mathcal{C}$ is closed, if the following holds:

1. $\mathbf{D}_n(\mathcal{C})$ is a universal class of division rings for all $n > 0$;
2. $\mathbf{D}_n(\mathcal{C}) \subseteq \mathbf{D}_m(\mathcal{C})$ for all $n \geqslant m > 0$;
3. if $n = mk > 0$, $F \in \mathbf{D}_n(\mathcal{C})$, and $D \in \mathrm{S}(F^{k\times k})$ is a division ring, then $D \in \mathbf{D}_m(\mathcal{C})$;
4. p is a prime;
   if for any $n > 0$, there is $D \in \mathbf{D}_n(\mathcal{C})$ with char $D = p$, then $F \in \bigcap_{n>0} \mathbf{D}_n(\mathcal{C})$ for any F with char $F = p$;
5. $\mathbf{D}_1(\mathcal{C})$ is the class of all division rings.

Theorem

Let $\mathcal{C}$ be a class of simple Artinian regular rings.
$\mathcal{C}$ is closed if and only if $\mathcal{C} = C(\mathcal{V})$ for an $\exists$-variety of regular rings.

Sectionally complemented modular lattices

Sectionally complemented modular lattices

Theorem

Let L be a SI modular SCL of infinite height.

Sectionally complemented modular lattices

Theorem

Let L be a SI modular SCL of infinite height. There is a unique prime field $\mathbb{F}$ *such that* $\mathsf{V}_\exists(L) = \mathsf{V}_\exists(\mathbb{L}(\mathbb{F}^{n\times n}) \mid n_0 < n < \omega)$.

Sectionally complemented modular lattices

Theorem

Let L be a SI modular SCL of infinite height. There is a unique prime field $\mathbb{F}$ *such that* $\mathsf{V}_\exists(L) = \mathsf{V}_\exists(\mathbb{L}(\mathbb{F}^{n\times n}) \mid n_0 < n < \omega)$.

Corollary

Any $\exists$*-variety of SCML is generated by its simple finite height members.*

Corollary

1. $\mathsf{V}_{\exists}(\mathbb{L}(\mathbb{F}_p^{n\times n}) \mid n_0 < n < \omega,\ p \text{ is prime})$ *is the variety of Arguesian SCL.*

Corollary

1. $\mathsf{V}_\exists(\mathbb{L}(\mathbb{F}_p^{n\times n}) \mid n_0 < n < \omega,\ p \text{ is prime})$ *is the variety of Arguesian SCL.*
2. *Free Arguesian SCL are residually finite.*

Corollary

1. $\mathsf{V}_\exists(\mathbb{L}(\mathbb{F}_p^{n\times n}) \mid n_0 < n < \omega,\ p \text{ is prime})$ *is the variety of Arguesian SCL.*
2. *Free Arguesian SCL are residually finite.*
3. *Equational theory of Arguesian lattices with sectional complementation as a fundamental operation is decidable.*

Corollary

1. $\mathsf{V}_\exists(\mathbb{L}(\mathbb{F}_p^{n\times n}) \mid n_0 < n < \omega,\ p \text{ is prime})$ *is the variety of Arguesian SCL.*
2. *Free Arguesian SCL are residually finite.*
3. *Equational theory of Arguesian lattices with sectional complementation as a fundamental operation is decidable.*

Corollary

Equational theory of modular lattices with sectional complementation is decidable.

For an $\exists$-variety $\mathcal{V}$ of Arguesian SCL, $C(\mathcal{V})$ is the class of its simple Arguesian finite height members.

For an $\exists$-variety $\mathcal{V}$ of Arguesian SCL, $C(\mathcal{V})$ is the class of its simple Arguesian finite height members.

By the von-Neumann-Jónsson coordinatization theorem, any $L \in C(\mathcal{V})$ with $\operatorname{ht} L \geqslant 3$ is of the form $\mathbb{L}(D_D^n)$.

For a class $\mathcal{C}$ of simple Arguesian finite height SCL and for $n > 0$:

$$D \in \mathbf{D}_n(\mathcal{C}) \quad \text{if and only if} \quad \mathbb{L}(D_D^n) \in \mathcal{C}\,.$$

Definition

$\mathcal{C}$ is closed, if the following holds:

Definition

$\mathcal{C}$ is closed, if the following holds:

1. $\mathbf{D}_n(\mathcal{C})$ is a universal class of division rings for all $n > 0$;

Definition

$\mathcal{C}$ is closed, if the following holds:

1. $\mathbf{D}_n(\mathcal{C})$ is a universal class of division rings for all $n > 0$;
2. $\mathbf{D}_n(\mathcal{C}) \subseteq \mathbf{D}_m(\mathcal{C})$ for all $n \geqslant m > 0$;

Definition

$\mathcal{C}$ is closed, if the following holds:

1. $\mathbf{D}_n(\mathcal{C})$ is a universal class of division rings for all $n > 0$;
2. $\mathbf{D}_n(\mathcal{C}) \subseteq \mathbf{D}_m(\mathcal{C})$ for all $n \geqslant m > 0$;
3. if $n = mk > 0$, $F \in \mathbf{D}_n(\mathcal{C})$, and $D \in \mathsf{S}(F^{k \times k})$ is a division ring, then $D \in \mathbf{D}_m(\mathcal{C})$;

Definition

$\mathcal{C}$ is closed, if the following holds:

1. $\mathbf{D}_n(\mathcal{C})$ is a universal class of division rings for all $n > 0$;
2. $\mathbf{D}_n(\mathcal{C}) \subseteq \mathbf{D}_m(\mathcal{C})$ for all $n \geqslant m > 0$;
3. if $n = mk > 0$, $F \in \mathbf{D}_n(\mathcal{C})$, and $D \in \mathrm{S}(F^{k \times k})$ is a division ring, then $D \in \mathbf{D}_m(\mathcal{C})$;
4. p is a prime;
   if for any $n > 0$, there is $D \in \mathbf{D}_n(\mathcal{C})$ with char $D = p$, then $F \in \bigcap_{n>0} \mathbf{D}_n(\mathcal{C})$ for any F with char $F = p$;

Definition

$\mathcal{C}$ is closed, if the following holds:

1. $\mathbf{D}_n(\mathcal{C})$ is a universal class of division rings for all $n > 0$;
2. $\mathbf{D}_n(\mathcal{C}) \subseteq \mathbf{D}_m(\mathcal{C})$ for all $n \geqslant m > 0$;
3. if $n = mk > 0$, $F \in \mathbf{D}_n(\mathcal{C})$, and $D \in \mathsf{S}(F^{k \times k})$ is a division ring, then $D \in \mathbf{D}_m(\mathcal{C})$;
4. p is a prime;
   if for any $n > 0$, there is $D \in \mathbf{D}_n(\mathcal{C})$ with char $D = p$, then $F \in \bigcap_{n>0} \mathbf{D}_n(\mathcal{C})$ for any F with char $F = p$;
5. if $D \in \mathbf{D}_2(\mathcal{C})$ and $|F| \leqslant |D|$, then $F \in \mathbf{D}_2(\mathcal{C})$;

Definition

$\mathcal{C}$ is closed, if the following holds:

1. $\mathbf{D}_n(\mathcal{C})$ is a universal class of division rings for all $n > 0$;
2. $\mathbf{D}_n(\mathcal{C}) \subseteq \mathbf{D}_m(\mathcal{C})$ for all $n \geqslant m > 0$;
3. if $n = mk > 0$, $F \in \mathbf{D}_n(\mathcal{C})$, and $D \in \mathrm{S}(F^{k\times k})$ is a division ring, then $D \in \mathbf{D}_m(\mathcal{C})$;
4. p is a prime;
   if for any $n > 0$, there is $D \in \mathbf{D}_n(\mathcal{C})$ with $\operatorname{char} D = p$, then $F \in \bigcap_{n>0} \mathbf{D}_n(\mathcal{C})$ for any F with $\operatorname{char} F = p$;
5. if $D \in \mathbf{D}_2(\mathcal{C})$ and $|F| \leqslant |D|$, then $F \in \mathbf{D}_2(\mathcal{C})$;
   if $M_k \in \mathcal{C}$ for $k < \omega$, then $M_n \in \mathcal{C}$ for all $2 < n \leqslant k$;

Definition

$\mathcal{C}$ is closed, if the following holds:

1. $\mathbf{D}_n(\mathcal{C})$ is a universal class of division rings for all $n > 0$;
2. $\mathbf{D}_n(\mathcal{C}) \subseteq \mathbf{D}_m(\mathcal{C})$ for all $n \geqslant m > 0$;
3. if $n = mk > 0$, $F \in \mathbf{D}_n(\mathcal{C})$, and $D \in \mathsf{S}(F^{k\times k})$ is a division ring, then $D \in \mathbf{D}_m(\mathcal{C})$;
4. p is a prime;
   if for any $n > 0$, there is $D \in \mathbf{D}_n(\mathcal{C})$ with char $D = p$, then $F \in \bigcap_{n>0} \mathbf{D}_n(\mathcal{C})$ for any F with char $F = p$;
5. if $D \in \mathbf{D}_2(\mathcal{C})$ and $|F| \leqslant |D|$, then $F \in \mathbf{D}_2(\mathcal{C})$;
   if $M_k \in \mathcal{C}$ for $k < \omega$, then $M_n \in \mathcal{C}$ for all $2 < n \leqslant k$;

Definition

$\mathcal{C}$ is closed, if the following holds:

1. $\mathbf{D}_n(\mathcal{C})$ is a universal class of division rings for all $n > 0$;
2. $\mathbf{D}_n(\mathcal{C}) \subseteq \mathbf{D}_m(\mathcal{C})$ for all $n \geqslant m > 0$;
3. if $n = mk > 0$, $F \in \mathbf{D}_n(\mathcal{C})$, and $D \in \mathsf{S}(F^{k\times k})$ is a division ring, then $D \in \mathbf{D}_m(\mathcal{C})$;
4. p is a prime;
   if for any $n > 0$, there is $D \in \mathbf{D}_n(\mathcal{C})$ with $\operatorname{char} D = p$, then $F \in \bigcap_{n>0} \mathbf{D}_n(\mathcal{C})$ for any F with $\operatorname{char} F = p$;
5. if $D \in \mathbf{D}_2(\mathcal{C})$ and $|F| \leqslant |D|$, then $F \in \mathbf{D}_2(\mathcal{C})$;
   if $M_k \in \mathcal{C}$ for $k < \omega$, then $M_n \in \mathcal{C}$ for all $2 < n \leqslant k$;
6. $\mathbf{D}_1(\mathcal{C})$ is the class of all division rings.

Theorem

Let $\mathcal{C}$ be a class of simple Arguesian finite height SCL.
$\mathcal{C}$ is closed if and only if $\mathcal{C} = C(\mathcal{V})$ for an $\exists$-variety of Arguesian SCL.

Problem

Is the class of lattices embeddable into SCML-s a variety?

Problem

Is the class of lattices embeddable into SCML-s a variety?

Corollary

If L embeds into a SCML, then $\mathrm{Id}(L)$ does.