Finite Coxeter lattices: some trong properties

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8FCC, Orsay, June-July 2010

My colleague (and friend)

Fig.: C. le Conte de Poly-Barbut, CAMS, EHESS, Paris.

Sketch of the talk

1. Bounded lattices and the interval doubling operation
2. Finite Coxeter groups and lattices
   - Definition and classification
   - The lattice structure of Coxeter groups
3. Finite Coxeter lattices are bounded : the proof
   - The class $\mathcal{HH}$ of lattices
   - All lattices of $\mathcal{HH}$ are bounded
   - Finite Coxeter lattices are in $\mathcal{HH}$

Outline

Bounded lattices

Definition (McKenzie, 1972)

A lattice is *lower bounded* if it is the lower bounded homomorphic image of a free lattice.

A homomorphism $\alpha : L \to L'$ is called *lower bounded* if the inverse image of each element of L' is either empty or has a minimum.

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An *upper bounded* lattice is defined dually.

A lattice is *bounded* if it is lower and upper bounded.

Characterization of bounded lattices

Theorem (Day, 1979)

Let L be a lattice. The following are equivalent :

- *L is bounded,*
- *L can be constructed starting from $\underline{2}$ by a finite sequence of interval doublings.*

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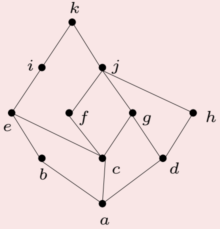
What is an INTERVAL DOUBLING ?

Interval doubling construction (Day, 1970)



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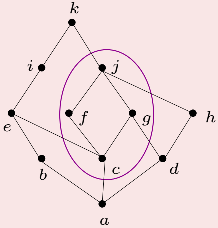

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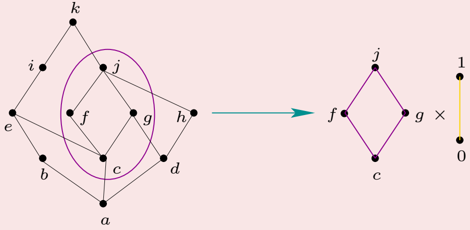

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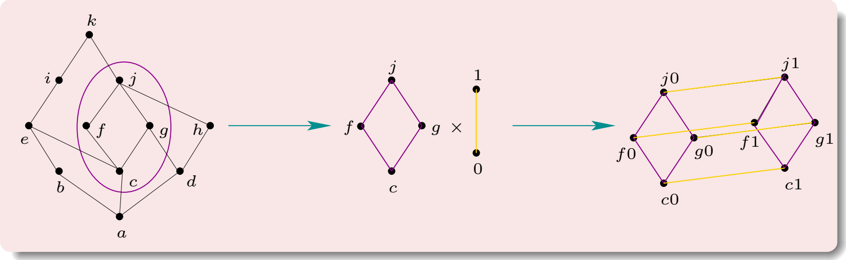

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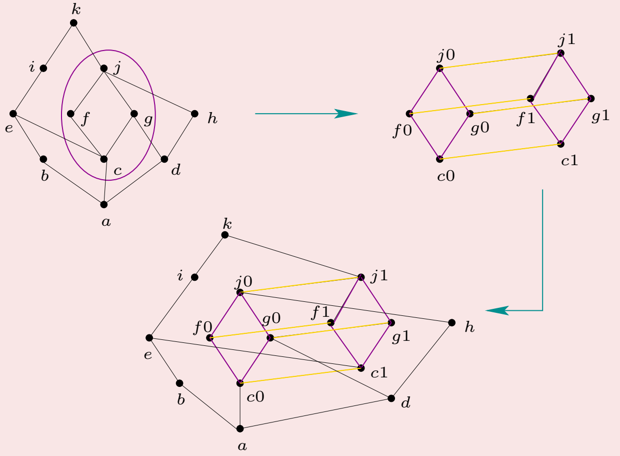

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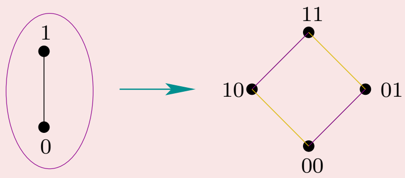

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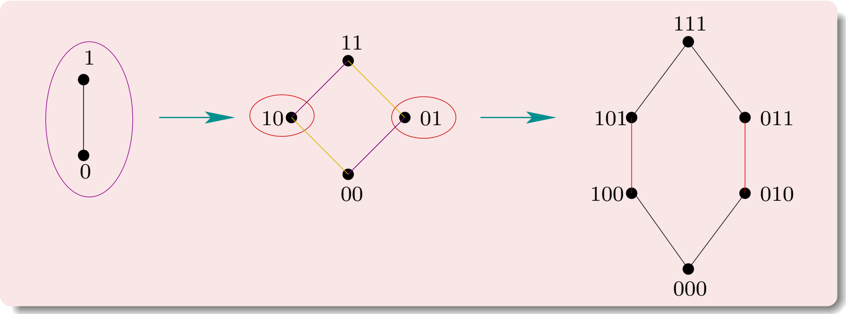
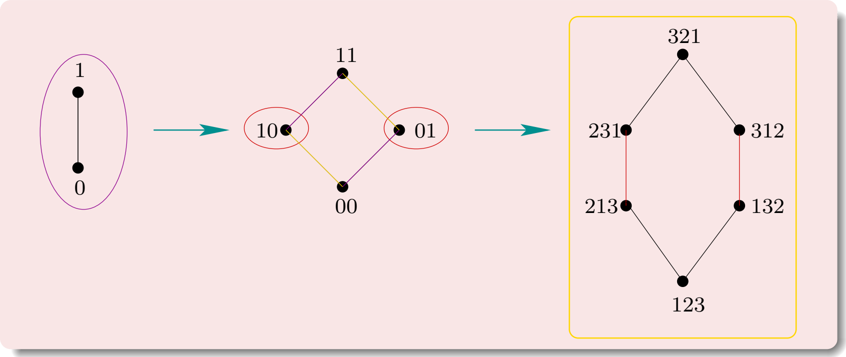
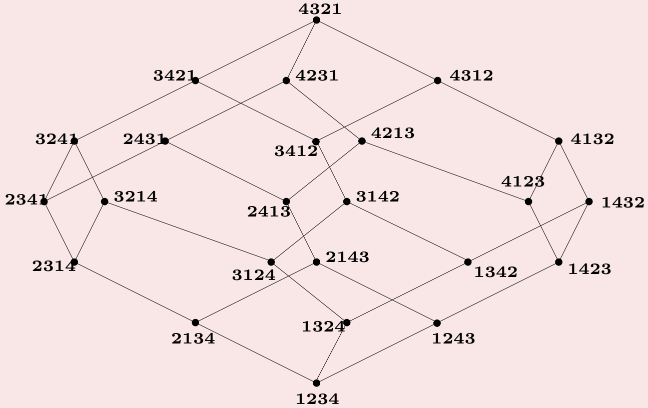


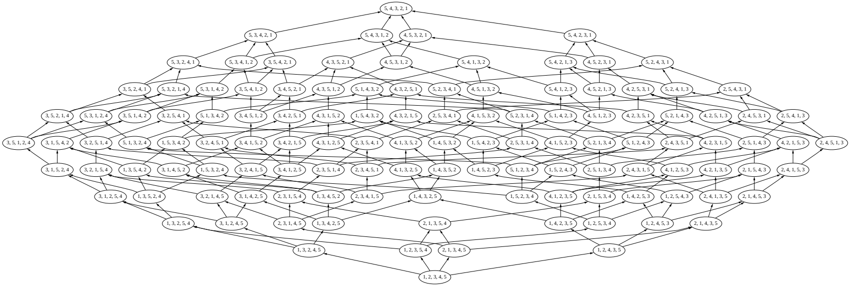
Perm(3) is bounded



Perm(4) : bounded too



Perm(5) : bounded again



In fact...

Theorem (2000)

Permutohedron is bounded

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Permutohedron is bounded

And in fact...

Theorem (2004)

All finite Coxeter lattices are bounded

Outline

What is a Coxeter group ?

Definition

A Coxeter group : group W with a set of generators $S \subset W$, satisfying relations of the form :

$$(ss')^{m(s,s')} = e$$

with :

- $m(s,s) = 1$ for any $s \in S$ (all generators have order 2),
- $m(s,s') = m(s',s) \geq 2$ for $s \neq s'$ in S.

Classification of finite irreducible Coxeter groups

1. Four infinite families :
   - A_n (symmetric groups),
   - B_n,
   - D_n,
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   - A_n (symmetric groups),
   - B_n,
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2. and six isolated groups : E_6, E_7, E_8, F_4, H_3 and H_4.

The (right) weak order on a Coxeter group

Definition

$w <_R w'$ if there exist $s_1, \ldots, s_r \in S$ with

- $w' = ws_1 \ldots s_r$ and
- $\ell(w') = \ell(w) + r$

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Theorem (Björner, 1984)

The (right) weak order on any finite Coxeter group is a (autodual) lattice.

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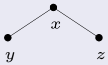
Hat, antihat and 2-facet

Definition

- a *Hat* $(y, x, z)^\wedge$:



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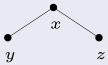

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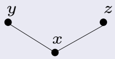

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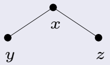

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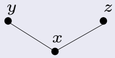
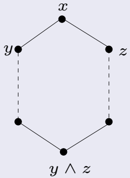


- a *2-facet* $F^{y,x,z}$:



2-facet labelling

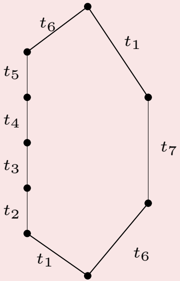
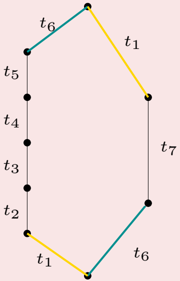


Fig.: Example of a 2-facet labelling

2-FACET LABELLING



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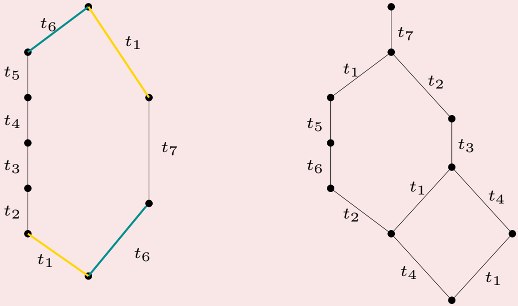

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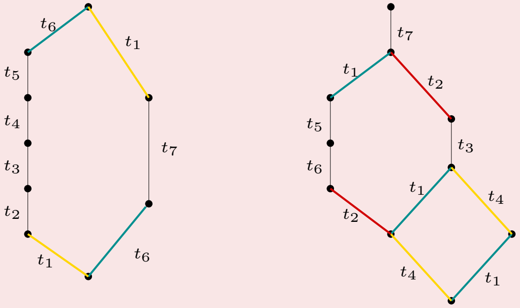

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2-FACET RANK FUNCTION ON A 2-FACET LABELLING

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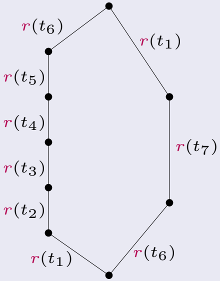

Function r from a 2-facet labelling onto $\mathbb{R}$

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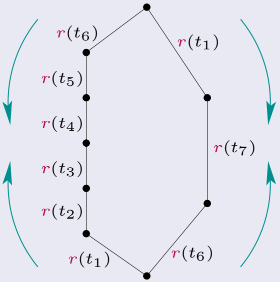

Function r from a 2-facet labelling onto $\mathbb{R}$ such that :

So : $r(t_1) < r(t_2) < r(t_3)$
and $r(t_6) < r(t_5) < r(t_4)$
and $r(t_1), r(t_6) < r(t_7)$

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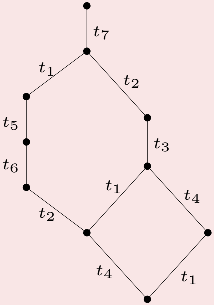
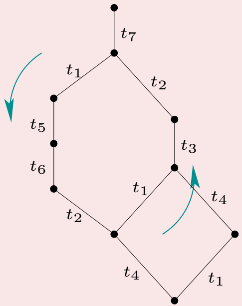


2-facet rank function on a 2-facet labelling



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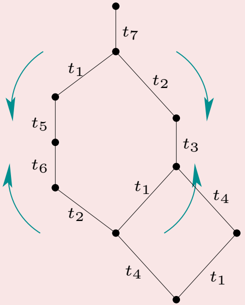

Here $r(t_1) < r(t_5), r(t_3)$ and $r(t_2) < r(t_6), r(t_3)$

On semidistributivity

Definition

A lattice is *semidistributive* if, for all $x, y, z \in L$:

- $x \wedge y = x \wedge z$ implies $x \wedge y = x \wedge (y \vee z)$
- $x \vee y = x \vee z$ implies $x \vee y = x \vee (y \wedge z)$

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Proposition (Day, Nation, Tschantz, 1989)

Bounded lattices are semidistributive.

The class $\mathcal{HH}$ of lattices

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4. to every anti-hat $(y, x, z)_\vee$ of L is associated a hat $(y', y \vee z, z')^\wedge$ of L such that $[x, y \vee z]$ is a 2-facet.

First part of the theorem

All lattices of $\mathcal{HH}$ are bounded

HOW DO WE PROVE THIS ?

First part of the theorem

All lattices of $\mathcal{HH}$ are bounded

How do we prove this ?

Technical proof : not presented here.

Second part of the theorem

Finite Coxeter lattices are in $\mathcal{HH}$

How do we prove this ?

A STRONG RESULT

Proposition (L.C.D.P.-B., 1994)

Finite Coxeter lattices are semidistributive.

Reflections of Coxeter groups

Definition

Set of the *reflections* of the Coxeter group W : the conjugates of the generators.

$$T_W = \{t \in W : \exists s \in S, \exists w \in W \text{ such that } t = wsw^{-1}\}$$

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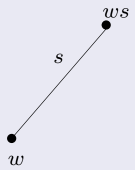
Two labellings of the arcs : the g-labelling



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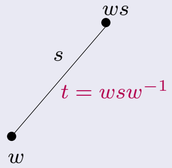
Two labellings of the arcs : the g-labelling and the r-labelling



Properties of the reflections

Proposition (L.C.d.P.-B.)

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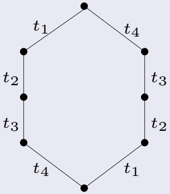

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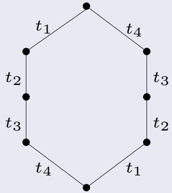

Corollary

The r-labelling on the arcs of any finite Coxeter lattice is a 2-facet labelling.

Properties of the length function

Theorem

The length function ℓ on every Coxeter lattice L_W is a 2-facet rank function when defined on the r-labelling of the arcs of L_W.

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The length function ℓ on every Coxeter lattice L_W is a 2-facet rank function when defined on the r-labelling of the arcs of L_W.

Theorem

Every Coxeter lattice is in the class $\mathcal{HH}$ and therefore is bounded.

Two additional results

Theorem

Let L_W be a Coxeter lattice and W_H a parabolic subgroup of W. There exists an interval doubling series that leads from the lattice L_{W_H} to the lattice L_W.

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Let L_W be a Coxeter lattice and W_H a parabolic subgroup of W. There exists an interval doubling series that leads from the lattice L_{W_H} to the lattice L_W.

Proposition

There exists a particular interval doubling series from a given Coxeter lattice generated by n generators to the Coxeter lattice of the same family, generated by $n+1$ generators.

Any question ?

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