Derived Semidistributive Lattices

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Let $\mathbb{C}(L)$ denote the set of covers of a poset L: $\gamma \in \mathbb{C}(L)$ if and only $\gamma = (\gamma_0, \gamma_1) \in L \times L$ and the interval $\{ x \mid \gamma_0 \leq x \leq \gamma_1 \}$ is a two elements poset. If L is a lattice then there is a natural ordering of $\mathbb{C}(L)$: $\gamma \leq \delta$ if and only if $\gamma_0 \leq \delta_0$, $\gamma_1 \not\leq \delta_0$, and $\gamma_1 \leq \delta_1$. That is, $\gamma \leq \delta$ if and only if the cover γ transposes up to δ.

For $\alpha \in \mathbb{C}(L)$ let $\mathbb{C}(L, \alpha)$ denote the component of the poset $\mathbb{C}(L)$ connected to α. For example, if L is finite join semidistributive and $\alpha = (j_*, j)$ for a join irreducible j and its unique lower cover j_*, then $\mathbb{C}(L, \alpha) = \{ \beta \mid \alpha \leq \beta \}$. The main result we wish to present is the following:

Theorem 1. *If L is a finite semidistributive lattice and $\alpha \in \mathbb{C}(L)$, then $\mathbb{C}(L, \alpha)$ is a semidistributive lattice.*

We call $\mathbb{C}(L, \alpha)$ the semidistributive lattice derived from L and α. Theorem 1 can be lifted to bounded lattices:

Theorem 2. *If L is a finite bounded lattice and $\alpha \in \mathbb{C}(L)$, then $\mathbb{C}(L, \alpha)$ is a bounded lattice.*

We are interested in the explicit computation of semidistributive lattices derived from the Newman lattices of [1]. To this goal, let $\mathcal{S}_n$ be the permutohedron on n letters (i.e., the weak Bruhat order on permutations of n), and let $\mathcal{T}_n$ be the associahedron on $n+1$ letters (i.e., binary trees with $n+1$ leaves and n internal nodes).

Proposition 3. *Let a be an atom of $\mathcal{S}_n$ (resp. of $\mathcal{T}_n$) and consider the cover $\alpha = (\bot, a)$. The following relations hold (up to isomorphism) for $n \geq 2$:*

$$\mathbb{C}(\mathcal{S}_n, \alpha) = \mathcal{S}_{n-1}\,, \qquad\qquad \mathbb{C}(\mathcal{T}_n, \alpha) = \mathcal{T}_{n-1}\,.$$

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The Proposition above shows that $\mathbb{C}(L,\alpha)$ does not depend on the choice of the atom a for L either $\mathcal{S}_n$ or $\mathcal{T}_n$. It is possible, on the other hand, to exhibit a multinomial lattice – not a complemented lattice – and two distinct atoms giving rise to non isomorphic derived lattices.

It might be conjectured that the lattice $\mathbb{C}(L,\alpha)$ is related to the quotient lattice $L/\theta(\alpha_0,\alpha_1)$, where $\theta(\alpha_0,\alpha_1)$ is the congruence generated by the pair (α_0,α_1). Proposition 3 shows that these lattices are not in general isomorphic. Using the characterization of the join dependency relation in permutohedra, see [8, 3.10], it is relatively easy to argue that

$$\mathcal{S}_n/\theta_i = \mathcal{S}_i \times \mathcal{S}_{n-i}\,,$$

where $i = 1,\ldots,n-1$ and $\theta_i = \theta(\perp,(i,i+1))$.

These results are part of a general investigation relating rewrite systems to lattices, following [7] and [1]. The examples at hand have directed us to consider join semidistributive lattices, i.e. lattices satisfying the Horn sentence

$$x \vee y = x \vee z \ \text{ implies } \ x \vee (y \wedge z) = x \vee y\,.$$

There are already many characterization of finite join semidistributive lattices, see for example [4] and [5, Theorem 2.56]. Proposition 4, which is a refinement of the latter characterization, allow us to derive Theorem 1.

To state the Proposition, remark that the projections $(\,\cdot\,)_i : \mathbb{C}(L) \longrightarrow L$, sending $\gamma \in \mathbb{C}(L)$ to γ_i, $i = 0,1$, are order preserving. An order preserving function $f : P \longrightarrow Q$ is said to create pullbacks if whenever $y, z \leq w \in P$ and the meet $x = f(y) \wedge f(z)$ exists in Q, then there exists a unique $x' \in P$ such that $f(x') = x$, and moreover $x' = y \wedge z$.

Proposition 4. *A finite lattice is join semidistributive if and only if the projection $(\,\cdot\,)_0 : \mathbb{C}(L) \longrightarrow L$ creates pullbacks.*

Consideration of pullbacks, i.e. meets of the form $x \wedge y$ where x, y have an upper bound, is suggested by recent work on Cayley lattices of Coxeter groups. We derive Theorem 2 by means of a new characterization – Proposition 6 – of finite lower bounded lattices. The characterization relies on the tools used in [2] to prove that $\mathcal{HH}$ lattices are bounded.

A *hat* in a finite lattice L is a triple $(x,\delta_1,\delta_0) \in L^3$ such that $x \neq \delta_0$ and $(x,\delta_1),(\delta_0,\delta_1) \in \mathbb{C}(L)$. If L is join semidistributive, then there exists a unique $\gamma_1 \in L$ such that $x \wedge \delta_0 \prec \gamma_1 \leq x$. Let $\gamma = (x \wedge \delta_0, \gamma_1)$ and $\delta = (\delta_0,\delta_1)$, we denote this relation by $\gamma \prec^x \delta$, since the covers of the poset $\mathbb{C}(L)$ have exactly this form (provided L is join semidistributive). Note also that γ_0 is the pullback of the hat (x,δ_1,δ_0).

Dually, an *antihat* is a triple $(x,\gamma_0,\gamma_1) \in L^3$ such that $(\gamma_0,x),(\gamma_0,\gamma_1) \in \mathbb{C}(L)$ and $x \neq \gamma_1$. If L is join semidistributive, then there exists a unique $\delta_0 \in L$ such that $x \leq \delta_0 \prec x \vee \gamma_1$. Let $\gamma = (\gamma_0,\gamma_1)$ and $\delta = (\delta_0, x \vee \gamma_1)$, and note this relation by $\gamma \xrightarrow{x} \delta$. Observe that $\gamma \xrightarrow{x} \delta$ implies $\gamma < \delta$ but this might not be a cover.

A *facet* is an interval of the form $[\gamma_0,\delta_1]$, where $\gamma \prec^x \delta$ or $\gamma \xrightarrow{x} \delta$.

Definition 5. Let L be a finite join semidistributive lattice. A function $f : \mathbb{C}(L) \longrightarrow \mathbb{N}$ is:

(i) a *strict lower facet labeling* if $f(\delta) = f(\gamma) < f(\epsilon)$ whenever $\gamma \prec^x \delta$ and $\gamma_1 \leq \epsilon_0 \prec \epsilon_1 \leq x$.

(ii) a *strict upper facet labeling* if $f(\delta) = f(\gamma) < f(\epsilon)$ whenever $\gamma \xrightarrow{x} \delta$ and $x \leq \epsilon_0 \prec \epsilon_1 \leq \delta_0$.

(iii) a *strict facet labeling* if it is both a strict lower facet labeling and a strict upper facet labeling.

Proposition 6. *A finite join semidistributive lattice is lower bounded if and only if it has a strict facet labeling.*

Among the existing characterizations of finite lower bounded lattices, see for example [5, Corollary 2.39, Theorem 2.43] or [6, I.2], Proposition 6 has already shown its use for lattices of combinatorial presentation [2, 3] and, we recall, it is the tool by which Theorem 2 is derived.

References

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