

SENSITIVITY OF m -ARY FUNCTIONS AND LOW DEGREE PARTITIONS OF HAMMING GRAPHS

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ABSTRACT. The study of complexity measures of Boolean functions led Nisan and Szegedy to state the sensitivity conjecture in 1994, claiming a polynomial relation between degree and sensitivity. This problem remained unsolved until 2019, when Huang proved the conjecture via an equivalent graph theoretical reformulation due to Gotsman and Linial.

We study m -ary functions, i.e., functions $f : T^m \rightarrow T$ where $T \subseteq \mathbb{C}$ is a finite alphabet of cardinality $|T| = m$. We extend the notions of *degree* $\deg(f)$ and *sensitivity* $s(f)$ to m -ary functions and show $s(f) \in O(\deg(f)^2)$. This generalizes results of Nisan and Szegedy. Conversely, we introduce the m -ary sensitivity conjecture, claiming a polynomial upper bound for $\deg(f)$ in terms of $s(f)$. Analogously to results of Gotsman and Linial, we provide a formulation of the conjecture in terms of imbalanced partitions of Hamming graphs into low degree subgraphs. Combining this with ideas of Chung, Füredi, Graham and Seymour, we show that for any prime p the bound in the p -ary sensitivity conjecture has to be at least quadratic: there exist p -ary functions f of arbitrarily large degree and $\deg(f) \in \Omega(s(f)^2)$.

1. INTRODUCTION

Many computational combinatorial problems can be thought of as evaluating a *Boolean function*, i.e., a function $f : \{0, 1\}^n \rightarrow \{0, 1\}$. Hence, determining the complexity of (evaluating) a Boolean function is one of the most fundamental algorithmic problems of Theoretical Computer Science. This framework was proposed in the 60s [26] and first measures of complexity were studied and compared in the 80s [20] and 90s [28]. Nowadays, Boolean functions intervene in such diverse areas as Cryptography [5], Quantum Computing [1, 21] and Deep Learning [16, 27].

A variety of complexity measures of Boolean functions has been considered in the literature, see [4] for a survey. Already since the 90s it has been known that most deterministic measures are polynomially equivalent except (back then) possibly the sensitivity. This led Nisan and Szegedy [28] to the *sensitivity conjecture*, which claims that the sensitivity is polynomially equivalent to the other measures. A first step is an equivalence theorem due to Gotsman and Linial [13], which translates the sensitivity conjecture to a problem in graph theory. In 2019, Huang [17] gave a one-page proof for this statement based on elementary linear algebra arguments. Namely, he showed that any induced subgraph on more than half of the vertices of the n -dimensional hypercube has maximum degree at least \sqrt{n} . See [3] for an extended summary of this story (in Spanish).

Studying Huang's result in other families of graphs is an active area of research. His result has been generalized to Cartesian powers of directed cycles (Tikaradze, [33]), Cartesian powers of paths (Zeng and Hou, [36]), and other Cartesian and semistrong products of graphs (Hong, Lai and Liu, [15]). Alon and Zheng showed that Huang's result implies a similar result for Cayley graphs over \mathbb{Z}_2^n [2], which was later generalized to arbitrary abelian Cayley graphs by Potechin and Tsang [29], and to Cayley graphs of Coxeter groups and expander graphs by García-Marco and Knauer [14]. Similar results on Kneser graphs have been developed by Frankl and Kupavskii [12], and Chau, Ellis, Friedgut and Lifshitz [6]. On the negative side, infinite families of Cayley graphs with low-degree induced subgraphs on many (more than the independence number) vertices were constructed by Lehner and Verret [22], and García-Marco and Knauer [14]. The existence of such subgraphs in Hamming graphs $H(n, m)$ is the topic of several recent works, see [10, 14, 30, 32].

The study of sensitivity has also been extended beyond Boolean functions without going through graphs. Dafni, Filmus, Lifshitz, Lindzey and Vinyals [9] consider $f : \mathcal{X} \rightarrow \{0, 1\}$ on different domains

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such as the symmetric group $\mathcal{X} = S_n$. They show that in this case all classical complexity measures of Boolean functions can also be defined and are polynomially equivalent. In particular, they prove the analogous result to the sensitivity conjecture.

Another natural generalization of Boolean functions are m -ary functions, i.e., functions $f : T^m \rightarrow T$ where $T \subseteq \mathbb{C}$ is a finite set of cardinality m . For p prime, p -ary functions have been studied in relation to combinatorial structures such as association schemes [35] and strongly regular graphs [7, 18, 25]. Moreover, they provide connections to Cryptography and Coding Theory, see [24]. Going back to the initial motivation of Boolean functions, m -ary functions can be viewed as combinatorial problems over a multi-valued logic.

Our results: The aim of the present paper is to study the sensitivity of m -ary functions. In Section 2 we define complexity measures of an m -ary function f , namely the sensitivity $s(f)$, the block sensitivity $bs(f)$ and the degree $\deg(f)$. After this, in Section 3 we prove a polynomial upper bound for the sensitivity of an m -ary function in terms of its degree; more precisely, we prove

Theorem 1.1. *For every m -ary function f , we have $s(f) \leq 2(m-1)^3 \deg(f)^2$.*

As in the Boolean case, the most difficult question seems to be the proof of a polynomial bound in the other direction, prompting us to put forward the

Conjecture 1.2 (m -ary Sensitivity Conjecture). *For every m there exists a constant c such that $\deg(f) \in \mathcal{O}(s(f)^c)$ for every m -ary function f .*

As a first step towards the conjecture, we generalize the equivalence theorem by Gotsman and Linial [13]. This leads to a graph theoretical formulation of Conjecture 1.2 in terms of the Hamming graph $H(n, m)$, see Theorem 4.9. Even if Theorem 4.9 is more technical than the Boolean case, we believe that it will be crucial in an eventual resolution of Conjecture 1.2. As of now, Theorem 4.9 has two main consequences. First, we present a far generalization of a result by Chung, Füredi, Graham and Seymour [8]. For every prime p , we present a construction proving that the polynomial upper bound in Conjecture 1.2 has to be at least quadratic for p -ary functions.

Theorem 1.3. *For every set $T \subseteq \mathbb{C}$ of prime cardinality $|T| = p$ and every positive integer D , there exists a p -ary function $f : T^n \rightarrow T$ with $\deg(f) > D$ and $s(f)^2/(p-1)^3 \leq \deg(f)$.*

Second, in Section 6, based on Theorem 4.9 we obtain a natural graph theoretical strengthening of Conjecture 1.2 in terms of imbalanced partitions of the Hamming graph $H(n, m)$ into induced subgraphs of low maximum degree, see Conjecture 6.1. Already the first open case is non-trivial and tempting, where Δ denotes the maximum degree of a graph.

Conjecture 1.4. *There exists $\mu > 0$ such that for any partition of the Hamming graph $H(n, 3)$ into three (possibly empty) induced subgraphs H_1, H_2, H_3 not all of the same order, it holds:*

$$\max\{\Delta(H_1), \Delta(H_2), \Delta(H_3)\} \in \Omega(n^\mu) .$$

2. SOME COMPLEXITY MEASURES OF m -ARY FUNCTIONS

For the entire paper let m, n be positive integers and $T \subseteq \mathbb{C}$ a set of size m . For convenience, we consider m -ary functions over $T \subseteq \mathbb{C}$, i.e., maps $f : T^n \rightarrow T$. In this section we introduce several complexity measures of m -ary functions, which naturally extend those of Boolean functions. We begin with the first ingredient of the sensitivity conjecture.

Definition 2.1 (Degree). A polynomial $F \in \mathbb{C}[x_1, \dots, x_n]$ represents an m -ary function $f : T^n \rightarrow T$ if $F(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x} \in T^n$. The *degree* of f is $\deg(f) = \min\{\deg(F) \mid F \in \mathbb{C}[x_1, \dots, x_n] \text{ represents } f\}$, the smallest degree of a polynomial representing f .

The following polynomial interpolation result shows a way to compute the degree of an m -ary function:

Proposition 2.2. *Let $T \subseteq \mathbb{C}$ be a set of size m and $f : T^n \rightarrow T$ an m -ary function. There is a unique polynomial $F \in \mathbb{C}[x_1, \dots, x_n]$ of degree at most $m-1$ in each variable representing f . Moreover, $\deg(f) = \deg(F)$.*

Proof. Let $f : T^n \rightarrow T$ be an m -ary function. By, e.g, [19, Chapter 6.6] or [31, Section 19], there is a unique polynomial $F \in \mathbb{C}[x_1, \dots, x_n]$ of degree at most $m-1$ in each variable representing f . Let now G be a polynomial representing f and consider the polynomial ideal $I = \langle p(x_1), \dots, p(x_n) \rangle \subseteq \mathbb{C}[x_1, \dots, x_n]$

with $p(x) = \prod_{t \in T} (x - t)$. One can reduce G modulo the ideal I to get a polynomial H of degree at most $m - 1$ in each variable such that $G - H \in I$ and $\deg(G) \geq \deg(H)$. Since all the polynomials in I vanish on T^n , we have that H also represents f ; thus $F = H$ and $\deg(G) \geq \deg(F)$. \square

It is worth pointing out that there might be several polynomials of degree $\deg(f)$ representing f . However, only one of them can have degree at most $m - 1$ in each variable. For example, consider the set $T = \{1, \varepsilon, \varepsilon^2\}$, with ε a primitive third root of unity, and the 3-ary function $f : T^3 \rightarrow T$ defined as $f(a, b, c) = a^2b$ for all $a, b, c \in T$. Clearly, the polynomial $F(x, y, z) = x^2y \in \mathbb{C}[x, y, z]$ represents f and has degree at most 2 in each variable and, by Proposition 2.2, $\deg(f) = \deg(F) = 3$. Also the polynomial $G(x, y, z) = x^2y + z^3 - 1$ of degree 3 represents f , but G has degree 3 in the variable z .

The other ingredient of the sensitivity conjecture is:

Definition 2.3 (Sensitivity). The *local sensitivity* $s_{\mathbf{x}}(f)$ of an m -ary function f at a vector $\mathbf{x} \in T^n$ is the number of elements $\mathbf{y} \in T^n$ which differ from \mathbf{x} in exactly one entry and $f(\mathbf{x}) \neq f(\mathbf{y})$. The *sensitivity* of f is $s(f) = \max_{\mathbf{x} \in T^n} \{s_{\mathbf{x}}(f)\}$.

Consider $T = \{t_0, \dots, t_{m-1}\}$ a set with m elements, $f : T^n \rightarrow T$ an m -ary function, and $\mathbf{x} = (t_{i_1}, \dots, t_{i_n}) \in T^n$. For every multisubset S of $[n] := \{1, \dots, n\}$, we denote by $\text{mult}_S(j)$ the multiplicity of the element j in S and by \mathbf{x}^S the vector obtained from \mathbf{x} by replacing its j -th entry t_{i_j} with $t_{i_j + \text{mult}_S(j)} \in T$ for all $1 \leq j \leq n$. Here the indices are taken modulo m .

For example, let $T = \{0, 1, 2\}$, $n = 3$, and $\mathbf{x}_1 = (0, 0, 0)$, $\mathbf{x}_2 = (1, 0, 2) \in T^3$. If $S_1 = \{1, 1, 2\}$, then $\text{mult}_{S_1}(1) = 2$, $\text{mult}_{S_1}(2) = 1$, $\text{mult}_{S_1}(3) = 0$, and hence $\mathbf{x}_1^{S_1} = (2, 1, 0)$ and $\mathbf{x}_2^{S_1} = (0, 1, 2)$. If $S_2 = \{1\}$, then $\mathbf{x}_1^{S_2} = (1, 0, 0)$ and $\mathbf{x}_2^{S_2} = (2, 0, 2)$.

We say that an element t belongs to a multiset A , and write $t \in A$, whenever $\text{mult}_A(t) > 0$. We say that two multisubsets A and B of $[n]$ are disjoint if they do not contain common elements, regardless of their multiplicities. In the previous example, $S_1 = \{1, 1, 2\}$ and $S_2 = \{1\}$ are not disjoint, since both contain the element 1.

Definition 2.4 (Block sensitivity). The *local block sensitivity* $bs_{\mathbf{x}}(f)$ of an m -ary function f at $\mathbf{x} \in T^n$ is the maximum k for which there exist k pairwise disjoint multisubsets B_1, \dots, B_k of $[n]$ (the *sensitive blocks*) such that $f(\mathbf{x}) \neq f(\mathbf{x}^{B_i})$ for all $i \in \{1, \dots, k\}$. The *block sensitivity* of f is $bs(f) = \max_{\mathbf{x} \in T^n} \{bs_{\mathbf{x}}(f)\}$.

Remark 2.5. Definitions 2.3 and 2.4 imply that $0 \leq \frac{s(f)}{m-1} \leq bs(f) \leq n$. This is because it is possible to change every entry of a vector $\mathbf{x} \in T^n$ in $m - 1$ different ways, but only one of them can contribute to the computation of the block sensitivity (because the sensitive blocks are pairwise disjoint).

Example 2.6. Let $T = \{0, 1, 2\}$, $n = 2$, and $f : T^n \rightarrow T$ the 3-ary function given by

$$\begin{aligned} f^{-1}(0) &= \{(0, 1)\}, \\ f^{-1}(1) &= \{(0, 2), (1, 2), (2, 2)\}, \\ f^{-1}(2) &= \{(0, 0), (1, 0), (1, 1), (2, 0), (2, 1)\}. \end{aligned}$$

To compute the degree of f , following Proposition 2.2, we are going to find a polynomial $F \in \mathbb{C}[x_1, x_2]$ of degree at most 2 in each variable which coincides with f at every $\mathbf{x} \in \{0, 1, 2\}^2$. We can write F as

$$F(x_1, x_2) = a_{00} + a_{10}x_1 + a_{11}x_1x_2 + a_{01}x_2 + a_{20}x_1^2 + a_{21}x_1^2x_2 + a_{22}x_1^2x_2^2 + a_{12}x_1x_2^2 + a_{02}x_2^2,$$

with $a_{ij} \in \mathbb{C}$. Imposing that F represents f leads us to the following system of linear equations:

$$\left\{ \begin{array}{l} a_{00} = 2 \\ a_{00} + \quad \quad \quad + a_{01} + \quad \quad \quad + a_{02} = 0 \\ a_{00} + \quad \quad \quad + 2a_{01} + \quad \quad \quad + 4a_{02} = 1 \\ a_{00} + a_{10} + \quad \quad \quad + a_{20} = 2 \\ a_{00} + 2a_{10} + \quad \quad \quad + 4a_{20} = 2 \\ a_{00} + a_{10} + 2a_{11} + 2a_{01} + a_{20} + 2a_{21} + 4a_{22} + 4a_{12} + 4a_{02} = 1 \\ a_{00} + 2a_{10} + 4a_{11} + 2a_{01} + 4a_{20} + 8a_{21} + 16a_{22} + 8a_{12} + 4a_{02} = 1 \\ a_{00} + a_{10} + a_{11} + a_{01} + a_{20} + a_{21} + a_{22} + a_{12} + a_{02} = 2 \\ a_{00} + 2a_{10} + 2a_{11} + a_{01} + 4a_{20} + 4a_{21} + 4a_{22} + 2a_{12} + a_{02} = 2 \end{array} \right. .$$

Solving the system we get

$$F(x_1, x_2) = 2 + 6x_1x_2 - \frac{7}{2}x_2 - 2x_1^2x_2 + x_1^2x_2^2 - 3x_1x_2^2 + \frac{3}{2}x_2^2,$$

and hence $\deg(f) = 4$.

By Remark 2.5 one has that $s(f) \leq 4$. If we consider now $\mathbf{x} = (0, 1)$, then

$$s_{\mathbf{x}}(f) = |\{(0, 0), (0, 2), (1, 1), (2, 1)\}| = 4,$$

and this implies that $s(f) = 4$.

Finally, we compute the block sensitivity of f . By Remark 2.5, $bs(f) \leq 2$. If we consider $\mathbf{x}_1 = (1, 0)$, then we cannot have a sensitive block consisting only of 1. Consequently, every sensitive block must contain 2 and this yields $bs_{\mathbf{x}_1}(f) = 1$ taking for example $B = \{2, 2\}$ as the sensitive block. In contrast, if we consider $\mathbf{x}_2 = (0, 1)$, then $B_1 = \{1\}$ and $B_2 = \{2\}$ are two disjoint sensitive blocks, and $bs_{\mathbf{x}_2}(f) = 2$. This allows us to conclude that $bs(f) = 2$.

As mentioned in the introduction, the degree, the sensitivity and the block sensitivity are polynomially equivalent for Boolean functions. Before studying such properties for m -ary functions, a comment on the choice of the set $T \subseteq \mathbb{C}$ is in order. When working with Boolean functions, the set T is traditionally $T = \{0, 1\}$, although another commonly used representation is $T = \{-1, 1\}$. Most complexity measures of Boolean functions (including degree, sensitivity and block sensitivity) do not depend on the choice of the two-element-set $T \subseteq \mathbb{C}$. If one neglects constant factors, the same holds for general m -ary functions.

Proposition 2.7. *Let $T = \{t_0, \dots, t_{m-1}\}, T' = \{t'_0, \dots, t'_{m-1}\} \subseteq \mathbb{C}$ sets of size m . If $f : T^n \rightarrow T$ is an m -ary function, then $g : (T')^n \rightarrow T'$ obtained from f by identifying t_i with t'_i is an m -ary function and:*

- $s(f) = s(g)$,
- $bs(f) = bs(g)$,
- $(m-1)^{-2} \deg(f) \leq \deg(g) \leq (m-1)^2 \deg(f)$.

Proof. It is clear that $s(f) = s(g)$ and $bs(f) = bs(g)$. To see the claim about the degree, let F and G be polynomials representing f and g with $\deg(f) = \deg(F)$ and $\deg(g) = \deg(G)$, respectively. Denote by $p, q \in \mathbb{C}[x]$ the unique univariate polynomials of degree at most $m-1$ such that $p(t_i) = t'_i$ and $q(t'_i) = t_i$ for all i . Then

$$F(x_1, \dots, x_n) = q(G(p(x_1), \dots, p(x_n))) \text{ for all } (x_1, \dots, x_n) \in T^n,$$

and $\deg(f) = \deg(F) \leq (m-1)^2 \deg(G) = (m-1)^2 \deg(g)$. Similarly, one has that

$$G(x_1, \dots, x_n) = p(F(q(x_1), \dots, q(x_n))) \text{ for all } (x_1, \dots, x_n) \in (T')^n,$$

and $\deg(g) = \deg(G) \leq (m-1)^2 \deg(F) = (m-1)^2 \deg(f)$. \square

As a consequence, when changing the set T the degree can only change by a constant factor, and hence, for proving equivalence between complexity measures, one can consider any set $T \subseteq \mathbb{C}$ of cardinality m .

3. AN UPPER BOUND FOR THE SENSITIVITY IN TERMS OF THE DEGREE

The goal of this section is to provide an upper bound for the sensitivity of an m -ary function in terms of its degree. This is achieved in Theorem 1.1, which follows from generalizing [28, Lemma 7], a result for Boolean functions, to the m -ary case:

Lemma 3.1. *Let $T = \{t_0, t_1, \dots, t_{m-1}\} \subseteq \mathbb{C}$ and let $f : T^n \rightarrow T$ be an m -ary function. Then,*

$$(m-1) \deg(f) \geq \sqrt{\frac{bs(f)}{2}}.$$

Proof. Denote $k := bs(f)$ and take $\mathbf{t} = (t_{i_1}, t_{i_2}, \dots, t_{i_n}) \in T^n$ such that $bs(f) = bs_{\mathbf{t}}(f)$. We assume without loss of generality that $f(\mathbf{t}) = t_0$. Consider also S_1, \dots, S_k a collection of pairwise disjoint multisubsets of $[n]$ such that $f(\mathbf{t}) \neq f(\mathbf{t}^{S_i})$ for all $i \in \{1, \dots, k\}$.

We define $g : \{0, 1\}^k \rightarrow \{0, 1\}$ as follows: we consider the polynomial $P \in \mathbb{C}[x]$ of degree $m-1$ such that $P(t_0) = 0$ and $P(t_i) = 1$ for $i \in \{1, \dots, m-1\}$. For $\mathbf{y} := (y_1, \dots, y_k) \in \{0, 1\}^k$ we take the multiset $S_{\mathbf{y}} := \cup_{y_i=1} S_i$ and define

$$g(\mathbf{y}) := P(f(\mathbf{t}^{S_{\mathbf{y}}})) .$$

The following facts hold:

- $g : \{0, 1\}^k \rightarrow \{0, 1\}$ is a Boolean function,
- $g(\mathbf{0}) = P(f(\mathbf{t})) = P(t_0) = 0$, and
- if $\mathbf{y} \in \{0, 1\}^k$ has all its entries equal to 0 except the i -th one, which is equal to 1, then $g(\mathbf{y}) = P(f(\mathbf{t}^{S_{\mathbf{y}}})) = P(f(\mathbf{t}^{S_i})) = 1$.

By [28, Lemma 5], we have that

$$(1) \quad \deg(g) \geq \sqrt{\frac{k}{2}} = \sqrt{\frac{bs(f)}{2}}.$$

Moreover, if F denotes a polynomial representing f with $\deg(F) = \deg(f)$, and $\ell_1, \dots, \ell_n \in \mathbb{C}[x_1, \dots, x_k]$ are the unique linear forms such that $(\ell_1(\mathbf{y}), \dots, \ell_n(\mathbf{y})) = \mathbf{t}^{S_{\mathbf{y}}}$ for all $\mathbf{y} \in \{0, 1\}^k$, then the polynomial $G = P(F(\ell_1, \dots, \ell_n)) \in \mathbb{C}[x_1, \dots, x_k]$ represents g . Hence,

$$(2) \quad \deg(g) \leq \deg(G) \leq \deg(P) \cdot \deg(F) = (m-1)\deg(f).$$

The result follows directly from (1) and (2). \square

As a direct consequence of Lemma 3.1 together with $bs(f) \geq \frac{s(f)}{m-1}$ (see Remark 2.5), we have the following theorem:

Theorem 1.1. *For every m -ary function f , we have $s(f) \leq 2(m-1)^3 \deg(f)^2$.*

We conjecture a polynomial relation into the other direction:

Conjecture 1.2 (m -ary Sensitivity Conjecture). *For every m there exists a constant c such that $\deg(f) \in \mathcal{O}(s(f)^c)$ for every m -ary function f .*

As in the Boolean case, graph theory might play an important role for proving the m -ary sensitivity conjecture. In the next section we present a generalization of the Boolean equivalence theorem by Gotsman and Linial, which allows us to reformulate the sensitivity conjecture for m -ary functions in graph theoretical terms.

4. THE EQUIVALENCE THEOREM FOR m -ARY FUNCTIONS

All graphs we consider are simple, undirected and finite. For a graph $G = (V, E)$ and a vertex $v \in V$, we denote by $\deg_G(v)$ the degree of v in G . We denote by $\Delta(G)$ (resp. $\delta(G)$) the maximum (resp. minimum) degree of G . The maximum (resp. minimum) degree of the graph without vertices is $-\infty$ (resp. $+\infty$).

Huang's proof of the sensitivity conjecture for Boolean functions heavily relies on the equivalence theorem by Gotsman and Linial [13]. In this section we present an equivalence theorem for the m -ary case, which generalizes the Boolean one. Before presenting its statement we introduce some definitions, notations and some preliminary results that will be used in the proof.

For convenience, from now on we consider $T = \mathcal{U}_m := \{1, \varepsilon, \varepsilon^2, \dots, \varepsilon^{m-1}\}$ the set of m -th roots of unity, where ε is an m -th primitive root of unity. In the case of Boolean functions, the set $\{0, 1\}^n$ can be seen as the vertex set of the n -dimensional hypercube Q_n . In the context of m -ary functions, a natural generalization is the *Hamming graph* $H(n, m)$ whose vertex set is \mathcal{U}_m^n (or $[0, m-1]^n := \{0, 1, \dots, m-1\}^n$) and where two vertices are adjacent if and only if they differ in exactly one of their entries. This graph can be seen as the n -th Cartesian power of the complete graph on m vertices K_m . Some Hamming graphs are shown in Figure 1. It is easy to see that the Hamming graph $H(n, m)$ is $(m-1)n$ -regular and the product of the entries of every vertex provides a proper m -coloring of it. We denote each set of the resulting m -partition by C_{ε^k} for $k \in [0, m-1]$, i.e.,

$$C_{\varepsilon^k} = \left\{ \mathbf{x} = (x_1, \dots, x_n) \in \mathcal{U}_m^n \mid \prod_{j=1}^n x_j = \varepsilon^k \right\}.$$

One can associate to every m -ary function $f : \mathcal{U}_m^n \rightarrow \mathcal{U}_m$ a partition of the vertex set of $H(n, m)$ (where some parts might be empty) by just considering $V_{\varepsilon^k}(f) := f^{-1}(\varepsilon^k)$ for all $k \in [0, m-1]$. If we denote by H_{ε^k} the induced subgraph of $H(n, m)$ with vertex set $V_{\varepsilon^k}(f)$ for all $k \in [0, m-1]$, then for every vertex $\mathbf{x} \in V_{\varepsilon^k}(f)$ one has that $s_{\mathbf{x}}(f) = (m-1)n - \deg_{H_{\varepsilon^k}}(\mathbf{x})$. Consequently,

$$(3) \quad s(f) = (m-1)n - \min\{\delta(H_{\varepsilon^k}) \mid 0 \leq k \leq m-1\}.$$

Recall that the minimum degree of the empty graph is $+\infty$.

Example 4.1. *Figure 2 shows an example of three induced subgraphs $H_1, H_{\varepsilon}, H_{\varepsilon^2}$ whose vertices partition \mathcal{U}_3^3 . We observe that $\delta(H_1) = \delta(H_{\varepsilon}) = \delta(H_{\varepsilon^2}) = 3$. If we consider the 3-ary function $f : \mathcal{U}_3^3 \rightarrow \mathcal{U}_3$ such that $f^{-1}(\varepsilon^j) = V(H_{\varepsilon^j})$, then $s(f) = 2 \cdot 3 - 3 = 3$.*

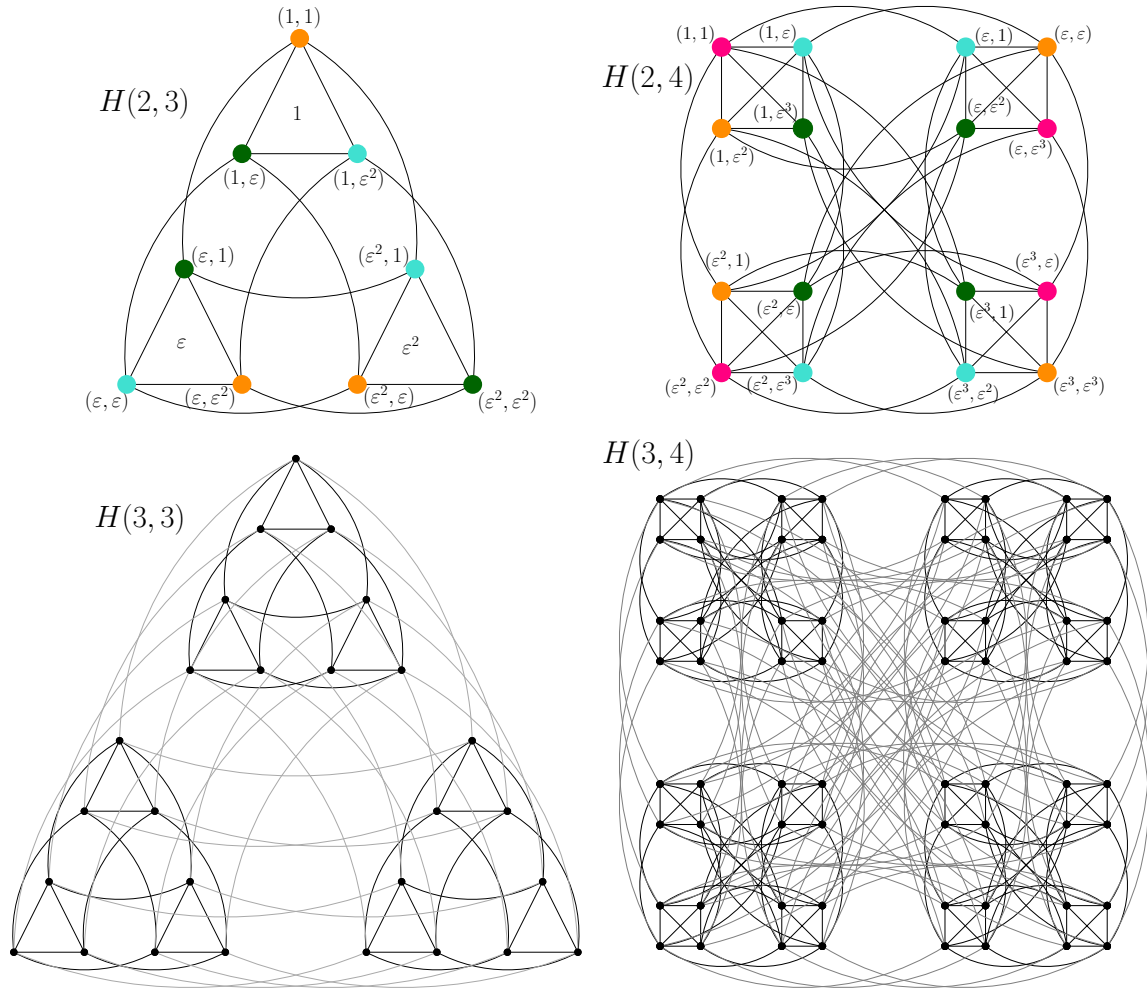


FIGURE 1. Some Hamming graphs, the upper two with their proper 3 and 4-coloring, respectively

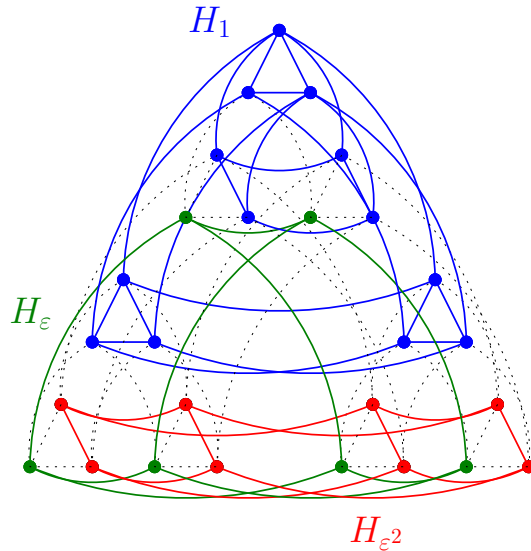


FIGURE 2. Three induced subgraphs of $H(3,3)$ whose vertices partition \mathcal{U}_3^3

Let $f : \mathcal{U}_m^n \rightarrow \mathcal{U}_m$ be an m -ary function. For every $i \in [0, m - 1]$, denote by f_{ϵ^i} the m -ary function given by $f_{\epsilon^i}(\mathbf{x}) = f(\mathbf{x}) \left(\prod_{j=1}^n x_j^i \right)$ for all $\mathbf{x} = (x_1, \dots, x_n) \in \mathcal{U}_m^n$. The following result relates the vertex partitions of $H(n, m)$ associated to f and f_{ϵ} .

Lemma 4.2. Let $f : \mathcal{U}_m^n \rightarrow \mathcal{U}_m$ be an m -ary function. For all $k \in [0, m-1]$ one has:

$$f_\varepsilon^{-1}(\varepsilon^k) = \bigcup_{j \in [0, m-1]} (C_{\varepsilon^j} \cap f^{-1}(\varepsilon^{k-j})).$$

Proof. For all $\mathbf{x} = (x_1, \dots, x_n) \in \mathcal{U}_m^n$ we have that $f_\varepsilon(\mathbf{x}) = f(\mathbf{x}) \cdot (x_1 \cdots x_n)$. Taking $\varepsilon^j := x_1 \cdots x_n \in \mathcal{U}_m$, we have that $\mathbf{x} \in C_{\varepsilon^j}$, and $f_\varepsilon(\mathbf{x}) = \varepsilon^k$ if and only if $f(\mathbf{x}) = \varepsilon^{k-j} \in \mathcal{U}_m$. \square

Definition 4.3. Let $\{V_{\varepsilon^k} \mid k \in [0, m-1]\}$ be a partition of \mathcal{U}_m^n into m subsets (where some of them might be empty). We denote by $\{\rho(V_{\varepsilon^k}) \mid k \in [0, m-1]\}$ the new partition of \mathcal{U}_m^n given by

$$\rho(V_{\varepsilon^k}) := \bigcup_{j \in [0, m-1]} (C_{\varepsilon^j} \cap V_{\varepsilon^{k-j}}) \text{ for all } k \in [0, m-1].$$

Figure 3 illustrates how the sets $\rho(V_{\varepsilon^k})$ can be visualized as a certain cyclic rotation of the vertices of V_{ε^k} through the m -partition given by the C_{ε^j} 's.

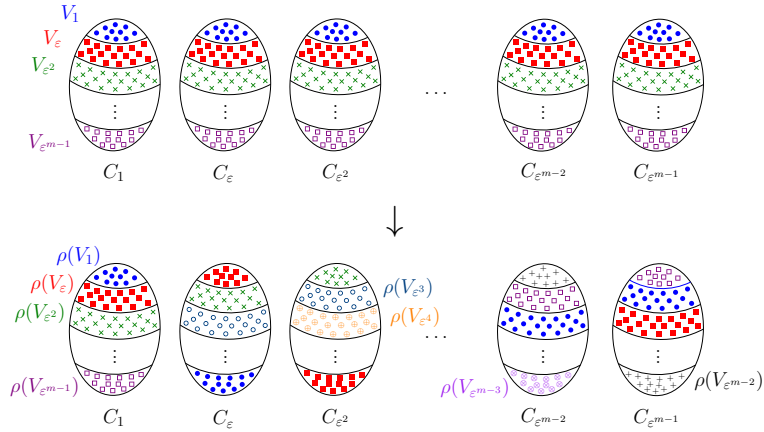


FIGURE 3. Vertices of $\rho(V_{\varepsilon^k})$ as a certain cyclic rotation of the vertices of V_{ε^k} through the partition given by the C_{ε^j} 's

For an m -ary function $f : \mathcal{U}_m^n \rightarrow \mathcal{U}_m$, we consider the partition of $H(n, m)$ with $V_{\varepsilon^k}(f) = f^{-1}(\varepsilon^k)$ for all $k \in [0, m-1]$. By Lemma 4.2, we have that $\rho(V_{\varepsilon^k}(f)) = f_\varepsilon^{-1}(\varepsilon^k)$ for all $k \in [0, m-1]$. Hence, $\{\rho(V_{\varepsilon^k}(f)) \mid k \in [0, m-1]\}$ is the partition of $H(n, m)$ associated to f_ε .

Example 4.4. If we consider the 3-ary function $f : \mathcal{U}_3^3 \rightarrow \mathcal{U}_3$ described in Example 4.1, then Figure 4 shows the partition of $H(3, 3)$ associated to f_ε . It turns out that $|f_\varepsilon^{-1}(1)| = |\rho(V(H_1))| = 8$, $|f_\varepsilon^{-1}(\varepsilon)| = |\rho(V(H_\varepsilon))| = 10$ and $|f_\varepsilon^{-1}(\varepsilon^2)| = |\rho(V(H_{\varepsilon^2}))| = 9$.

Remark 4.5. Note that if we consider $[0, m-1]^n$ as the vertex set of $H(n, m)$, then a proper m -coloring of $H(n, m)$ is given by

$$D_k = \left\{ \mathbf{x} = (x_1, \dots, x_n) \in [0, m-1]^n \mid \sum_{j=1}^n x_j \equiv k \pmod{m} \right\},$$

for $k \in [0, m-1]$.

Moreover, for a given m -ary function $g : [0, m-1]^n \rightarrow [0, m-1]$ and for every $i \in [0, m-1]$, we denote by g_i the m -ary function given by $g_i(\mathbf{x}) = g(\mathbf{x}) + i(\sum_{j=1}^n x_j) \pmod{m}$ for all $\mathbf{x} = (x_1, \dots, x_n) \in [0, m-1]^n$. For each $k \in [0, m-1]$ we take $V_k(g) := g^{-1}(k)$ and denote

$$\rho(V_k(g)) = \bigcup_{j \in [0, m-1]} (D_j \cap V_{((k-j) \bmod m)}(g)).$$

Then, $\rho(V_k(g)) = g_1^{-1}(k)$ for all $k \in [0, m-1]$.

In the proof of the equivalence theorem we use the following lemma, which explicitly describes the constant term of the unique polynomial of degree at most $m-1$ in each variable representing an m -ary function $f : \mathcal{U}_m^n \rightarrow \mathcal{U}_m$. This result depends on the fact that $T = \mathcal{U}_m$ and does not hold for other choices of T .

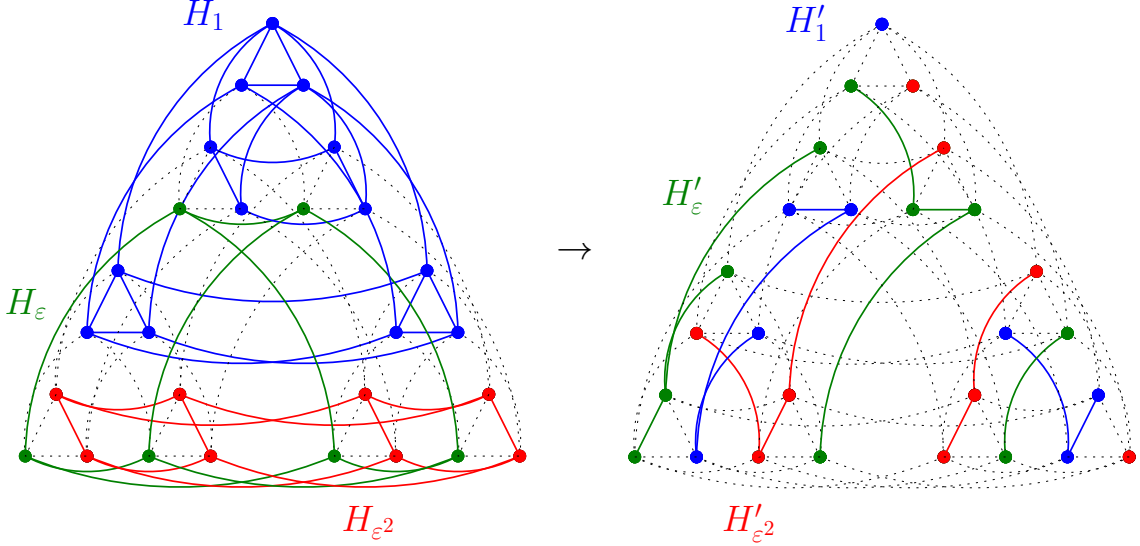


FIGURE 4. Partition associated to f_ε for the 3-ary function f of Example 4.1, where H'_i is the induced subgraph of $H(3, 3)$ on $\rho(V(H_i))$

Lemma 4.6. *If $f : \mathcal{U}_m^n \rightarrow \mathcal{U}_m$ is an m -ary function and $F \in \mathbb{C}[x_1, \dots, x_n]$ is the unique polynomial of degree at most $m-1$ in each variable representing it, then $F(\mathbf{0}) = E(f)$, where $E(f) = \frac{1}{m^n} \sum_{\mathbf{x} \in \mathcal{U}_m^n} f(\mathbf{x})$ is the average value of f on \mathcal{U}_m^n .*

Proof. We denote by $\mathcal{P}([n])_{\leq(m-1)}$ the set of all multisubsets of $[n]$ in which every element has multiplicity at most $m-1$. Since F has degree at most $m-1$ in each variable, it can be written as $F(x_1, \dots, x_n) = \sum_{I \in \mathcal{P}([n])_{\leq(m-1)}} \hat{f}(I) \prod_{i \in I} x_i^{\text{mult}_I(i)}$, where $\hat{f}(I) \in \mathbb{C}$ for all $I \in \mathcal{P}([n])_{\leq(m-1)}$, and we have that

$$\begin{aligned}
E(f) &= \frac{1}{m^n} \sum_{\mathbf{x} \in \mathcal{U}_m^n} f(\mathbf{x}) = \frac{1}{m^n} \sum_{\mathbf{x} \in \mathcal{U}_m^n} F(\mathbf{x}) = \\
&= \frac{1}{m^n} \sum_{\mathbf{x} \in \mathcal{U}_m^n} \left[\sum_{I \in \mathcal{P}([n])_{\leq(m-1)}} \hat{f}(I) \prod_{i \in I} x_i^{\text{mult}_I(i)} \right] = \\
&= \frac{1}{m^n} \left[\sum_{\mathbf{x} \in \mathcal{U}_m^n} \hat{f}(\emptyset) + \sum_{\mathbf{x} \in \mathcal{U}_m^n} \sum_{\substack{I \in \mathcal{P}([n])_{\leq(m-1)} \\ I \neq \emptyset}} \hat{f}(I) \prod_{i \in I} x_i^{\text{mult}_I(i)} \right] = \\
&= \frac{1}{m^n} \left[m^n F(\mathbf{0}) + \sum_{\substack{I \in \mathcal{P}([n])_{\leq(m-1)} \\ I \neq \emptyset}} \hat{f}(I) \sum_{\mathbf{x} \in \mathcal{U}_m^n} \prod_{i \in I} x_i^{\text{mult}_I(i)} \right],
\end{aligned}$$

so it is enough to prove that $\sum_{\mathbf{x} \in \mathcal{U}_m^n} \prod_{i \in I} x_i^{\text{mult}_I(i)} = 0$ for every nonempty $I \in \mathcal{P}([n])_{\leq(m-1)}$.

Take $I \neq \emptyset$, assume without loss of generality that $1 \in I$, and take $M := \text{mult}_I(1)$. Then

$$\begin{aligned}
\sum_{\mathbf{x} \in \mathcal{U}_m^n} \prod_{i \in I} x_i^{\text{mult}_I(i)} &= \sum_{\ell=0}^{m-1} \left(\sum_{(\varepsilon^\ell, x_2, \dots, x_n) \in \mathcal{U}_m^n} \varepsilon^{\ell M} \prod_{i \in I \setminus \{1\}} x_i^{\text{mult}_I(i)} \right) = \\
&= \sum_{\mathbf{y}=(x_2, \dots, x_n) \in \mathcal{U}_m^{n-1}} \left(\prod_{i \in I \setminus \{1\}} x_i^{\text{mult}_I(i)} \right) \left(\sum_{\ell=0}^{m-1} \varepsilon^{\ell M} \right) = 0,
\end{aligned}$$

where the last equality follows from the fact that $\sum_{\ell=0}^{m-1} \mu^\ell = 0$ for every $\mu \in \mathcal{U}_m$. Thus, we conclude the result. \square

Lemma 4.7. *Let $f : \mathcal{U}_m^n \rightarrow \mathcal{U}_m$ be an m -ary function and denote by $F, G \in \mathbb{C}[x_1, \dots, x_n]$ the unique polynomials of degree at most $m-1$ in each variable representing f and $f_{\varepsilon^{m-1}}$, respectively. Then, $F(\mathbf{0})$ coincides with the coefficient of $(x_1 \cdots x_n)^{m-1}$ in G . In particular, $\deg(f_{\varepsilon^{m-1}}) = (m-1)n$ if and only if $E(f) \neq 0$.*

Proof. Clearly $H := (x_1 \cdots x_n)^{m-1}F$ is a polynomial representing $f_{\varepsilon^{m-1}}$, and hence G is the reduction of H modulo the ideal $I = \langle x_1^m - 1, \dots, x_n^m - 1 \rangle$. The result follows from the fact that reducing a polynomial modulo I consists of just taking all the exponents appearing in the expression modulo m . In particular, $\deg(f_{\varepsilon^{m-1}}) = (m-1)n$ if and only if $F(\mathbf{0}) \neq 0$, and the latter is equivalent to $E(f) \neq 0$ by Lemma 4.6. \square

The next result shows how to graph theoretically construct an m -ary function $f : \mathcal{U}_m^n \rightarrow \mathcal{U}_m$ of degree $\deg(f) = (m-1)n$ (the maximum possible degree) with prescribed sensitivity. This result will be particularly useful in the next section.

Proposition 4.8. *Let $m \geq 2$ and $n, s \geq 1$. The following are equivalent:*

- (1) *There is an m -ary function $f : \mathcal{U}_m^n \rightarrow \mathcal{U}_m$ of degree $\deg(f) = (m-1)n$ and sensitivity $s(f) = s$.*
- (2) *There are (possibly empty) induced subgraphs $\{H_{\varepsilon^k} \mid 0 \leq k \leq m-1\}$ of $H(n, m)$ such that their vertex sets partition $V(H(n, m))$, $\sum_{k=0}^{m-1} |\rho(V(H_{\varepsilon^k}))| \varepsilon^k \neq 0$, and*

$$(m-1)n - \min\{\delta(H_{\varepsilon^k}) \mid 0 \leq k \leq m-1\} = s.$$

Proof. For an m -ary function $f : \mathcal{U}_m^n \rightarrow \mathcal{U}_m$, one can consider the family of induced subgraphs $\{H_{\varepsilon^k} \mid 0 \leq k \leq m-1\}$ of $H(n, m)$ such that $V(H_{\varepsilon^k}) = f^{-1}(\varepsilon^k)$. Conversely, for any induced subgraphs $\{H_{\varepsilon^k} \mid 0 \leq k \leq m-1\}$ of $H(n, m)$ such that their vertex sets partition $V(H(n, m))$, one can consider the m -ary function $f : \mathcal{U}_m^n \rightarrow \mathcal{U}_m$ defined as $f(\mathbf{x}) = \varepsilon^k$ if and only if $\mathbf{x} \in V(H_{\varepsilon^k})$.

In both cases, by Formula (3), we have that

$$s(f) = (m-1)n - \min\{\delta(H_{\varepsilon^k}) \mid 0 \leq k \leq m-1\}.$$

Also, taking $g = f_{\varepsilon}$ we have that $f = (f_{\varepsilon})_{\varepsilon^{m-1}} = g_{\varepsilon^{m-1}}$. By Lemma 4.7, it follows that $\deg(f) = (m-1)n$ if and only if $E(g) \neq 0$. Finally, by Lemma 4.2, we have that $E(g) = \frac{1}{m^n} \sum_{k=0}^{m-1} |\rho(V(H_{\varepsilon^k}))| \varepsilon^k$, and the result follows. \square

Now we can prove the equivalence theorem.

Theorem 4.9 (Equivalence theorem for m -ary functions). *Let $h : \mathbb{N} \rightarrow \mathbb{R}$ be a function and m a positive integer. The following are equivalent:*

- (1) *For any collection of (possibly empty) induced subgraphs $\{H_{\varepsilon^k} \mid 0 \leq k \leq m-1\}$ of $H(n, m)$ whose vertex sets partition $V(H(n, m))$ and $\sum_{k=0}^{m-1} |\rho(V(H_{\varepsilon^k}))| \varepsilon^k \neq 0$, it holds*

$$(m-1)n - \min\{\delta(H_{\varepsilon^k}) \mid 0 \leq k \leq m-1\} \geq h(n).$$

- (2) *For any m -ary function $f : \mathcal{U}_m^n \rightarrow \mathcal{U}_m$, $s(f) \geq h\left(\frac{1}{m-1} \deg(f)\right)$.*

Proof. First of all, we transform statement (1) into the equivalent statement (1') concerning m -ary functions:

- (1') *For any m -ary function $f : \mathcal{U}_m^n \rightarrow \mathcal{U}_m$ with $E(f) \neq 0$, there is a vector $v \in \mathcal{U}_m^n$ such that $h(n) \leq s_v(f_{\varepsilon^{m-1}})$.*

We begin by proving that (1') \Rightarrow (1). Let $H_1, H_{\varepsilon}, \dots, H_{\varepsilon^{m-1}}$ be m induced subgraphs of $H(n, m)$ such that their vertex sets partition \mathcal{U}_m^n and $\sum_{k=0}^{m-1} |\rho(V(H_{\varepsilon^k}))| \varepsilon^k \neq 0$.

We define the m -ary function $f : \mathcal{U}_m^n \rightarrow \mathcal{U}_m$ given by $f^{-1}(\varepsilon^k) = V(H_{\varepsilon^k})$ for all $k \in [0, m-1]$. Then, by Lemma 4.2 we have that $f_{\varepsilon^{-1}(\varepsilon^k)} = \rho(V(H_{\varepsilon^k}))$ for all $k \in [0, m-1]$. Moreover,

$$0 \neq \sum_{k=0}^{m-1} |\rho(V(H_{\varepsilon^k}))| \varepsilon^k = \sum_{\mathbf{x} \in \mathcal{U}_m^n} f_{\varepsilon}(\mathbf{x}) = m^n E(f_{\varepsilon}),$$

being $E(f_{\varepsilon})$ the average value of f_{ε} on \mathcal{U}_m^n . Then $E(f_{\varepsilon}) \neq 0$ and, by (1'), there is a vector $v \in \mathcal{U}_m^n$ satisfying $h(n) \leq s_v((f_{\varepsilon})_{\varepsilon^{m-1}}) = s_v(f)$.

Now, by Equation (3),

$$(m-1)n - \min\{\delta(H_{\varepsilon^k}) \mid 0 \leq k \leq m-1\} = s(f) \geq s_v(f) \geq h(n).$$

Now we prove that (1) \Rightarrow (1'). Let $f : \mathcal{U}_m^n \rightarrow \mathcal{U}_m$ be an m -ary function with $E(f) \neq 0$. We define H_{ε^k} as the induced subgraph of $H(n, m)$ on $(f_{\varepsilon^{m-1}})^{-1}(\varepsilon^k)$ for all $k \in [0, m-1]$. By Lemma 4.2, $\rho(V(H_{\varepsilon^k})) = ((f_{\varepsilon^{m-1}})_{\varepsilon})^{-1}(\varepsilon^k) = f^{-1}(\varepsilon^k)$ for all $k \in [0, m-1]$, and hence $\sum_{i=0}^{m-1} |\rho(V(H_{\varepsilon^i}))| \varepsilon^i = m^n E(f) \neq 0$.

By (1), we get that

$$\min\{\delta(H_{\varepsilon^k}) \mid 0 \leq k \leq m-1\} \leq (m-1)n - h(n),$$

so there must be some $\varepsilon^j \in \mathcal{U}_m$ and a vertex $v \in V(H_{\varepsilon^j})$ such that $\deg_{H_{\varepsilon^j}}(v) \leq (m-1)n - h(n)$. Furthermore, since $V(H_{\varepsilon^j}) = (f_{\varepsilon^{m-1}})^{-1}(\varepsilon^j)$ for all $j \in [0, m-1]$, we have that $\deg_{H_{\varepsilon^j}}(v) = (m-1)n - s_v(f_{\varepsilon^{m-1}})$. Hence $(m-1)n - s_v(f_{\varepsilon^{m-1}}) \leq (m-1)n - h(n)$, and we conclude that $h(n) \leq s_v(f_{\varepsilon^{m-1}})$.

We now rewrite statement (2) in an equivalent way:

(2') For any m -ary function $f : \mathcal{U}_m^n \rightarrow \mathcal{U}_m$, $s(f) < h(n)$ implies that $\deg(f) < (m-1)n$.

We start by showing that (2) \Rightarrow (2'). Let f be an arbitrary m -ary function with $\deg(f) \geq (m-1)n$. Then we have that $\deg(f) = (m-1)n$ (because $\deg(f) \leq (m-1)n$). Let us prove that $s(f) \geq h(n)$. By (2), we have that $s(f) \geq h\left(\frac{1}{m-1}\deg(f)\right)$, and thus

$$s(f) \geq h\left(\frac{1}{m-1}\deg(f)\right) = h\left(\frac{1}{m-1}(m-1)n\right) = h(n).$$

Now we prove that (2') \Rightarrow (2) by contradiction. Assume that (2') holds and that there exists an m -ary function f with $s(f) < h\left(\frac{1}{m-1}\deg(f)\right)$. Then, by (2'), $\deg(f) < (m-1)\frac{1}{m-1}\deg(f) = \deg(f)$, a contradiction.

At this point, it is enough to show that statements (1') and (2') are equivalent for any function $h : \mathbb{N} \rightarrow \mathbb{R}$ and any natural number m .

(1') For any m -ary function $f : \mathcal{U}_m^n \rightarrow \mathcal{U}_m$ with $E(f) \neq 0$, there is a vector $v \in \mathcal{U}_m^n$ satisfying $h(n) \leq s_v(f_{\varepsilon^{m-1}})$.

(2') For any m -ary function $f : \mathcal{U}_m^n \rightarrow \mathcal{U}_m$, $s(f) < h(n)$ implies that $\deg(f) < (m-1)n$.

We begin by proving (2') \Rightarrow (1'). Let us see that $E(f) = 0$ for any m -ary function f such that $h(n) > s_v(f_{\varepsilon^{m-1}})$ for all $v \in \mathcal{U}_m^n$. We have that $s(f_{\varepsilon^{m-1}}) = \max_{v \in \mathcal{U}_m^n} s_v(f_{\varepsilon^{m-1}}) < h(n)$, and by (2') this implies that $\deg(f_{\varepsilon^{m-1}}) < (m-1)n$. Then, by Lemma 4.7 we are done.

It only remains to prove (1') \Rightarrow (2'). Since $\deg(f) \leq (m-1)n$ for every m -ary function f , $\deg(f) < (m-1)n$ is equivalent to $\deg(f) \neq (m-1)n$. Take an m -ary function f with $\deg(f) = (m-1)n$, and let us prove that $s(f) \geq h(n)$. Taking $g := f_{\varepsilon}$, we have that $g_{\varepsilon^{m-1}} = f$. Hence, applying Lemma 4.7 to g , we have that $E(g) = E(f_{\varepsilon}) \neq 0$. Then, by (1') there is a vector $v \in \mathcal{U}_m^n$ satisfying that $h(n) \leq s_v((f_{\varepsilon})_{\varepsilon^{m-1}}) = s_v(f)$, and we directly conclude that $s(f) = \max_{\mathbf{x} \in \mathcal{U}_m^n} s_{\mathbf{x}}(f) \geq s_v(f) \geq h(n)$. \square

Example 4.10. The three induced subgraphs $H_1, H_{\varepsilon}, H_{\varepsilon^2}$ whose vertices partition \mathcal{U}_3^3 and were studied in Examples 4.1 and 4.4 satisfy that $\sum_{k=0}^2 |\rho(V(H_{\varepsilon^k}))| \varepsilon^k = 9\varepsilon^2 + 10\varepsilon + 8 = \varepsilon - 1 \neq 0$, and $2 \cdot 3 - \min\{\delta(H_1), \delta(H_{\varepsilon}), \delta(H_{\varepsilon^2})\} = 3$. As a consequence, if there exists $h : \mathbb{N} \rightarrow \mathbb{R}$ satisfying (any of) the conditions of Theorem 4.9 for 3-ary functions, then $h(3) \leq 3$.

Remark 4.11. Although the equivalence theorem for m -ary functions looks different from its Boolean version, it is a generalization of it. Indeed, if $m = 2$, then $\mathcal{U}_m = \{-1, 1\}$, and if one denotes H_1' (respect. H_{-1}') the induced subgraphs with vertices $\rho(V(H_1))$ (respect. $\rho(V(H_{-1}))$), in Theorem 4.9 one has that

$$n - \min\{\delta(H_1), \delta(H_{-1})\} = \max\{\Delta(H_1'), \Delta(H_{-1}')\}$$

(see Figure 5), and the condition $\sum_{i=0}^{m-1} |\rho(V(H_{\varepsilon^i}))| \varepsilon^i \neq 0$ is translated to $|V(H_1')| \neq |V(H_{-1}')|$. Hence, for $m = 2$ and taking $G := H_1'$, condition (1) can be equivalently restated as:

For any induced subgraph G of Q_n such that $|V(G)| \neq 2^{n-1}$, then $\max\{\Delta(G), \Delta(Q_n - G)\} \geq h(n)$, where $Q_n - G$ denotes the induced subgraph of Q_n with $V(Q_n - G) = V(Q_n) \setminus V(G)$.

The equivalence theorem allows us to translate the sensitivity conjecture for m -ary functions into an equivalent graph theoretical problem.

Conjecture 4.12. For every positive integer m there exists $\mu > 0$ such that for any collection of (possibly empty) induced subgraphs $H_1, H_{\varepsilon}, \dots, H_{\varepsilon^{m-1}}$ of $H(n, m)$ whose vertex sets partition \mathcal{U}_m^n and $\sum_{k=0}^{m-1} |\rho(V(H_{\varepsilon^k}))| \varepsilon^k \neq 0$, it holds

$$(m-1)n - \min\{\delta(H_{\varepsilon^k}) \mid 0 \leq k \leq m-1\} \in \Omega(n^{\mu}).$$

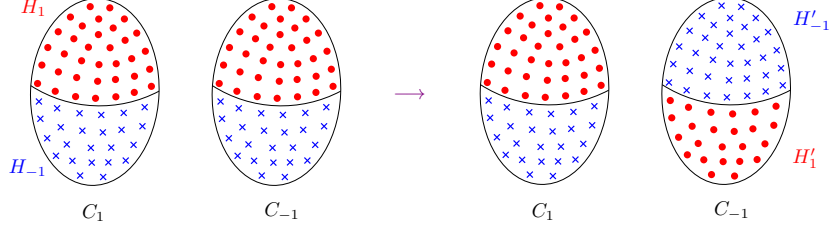


FIGURE 5. Theorem 4.9 generalizes Gotsman-Linial's Boolean equivalence theorem

Remark 4.13. Note that when $m = p$ is a prime number, the m -th cyclotomic polynomial is $\Phi_p(x) = x^{p-1} + x^{p-2} + \dots + x + 1 \in \mathbb{Q}[x]$. Hence, $\sum_{i=0}^{p-1} a_i \varepsilon^i = 0$ with $a_0, \dots, a_{p-1} \in \mathbb{Q}$ if and only if $a_0 = \dots = a_{p-1}$. As a consequence, when m is a prime number, the condition “ $\sum_{k=0}^{m-1} |\rho(V(H_{\varepsilon^k}))| \varepsilon^k \neq 0$ ” in the statement of Conjecture 4.12 can be replaced by the equivalent condition “there exists $k \in [0, p-1]$ such that $|\rho(V(H_{\varepsilon^k}))| \neq p^{n-1}$ ”.

In the next section, we prove that if m is a prime number and Conjecture 4.12 holds for m -ary functions, then $\mu \leq 1/2$.

5. QUADRATIC SEPARATION BETWEEN THE SENSITIVITY AND THE DEGREE

Chung, Füredi, Graham and Seymour [8] provided an induced subgraph of the n -dimensional hypercube with $2^{n-1} + 1$ vertices whose maximum degree is strictly smaller than $\sqrt{n} + 1$. By Gotsman-Linial's equivalence theorem, this resulted in the existence of Boolean functions with $s(f) < \sqrt{\deg(f)} + 1$. Huang [17] later proved that $s(f) \geq \sqrt{\deg(f)}$ for every Boolean function f . The goal of this section is to extend the construction in [8] to the Hamming graph $H(n, m)$ and use the equivalence theorem for m -ary functions to produce m -ary functions with low sensitivity and high degree. For convenience, in this section we work with the set $[0, m-1]$ instead of $\{1, \varepsilon, \dots, \varepsilon^{m-1}\}$; consequently, we denote the vertex set of $H(n, m)$ by $[0, m-1]^n$.

Let $m, n \geq 2$. There is a natural bijection between the set $\mathcal{P}([n])_{\leq (m-1)}$ of multisubsets of $[n]$ with multiplicity at most $m-1$ and $[0, m-1]^n$. This bijection sends $I \in \mathcal{P}([n])_{\leq (m-1)}$ to the n -tuple $(\text{mult}_I(1), \dots, \text{mult}_I(n)) \in [0, m-1]^n$, and lets us identify the vertices of $H(n, m)$ with elements of $\mathcal{P}([n])_{\leq (m-1)}$.

For $A \subset [n]$ and $j \geq 1$, we denote by $A^{(j)}$ the multisubset of $[n]$ with the same elements as A , all of them with multiplicity j . In $\mathcal{P}([n])_{\leq (m-1)}$ we consider the partial order given by inclusion, where $P \subseteq Q$ if the multiplicity of every element in P is smaller than or equal to its multiplicity in Q . For every multisubset $P \in \mathcal{P}([n])_{\leq (m-1)}$ we denote by $\uparrow\{P\}$ its filter, i.e., the set $\uparrow\{P\} = \{Q \in \mathcal{P}([n])_{\leq (m-1)} : P \subseteq Q\}$, and for $F \subseteq \mathcal{P}([n])_{\leq (m-1)}$ we denote $\uparrow F = \cup_{P \in F} \uparrow\{P\}$.

Lemma 5.1. Let $m, n \geq 2$, and let $\{A_1, \dots, A_k\}$ be a partition of $[n]$ into k nonempty sets. We denote $F_i := \uparrow\{A_1^{(i)}, A_2^{(i)}, \dots, A_k^{(i)}\}$ and consider the following induced subgraphs of $H(n, m)$:

- G_{m-1} is the induced subgraph on F_{m-1} ,
- G_i is the induced subgraph on $F_i \setminus F_{i+1}$ for $i \in \{1, 2, \dots, m-2\}$, and
- G_0 is the induced subgraph on $\mathcal{P}([n])_{\leq (m-1)} \setminus F_1$.

Then $\{V(G_i) \mid 0 \leq i \leq m-1\}$ constitute a partition of $V(H(n, m))$, and

$$\delta(G_i) = (m-1)(n-k) + i(k - \max(|A_1|, \dots, |A_k|)) \text{ for } 0 \leq i \leq m-1.$$

In particular, $(m-1)n - \min\{\delta(G_0), \dots, \delta(G_{m-1})\} = (m-1) \max(k, |A_1|, \dots, |A_k|)$.

Proof. It is clear that $V(G_0), V(G_1), \dots, V(G_{m-1})$ constitute a partition of $V(H(n, m))$. Let us compute the values $\delta(G_0), \delta(G_1), \dots, \delta(G_{m-1})$.

Take $B \in V(G_{m-1})$. Then there is at least one index $i \in [k]$ such that $A_i^{(m-1)} \subseteq B$. If there are two indices verifying the previous property, then all the neighbors of B are in G_{m-1} and $\deg_{G_{m-1}}(B) = (m-1)n$. If there is only one index i_0 such that $A_{i_0}^{(m-1)} \subseteq B$, then changing the multiplicity in B of any element of A_{i_0} (from $m-1$ to any value in $[0, m-2]$) one gets a multiset not containing $A_{i_0}^{(m-1)}$ anymore, and hence $\deg_{G_{m-1}}(B) = (m-1)(n - |A_{i_0}|)$. As a consequence, $\delta(G_{m-1}) = (m-1)(n - \max(|A_1|, \dots, |A_k|))$.

Take now $B \in V(G_0)$. Then B does not contain A_i for any $i \in [k]$. If there are at least two elements in each A_i which do not belong to B , then all the neighbors of B are in G_0 and $\deg_{G_0}(B) = (m-1)n$. On the other hand, for each $i \in [k]$ such that B does not contain exactly one element of A_i , then we can add to B this missing element with all possible multiplicities to get $(m-1)$ vertices adjacent to B and not in G_0 . Since this can happen for all $i \in [k]$, B has at most $(m-1)k$ neighbors not in G_0 , and there are vertices attaining this bound. Hence, $\delta(G_0) = (m-1)(n-k)$.

Now let $i \in \{1, \dots, m-2\}$ and $B \in V(G_i)$. Then there is some $j \in [k]$ such that $A_j^{(i)} \subseteq B$ but $A_s^{(i+1)} \not\subseteq B$ for all $s \in [k]$. Let us compute how many neighbors of B are not in G_i .

First, we compute the number of neighbors of B not in F_i . If B contains more than one multiset $A_j^{(i)}$, then B has all its neighbors in F_i . Assume that there is only one index j_0 such that $A_{j_0}^{(i)} \subseteq B$. In order to obtain a neighbor of B which does not contain $A_{j_0}^{(i)}$, we have to change the multiplicity in B of any element of A_{j_0} to any value in $[0, i-1]$. Hence B has at most $i|A_{j_0}|$ neighbors not in F_i .

Now we compute the number of vertices adjacent to B in F_{i+1} . If we change the multiplicity of only one element of B and as a result we obtain a new multiset B' such that $A_s^{(i+1)} \subseteq B'$ for some $s \in [k]$, then all the elements of A_s already have multiplicity at least $i+1$ in B except one, which is the one whose multiplicity we have to change to any value in $[i+1, m-1]$ ($m-i-1$ different possibilities). Since this can happen with every A_j , we conclude that B has at most $k(m-i-1)$ neighbors in F_{i+1} . As a consequence,

$$\delta(G_i) \geq (m-1)n - i(\max(|A_1|, \dots, |A_k|)) - k(m-1-i)$$

for all $i \in [0, m-1]$, and there are vertices achieving this bound. Hence, $\delta(G_i) = (m-1)(n-k) + i(k - \max(|A_1|, \dots, |A_k|))$ and the result follows. \square

If one takes a partition $\{A_1, \dots, A_k\}$ of $[n]$ such that $\max(k, |A_1|, \dots, |A_k|) = \lceil \sqrt{n} \rceil$, applying the previous lemma one gets that

$$(m-1)n - \min\{\delta(G_0), \delta(G_1), \dots, \delta(G_{m-1})\} = (m-1)\lceil \sqrt{n} \rceil.$$

Now we are going to prove that, when m is a prime number and n is a perfect square which is a multiple of $m^2(m-1)^2$, then this construction also satisfies that $\sum_{k=0}^{m-1} |\rho(V(G_k))| \varepsilon^k \neq 0$, and hence one may apply Proposition 4.8.

Proposition 5.2. *For every prime $p \geq 3$ and every perfect square $n = N^2$ with $N \equiv 0 \pmod{p(p-1)}$, there exist p induced subgraphs G_0, G_1, \dots, G_{p-1} of $H(n, p)$ whose vertex sets partition $V(H(n, p))$ such that $\sum_{i=0}^{p-1} |\rho(V(G_k))| \varepsilon^k \neq 0$ and $(p-1)n - \min\{\delta(G_0), \delta(G_1), \dots, \delta(G_{p-1})\} = (p-1)\sqrt{n}$.*

For proving this result we will use some lemmas:

Lemma 5.3. *Let $m, \ell \geq 2$ and $s \in \mathbb{Z}$, and denote*

$$H_s^\ell := \left| \left\{ (x_1, \dots, x_\ell) \in [0, m-2]^\ell : \sum_{i=1}^{\ell} x_i \equiv s \pmod{m} \right\} \right|.$$

Then,

$$H_s^\ell = \begin{cases} \left(\sum_{i=0}^{(\ell-2)/2} \binom{m-1}{2i} \right) (m-2) & \text{if } \ell \text{ is even and } s + \ell \not\equiv 0 \pmod{m}, \\ \left(\sum_{i=0}^{(\ell-2)/2} \binom{m-1}{2i} \right) (m-2) + 1 & \text{if } \ell \text{ is even and } s + \ell \equiv 0 \pmod{m}, \\ \left(\sum_{i=0}^{(\ell-3)/2} \binom{m-1}{2i+1} \right) (m-2) + 1 & \text{if } \ell \text{ is odd and } s + \ell \not\equiv 0 \pmod{m}, \\ \left(\sum_{i=0}^{(\ell-3)/2} \binom{m-1}{2i+1} \right) (m-2) & \text{if } \ell \text{ is odd and } s + \ell \equiv 0 \pmod{m}. \end{cases}$$

Proof. The result follows by induction on ℓ applying the recursive formula

$$H_s^\ell = \sum_{k=0}^{m-2} H_{s-k}^{\ell-1}$$

and the fact that for $\ell = 1$ one has that $H_s^1 = 1$ if $s \not\equiv -1 \pmod{m}$, and $H_s^1 = 0$ if $s \equiv -1 \pmod{m}$, which lead us to the table in Figure 6, in which the entry in the i -th row and the j -th column is equal to H_j^i . It is clear that each row satisfies that its entries are all equal except one, and the one which is different corresponds to the case in which $i + j \equiv 0 \pmod{m}$. \square

	0	1	2	\dots	$m-3$	$m-2$	$m-1$
1	$H_0^1 = 1$	1	1	\dots	1	1	0
2	$H_0^2 = m-2$	$m-2$	$m-2$	\dots	$m-2$	$m-1$	$m-2$
3	$H_0^3 = (m-2)(m-1)+1$	$(m-2)(m-1)+1$	$(m-2)(m-1)+1$	\dots	$(m-2)(m-1)$	$(m-2)(m-1)+1$	$(m-2)(m-1)+1$
\vdots							
$m-1$							
m	H_0^m	H_1^m	H_2^m	\dots	H_{m-3}^m	H_{m-2}^m	H_{m-1}^m
$m+1$							
\vdots							

 FIGURE 6. Table T with $T_{ij} = H_j^i$

Lemma 5.3 provides the exact values of H_s^ℓ . However, in the proof of Proposition 5.2 we only use that $H_s^\ell - H_t^\ell \in \{-1, 0, 1\}$, and that $H_s^\ell = H_t^\ell$ whenever $s + \ell \not\equiv 0 \pmod m$ and $t + \ell \not\equiv 0 \pmod m$.

A theorem due to Lucas (see, e.g., [11, 23]) provides the value of the binomial coefficient $\binom{M}{N}$ modulo a prime in terms of the p -adic expansion of M and N . A direct consequence of this result is the following:

Corollary 5.4. *Let p be a prime and let $M \in \mathbb{N}$ be a multiple of p . If $N \in \mathbb{N}$ is not a multiple of p , then $\binom{M}{N} \equiv 0 \pmod p$. As a consequence, if $N_1, \dots, N_s \in \mathbb{N}$ satisfy that $N_1 + \dots + N_s = N$ and N_i is not a multiple of p for some $i \in [1, s]$, then*

$$\binom{M}{N_1, \dots, N_s} \equiv 0 \pmod p.$$

Lemma 5.5. *Let $p \geq 3$ be a prime number, $t \in [p-2]$, N a multiple of $p(p-1)$ and $s \in \mathbb{Z}$. Then,*

$$\Delta_{t,s}^N := \left| \left\{ (x_1, \dots, x_N) \in [0, t]^N \setminus [1, t]^N : \sum_{i=1}^N x_i \equiv s \pmod p \right\} \right|$$

is a multiple of p .

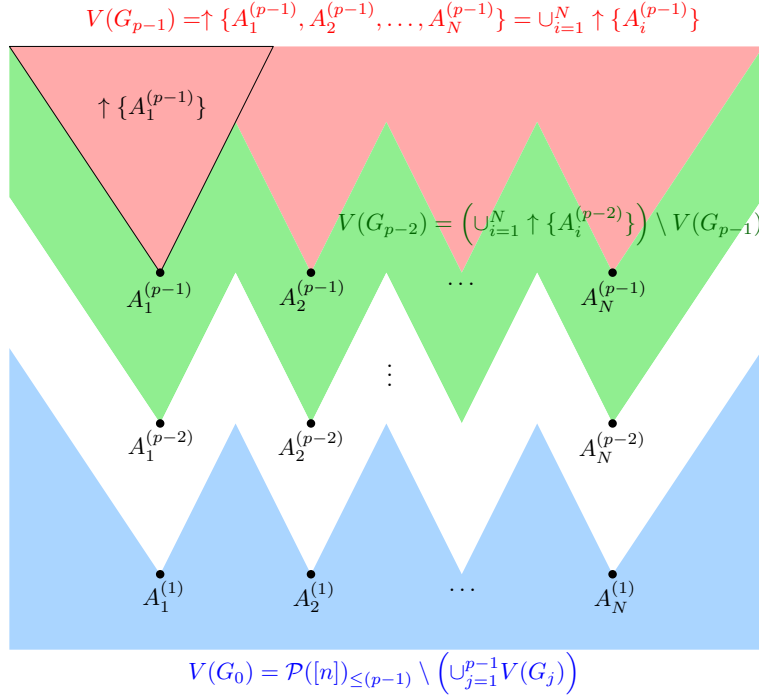
Proof. We observe that $\Delta_{t,s}^N = \Delta_{t,(s \bmod p)}^N$, so we may assume that $0 \leq s \leq p-1$. We separate the proof in two cases.

We first assume that $s \neq 0$. We observe that for every $(x_1, \dots, x_N) \in \Delta_{t,s}^N$ and every permutation $\sigma : \{1, \dots, N\} \rightarrow \{1, \dots, N\}$ one has that $(x_{\sigma(1)}, \dots, x_{\sigma(N)}) \in \Delta_{t,s}^N$. Hence, if y_i denotes the number of entries among x_1, \dots, x_N that are equal to i , then there are $\binom{N}{y_0, \dots, y_t}$ different elements that can be obtained permuting the entries of (x_1, \dots, x_N) . Since $x_1 + \dots + x_N = 0y_0 + \dots + ty_t \equiv s \pmod p$, then there exists j such that $y_j \not\equiv 0 \pmod p$. Since N is a multiple of p , applying Corollary 5.4 we have that $\binom{N}{y_0, \dots, y_t} \equiv 0 \pmod p$.

We assume now that $s = 0$. We have that $\sum_{i=0}^{p-1} \Delta_{t,i}^N = (t+1)^N - t^N$. Since N is a multiple of $p-1$ and $1 \leq t < t+1 \leq p-1$, by Fermat's little Theorem one has that $(t+1)^N \equiv t^N \equiv 1 \pmod p$. Then $\Delta_{t,0}^N \equiv -\sum_{i=1}^{p-1} \Delta_{t,i}^N \equiv 0 \pmod p$. \square

Proof of Proposition 5.2. Consider the partition of $[n]$ into the sets $A_1 = \{1, \dots, N\}, A_2 = \{N+1, \dots, 2N\}, \dots, A_N = \{N^2 - N + 1, \dots, N^2\}$ and consider the induced subgraphs G_0, \dots, G_{p-1} described in Lemma 5.1 (see Figure 7).

By Lemma 5.1 we have that $V(G_0), \dots, V(G_{p-1})$ partition $V(H(n, p)) \cong \mathcal{P}([n])_{\leq (p-1)}$, and $(p-1)n - \min\{\delta(G_0), \delta(G_1), \dots, \delta(G_{p-1})\} = (p-1)\sqrt{n}$.

FIGURE 7. $V(G_0), \dots, V(G_{p-1})$ as elements of the poset $\mathcal{P}([n]_{\leq (p-1)})$

It only remains to prove that $\sum_{i=0}^{p-1} |\rho(V(G_i))| \varepsilon^i \neq 0$. Since p is prime, by Remark 4.13 this is equivalent to proving that $|\rho(V(G_0))|, \dots, |\rho(V(G_{p-1}))|$ are not all equal. Indeed, we are going to show that $|\rho(V(G_{p-1}))| \not\equiv |\rho(V(G_1))| \pmod{p}$. This will follow from:

- (i) $|\rho(V(G_{p-1})) \cap V(G_{p-1})| - |\rho(V(G_1)) \cap V(G_{p-1})| \in \{1, -1\}$,
- (ii) $|\rho(V(G_{p-1})) \cap V(G_0)| - |\rho(V(G_1)) \cap V(G_0)| = 0$, and
- (iii) $|\rho(V(G_r)) \cap V(G_i)| \equiv 0 \pmod{p}$ for $1 \leq i \leq p-2$ and $0 \leq r \leq p-1$.

We recall that, by Remark 4.5, we have that

$$\rho(V(G_i)) = \cup_{j=0}^{p-1} (D_j \cap V(G_{i-j}))$$

for every $i \in [0, p-1]$, where the indices are taken modulo p and $D_j = \{\mathbf{x} = (x_1, x_2, \dots, x_n) \in [0, p-1]^n \mid \sum_{i=1}^n x_i \equiv j \pmod{p}\}$.

Let us prove (i), taking into account that

$$|\rho(V(G_{p-1})) \cap V(G_{p-1})| - |\rho(V(G_1)) \cap V(G_{p-1})| = |D_0 \cap V(G_{p-1})| - |D_2 \cap V(G_{p-1})|.$$

We observe that if $B \in \uparrow \{A_1^{(p-1)}\}$, then its last entry is free in the following sense: if we change its last entry by any value (or, in other words, if we change the multiplicity of n in B), then we obtain new elements of $\uparrow \{A_1^{(p-1)}\}$, and each of them belongs to a different D_i . In particular, $|\uparrow \{A_1^{(p-1)}\} \cap D_0| - |\uparrow \{A_1^{(p-1)}\} \cap D_2| = 0$. The same happens if we take an element of $\uparrow \{A_i^{(p-1)}\} \setminus (\cup_{j=1}^{i-1} \uparrow \{A_j^{(p-1)}\})$ for all $i \in \{2, \dots, N-1\}$, so we just have to focus on what happens with the elements of $\uparrow \{A_N^{(p-1)}\} \setminus (\cup_{i=1}^{N-1} \uparrow \{A_i^{(p-1)}\})$.

The entries of every element of $V(H(n, p))$ can be separated in N blocks according to the different sets A_1, A_2, \dots, A_N to which their indices belong. If the first element of the block corresponding to A_{N-1} is not equal to $p-1$, then the last entry of that block is free. So we can assume that this first element is equal to $p-1$, and repeating this reasoning we get that the elements of $V(G_{p-1})$ that are not equally distributed among D_0, D_1, \dots, D_{p-1} must be of the form

$$\underbrace{(p-1, \dots, p-1, *_1)}_N, \underbrace{(p-1, \dots, p-1, *_2)}_N, \dots, \underbrace{(p-1, \dots, p-1, *_{N-1})}_N, \underbrace{(p-1, \dots, p-1, p-1)}_N.$$

To compute which of these elements belong to D_0 , we want that $\sum_{j=1}^{N-1} *_j + (p-1)(N-1)^2 + (p-1)N \equiv 0 \pmod{p}$. Since $N \equiv 0 \pmod{p}$, this is equivalent to $\sum_{j=1}^{N-1} *_j \equiv 1 \pmod{p}$, and following the notation of

Lemma 5.3 there are exactly H_1^{N-1} such vertices. Furthermore, for D_2 , we want that $\sum_{j=1}^{N-1} *j + (p-1)(N-1)^2 + (p-1)N \equiv 2 \pmod{p}$; i.e., we want that $\sum_{j=1}^{N-1} *j \equiv 3 \pmod{p}$, and there are exactly H_3^{N-1} such vertices. Moreover, since N is a multiple of p one has that $N-1+1 \equiv 0 \pmod{p}$ and, since $p \geq 3$, $N-1+3 \not\equiv 0 \pmod{p}$; thus, by Lemma 5.3,

$$|\rho(V(G_{p-1})) \cap V(G_{p-1})| - |\rho(V(G_1)) \cap V(G_{p-1})| = H_1^{N-1} - H_3^{N-1} \in \{+1, -1\}.$$

Let us prove (ii), starting with the fact that

$$|\rho(V(G_{p-1})) \cap V(G_0)| - |\rho(V(G_1)) \cap V(G_0)| = |D_{p-1} \cap V(G_0)| - |D_1 \cap V(G_0)|.$$

As in the previous case, we separate the entries of these vertices in N blocks. The vertices of G_0 have at least one entry equal to 0 in each of the N blocks. If the first entry of the first block is equal to 0, then the last entry of that block is free. Hence we can assume that this entry is not equal to 0. Repeating this idea we get that the elements of $V(G_0)$ that are not equally distributed among D_0, D_1, \dots, D_{p-1} must be of the form

$$\left(\underbrace{*^1_1, \dots, *^1_{N-1}, 0}_N, \underbrace{*^2_1, \dots, *^2_{N-1}, 0}_N, \dots, \underbrace{*^N_1, \dots, *^N_{N-1}, 0}_N \right),$$

where $*^j_i \in [1, p-1]$. If we subtract one unit to every nonzero entry of the previous vector, then we get that

$$|D_{p-1} \cap V(G_0)| - |D_1 \cap V(G_0)| = H_{p-1}^{N(N-1)} - H_1^{N(N-1)},$$

which equals 0 by Lemma 5.3.

Let us prove (iii). We are going to show that $|\rho(V(G_r)) \cap V(G_j)| \equiv 0 \pmod{p}$ for all $r \in [0, p-1]$ and all $j \in [1, p-2]$ or, equivalently, that $|D_s \cap V(G_j)| \equiv 0 \pmod{p}$ for all $s \in [0, p-1]$ and all $j \in [1, p-2]$. We observe that all the elements $\mathbf{x} \in V(G_j)$ can be uniquely built by choosing:

- $k \in [1, N]$ (here k indicates the smallest index such that \mathbf{x} contains $A_k^{(j)}$),
- $c_1, \dots, c_{k-1} \in [0, p-1]^N \setminus [j, p-1]^N$,
- $b \in [j, p-1]^N \setminus [j+1, p-1]^N$,
- $c_{k+1}, \dots, c_N \in [0, p-1]^N \setminus [j+1, p-1]^N$, and

considering $\mathbf{x} = (c_1, \dots, c_{k-1}, b, c_{k+1}, \dots, c_N)$.

Hence, by Lemma 5.5, for a given $k \in [1, N]$, $c_1, \dots, c_{k-1} \in [0, p-1]^N \setminus [j, p-1]^N$, $c_{k+1}, \dots, c_N \in [0, p-1]^N \setminus [j+1, p-1]^N$, the number of elements $\mathbf{x} = (c_1, \dots, c_{k-1}, b, c_{k+1}, \dots, c_N) \in D_s \cap V(G_j)$ is a multiple of p . Thus, $|D_s \cap V(G_j)| \equiv 0 \pmod{p}$.

By (i), (ii) and (iii) we can conclude that

$$|\rho(V(G_{p-1}))| - |\rho(V(G_1))| = \sum_{i=0}^{p-1} (|\rho(V(G_{p-1})) \cap V(G_i)| - |\rho(V(G_1)) \cap V(G_i)|) \equiv \pm 1 \pmod{p},$$

and hence $|\rho(V(G_{p-1}))| \neq |\rho(V(G_1))|$. \square

Proposition 5.2 imposes several conditions on the values of m and n ($m \geq 3$ is a prime and $n = N^2$ is a perfect square which is a multiple of $p^2(p-1)^2$), and our proof heavily relies on these hypotheses. However, we believe that the same type of results can be proved for general values of m and n .

As a consequence of Proposition 5.2 together with Proposition 4.8 we have the following result, which in particular implies that if the p -ary sensitivity conjecture (Conjecture 1.2) holds, then $c \geq 2$. Equivalently, this also implies that if Conjecture 4.12 holds for p -ary functions, then $\mu \leq 1/2$.

Proposition 5.6. *For every prime p and every positive integer D , there exists n_0 such that for all $n \geq n_0$ there is a p -ary function $f : \{1, \varepsilon, \dots, \varepsilon^{p-1}\}^n \rightarrow \{1, \varepsilon, \dots, \varepsilon^{p-1}\}$ with $\deg(f) > D$ and*

$$s(f) = \sqrt{(p-1)\deg(f)}.$$

Proof. Take $n_0 = N^2$ a perfect square such that $N \equiv 0 \pmod{p(p-1)}$ and $n_0 > D/(p-1)$. By Proposition 5.2 together with Proposition 4.8, there exists a p -ary function $g : \{1, \varepsilon, \dots, \varepsilon^{p-1}\}^{n_0} \rightarrow \{1, \varepsilon, \dots, \varepsilon^{p-1}\}$ of degree $\deg(g) = (p-1)n_0 > D$ and sensitivity $s(g) = \sqrt{(p-1)\deg(g)}$. Now, for $n \geq n_0$, we consider $f : \{1, \varepsilon, \dots, \varepsilon^{p-1}\}^n \rightarrow \{1, \varepsilon, \dots, \varepsilon^{p-1}\}$ given by $f(x_1, \dots, x_n) = g(x_1, \dots, x_{n_0})$ for every $(x_1, x_2, \dots, x_n) \in \{1, \varepsilon, \dots, \varepsilon^{p-1}\}^n$. Since f and g have the same sensitivity and degree, the result follows. \square

This result together with Proposition 2.7 yield the following.

Theorem 1.3. *For every set $T \subseteq \mathbb{C}$ of prime cardinality $|T| = p$ and every positive integer D , there exists a p -ary function $f : T^n \rightarrow T$ with $\deg(f) > D$ and $s(f)^2/(p-1)^3 \leq \deg(f)$.*

6. CONCLUSIONS

The main open problem is the m -ary sensitivity conjecture (Conjecture 1.2) and its reformulation in graph theoretical terms (Conjecture 4.12). The statement of Conjecture 4.12 is a bit intricate due to the additional property concerning the values $|\rho(V(H_{\varepsilon^k}))|$. We present a stronger conjecture in more natural terms:

Conjecture 6.1 (Strong m -ary Sensitivity Conjecture). *Let m be a positive integer and ε an m -th primitive root of unity. There exists $\mu > 0$ such that for any partition of $H(n, m)$ into (possibly empty) induced subgraphs H_1, \dots, H_m with $\sum_{i=1}^m |V(H_i)| \varepsilon^i \neq 0$, we have*

$$\max\{\Delta(H_1), \dots, \Delta(H_m)\} \in \Omega(n^\mu) .$$

Let us quickly see that indeed:

Proposition 6.2. *Conjecture 6.1 implies Conjecture 4.12.*

Proof. Assume that Conjecture 6.1 holds, and consider m induced subgraphs $H_1, H_\varepsilon, \dots, H_{\varepsilon^{m-1}}$ of $H(n, m)$ such that their vertex sets partition $V(H(n, m))$ and $\sum_{i=0}^{m-1} |\rho(V(H_{\varepsilon^i}))| \varepsilon^i \neq 0$. Denote by H'_{ε^i} the induced subgraph with $V(H'_{\varepsilon^i}) = \rho(V(H_{\varepsilon^i}))$. Since we are assuming that Conjecture 6.1 holds, then

$$\max\{\Delta(H'_1), \dots, \Delta(H'_{\varepsilon^{m-1}})\} \in \Omega(n^\mu)$$

for some $\mu > 0$.

We observe that for all $v \in V(H(n, m))$ there exist $r, s \in [0, m-1]$ such that $v \in V(H_{\varepsilon^r})$, $v \in V(H'_{\varepsilon^s})$, and $\deg_{H_{\varepsilon^r}}(v) + \deg_{H'_{\varepsilon^s}}(v) \leq (m-1)n$ because the neighborhoods of v in H_{ε^r} and H'_{ε^s} are disjoint. Hence,

$$(m-1)n - \min\{\delta(H_{\varepsilon^k}) \mid 0 \leq k \leq m-1\} \geq \max\{\Delta(H'_1), \dots, \Delta(H'_{\varepsilon^{m-1}})\}$$

and the result follows. \square

Note that if m is a prime number, then the condition $\sum_{i=1}^m |V(H_i)| \varepsilon^i \neq 0$ in Conjecture 6.1 reduces to requiring that not all H_i have the same order (see Remark 4.13). For $m = 3$ this leads to the conjecture in the introduction:

Conjecture 1.4. *There exists $\mu > 0$ such that for any partition of the Hamming graph $H(n, 3)$ into three (possibly empty) induced subgraphs H_1, H_2, H_3 not all of the same order, it holds:*

$$\max\{\Delta(H_1), \Delta(H_2), \Delta(H_3)\} \in \Omega(n^\mu) .$$

Concerning Conjecture 6.1, there are some recent works in which the authors study the maximum degrees of induced subgraphs of the Hamming graph $H(n, m)$. For example, denoting by $\alpha(G)$ the independence number of a graph G , Dong [10] constructed an induced subgraph of $H(n, m)$ on more than $\alpha(H(n, m)) = m^{n-1}$ vertices whose maximum degree is strictly smaller than $\sqrt{n} + 1$. This construction was later improved by Tandy [32], who was able to prove that $H(n, m)$ has an induced subgraph on $m^{n-1} + 1$ vertices with maximum degree 1 whenever $m \geq 3$.

For $m = 3$, Potechin and Tsang [30] provide upper bounds on the maximum number of vertices of an induced subgraph with maximum degree at most 1. They show that if $U \subseteq V(H(n, 3))$ is disjoint from a maximum size independent set of $H(n, 3)$ and the induced subgraph on U has maximum degree at most 1, then $|U| \leq 3^{n-1} + 1$. This shows that a construction by García-Marco and Knauer [14] is largest-possible. Potechin and Tsang also provide an induced subgraph of $H(n, 3)$ with $3^{n-1} + 18$ vertices with maximum degree equal to 1 and which is almost optimal in several senses. However, it seems that from these constructions no partition into induced subgraphs of small maximum degrees can be obtained, if they are not allowed to be all of the same order. Finally, note that our graph theoretical Conjectures 1.4, 4.12, 6.1 can all be seen as special variants of so-called defective colorings, see [34] for a survey. However, our imbalance requirements on the color classes seem to be novel.

REFERENCES

- [1] A. Abbott, M. Mhalla and P. Pocreau, *Quantum query complexity of Boolean functions under indefinite causal order*, Physical Review Research, **6**(3), 2024.
- [2] N. Alon and K. Zheng, *Unitary signings and induced subgraphs of Cayley graphs of \mathbb{Z}_2^n* , Advances in Combinatorics, 12 pp., 2020.
- [3] S. Asensio, *La conjetura de la sensibilidad y su resolución vía teoría de grafos*, La Gaceta de la RSME, **26**(1), pp. 131-148, 2023.
- [4] H. Buhrman and R. de Wolf, *Complexity measures and decision tree complexity: a survey*, Theoretical Computer Science, **288**(1), pp. 21-43, 2002.

- [5] C. Carlet, *Boolean functions for cryptography and error-correcting codes*, Boolean models and methods in mathematics, computer science, and engineering. Encyclopedia of Mathematics and its Applications, **134**, pp. 257-397. Y. Crama (ed.) et al., Cambridge University Press, 2010.
- [6] H. T. Chau, D. Ellis, E. Friedgut and N. Lifshitz, *On the maximum degree of induced subgraphs of the Kneser graph*, 2023. arXiv:2312.06370.
- [7] Y. M. Chee, Y. Tan and X. D. Zhang, *Strongly regular graphs constructed from p -ary bent functions*, Journal of Algebraic Combinatorics, **34**, pp. 251-266, 2011.
- [8] F. R. K. Chung, Z. Füredi, R. L. Graham and P. Seymour, *On induced subgraphs of the cube*, Journal of Combinatorial Theory, Series A, **49**(1), pp. 180-187, 1988.
- [9] N. Dafni, Y. Filmus, N. Lifshitz, N. Lindzey and M. Vinyals, *Complexity measures on the symmetric group and beyond* (Extended Abstract). In 12th Innovations in Theoretical Computer Science Conference (ITCS 2021). Leibniz International Proceedings in Informatics (LIPIcs), **185**, 87:1-87:5, Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2021.
- [10] D. Dong, *On induced subgraphs of the Hamming graph*, Journal of Graph Theory, **96**(1), pp. 160-166, 2021.
- [11] N. J. Fine, *Binomial coefficients modulo a prime*, The American Mathematical Monthly, **54**(10), pp. 589-592, 1947.
- [12] P. Frankl and A. Kupavskii, *Maximal degrees in subgraphs of Kneser graphs*, 2020. arXiv:2004.08718.
- [13] C. Gotsman and N. Linial, *The equivalence of two problems on the cube*, Journal of Combinatorial Theory, Series A, **61**(1), pp. 142-146, 1992.
- [14] I. García-Marco and K. Knauer, *On sensitivity in bipartite Cayley graphs*, Journal of Combinatorial Theory, Series B, **154**, pp. 211-238, 2022.
- [15] Z. Hong, H. Lai and J. Liu, *Induced subgraphs of product graphs and a generalization of Huang's theorem*, Journal of Graph Theory, **98**(2), pp. 285-308, 2021.
- [16] X. Hu, L. Chu, J. Pei, W. Lui and J. Bian, *Model complexity of deep learning: a survey*. Knowledge and Information Systems, **63**(10), pp. 2585–2619, 2021.
- [17] H. Huang, *Induced subgraphs of hypercubes and a proof of the sensitivity conjecture*, Annals of Mathematics, **190**(3), pp. 949-955, 2019.
- [18] J. Y. Hyun and Y. Lee, *Characterization of p -ary bent functions in terms of strongly regular graphs*, IEEE Transactions on Information Theory, **65**(1), pp. 676-684, 2019.
- [19] E. Isaacson and H. B. Keller, *Analysis of Numerical Methods*, John Wiley & Sons, 1966.
- [20] J. Kahn, G. Kalai and N. Linial, *The influence of variables on Boolean functions*, Proceedings of the 29th Annual Symposium on Foundations of Computer Science, pp. 68-80, 1988.
- [21] P. Kaye, R. Laflamme and M. Mosca, *An introduction to quantum computing*, Oxford University Press, 2007.
- [22] F. Lehner and G. Verret, *Counterexamples to “A conjecture on induced subgraphs of Cayley graphs”*, Ars Mathematica Contemporanea, **19**, pp. 77-82, 2020.
- [23] E. Lucas, *Théorie des nombres*, Gauthier-Villars, Paris, 1891.
- [24] W. Meidl, *A survey on p -ary and generalized bent functions*, Cryptography and Communications, **14**, pp. 737-782, 2022.
- [25] C. Melles and D. Joyner, *On p -ary bent functions and strongly regular graphs*, 2019. arXiv:1904.09359.
- [26] M. Minsky and S. Papert, *Perceptrons. An introduction to computational geometry*, The MIT Press, Cambridge, 1969.
- [27] M. Nicolau, A. R. Tavares, Z. Zhang, P. Avelar, J. M. Flach, L. C. Lamb and M. Y. Vardi, *Understanding Boolean Function Learnability on Deep Neural Networks: PAC Learning Meets Neurosymbolic Models*, arXiv:2009.05908, 2022.
- [28] N. Nisan and M. Szegedy, *On the degree of Boolean functions as real polynomials*, Computational Complexity, **4**, pp. 301-313, 1994.
- [29] A. Potechin and H. Y. Tsang, *A conjecture on induced subgraphs of Cayley graphs*, 2020. arXiv:2003.13166.
- [30] A. Potechin and H. Y. Tsang, *On induced subgraphs of $H(n, 3)$ with maximum degree 1*, 2024. arXiv:2405.15004.
- [31] I. F. Steffensen. *Interpolation*. Chelsea Publications, New York, 1927.
- [32] V. Tandy, *An induced subgraph of the Hamming graph with maximum degree 1*, Journal of Graph Theory, **101**(2), pp. 311-317, 2022.
- [33] A. Tikaradze, *Induced subgraphs of powers of oriented cycles*, Linear and Multilinear Algebra, **70**(17), pp. 3301-3303, 2022.
- [34] D. Wood, *Defective and Clustered Graph Colouring*, The Electronic Journal of Combinatorics, **1**, DS23, 2018.
- [35] Y. Wu, J. Y. Hyun and Y. Lee, *Characterization of p -ary functions in terms of association schemes and its applications*, Journal of Combinatorial Theory, Series A, **187**, 105576, 2022.
- [36] J. Zeng and X. Hou, *On induced subgraph of Cartesian product of paths*, Journal of Graph Theory, **107**(1), pp. 169-180, 2024.

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