

MAKING TRIANGLES COLORFUL*

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ABSTRACT. We prove that for any finite point set P in the plane, a triangle T , and a positive integer k , there exists a coloring of P with k colors such that any homothetic copy of T containing at least $144k^8$ points of P contains at least one of each color. This is the first polynomial bound for range spaces induced by homothetic polygons. The only previously known bound for this problem applies to the more general case of octants in \mathbb{R}^3 , but is doubly exponential.

1 Introduction

Covering and packing problems are ubiquitous in discrete geometry. In this context, the notion of ϵ -nets captures the idea of finding a small but representative sample of a data set (see for instance Chapter 10 in Matoušek's lectures [14]). Given a set system, or *range space*, on n elements, an ϵ -net for this system is a subset of the elements such that any set, or *range*, containing at least ϵn elements contains at least one element of the subset.

In this paper, we are interested in *coloring* the elements so that any range containing *sufficiently many* elements contains at least *one element of each color*. Hence instead of finding a single subset of representative elements, we wish to partition the elements into representative classes.

For a given class of range spaces, we define the function $p(k)$ as the minimum number p such that the following holds: the elements of every range space in that class can be colored with k colors so that any range containing at least p elements contains at least one of each color. It is not difficult to show that if $p(k) = O(k)$ for a class of range spaces, then this class admits ϵ -nets of size $O(1/\epsilon)$.

We are interested in range spaces defined by a collection \mathcal{B} of subsets of \mathbb{R}^d . In what follows, we are mainly concerned with the case where \mathcal{B} is a collection of *convex bodies*, that is, compact convex subsets of \mathbb{R}^d . For given \mathcal{B} we obtain a range space whose ground set is a (countable or finite) point set $P \subseteq \mathbb{R}^d$ by considering all subsets of P formed by

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intersecting P with a member of \mathcal{B} . This construction yields so-called *primal range spaces induced by \mathcal{B}* . For instance, if P is a set of points in the plane and \mathcal{B} the set of all disks, then the ranges are all possible intersections of P with a disk. Such range spaces and their ϵ -nets appear frequently in discrete geometry and in applications such as sensor networks [8].

One can also consider *dual range spaces induced by \mathcal{B}* , where the ground set is a (countable or finite) subcollection \mathcal{B}' of \mathcal{B} , and the ranges are all subsets X of \mathcal{B}' such that there exists some $p \in \mathbb{R}^d$ with $X = \{B \in \mathcal{B}' \mid p \in B\}$. For instance, if \mathcal{B}' is a set of disks in the plane, then the ranges are all maximal sets of disks containing a common point.

In general, those are also referred to as (primal and dual) *geometric hypergraphs*.

In the case of dual range spaces induced by a collection \mathcal{B} of objects, the problem of bounding $p(k)$ is known as the *covering decomposition problem of \mathcal{B}* . In this setting, we are given a subcollection of these objects, and we wish to partition them into k color classes, so that whenever a point is contained in sufficiently many objects of the initial collection, it is contained in at least one object of each class.

We prove a polynomial upper bound on $p(k)$ for primal range spaces induced by homothetic triangles in the plane.

1.1 Previous Work

These questions were first studied by János Pach in the early eighties [15]. An account of early related results and conjectures can be found in Chapter 2 of the survey on open problems in discrete geometry by Brass, Moser, and Pach [4].

In the past five years, tremendous progress has been made in this area, for range spaces induced by various families of convex bodies. One of the most striking achievements is the recent proof that $p(k) = O(k)$ for translates of convex polygons, the culmination of a series of intermediate results for various special cases. We remark that convex bodies are considered because $p(k) = \infty$ for range spaces induced by translates of concave polygons [16]. We refer the reader to Table 1 for a summary of the known bounds.

The specific case of translates of a triangle with $k = 2$ was tackled by Tardos and Tóth in 2007 [21]. They proved that every point set can be colored red and blue so that every translate of a given triangle containing at least 43 points contains at least one red and one blue. We generalize this result in two ways: we consider *homothetic* triangles, and an arbitrary number of colors.

The only previously known results applying to our problem are due to Keszegh and Pálvölgyi [11, 12]. They actually apply to the more general case of translates of (say) the positive octant in a cartesian representation of \mathbb{R}^3 . The special case of triangles homothetic to the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$ occurs when all points lie on a plane orthogonal to the vector $(1, 1, 1)$. The bound that was proven for arbitrary k is of the order of 12^{2^k} , and is most probably far from being tight.

Range spaces	primal	dual
halfplanes	$p(k) = 2k - 1$ [2, 10, 20]	$p(2) = 3$ [7] $p(k) \leq 3k - 2$ [2, 20]
translates of a convex polygon	$p(k) = O(k)$ [21, 19, 17, 1, 9]	
translates of an octant in \mathbb{R}^3	$p(2) \leq 12$ [11] $p(k) \leq 12^{2^k}$ [12]	
unit disks	∞ [18]	
bottomless rectangles	$p(2) = 4$ [10] $1.6k \leq p(k) \leq 3k - 2$ [3]	$p(2) = 3$ [10] $p(k) \leq 12^{2^k}$ [12] (from octants in \mathbb{R}^3)
axis-aligned rectangles	∞ [6]	∞ [16]
disks and halfspaces in \mathbb{R}^3	∞ [16]	∞ [18]

Table 1: Known results for various families of range spaces. For range spaces induced by translates of a set, the primal problem is the same as the dual. When more than one reference is given, they correspond to successive improvements, but only the best known bound is indicated. The symbol ∞ indicates that $p(k)$ does not exist.

1.2 Our Result

Theorem 1.1. *Given a finite point set $P \subseteq \mathbb{R}^2$, a triangle $T \subseteq \mathbb{R}^2$ and a positive integer k , there exists a coloring of P with k colors such that any homothetic copy of T containing at least $144 \cdot k^8$ points of P contains at least one of each color.*

The proof is elementary, and builds on the previous work by Keszegh and Pálvölgyi [11, 12]. The degree of the polynomial depends on $p(2)$. Hence any improvement on $p(2)$ would yield a polynomial improvement in the bound. For the same reason, it can be shown that the same coloring method cannot be used to prove any upper bound better than $O(k^4)$ (as $p(2) \geq 4$).

2 Proof

Let \mathcal{B} be the collection of all homothetic copies of a fixed closed triangle T in the plane. We consider the class of primal range spaces induced by \mathcal{B} . From now on we denote by $p(k)$ the minimum p such that every finite set of points in the plane can be colored with k colors so that any homothetic copy of T containing at least p points contains at least one point of each color.

Lemma 2.1. *If $p(2) \leq c$, for some constant c , then $p(2k) \leq c^2 p(k)$, for all $k \geq 2$.*

Proof. It suffices to prove the lemma for any fixed triangle T and then argue for all others using an affine transformation of the plane. Let T be the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$.

Consider a finite point set P and a k -coloring $\phi : P \rightarrow \{1, \dots, k\}$ such that any homothetic copy of T containing at least $p(k)$ points contains one of each color. Note that $p(k) < \infty$ [12]. We suppose without loss of generality that no two points of P lie on a line of slope -1 , otherwise we can slightly perturb the points, and a suitable coloring for the perturbed version will also work for P .

We now describe a simple procedure to double the number of colors. For $1 \leq i \leq k$ let $P_i = \phi^{-1}(i)$ that is the set of points with color i . Provided $p(2) \leq c$ there is a 2-coloring $\phi_i : P_i \rightarrow \{i', i''\}$ of P_i such that for any homothetic copy T' of T containing at least c points of P_i , T' contains at least one point of each color. We define ϕ' to be the disjoint union of all ϕ_i , and claim that ϕ' is a $2k$ -coloring of P such that for any homothetic copy T' of T containing at least $c^2 p(k)$ points, T' contains at least one point of each of the $2k$ colors.

Consider a homothetic copy T' of a triangle T containing at least $c^2 p(k)$ points from P , and in order to get a contradiction suppose that one of the $2k$ colors used by ϕ' is missing in T' . Let i' be this color. Note that if there are at least c points in T' with color i then i' and i'' must be present in T' , from the correctness of the 2-coloring ϕ_i . Hence we conclude that there are less than c points in T' with color i .

Order the points in $T' \cap P = \{p_1, p_2, \dots\}$ in such a way that the sum of their x - and y -coordinates is non-decreasing. Hence the order corresponds to a sweep of the points in $T' \cap P$ by a line of slope -1 . By the pigeonhole principle, since there are less than c points colored with i , there must exist a subsequence $Q = (p_j, p_{j+1}, \dots, p_{j+\ell-1})$ of points of color distinct from i , of length $\ell := c^2 p(k)/c = cp(k)$.

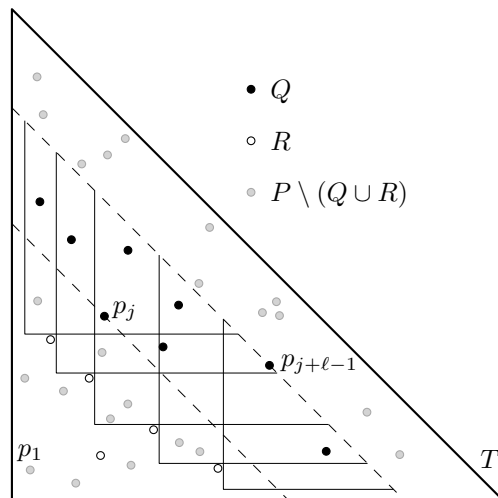


Figure 1: Illustration of the proof of Lemma 2.1.

Let $R := P_i \cap \{p_1, p_2, \dots, p_{j-1}\}$ be the set of points of color i that come before Q in the sweep order. By assumption, we have $|R| < c$. Hence the points of Q can be covered with c translates of the first quadrant, such that none of them intersects R ; see Figure 1. For example, it is enough to consider all inclusion-wise maximal quadrants with apex in T' that avoid points in R . Applying the pigeonhole principle a second time, one of these quadrants must contain at least $|Q|/c = cp(k)/c = p(k)$ points, none of which is colored i .

This quadrant, together with the sweepline containing the last point $p_{j+\ell-1}$ of Q , forms a triangle that is homothetic to T , contains at least $p(k)$ points, none of which has color i . This is a contradiction with the correctness of the initial k -coloring ϕ . \square

Proof of Theorem 1.1. It was shown by Keszegh and Pálvölgyi that $p(2) \leq 12$ [11]. Hence it remains to solve the recurrence of the previous lemma with $c = 12$. We look for an upper bound on $p(k)$ satisfying $p(2k) \leq 144 \cdot p(k)$ for any positive integer k , and $p(2) \leq 12$. This yields $p(2^i) \leq 144^i$ for any positive integer i , and $p(k) \leq 144^{\lceil \log_2 k \rceil} < 144 \cdot k^8$ for any positive integer k . \square

3 Open Problems

The only lower bounds on $p(k)$ the authors are aware of is the bound $p(k) \geq 1.6k$ for bottomless rectangles [3] (which improves the bound $p(k) \geq 4k/3$ for translates of squares [17]) and the tight bound $p(k) \geq 2k - 1$ for halfplanes [20].

No bound at all is known for the primal range space induced by axis-aligned squares: does there exist a function $p(k)$ such that for any point set P there is a k -coloring of P such that any axis-aligned square containing at least $p(k)$ points of P contains at least one point of each color?

We remark that after this paper has been submitted the bound of $144k^8$ was improved to $O(k^6)$ even in the more general setting of translates of octants in \mathbb{R}^3 [5], and also by Keszegh and Pálvölgyi [13] to $O(k^{4.58})$ again only in the case of homothetic triangles. Both results rely on the same idea as the one in Lemma 2.1, namely defining a $2k$ -coloring from a k -coloring by splitting each color class into two.

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