THE SIGNED VARCHENKO DETERMINANT FOR COMPLEXES OF ORIENTED MATROIDS

WINFRIED HOCHSTÄTTLER, SOPHIA KEIP, AND KOLJA KNAUER

Abstract. We generalize the (signed) Varchenko matrix of a hyperplane arrangement to complexes of oriented matroids and show that its determinant has a nice factorization. This extends previous results on hyperplane arrangements and oriented matroids.

1. Introduction

Schechtman and Varchenko [11] considered a symmetric matrix which may be viewed as a bilinear form on the vector space of linear forms of the set of regions of a hyperplane arrangement $\mathcal{A}$ over some ordered field $\mathbb{K}$. The value of the product of the characteristic vectors of regions Q_i and Q_j is given by a product $\prod_{e\in S(Q_i,Q_j)} w_e$, where the w_e are weights on the hyperplanes H_e of the arrangements and $S(Q_i,Q_j)$ is the set of hyperplanes that have to be crossed on a shortest path from Q_i to Q_j. The corresponding *Varchenko Matrix* $B_{\mathcal{A}}$ has entries of the form $\prod_{e\in S(Q_i,Q_j)} w_e$ for any pair of regions Q_i and Q_j, also see Theorem 2.9. In order to determine when the bilinear form is degenerate, Varchenko [13] gave an elegant factorization of the determinant of that matrix, considering the weights as variables.

Theorem 1.1 (Varchenko 1993 [13]). *Let $\mathcal{A}$ be a real hyperplane arrangement, $B_{\mathcal{A}}$ its Varchenko matrix, and $L(\mathcal{A})$ the* geometric lattice *formed by the intersections of hyperplanes in $\mathcal{A}$, then*

$$\det(B_{\mathcal{A}}) = \prod_{F\in L(\mathcal{A})} \left(1-w_F^2\right)^{m_F}$$

where $w_F = \prod_{F\subset H_e} w_e$ and m_F are positive integers depending only on $L(\mathcal{A})$.

After the original proof of Varchenko there were several approaches to provide cleaner proofs of this result. Denham and Henlon [5] sketched an elegant alternative way to prove the result. Gente [6] provided some more details for that proof and claimed to have generalized the result to cones, which are also called topcones or in our notation supertopes, i.e. convex sets of regions. This method was generalized by Hochstättler and Welker [8] to oriented matroids, which form a combinatorial model for hyperplane arrangements reflecting their local linear structure but allowing for some global non-linearities.

Aguiar and Mahajan [1] generalized the original proof of Varchenko to a signed version of the matrix and also derived the result for topcones. Here one considers an oriented hyperplane arrangement and the entries of the signed Varchenko matrix depend on which side of a hyperplane a cell lies, see Definition 2.7. Randriamaro [10] generalized their proof to oriented matroids.

Bandelt et al. generalized oriented matroids to *complexes of oriented matroids* by relaxing the global symmetry while maintaining convexity and local symmetry. This framework captures a variety of classes beyond oriented matroids, e.g., distributive lattices, CAT(0)-cube complexes, lopsided sets, linear extension graphs, and affine oriented matroids, see [2]. The purpose of this paper is to show, that the latter requirements are still sufficient for the factorization formula to hold. The presentation as well as the proof follow the lines of Hochstättler and Welker [8]. We furthermore achieve a generalization to the signed version of the Varchenko matrix, thus generalizing Randriamaro [10].

The paper is organized as follows. In Section 2 we introduce the considered structures. In Section 3 we present some tools from algebraic topology that we need for the proof of the main theorem. The latter is presented in Section 4. We give some examples and applications in Section 5 and conclude the paper with some further remarks in Section 6.

2. The Varchenko Determinant and Complexes of Oriented Matroids

Before we introduce the Varchenko Determinant, we need to get familiar with complexes of oriented matroids (COMs). COMs have been introduced in [2] as a common generalization of oriented matroids, affine oriented matroids, and lopsided sets. We will use the notation from [2] and [4]. Note that the symbols $+, -$ and 0 act like $1, -1$ and 0 when it comes to negation and multiplication. We start with the following definitions and axioms.

Definition 2.1. We consider *sign vectors* on a finite *ground set* E, i.e., elements of $\{0,+,-\}^E$. The *composition* of two sign vectors X and Y is defined as the sign-vector

$$(X \circ Y)_e = \begin{cases} X_e & \text{if } X_e \neq 0, \\ Y_e & \text{if } X_e = 0 \end{cases} \forall e \in E.$$

The *reorientation* of X with respect to $A \subseteq E$ is defined as the sign-vector

$${}_{A}X = \begin{cases} -X_e & \text{if } e \in A, \\ X_e & \text{if } e \notin A \end{cases} \forall e \in E.$$

The *separator* of X and Y is defined as

$$S(X,Y) = \{e \in E : X_e = -Y_e \neq 0\}.$$

The *support* of X is defined as

$$\underline{X} = \{e \in E : X_e \neq 0\}.$$

The *zero-set* of X is defined as

$$z(X) = E \backslash \underline{X} = \{e \in E : X_e = 0\}.$$

For a set $\mathcal{L} \subseteq \{0,+,-\}^E$ we introduce five axioms:

(FS): *Face Symmetry*

$$\forall X, Y \in \mathcal{L} : X \circ (-Y) \in \mathcal{L}.$$

(SE): *Strong Elimination*

$$\begin{aligned} &\forall X, Y \in \mathcal{L}\, \forall e \in S(X,Y)\, \exists Z \in \mathcal{L} : \\ &Z_e = 0 \text{ and } \forall f \in E \setminus S(X,Y) : Z_f = (X \circ Y)_f. \end{aligned}$$

(C): *Composition*

$$\forall X, Y \in \mathcal{L} : X \circ Y \in \mathcal{L}.$$

(Z): *Zero*

$$\text{The all zeros vector } \mathbf{0} \in \mathcal{L}.$$

(Sym): *Symmetry*

$$\forall X \in \mathcal{L} : -X \in \mathcal{L}.$$

Now we can define the term COM.

Definition 2.2 (Complex of Oriented Matroids (COM))**.** Let E be a finite set and $\mathcal{L} \subseteq \{0,+,-\}^E$. The pair $\mathcal{M} = (E, \mathcal{L})$ is called a COM, if $\mathcal{L}$ satisfies (FS) and (SE). The elements of $\mathcal{L}$ are called *covectors*.

Let us first present OMs as special COMs.

Definition 2.3 (Oriented Matroid (OM))**.** Let E be a finite set and $\mathcal{L} \subseteq \{0,+,-\}^E$. The pair $\mathcal{M} = (E, \mathcal{L})$ is called an OM, if it is a COM that satisfies (Z).

Remark 2.4. Usually OMs are defined satisfying (C),(Sym),(SE). But note that (FS) implies (C). Indeed, by (FS) we first get $X \circ -Y \in \mathcal{L}$ and then $X \circ Y = (X \circ -X) \circ Y = X \circ -(X \circ -Y) \in \mathcal{L}$ for all $X, Y \in \mathcal{L}$. Further, (Z) together with (FS) clearly implies (Sym). Conversely, (Sym) and (C) imply (FS) while (Sym) and (SE) imply (Z).

Let $\mathcal{M} = (E, \mathcal{L})$ be a COM. In the following we assume that $\mathcal{M} = (E, \mathcal{L})$ is *simple*, i.e.

$$\forall e \in E : \{X_e \mid X \in \mathcal{L}\} = \{+,-,0\} \quad \text{and} \quad \forall e \neq f \in E : \{X_e X_f \mid X \in \mathcal{L}\} = \{+,-,0\}.$$

In this setting the sign-vectors in $\mathcal{L}$ of full support are called *topes* and their collection is denoted by $\mathcal{T}$.

The *restriction* of a sign-vector $X \in \{0,+,-\}^E$ to $E\backslash A$, $A \subseteq E$, denoted by $X\backslash A \in \{0,+,-\}^{E\backslash A}$, is defined by $(X\backslash A)_e = X_e$ for all $e \in E\backslash A$. We also write $X|_{E\backslash A}$. The *deletion* of a COM is defined by $(E\backslash A, \mathcal{L}\backslash A)$, where $\mathcal{L}\backslash A = \{X\backslash A,\ X \in \mathcal{L}\}$, also written as $\mathcal{L}|_{E\backslash A}$. Let $T \in \mathcal{T}$ a tope of a COM $\mathcal{M} = (E, \mathcal{L})$ and $S^+, S^- \subseteq E$ be subsets of the positive respectively negative elements of T. The *topal fiber* $\rho_{(S^+,S^-)}(\mathcal{L})$ has ground set $E \setminus (S^+ \cup S^-)$ and covectors $\{X \setminus (S^+ \cup S^-) \mid X \in \mathcal{L}, e \in S^\pm \implies X_e = \pm\}$. We denote by $\mathcal{T}(S^+, S^-)$ the set of topes of $\rho_{(S^+,S^-)}(\mathcal{L})$.

We will make use of the fact (shown in [2]) that the class of simple COMs is closed under deletion and under taking topal fibers.

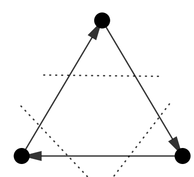
For a covector $X \in \mathcal{L}$, the set $F(X) = \{X \circ Y \mid Y \in \mathcal{L}\}$ is usually called the *face* of X. We define $\text{star}(X) = \{T \in \mathcal{T} \mid X \leq T\}$, where the componentwise ordering with respect to $0 < +, -$ is used. Note that $\text{star}(X) \backslash \underline{X}$ is the set of topes of $(E \backslash \underline{X}, F(X) \backslash \underline{X})$, which is well-known and easily seen to be an oriented matroid.

Let us look at a special OM which we will need in the next chapter.

Definition 2.5 (Graphic OM of a directed n-cycle)**.** This OM has a ground set E of size n and its set of covectors $\mathcal{C}_n$ consists of $\mathbf{0}$ and all compositions of sign-vectors from $\{0, +, -\}^E$ with exactly one positive and exactly one negative entry.

It can easily be checked that $\mathcal{C}_n$ is the set of covectors of an OM. We use $\mathcal{C}_3$ as an example:

Example 2.6 (Graphic OM of a directed triangle)*.* We look at a digraph with three vertices which just consists of a directed cycle, i.e.



The ground set E of this OM corresponds to the three arcs. One gets the covectors of such an OM by looking at the sign vectors of directed cuts (indicated with dotted lines). These sign vectors are $(+, -, 0)$, $(-, +, 0)$, $(+, 0, -)$, $(-, 0, +)$, $(0, +, -)$ and $(0, -, +)$. Their compositions additionally yield the covectors $(+, +, -)$, $(+, -, +)$, $(-, +, +)$, $(-, -, +)$, $(-, +, -)$ and $(+, -, -)$. We see that $\mathcal{T}$ consists of all full support sign vectors, except $(+, +, +)$ and $(-, -, -)$.

We define the signed Varchenko matrix for COMs analogously to this matrix for hyperplane arrangements in [1]. For this purpose we introduce two variables x_{e^+}, x_{e^-} for each element $e \in E$. Let $\mathbb{K}$ be a field and let $\mathbb{K}[x_{e^*} \mid * \in \{+, -\}, e \in E]$ the polynomial ring in the set of variables x_{e^*}, $* \in \{+, -\}$, $e \in E$.

Definition 2.7 (Signed Varchenko Matrix of a COM)**.** Let $\mathcal{M} = (E, \mathcal{L})$ be a COM. The signed Varchenko matrix $\mathfrak{V}$ of a COM is defined by a $\#\mathcal{T} \times \#\mathcal{T}$-Matrix over

$$\mathbb{K}[x_{e^*} \mid * \in \{+, -\}, e \in E].$$

Its rows and columns are indexed by the topes $\mathcal{T}$ in a fixed linear order. For $P, Q \in \mathcal{T}$

$$\mathfrak{V}_{P,Q} = \prod_{e \in S(P,Q)} x_{e^{P_e}}.$$

Note that the diagonal entries $\mathfrak{V}_{P,P}$ of the matrix are equal to 1. Let us illustrate this definition with the graphic OM of a directed triangle.

Example 2.8 (continued). The signed Varchenko Matrix of the graphic OM of a directed triangle is

$$\mathfrak{V} = \begin{pmatrix} 1 & x_2^+x_3^- & x_1^+x_3^- & x_1^+x_2^+x_3^- & x_1^+ & x_2^+ \\ x_2^-x_3^+ & 1 & x_1^+x_2^- & x_1^+ & x_1^+x_2^-x_3^+ & x_3^+ \\ x_1^-x_3^+ & x_1^-x_2^+ & 1 & x_2^+ & x_3^+ & x_1^-x_2^+x_3^+ \\ x_1^-x_2^-x_3^+ & x_1^- & x_2^- & 1 & x_2^-x_3^+ & x_1^-x_3^+ \\ x_1^- & x_1^-x_2^+x_3^- & x_3^- & x_2^+x_3^- & 1 & x_1^-x_2^+ \\ x_2^- & x_3^- & x_1^+x_2^-x_3^- & x_1^+x_3^- & x_1^+x_2^- & 1 \end{pmatrix}$$

In this work we will prove the following theorem.

Theorem 2.9. *Let $\mathfrak{V}$ be the signed Varchenko matrix of the COM $\mathcal{M} = (E, \mathcal{L})$. Then*

$$\det(\mathfrak{V}) = \prod_{Y \in \mathcal{L}} (1 - a(Y))^{b_Y}.$$

where $a(Y) := \prod_{e \in z(Y)} x_{e^+}x_{e^-}$ and b_Y are nonnegative integers that can be explicitly computed, see Remark 4.8.

Example 2.10 (continued). For our example the determinant of the signed Varchenko matrix factorizes to

$$\det(\mathfrak{V}) = (1 - x_1^+x_1^-)^2(1 - x_2^+x_2^-)^2(1 - x_3^+x_3^-)^2(1 - x_1^+x_1^-x_2^+x_2^-x_3^+x_3^-).$$

A corollary of this result, namely the case where $x_{e^-} = x_{e^+}$, which is the original version of the Varchenko matrix, has been already proven for OMs in [8]. We formulate it for COMs.

Corollary 2.11. *Let $\mathbf{V}$ be the (unsigned) Varchenko matrix (i.e. $x_{e^-} = x_{e^+} = x_e$) of the COM $\mathcal{M} = (E, \mathcal{L})$. Then*

$$\det(\mathbf{V}) = \prod_{Y \in \mathcal{L}} (1 - c(Y)^2)^{b_Y}.$$

where $c(Y) := \prod_{e \in z(Y)} x_e$ and b_Y are nonnegative integers.

Example 2.12 (continued). For our example the determinant of the (unsigned) Varchenko matrix factorizes to

$$\det(\mathfrak{V}) = (1 - x_1^2)^2(1 - x_2^2)^2(1 - x_3^2)^2(1 - x_1^2x_2^2x_3^2).$$

3. Preparation

We start with some basics about partially ordered sets $\mathcal{P}$ (*posets*). For an introduction we recommend [14]. One can associate an abstract simplicial complex $\Delta(\mathcal{P})$, called *order complex*, to every poset. The elements of $\mathcal{P}$ are the vertices of this complex and the chains (i.e. totally ordered subsets) the faces. Two posets are *homotopy equivalent* if their order complexes are homotopy equivalent. A poset is called *contractible* if its order complex is homotopy equivalent to a point. Clearly a poset is contractible if it has a unique minimal or a unique maximal element, since this element is contained in every chain. We introduce now the *Möbius function* μ of a poset:

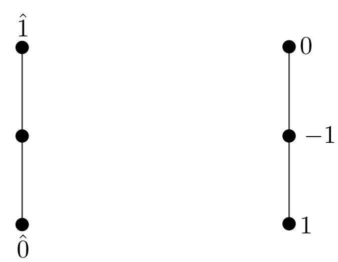
$$\mu(x,x) = 1 \text{ for all } x \in \mathcal{P}$$
$$\mu(x,y) = -\sum_{x \le z < y} \mu(x,z) \text{ for all } x < y \in \mathcal{P}.$$

The *bounded extension* $\hat{\mathcal{P}}$ of a poset is the poset together with a new maximal element $\hat{1}$ and a new minimal element $\hat{0}$. The *Möbius number* of $\mathcal{P}$ is defined by

$$\mu(\mathcal{P}) = \mu(\hat{0},\hat{1}),$$

where the right-hand-side is evaluated in $\hat{\mathcal{P}}$.

Example 3.1. Let us look at the poset $\mathcal{P}$ which consists only of one element. In the following its bounded extension and the value of the Möbius function of the elements of the bounded extension are depicted.



Hence, the Möbius number of the poset consisting of only one element is

$$\mu(\mathcal{P}) = \mu(\hat{0},\hat{1}) = 1 + (-1) = 0.$$

It follows from the following fact that the Möbius number is a topological invariant with respect to homotopic equivalence.

Theorem 3.2. [14, Philip Hall Theorem] *The Möbius number of a poset equals the reduced Euler characteristic of its order complex, i.e.*

$$\mu(\mathcal{P}) = \chi(\Delta(\mathcal{P})) - 1.$$

In particular we get the following corollary, whose second part follows from the definition of contractability and Example 3.1.

Corollary 3.3. *For two homotopy equivalent posets $\mathcal{P}$ and $\mathcal{Q}$ we have $\mu(\mathcal{P}) = \mu(\mathcal{Q})$. In particular, if $\mathcal{P}$ is contractible then $\mu(\mathcal{P}) = 0$.*

We denote for a poset $\mathcal{P}$ and $p \in \mathcal{P}$ by $\mathcal{P}_{\leq p}$ the subposet $\{q \in \mathcal{P} \mid q \leq p\}$.

Proposition 3.4 (Quillen Fiber Lemma)**.** *Let $\mathcal{P}$ and $\mathcal{Q}$ be posets and $f : \mathcal{P} \to \mathcal{Q}$ order preserving. If for all $q \in Q$ we have that $f^{-1}(\mathcal{Q}_{\leq q})$ is contractible, then $\mathcal{P}$ and $\mathcal{Q}$ are homotopy equivalent.*

We will now associate posets with COMs, so let $\mathcal{M} = (E, \mathcal{L})$ be a COM and let $R \in \{+,-\}^E$ be a fixed sign vector. We consider $\mathcal{T}$ as a poset with order relation

$$P \preceq_R Q \quad \text{if} \quad S(R,P) \subseteq S(R,Q).$$

We write $\mathcal{T}_R$ if we consider $\mathcal{T}$ with this partial order and we call R the *base pattern* of the poset.

Now we will introduce a theorem which will help us with our crucial Lemma 3.9.

Theorem 3.5. *Let $\mathcal{M} = (E, \mathcal{L})$ be a topal fiber of a COM $\mathcal{M}' = (E', \mathcal{L}')$, $R' \in \mathcal{T}'$ a tope of $\mathcal{M}'$ and $R = R'_{|E}$ its restriction to E. Then the order complex of $\mathcal{T}_R$ is contractible.*

Note that the restriction R in the statement of Theorem 3.5 is not necessarily a tope of $\mathcal{M}$. In order to apply the Quillen Fiber Lemma in the proof of Theorem 3.5 we need the following lemma. For this given $f \in E$

Lemma 3.6. *Let $f \in E$ and $R = \{+\}^E$. Let $\mathcal{T} \setminus f$ denote the set of topes of $\mathcal{M} \setminus f$ and $\mathcal{T} \backslash f_{R\backslash f}$ the corresponding tope poset with base pattern $R \backslash \{f\}$. Consider the order-preserving map $\pi_f : \mathcal{T}_R \to \mathcal{T} \backslash f_{R\backslash f}$ given by restriction. Let $Q \in \mathcal{T} \backslash f$. Then*

$$\pi_f^{-1}((\mathcal{T}\backslash f_{R\backslash f})_{\preceq Q}) = \mathcal{T}(Q^+, \emptyset).$$

Proof. Let $\tilde{Q} \in \mathcal{T} \setminus f_{\preceq Q}$. As $R \setminus f$ is all positive, we must have $\tilde{Q}^- \subseteq Q^-$ and hence $Q^+ \setminus f \subseteq \tilde{Q}^+$ implying $\pi_f^{-1}(\tilde{Q}) \subseteq \mathcal{T}(Q^+, \emptyset)$. If on the other hand $\hat{Q} \in \mathcal{T}(Q^+, \emptyset)$, then $\hat{Q}^- \subseteq Q^- \cup \{f\}$, $Q^+ \subseteq \hat{Q}^+ \cup \{f\}$. Hence $\pi_f(\hat{Q}) \preceq_{R\backslash f} Q$. ☐

We need two preparatory results. For the proof of Theorem 3.5, for the first one also see [7, Lemma 10]. We reprove it here, since in the presentation in [7] the signs are chosen the opposite way. Recall from Example 2.6 that $\mathcal{C}_n$ is the set of covectors of the OM of the directed cycle on n vertices.

Proposition 3.7. *Let $\mathcal{M} = (E, \mathcal{L})$ be a COM with tope set $\mathcal{T}$ and let $R = \{+\}^E$. If for all $f \in E$ we have $-_f R \in \mathcal{T}$, then the poset $\mathcal{T}_R$ is contractible.*

Proof. We will show by induction that all covectors which contain exactly one plus-entry and at least one minus-entry are in $\mathcal{L}$. Since then in particular all covectors which contain exactly one minus-entry and one plus-entry (i.e. the cocircuits) exist in $\mathcal{L}$, we get by (SE) that the all zero vector is in $\mathcal{L}$. Together, we can conclude that $\mathcal{L} = \mathcal{C}_n$, since we obtain all its covectors by composition of those vectors. Since $\mathcal{C}_n$ is uniform no other oriented matroid can contain these covectors.

So let $-_f R \in \mathcal{T}$ for all $f \in E$ and $-R \notin \mathcal{T}$. We will use induction over the number of zero-entries in the covectors, i.e. we want to show that for every $n = 0, \dots, |E| - 2$ all sign-vectors with n zero entries, one plus-entry and $|E| - (n+1)$ minus-entries are in $\mathcal{L}$.

$n = 0$: By the existence of $-_f R$ here is nothing to show. We fix $n \geq 0$ and assume that all covectors with n or less zero-entries, exactly one plus-entry and at least one minus-entry exist in $\mathcal{L}$.

$n \to n + 1 \leq |E| - 2$: We will show that there exists a covector with zero-entries in the i-th position, $i \in I \subset E$, $|I| = n + 1$, a plus-entry in the j-th position, $j \notin I$ and $-$ everywhere else. We choose an $\hat{i} \in I$ and consider two covectors with 0 in $I \backslash \hat{i}$, where one has a $+$ in the $\hat{i}$-th position and the other one in the j-th position and both have a $-$ everywhere else. These do exist by inductive assumption. W.l.o.g. those two covectors look like this:

$$\begin{gathered}(0, \dots, 0, \overbrace{+}^{\hat{i}}, - , -, \dots, -) \\ (\underbrace{0, \dots, 0}_{I \backslash \hat{i}}, - , \underbrace{+}_{j}, -, \dots, -).\end{gathered}$$

If we now perform strong elimination on $\hat{i}$ with those two covectors we get the covector

$$X = (\underbrace{0, \dots, 0}_{I \backslash \hat{i}}, \underbrace{0}_{\hat{i}}, \underbrace{*}_{j}, -, \dots, -).$$

If $*$ was $-$, then $X \circ T^j = \{-\}^E$. Since $\{-\}^E = -R \notin \mathcal{T}$ we have $* = +$ and have the covector we were looking for. We have shown that if $\mathcal{L} \neq \mathcal{C}_n$, then the poset $\mathcal{T}_R$ has a unique maximal element. In particular, it is contractible.

□

Lemma 3.8. *Let $\mathcal{M} = (E, \mathcal{L})$ be a topal fiber of a COM $\mathcal{M}' = (E', \mathcal{L}')$, $R' \in \mathcal{T}'$ a tope of $\mathcal{M}'$ and $R = R'_{|E}$ its restriction to E. If $\mathcal{L} = \mathcal{C}_n$, then $R \in \mathcal{L}$, in particular $R \neq \{+\}^E, \{-\}^E$.*

Proof. Let $\mathcal{M} = (E, \mathcal{L})$ be a COM such that there is a COM $\mathcal{M} = (E', \mathcal{L}')$, with $E \subset E'$ and $\mathcal{L} = \rho_{(S^+, S^-)}(\mathcal{L}')$ for some $S^+, S^- \subseteq E'$ and $\mathcal{L} = \mathcal{C}_n$. We saw in Example 2.6 that $\mathbf{0} \in \mathcal{C}_n$. By the definition of $\rho_{(S^+, S^-)}(\mathcal{L}')$ there exists $Z \in \mathcal{L}'$ with

$$Z_e = \begin{cases} + & \text{if } e \in S^+, \\ - & \text{if } e \in S^- \\ 0 & \text{else.} \end{cases}$$

Since the composition of Z with every other covector in $\mathcal{L}'$ is in $\mathcal{L}'$, we see that $\rho_{(S^+, S^-)}(\mathcal{L}') = \mathcal{L} \backslash \{S^+ \cup S^-\}$. So in this case $\mathcal{L} \backslash \{S^+ \cup S^-\} = \mathcal{C}_n$, so every tope $R' \in \mathcal{L}'$ restricted to E has to be in $\mathcal{C}_n$. Since $\{+\}^E, \{-\}^E \notin \mathcal{C}_n$, $R = R'|_E \neq \{+\}^E, \{-\}^E$.

□

Now we are in position to prove Theorem 3.5.

Proof of Theorem 3.5. Let $\mathcal{M} = (E, \mathcal{L})$ be a COM such that there is a COM $\mathcal{M} = (E', \mathcal{L}')$, with $E \subset E'$ and $\mathcal{L} = \rho_{(S^+,S^-)}(\mathcal{L}')$ for some $S^+, S^- \subseteq E'$, $R' \in \mathcal{T}'$ a tope of $\mathcal{M}'$ and $R = R'_{|E}$ its restriction to E. First we look at the case $\mathcal{L} = \mathcal{C}_n$. From Lemma 3.8 we know, R is a tope of $\mathcal{C}_n$ and hence different from $\{+\}^E$ and $\{-\}^E$. But since $\mathbf{0} \in \mathcal{C}_n$, $\mathbf{0} \circ -R = -R \in \mathcal{C}_n$. So we have a unique maximal element and $\mathcal{T}_R$ is contractible. Now let $\mathcal{L} \neq \mathcal{C}_n$. Possibly reorienting elements we may assume that $R = \{+\}^E$. We proceed by induction on $|E|$. If $|E| = 1$ then $\mathcal{T}_R$ either is a singleton or a chain of length 2 and thus contractible. Hence assume $|E| \geq 2$. If for all $f \in E$ there exists $-_fR$ as in Proposition 3.7, then $\mathcal{T}_R$ is contractible by Proposition 3.7. Hence we may assume that there exists $f \in E$ such that $-_fR \notin \mathcal{T}$. Let $\mathcal{T}\backslash f_{R\backslash f}$ denote the tope poset in $\mathcal{L} \setminus f$ with base pattern $R \setminus f$. Since the class of COMs is closed under deletion, we know that $\mathcal{L}' \setminus f$ is a COM. Since $\mathcal{L}$ evolved from $\mathcal{L}'$ by setting $\mathcal{L} = \rho_{(S^+,S^-)}\mathcal{L}'$ for some S^+ and S^-, $\mathcal{L} \setminus f$ evolves in the same way from $\mathcal{L}' \setminus f$, i.e. $\mathcal{L} \setminus f = \rho_{(S^+,S^-)}(\mathcal{L}' \setminus f)$ (note that f cannot be in $S^+ \cup S^-$, since $E = E' \setminus (S^+ \cup S^-)$ and $f \in E$). Also $R \setminus f$ is the restriction to E of the tope $R' \setminus f$ of $\mathcal{L}' \setminus f$. We see that $\mathcal{L} \setminus f$ together with $R \setminus f$ fulfills the assumptions of the theorem. Furthermore, $\mathcal{L} \setminus f \neq \mathcal{C}_n$, this follows from Lemma 3.8. Hence, $\mathcal{T}\backslash f_{R\backslash f}$ is contractible by inductive assumption. We now want to show that $\mathcal{T}_R$ and $\mathcal{T} \setminus f_{R\backslash f}$ are homotopy equivalent by using Proposition 3.4. So consider the order-preserving map $\pi_f : \mathcal{T}_R \to \mathcal{T}\backslash f_{R\backslash\{f\}}$ given by restriction. Let $Q \in \mathcal{T}\backslash f_{R\backslash\{f\}}$. By Lemma 3.6

$$\pi_f^{-1}((\mathcal{T} \setminus f_{R\backslash\{f\}})_{\preceq Q}) = \mathcal{T}(Q^+, \emptyset).$$

$\mathcal{T}(Q^+, \emptyset)$ is the set of topes of $\rho_{(S^+\cup Q^+,S^-)}\mathcal{L}'$. If $Q^+ \neq \emptyset$, then $\rho_{(S^+\cup Q^+,S^-)}\mathcal{L}'$ has fewer elements than $\mathcal{L}$. Furthermore, by Lemma 3.8, $\rho_{(S^+\cup Q^+,S^-)}\mathcal{L}' \neq \mathcal{C}_n$. Hence $\pi_f^{-1}(\mathcal{T} \setminus f_{\preceq Q})$ is contractible by inductive assumption. If $Q^+ = \emptyset$ then by the choice of f the preimage $\pi_f^{-1}(Q)$ is the all minus vector. Hence, this is the unique maximal element in $\pi_f^{-1}(\mathcal{T} \setminus f_{\preceq Q})$ and that fiber is also contractible. So by Proposition 3.4 $\mathcal{T}_R$ and $\mathcal{T} \setminus f_{R\backslash f}$ are homotopy equivalent and the claim follows. $\square$

For $e \in E$ and $R \in \mathcal{T}$ we write $\mathcal{T}_{R,e}$ for the poset $\{T \in \mathcal{T} \mid T_e = -R_e\} \cup \{\hat{0}\}$ with $\hat{0}$ as its artificial least element and the remaining poset structure induced from $\mathcal{T}_R$. For $P \in \mathcal{T}_{R,e}$ we write $(\hat{0}, P)_{R,e}$ for the interval from $\hat{0}$ to P in $\mathcal{T}_{R,e}$.

Lemma 3.9. *Let $\mathcal{M} = (E, \mathcal{L})$ be a COM, $R \in \mathcal{T}$ a tope, $e \in E$ an element, $P \in \mathcal{T}_{R,e}$ and S such that $e \notin S \subseteq E$. Then*

$$\sum_{\substack{Q \in \mathcal{T}(\emptyset,\{e\}) \\ S = S(P,Q) \cap S(Q,R)}} \mu((\hat{0}, Q)_{R,e}) = \begin{cases} -1 & \textit{if} \quad S = \emptyset \\ 0 & \textit{if} \quad S \neq \emptyset \end{cases} \tag{1}$$

and

$$\sum_{\substack{Q \in \mathcal{T}(\{e\},\emptyset) \\ S = S(P,Q) \cap S(Q,R)}} \mu((\hat{0}, Q)_{R,e}) = \begin{cases} -1 & \textit{if} \quad S = \emptyset \\ 0 & \textit{if} \quad S \neq \emptyset \end{cases}. \tag{2}$$

Proof. In order to prove (1) we assume $R = \{+\}^E$. We prove the assertion by induction on $|S|$. If $S = \emptyset$ then

$$\sum_{\substack{Q\in\mathcal{T}(\emptyset,\{e\})\\ S=S(P,Q)\cap S(Q,R)}} \mu((\hat{0},Q)_{R,e}) \quad = \quad \sum_{\hat{0}<_{R,e}Q\leq_{R,e}P} \mu((\hat{0},Q)_{R,e})$$

Note that $\sum_{\hat{0}<_{R,e}Q\leq_{R,e}P} \mu((\hat{0},Q)_{R,e}) = \mu\left(\{Q\in\mathcal{T}_{R,e}\mid \hat{0}\leq_{R,e} Q \leq_{R,e} P\}\right) - \mu((\hat{0},\hat{0})_{R,e})$. This poset has the maximal element P, so it is contractible and has Möbius number 0. Therefore we have

$$\sum_{\substack{Q\in\mathcal{T}(\emptyset,\{e\})\\ S=S(P,Q)\cap S(Q,R)}} \mu((\hat{0},Q)_{R,e}) = -\mu((\hat{0},\hat{0})_{R,e}) = -1.$$

Assume $|S| > 0$. Set

$$T^+ = \{f \in E\setminus(S\cup\{e\}) \mid P_f = +\}.$$

Then

$$\sum_{\substack{Q\in\mathcal{T}(\emptyset,\{e\})\\ S(P,Q)\cap S(Q,R)\subseteq S}} \mu((\hat{0},Q)_{R,e}) \quad = \quad \sum_{Q\in\mathcal{T}(T^+,\{e\})} \mu((\hat{0},Q)_{R,e}). \tag{3}$$

$\mathcal{T}(T^+,\{e\})$ is isomorphic to the set of topes of the COM $\rho_{(T^+,\{e\})}(\mathcal{L})$ which is contractible by Theorem 3.5. So the right hand side of (3) ranges over the elements of a contractible poset. By the same argument as above it is $-\mu(\hat{0},\hat{0}) = -1$ minus the Möbius number of the poset. Since the poset is contractible its Möbius number is 0 and we have shown that

$$\sum_{\substack{Q\in\mathcal{T}(\emptyset,\{e\})\\ S(P,Q)\cap S(Q,R)\subseteq S}} \mu((\hat{0},Q)_{R,e}) \quad = \quad -1. \tag{4}$$

Now rewrite the left hand side of (4) as

$$\sum_{\substack{Q\in\mathcal{T}(\emptyset,\{e\})\\ S(P,Q)\cap S(Q,R)\subseteq S}} \mu((\hat{0},Q)_{R,e}) \quad = \quad \sum_{T\subseteq S}\ \sum_{\substack{Q\in\mathcal{T}(\emptyset,\{e\})\\ S(P,Q)\cap S(Q,R)=T}} \mu((\hat{0},Q)_{R,e}) \tag{5}$$

By induction the summand $\displaystyle\sum_{\substack{Q\in\mathcal{T}(\emptyset,\{e\})\\ S(P,Q)\cap S(Q,R)=T}} \mu((\hat{0},Q)_{R,e})$ is 0 for $T \neq S,\emptyset$ and -1 for $T=\emptyset$.

Thus combining (4) and (5) we obtain:

$$\begin{aligned} -1 &= \sum_{\substack{Q\in\mathcal{T}(\emptyset,\{e\})\\ S(P,Q)\cap S(Q,R)\subseteq S}} \mu((\hat{0},Q)_{R,e}) \\ &= -1 + \sum_{\substack{Q\in\mathcal{T}(\emptyset,\{e\})\\ S(P,Q)\cap S(Q,R)=S}} \mu((\hat{0},Q)_{R,e}) \end{aligned}$$

From this we conclude

$$\sum_{\substack{Q\in\mathcal{T}(\emptyset,\{e\})\\ S(P,Q)\cap S(Q,R)=S}} \mu((\hat{0},Q)_{R,e})=0.$$

The second claim follows analogously by reorienting all the signs. $\square$

We conclude this section with another result on contractability needed in the main proof. We start with a lemma:

Lemma 3.10. *Let $\mathcal{M}=(E,\mathcal{L})$ be a COM, $X\in\mathcal{L}$ and $P\in\mathcal{T}$. The tope $Q=X\circ P\in$ $\mathrm{star}(X)$ is the only tope in $\mathrm{star}(X)$ such that for all $O\in\mathrm{star}(X)$ we have*

$$S(P,O)=S(P,Q)\cup S(Q,O) \tag{6}$$

$$\emptyset=S(P,Q)\cap S(Q,O). \tag{7}$$

Proof. It is easy to see that Q fulfills (6) and (7). Let us assume there is another tope $Q^*\neq Q$ in $\mathrm{star}(X)$ which has this property. By the definition of Q we have

$$S(Q,O)=S(P,O)\cap z(X) \text{ and } S(P,Q)=(P,O)\backslash z(X) \text{ for all } O\in\mathrm{star}(X).$$

Since $Q^*\neq Q$ and $S(Q^*,O)$ can only contain elements from $z(X)$, $S(P,Q^*)$ has to contain at least one element from $z(X)$. Now considering $O^*=(X\circ -Q^*)\in\mathrm{star}(X)$ we see that $S(P,Q^*)\cap S(Q^*,O^*)\neq\emptyset$, so Q^* does not fulfill the property and we have a contradiction. $\square$

For $e\in E$ and $P\in\mathcal{T}$ we say that e *defines a proper face* of P if there is a covector $X\in\mathcal{L}$ with $X\leq P$ and $X_e=0$ with $X\neq\mathbf{0}$. Note that in this case there is a unique maximal such covector, namely the composition of all of them. Otherwise, we say that e *does not define a proper face* of P.

Theorem 3.11. *Let $\mathcal{M}=(E,\mathcal{L})$ be a COM, $R\in\mathcal{T}$ a tope, and let $e\in E$ define a proper face of R. Let $Y\in\mathcal{L}$ be the maximal covector such that $Y\leq R$ and $Y_e=0$ and choose $P_{top}\in\mathcal{T}_{R,e}\setminus\mathrm{star}(Y)$. Then $(\hat{0},P_{top})_{R,e}$ is contractible. In particular, $\mu((\hat{0},P_{top})_{R,e})=0$.*

Proof. Let $P\in(\hat{0},P_{top})_{R,e}$. Then by Lemma 3.10 the tope $Q=Y\circ P\in\mathrm{star}(Y)$ is the unique tope in $\mathrm{star}(Y)$ such that for all $O\in\mathrm{star}(Y)$ we have

$$\begin{aligned} S(P,O) &= S(P,Q)\cup S(Q,O)\\ \emptyset &= S(P,Q)\cap S(Q,O). \end{aligned}$$

Since $Y_e=0$ and $P\in\mathcal{T}_{R,e}$ it also follows that $Q_e=-$. Since $Y\leq R$, clearly $S(R,Q)=S(R,Y\circ P)\subseteq S(R,P)$ and hence $Q\preceq_R P$. This shows $Q\in(\hat{0},P_{top})_{R,e}$. We now define the map

$$\begin{aligned} &\circ_Y:(\hat{0},P_{top})_{R,e}\to(\hat{0},P_{top})_{R,e}\\ &\circ_Y(P)=Y\circ P \end{aligned}$$

and prove that it is a closure operator by showing that it is order preserving and idempotent (i.e. $\circ_Y(\circ_Y(P))=\circ_Y(P)$). So let $Q\preceq_R Q'$. Then $Y\circ Q\preceq_R Y\circ Q'$. Since $Y\leq R$ it follows that $Y \circ Q \preceq_R Q$. Obviously $Y \circ (Y \circ Q) = Y \circ Q$. So $\circ_Y$ is a closure operator and it follows that $(\hat{0}, P_{top})_{R,e}$ is homotopy equivalent to its image (see e.g, [3, Corollary 10.12]).

Since $P_{top} \notin \mathrm{star}(Y)$ and $Y \circ P_{top} \in \mathrm{star}(Y) \cap (\hat{0}, P_{top})_{R,e}$, it also follows that $Y \circ Q \preceq_R Y \circ P_{top}$ for all $Q \in (\hat{0}, P_{top})_{R,e}$. Hence the image of $\circ_Y$ has a unique maximal element and hence is contractible. □

4. Main Proof

In this Section we assume that $\mathcal{M} = (E, \mathcal{L})$ is a COM with topes $\mathcal{T}$ and signed Varchenko matrix $\mathfrak{V}$. Recall, that we assume $\mathcal{T}$ to be linearly ordered. Note however that swapping two topes leads to a row swap and a columns swap at the same time, so we do not change the sign of our determinant. Hence, in this section we will rearrange the ordering on $\mathcal{T}$, whenever convenient for the proof. Moreover, for the proof we also fix a linear ordering on E, i.e., $E = \{e_1 \prec \cdots \prec e_r\}$.

For any sign vector $\epsilon = (\epsilon_1, \epsilon_2) \in \{+,-\}^2$ let $\mathfrak{V}^{e,\epsilon}$ be a matrix with rows indexed by $\mathcal{T}(\{e\}, \emptyset)$ for $\epsilon_1 = +$, $\mathcal{T}(\emptyset, \{e\})$ for $\epsilon_1 = -$ and columns indexed by $\mathcal{T}(\{e\}, \emptyset)$ for $\epsilon_2 = +$, $\mathcal{T}(\emptyset, \{e\})$ for $\epsilon_2 = -$. For a tope R indexing a row and a tope Q indexing a column we set $\mathfrak{V}^{e,\epsilon}_{R,Q} = \mathfrak{V}_{R,Q}$. After reordering $\mathcal{T}$ this yields a block decomposition of $\mathfrak{V}$ as

$$\mathfrak{V} = \begin{pmatrix} \mathfrak{V}^{e,(-,-)} & \mathfrak{V}^{e,(-,+)} \\ \mathfrak{V}^{e,(+,-)} & \mathfrak{V}^{e,(+,+)} \end{pmatrix}. \tag{8}$$

We fix such a linear ordering on $\mathcal{T}$ and set $\mathcal{M}^e$ to

$$\mathcal{M}^e_{Q,R} = \begin{cases} 1 & \text{if} & Q = R \\ -\mu((\hat{0}, Q)_{R,e})\,\mathfrak{V}_{Q,R} & \text{if} & e \text{ is the maximal element of } S(Q,R), \\ 0 & \text{otherwise} & \end{cases}.$$

Note that this matrix has the following form

$$\mathcal{M}^e = \begin{pmatrix} \mathcal{I}^e_\ell & U^e \\ L^e & \mathcal{I}^e_m \end{pmatrix},$$

where

$$\begin{aligned} U^e_{Q,R} &= -\mu((\hat{0}, Q)_{R,e})\,\mathfrak{V}_{Q,R}, && e \text{ is the maximal element of } \mathrm{S}(Q,R), \\ & && Q \in \mathcal{T}(\emptyset, \{e\}), R \in \mathcal{T}(\{e\}, \emptyset), \\ L^e_{Q,R} &= -\mu((\hat{0}, Q)_{R,e})\,\mathfrak{V}_{Q,R}, && e \text{ is the maximal element of } \mathrm{S}(Q,R), \\ & && Q \in \mathcal{T}(\{e\}, \emptyset), R \in \mathcal{T}(\emptyset, \{e\}) \end{aligned}$$

and $\mathcal{I}$ the identity matrix with $\ell = \#\mathcal{T}(\emptyset, \{e\})$ and $m = \#\mathcal{T}(\{e\}, \emptyset)$.

Lemma 4.1. *Let e be the maximal element of E. Then $\mathfrak{V}^{e,(-,+)}$ factors as*

$$\mathfrak{V}^{e,(-,+)} = \mathfrak{V}^{e,(-,-)} \cdot U^e \tag{9}$$

and $\mathfrak{V}^{e,(+,-)}$ *as*

$$\mathfrak{V}^{e,(+,-)} = \mathfrak{V}^{e,(+,+)} \cdot L^e. \tag{10}$$

Proof. Let us prove (9) first. For $P \in \mathcal{T}(\emptyset, \{e\})$ and $R \in \mathcal{T}(\{e\}, \emptyset)$ the entry in row P and column R on the left hand side of (9) is $\mathfrak{V}_{P,R}$. On the right hand side the corresponding entry is:

$$\sum_{Q \in \mathcal{T}(\emptyset,\{e\})} \mathfrak{V}_{P,Q} \cdot U^e_{Q,R} \quad = \quad - \sum_{Q \in \mathcal{T}(\emptyset,\{e\})} \mu((\hat{0}, Q)_{R,e}) \cdot \mathfrak{V}_{P,Q} \cdot \mathfrak{V}_{Q,R}$$

This follows from the fact that e is the maximal element of any separator of the topes indexing U^e. By definition we have for $Q \in \mathcal{T}(\emptyset, \{e\})$

$$\mathfrak{V}_{P,Q} \cdot \mathfrak{V}_{Q,R} \quad = \quad \mathfrak{V}_{P,R} \cdot \prod_{f \in S(P,Q) \cap S(Q,R)} x_{f^+} x_{f^-}.$$

We see that $\mathfrak{V}_{P,Q} \cdot \mathfrak{V}_{Q,R} = \mathfrak{V}_{P,R}$ if $S(P,Q) \cap S(Q,R) = \emptyset$. Thus the claim of the lemma is proved once we have shown that for a fixed subset $S \subseteq E$ and fixed P, R we have:

$$\sum_{\substack{Q \in \mathcal{T}(\emptyset,\{e\}) \\ S = S(P,Q) \cap S(Q,R)}} \mu((\hat{0}, Q)_{R,e}) = \begin{cases} 0 & \text{if } S \neq \emptyset \\ -1 & \text{otherwise.} \end{cases} . \tag{11}$$

But this is the content of Lemma 3.9 and we are done. For (10) the right hand side is

$$\sum_{Q \in \mathcal{T}(\{e\},\emptyset)} \mathfrak{V}_{P,Q} \cdot L^e_{Q,R} = \qquad - \sum_{Q \in \mathcal{T}(\{e\},\emptyset)} \mu((\hat{0}, Q)_{R,e}) \cdot \mathfrak{V}_{P,Q} \cdot \mathfrak{V}_{Q,R}$$

and we can proceed analogous to the proof above. □

Next we use the matrices $\mathcal{M}^e$ to factorize $\mathfrak{V}$. The following lemma yields the base case for the inductive step in the factorization.

Lemma 4.2. *Let e be the maximal element of E and let $\mathfrak{V}_{x_e=0}$ be the matrix $\mathfrak{V}$ after evaluating x_{e^+} and x_{e^-} to 0. Then*

$$\mathfrak{V} = \mathfrak{V}_{x_e=0} \cdot \mathcal{M}^e$$

Proof. Let $\mathcal{T}$ be in that order, that we get the block decomposition (8) of $\mathfrak{V}$. Using lemma Lemma 4.1, we see that

$$\mathfrak{V} = \begin{pmatrix} \mathfrak{V}^{e,(-,-)} & \mathfrak{V}^{e,(-,+)} \\ \mathfrak{V}^{e,(+,-)} & \mathfrak{V}^{e,(+,+)} \end{pmatrix} = \begin{pmatrix} \mathfrak{V}^{e,(-,-)} & 0 \\ 0 & \mathfrak{V}^{e,(+,+)} \end{pmatrix} \cdot \begin{pmatrix} \mathcal{I}^e_\ell & U^e \\ L^e & \mathcal{I}^e_m \end{pmatrix} \tag{12}$$

$$= \begin{pmatrix} \mathfrak{V}^{e,(-,-)} & 0 \\ 0 & \mathfrak{V}^{e,(+,+)} \end{pmatrix} \cdot \mathcal{M}^e. \tag{13}$$

Now the monomial $\mathfrak{V}_{P,Q}$ has a factor x_{e^+} or x_{e^-} if and only if $P \in \mathcal{T}(\emptyset, \{e\})$ and $Q \in \mathcal{T}(\{e\}, \emptyset)$ or $P \in \mathcal{T}(\{e\}, \emptyset)$ and $Q \in \mathcal{T}(\emptyset, \{e\})$. Hence

$$\mathfrak{V}_{x_e=0} \quad = \quad \begin{pmatrix} \mathfrak{V}^{e,(-,-)} & 0 \\ 0 & \mathfrak{V}^{e,(+,+)} \end{pmatrix}. \tag{14}$$

Combining (12) and (14) yields the claim. □

Now we are in position to state and prove the crucial factorization.

Proposition 4.3. *Let $E = \{e_1 \prec \cdots \prec e_r\}$ be a fixed ordering. Then*

$$\mathfrak{V} = \mathcal{M}^{e_1} \cdots \mathcal{M}^{e_r}.$$

Proof. We will prove by downward induction on i that

$$\mathfrak{V} = \mathfrak{V}_{x_i=\cdots=x_r=0} \cdot \mathcal{M}^{e_i} \cdots \mathcal{M}^{e_r}. \tag{15}$$

For $i = r$ the assertion follows directly from Lemma 4.2. For the inductive step assume $i > 1$ and (15) holds for i. We know from Lemma 4.2 that if we choose a linear ordering on E for which e_{i-1} is the largest element then

$$\mathfrak{V} = \mathfrak{V}_{x_{i-1}=0} \cdot \mathcal{N}, \tag{16}$$

where $\mathcal{N} = (N_{Q,R})_{Q,R\in\mathcal{T}}$ is defined as

$$N_{Q,R} = \begin{cases} 1 & \text{if} \quad Q = R \\ -\mu((\hat{0}, Q)_{R,e_{i-1}})\,\mathfrak{V}_{Q,R} & \text{if} \quad e_{i-1} \in \mathrm{Sep}(Q,R) \\ 0 & \text{otherwise} \end{cases},$$

Since $\mathcal{N} = \mathcal{M}^{e_{i-1}}$ for this particular ordering. Now we go back to the ordering in the assumption and set $x_i = \cdots = x_r = 0$ in $\mathcal{N}$. We see that

$$(N_{Q,R})_{x_i=\cdots=x_r=0} \begin{cases} 1 & \text{if } Q = R \\ -\mu((\hat{0}, Q)_{R,e_{i-1}})\,\mathfrak{V}_{Q,R} & \text{if } e_{i-1} \text{ is the largest element in } S(Q,R) \\ 0 & \text{otherwise} \end{cases}.$$

But then $\mathcal{N}_{x_i=\cdots=x_r=0} = \mathcal{M}^{e_{i-1}}$.

Now (16) implies

$$\begin{aligned} \mathfrak{V}_{x_i=\cdots=x_r=0} &= \mathfrak{V}_{x_{i-1}=\cdots=x_r=0} \cdot \mathcal{N}_{x_i=\cdots=x_r=0} \\ &= \mathfrak{V}_{x_{i-1}=\cdots=x_r=0} \cdot \mathcal{M}^{e_{i-1}} \end{aligned}$$

With the induction hypothesis this completes the induction step by

$$\begin{aligned} \mathfrak{V} &= \mathfrak{V}_{x_i=\cdots=x_r=0} \cdot \mathcal{M}^{e_i} \cdots \mathcal{M}^{e_r} \\ &= \mathfrak{V}_{x_{i-1}=x_i=\cdots=x_r=0} \cdot \mathcal{M}^{e_{i-1}} \cdots \mathcal{M}^{e_r}. \end{aligned}$$

For $i = 1$ the matrix $\mathfrak{V}_{x_1=\cdots=x_r=0}$ is the identity matrix. Thus (15) yields:

$$\mathfrak{V} = \mathcal{M}^{e_1} \cdots \mathcal{M}^{e_r}.$$

□

Before we prove the following proposition, we quote [8, Corollary 3], which is a result for oriented matroids.

Lemma 4.4. *Let $\mathbf{0} \in \mathcal{L}$ and let $P \in \mathcal{T}_{R,e}$ such that e does not define a proper face of P. Then the Möbius number $\mu((\hat{0}, P)_{R,e})$ is 0 if $-R \neq P$ and $(-1)^{\mathrm{rank}(\mathcal{L})}$ if $-R = P$.*

Now let $Y \in \mathcal{L}$ and $e \in z(Y)$ be the maximal element of $z(Y)$. Define $\mathcal{T}^{Y,e}$ as the set of topes $P \in \mathcal{T}$ such that Y is the maximal element of $\mathcal{L}$ for which $Y_e = 0$ and $Y \leq P$.

Proposition 4.5. *For any pair of topes $Q, R \in \mathcal{T}^{Y,e}$ we have*

$$\mu((\hat{0}, Q)_{R,e}) = \begin{cases} (-1)^{\operatorname{rank}(\mathcal{L}|_{z(Y)})} & \text{if} \quad Q_{z(Y)} = -R_{z(Y)} \\ 0 & \text{otherwise} \end{cases}.$$

Proof. By the definition of $\mathcal{T}^{Y,e}$ we have $Y \leq Q, R$. Thus, if we consider the poset $\mathcal{T}_{R|_{z(Y)},e}$ in the restriction $\mathcal{L}|_{z(Y)}$ we find that the interval $(\hat{0}, Q)_{R,e}$ is isomorphic to $(\hat{0}, Q|_{z(Y)})_{R|_{z(Y)},e}$. We saw in Section 2, that $(E\backslash\underline{Y}, F(Y)\backslash\underline{Y})$ is an OM. Further, $\mathcal{T}_{R|_{z(Y)},e}$ is a poset and $(\hat{0}, Q|_{z(Y)})_{R|_{z(Y)},e}$ is an interval in this particular OM. Furthermore, since Y is the maximal element satisfying $Y_e = 0$ and $Y \leq Q$, e does not define a proper face of $Q|_{z(Y)}$. Since our interval is in an OM, we can use Lemma 4.4 and the claim follows. □

We define $b_{Y,e} = 0$ if e is not the maximal element of $z(Y)$ and $\frac{1}{2}\#\mathcal{T}^{Y,e}$ otherwise. Since $P \mapsto Y \circ (-P)$ is a perfect pairing on $\mathcal{T}^{Y,e}$ it follows that $\mathcal{T}^{Y,e}$ contains an even number of topes. In particular, $b_{Y,e}$ is a nonnegative integer. We denote by $\mathcal{M}^{Y,e}$ the submatrix of $\mathcal{M}^e$ obtained by selecting rows and columns indexed by $\mathcal{T}^{Y,e}$.

Lemma 4.6. *Let $Y \in \mathcal{L}$ and $e \in z(Y)$. If $\mathcal{T}^{Y,e} \neq \emptyset$. then*

$$\det(\mathcal{M}^{Y,e}) = (1 - a(Y))^{b_{Y,e}}$$

where $a(Y) := \prod_{e \in z(Y)} x_{e^+} x_{e^-}$.

Proof. If $Q_{z(Y)} = -R_{z(Y)}$ then $\mathfrak{V}_{Q,R} = \prod_{e \in z(Y), Q_e = *} x_{e^*}$. Using the definition of $\mathcal{M}^e$ and Proposition 4.5 we find

$$\mathcal{M}^{Y,e}_{Q,R} = \begin{cases} 1 & \text{if} & Q = R \\ -(-1)^{\operatorname{rank}(\mathcal{L}|_{z(Y)})} \prod_{e \in z(Y), Q_e = *} x_{e^*} & \text{if} & Q = Y \circ (-R) \\ & & e \text{ largest element of } S(Q,R) \\ 0 & \text{otherwise} & \end{cases}.$$

We order rows and columns of $\mathcal{M}^{Y,e}$ so that the elements R and $Y \circ (-R)$ are paired in consecutive rows and columns. With this ordering $\mathcal{M}^{Y,e}$ is a block diagonal matrix having along its diagonal $b_{Y,e}$ two by two matrices

$$\begin{pmatrix} 1 & -(-1)^{\operatorname{rank}(\mathcal{L}|_{z(Y)})} \prod_{e \in z(Y), R_e = *} x_{e^*} \\ -(-1)^{\operatorname{rank}(\mathcal{L}|_{z(Y)})} \prod_{e \in z(Y), -R_e = *} x_{e^*} & 1 \end{pmatrix}$$

if e is the maximal element of $z(Y)$ and identity matrices otherwise. In any case we find $\det(\mathcal{M}^{Y,e}) = (1 - a(Y))^{b_{Y,e}}$ as desired. □

Lemma 4.7. *After suitably ordering $\mathcal{T}$ the matrix $\mathcal{M}^e$ is the block lower triangular matrix with the matrices $\mathcal{M}^{Y,e}$ for $Y \in \mathcal{L}$ with $Y_e = 0$ and $\mathcal{T}^{Y,e} \neq \emptyset$ on the main diagonal.*

Proof. If $Y = \mathbf{0}$, $\mathcal{T}^{Y,e}$ would be empty, so we assume that this is not the case. We fix a linear ordering of $\mathcal{T}$ such that for fixed $e \in E$ and $Y \in \mathcal{L}$ the topes from $\mathcal{T}^{Y,e}$ form an interval and such that the topes from $\mathcal{T}^{Y,e}$ precede those of $\mathcal{T}^{Y',e}$ if $Y < Y'$.

For this order the claim follows if we show that the entry $(\mathcal{M}^e)_{Q,R}$ is zero whenever $Q \in \mathcal{T}^{Y',e}$, $R \in \mathcal{T}^{Y,e}$ and $Y' < Y$.

If $Q_e = R_e$ then by $Q \neq R$ we have $(\mathcal{M}^e)_{Q,R} = 0$. Hence it suffices to consider the case $Q_e \neq R_e$. Since $Y \neq \mathbf{0}$, e is a proper face of R.

If $Q \notin \text{star}(Y)$, $Q \in \mathcal{T}(\emptyset, \{e\})$ and $R \in \mathcal{T}(\{e\}, \emptyset)$ then it follows from Theorem 3.11 that $\mu((\hat{0}, Q)_{R,e}) = 0$ and therefore $(\mathcal{M}^e)_{Q,R} = 0$. Analogously if $Q \notin \text{star}(Y)$, $Q \in \mathcal{T}(\{e\}, \emptyset)$ and $R \in \mathcal{T}(\emptyset, \{e\})$ then $\mu((\hat{0}, Q)_{R,e}) = 0$ and therefore $(\mathcal{M}^e)_{Q,R} = 0$.

On the other hand, if $Q \in \text{star}(Y)$, then in particular $Y \leq Q$. Since by definition of $\mathcal{T}^{Y',e}$ we have that Y' is the maximal covector such that $Y' \leq Q$ and $Y'_e = 0$ it follows that $Y \leq Y'$. Since $Y \neq Y'$ we must have that $Y < Y'$, i.e. $(\mathcal{M}^e)_{Q,R}$ is an entry below the diagonal and we are done.

□

Proof of Theorem 2.9. After fixing a linear order on E it follows from Proposition 4.3 that $\det \mathfrak{V}$ is the product of the determinants of $\mathcal{M}^e$ for $e \in E$. By Lemma 4.7 the determinant of each $\mathcal{M}^e$ is a product of determinants of $\mathcal{M}^{Y,e}$ for $e \in E$ and $Y \in \mathcal{L}$ for which $\mathcal{T}^{Y,e} \neq \emptyset$. Then Lemma 4.6 completes the proof. □

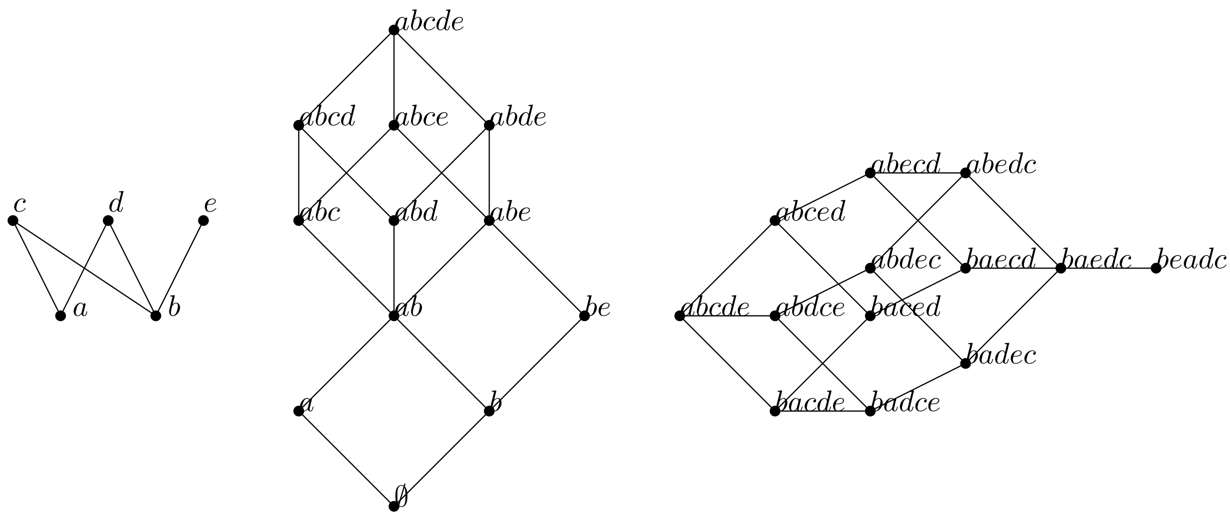
Remark 4.8 (Description of b_Y). In Theorem 2.9 we describe b_Y as a nonnegative integer, but this can be made more precise: Fix any linear order on E and let e_Y be the maximal element of $z(Y)$. From Lemma 4.6 we deduce, that $b_Y = b_{Y,e_Y}$. Thus $2b_Y$ counts the topes $P \in \mathcal{T}$ such that Y is the maximal element of $\mathcal{L}$ for which $Y_{e_Y} = 0$ and $Y \leq P$. In particular, b_Y does not depend on the choice of the linear ordering on E.

5. Applications

We give two applications of our formula for the Varchenko determinant on two COMs associated to a poset $\mathcal{P}$: its lattice of ideals and its set of linear extensions. As an example we will use the poset $\mathcal{Q}$ in Figure 1.

5.1. **Distributive Lattices.**

By the Fundamental Theorem of Finite Distributive Lattices, for every distributive lattice L there exists a poset $\mathcal{P}$, such that ordering the ideals (downward closed sets) of $\mathcal{P}$ by inclusion yields a lattice isomorphic to L. The topes of the COM associated to L correspond to the ideals of $\mathcal{P}$, the empty set can be seen as the all-plus vector, the ground set E of this COM is the ground set of $\mathcal{P}$, and the separator of two ideals I, I' is the symmetric difference $I \Delta I'$. So this allows, to quickly write down the (unsigned) Varchenko matrix $\mathbf{V}_L$ of L. In our example we indicate $\mathbf{V}_L(\mathcal{P})$ in the following way, where we just display the elements of the symmetric difference of two ideals to make it easier to read. Note that in order to get the Varchenko matrix itself one has to exchange a string $s_1 \dots s_k$ for the product $\prod_{s \in S} x_s$. The $\emptyset$ translates therefore to the empty product, which is 1.



$\mathcal{Q}$ $\qquad$ $L(\mathcal{Q})$ $\qquad$ $X(\mathcal{Q})$

FIGURE 1. A poset $\mathcal{Q}$, its lattice $L(\mathcal{Q})$ of ideals and its set $X(\mathcal{Q})$ of linear extensions. Edges in the graphs in the middle and on the right are drawn if endpoints correspond to topes with separator consisting of a single element. Edges corresponding to the same element are parallel.

$$\begin{pmatrix}
\emptyset & a & b & ab & be & abc & abd & abe & abcd & abce & abde & abcde \\
a & \emptyset & ab & b & abe & bc & bd & be & bcd & bce & bde & bcde \\
b & ab & \emptyset & a & e & ac & ad & ae & acd & ace & ade & acde \\
ab & b & a & \emptyset & ae & c & d & e & cd & ce & de & cde \\
be & abe & e & ae & \emptyset & ace & ade & a & acde & ac & ad & acd \\
abc & bc & ac & c & ace & \emptyset & cd & ce & d & e & cde & de \\
abd & bd & ad & d & ade & cd & \emptyset & de & c & cde & e & ce \\
abe & be & ae & e & a & ce & de & \emptyset & cde & c & d & cd \\
abcd & bcd & acd & cd & acde & d & c & cde & \emptyset & de & ce & e \\
abce & bce & ace & ce & ac & e & ced & c & de & \emptyset & cd & d \\
abde & bde & ade & de & ad & cde & e & d & ce & cd & \emptyset & c \\
abcde & bcde & acde & cde & acd & de & ce & cd & e & d & c & \emptyset
\end{pmatrix}$$

More generally, given two antichains $A' \subseteq A$, the set of ideals $\{\downarrow A'' \mid A' \subseteq A'' \subseteq A\}$ corresponds to the covector $Y(A', A)$, that is 0 on $A \setminus A'$, $-$ on all elements in or below A', and $+$ on all elements above A. In particular, when $A' = A$ we get a tope corresponding to the ideal $\downarrow A'$ and the all $-$-tope corresponds to the empty ideal. Now, if we pick a linear ordering on E, let e_Y be the largest element of $A \setminus A'$, then $2b_{Y(A',A)}$ counts those ideals I such that for the antichain of maxima $\mathrm{Max}(I)$ we have

- $A' \subseteq \mathrm{Max}(I) \subseteq A$,
- if $B' \subseteq \mathrm{Max}(I) \subseteq B$ and e_Y is the largest element of $B \setminus B'$, then $A \setminus A' \subsetneq B \setminus B'$.

But note that this condition is only satisfied if $A' = \mathrm{Max}(I)$ and $A = A' \cup \{e\}$ for some $e \in \mathcal{P}$ or $A = \mathrm{Max}(I)$ and $A' = A \setminus \{e\}$ for some $e \in \mathcal{P}$. Indeed, if otherwise $A' \subsetneq \mathrm{Max}(I) \subsetneq A$ and e_Y is the largest element of $A \setminus A'$ one can set $B' = \mathrm{Max}(I) \setminus \{e_Y\}$ and $B = \mathrm{Max}(I) \cup \{e_Y\}$, contradiction the above condition. Hence $b_{Y(A',A)}$ is 1 if $|A \setminus A'| = 1$ and 0 otherwise. Thus, Corollary 2.11 and Remark 4.8 yield that

$$\det(\mathbf{V}_L) = \prod_{A \in \mathcal{A}} \prod_{p \in A} (1 - x_p^2) = \prod_{p \in \mathcal{P}} (1 - x_p^2)^{m_p},$$

where $\mathcal{A}$ denotes the set of antichains of $\mathcal{P}$ and m_p denotes the number of antichains containing p. In our example we get the following formula for $\det(\mathbf{V}_{L(\mathcal{Q})})$:
$(1 - x_a^2) \cdot (1 - x_b^2) \cdot ((1 - x_a^2)(1 - x_b^2)) \cdot (1 - x_e^2) \cdot (1 - x_c^2) \cdot (1 - x_d^2) \cdot ((1 - x_a^2)(1 - x_e^2)) \cdot ((1 - x_c^2)(1 - x_d^2)) \cdot ((1 - x_c^2)(1 - x_e^2)) \cdot ((1 - x_d^2)(1 - x_e^2)) \cdot ((1 - x_c^2)(1 - x_d^2)(1 - x_e^2)) =$

$$(1 - x_a^2)^3 (1 - x_b^2)^2 (1 - x_c^2)^4 (1 - x_d^2)^4 (1 - x_e^2)^5.$$

5.2. **Linear extensions.** Another instance is the ranking COM of a poset $\mathcal{P}$, that was described in [2]. The topes are the linear extensions of $\mathcal{P}$, and the separator of two linear extensions L, L' is the set of pairs of elements of $\mathcal{P}$ that are ordered differently in L and L'. In particular, the ground set of this COM consists of the set $\mathrm{Inc}(\mathcal{P})$ of incomparable pairs of $\mathcal{P}$, e.g., $\mathrm{Inc}(\mathcal{Q}) = \{ab, ae, cd, ce, de\}$. We can thus define the (unsigned) Varchenko matrix $\mathbf{V}_{X(\mathcal{P})}$. We get a description of $\mathbf{V}_{X(\mathcal{Q})}$. We deem it too large to display it entirely, but for example the entry corresponding to extensions $abcde, beadc$ is $x_{ab}x_{ae}x_{cd}x_{ce}x_{de}$.

The covectors of the ranking COM are the weak extensions of $\mathcal{P}$, i.e., those poset extensions of $\mathcal{P}$ that are chains of antichains. The set $z(Y)$ of such an extension Y corresponds to its set of incomparable pairs $\mathrm{Inc}(Y)$. In order to properly define the signs of the covectors, one can pick an arbitrary linear extension L_0 of $\mathcal{P}$, and set at non-zero coordinate of Y to $+$ if the corresponding incomparable pair of $\mathcal{P}$ is ordered the same way in L_0 and Y and to $-$ otherwise. To define b_Y we can fix an arbitrary linear order on the set $\mathrm{Inc}(\mathcal{P})$ and let $e_Y = \{p, q\}$ be the largest element of $\mathrm{Inc}(Y)$. Then $2b_Y$ counts linear extensions L of $\mathcal{P}$ such that

- L is a linear extension of Y,
- if another weak extension Z of $\mathcal{P}$ has e_Y as largest incomparable pair, then either L is not an extension of Z or Z is not an extension of Y.

In this setting one can see that not such Z can exist if and only if Y is a chain of antichains only one which - say A - has size larger than 1. In this cases the feasible L are extensions of Y that extend A by starting and ending with an element among $\{p, q\}$. Hence, there are $2(|A| - 2)!$ such linear extensions. By Corollary 2.11 and Remark 4.8 we have

$$\det(\mathbf{V}_{\mathcal{P}}) = \prod_{A \in \mathcal{A}_{\geq 2}} (1 - \prod_{p \neq q \in A} x_{p,q}^2)^{(|A|-2)!},$$

where $\mathcal{A}_{\geq 2}$ denotes the set of antichains of size at least 2 of $\mathcal{P}$.

6. Conclusion

One might wonder, to which extent our result could be further generalized to other classes. A natural next class are partial cubes, i.e., isometric subgraphs of the hypercube Q_d. These generalize (tope graphs of) COMs and allow for an analogous definition of the Varchenko Matrix, where the (u, v) entry contains a product of monomials indexed by those coordinates in $\{1, \ldots, d\}$ where u and v differ. The smallest partial cube that is not the tope graph of a COM is the full subdivision of K_4, see [9]. In this case the Varchenko matrix looks like the following

$$\begin{pmatrix}
1 & x_1 & x_2 & x_3 & x_1x_4 & x_1x_3x_4 & x_3x_4 & x_2x_3x_4 & x_2x_4 & x_1x_2x_4 \\
x_1 & 1 & x_1x_2 & x_1x_3 & x_4 & x_3x_4 & x_1x_3x_4 & x_1x_2x_3x_4 & x_1x_2x_4 & x_2x_4 \\
x_2 & x_1x_2 & 1 & x_2x_3 & x_1x_2x_4 & x_1x_2x_3x_4 & x_2x_3x_4 & x_3x_4 & x_4 & x_1x_4 \\
x_3 & x_1x_3 & x_2x_3 & 1 & x_1x_3x_4 & x_1x_4 & x_4 & x_2x_4 & x_2x_3x_4 & x_1x_2x_3x_4 \\
x_1x_4 & x_4 & x_1x_2x_4 & x_1x_3x_4 & 1 & x_3 & x_1x_3 & x_1x_2x_3 & x_1x_2 & x_2 \\
x_1x_3x_4 & x_3x_4 & x_1x_2x_3x_4 & x_1x_4 & x_3 & 1 & x_1 & x_1x_2 & x_1x_2x_3 & x_2x_3 \\
x_3x_4 & x_1x_3x_4 & x_2x_3x_4 & x_4 & x_1x_3 & x_1 & 1 & x_2 & x_2x_3 & x_1x_2x_3 \\
x_2x_3x_4 & x_1x_2x_3x_4 & x_3x_4 & x_2x_4 & x_1x_2x_3 & x_1x_2 & x_2 & 1 & x_3 & x_1x_3 \\
x_2x_4 & x_1x_2x_4 & x_4 & x_2x_3x_4 & x_1x_2 & x_1x_2x_3 & x_2x_3 & x_3 & 1 & x_1 \\
x_1x_2x_4 & x_2x_4 & x_1x_4 & x_1x_2x_3x_4 & x_2 & x_2x_3 & x_1x_2x_3 & x_1x_3 & x_1 & 1
\end{pmatrix}$$

and its determinant is of the following form:

$$(x_4-1)^3(x_4+1)^3(x_3-1)^3(x_3+1)^3(x_2-1)^3(x_2+1)^3(x_1-1)^3(x_1+1)^3$$
$$(3x_1^2x_2^2x_3^2x_4^2 - x_1^2x_2^2x_3^2 - x_1^2x_2^2x_4^2 - x_1^2x_3^2x_4^2 - x_2^2x_3^2x_4^2 + 1)$$

Thus, in this case there is no nice factorization.

Problem 6.1. Are there classes of partial cubes beyond COMs, that allow for a factorization theorem of the Varchenko Matrix?

Hochstättler and Welker proved the factorization formula not only for the full oriented matroid but also for supertopes, i.e. topal fibers in oriented matroids. The main motivation for the work in the present paper is that COMs seem to capture convexity in oriented matroids. At the moment we do not know an example of a COM which cannot be extended to become the supertope of an oriented matroid. This supports the the suspicion that they might not exist. Therefore we pose the following problem in our language, that is equivalent to previous conjectures [2, Conjecture 1] and [9, Conjecture 1]:

Problem 6.2. Are supertopes of oriented matroids a proper subclass of the class of Complexes of Oriented Matroids?

Acknowledgements: KK was partially supported by the French *Agence nationale de la recherche* through project ANR-17-CE40-0015 and by the Spanish *Ministerio de Economía, Industria y Competitividad* through grant RYC-2017-22701 and grant PID2019-104844GB-I00.

References

[1] M. Aguiar, S. Mahajan, Topics in hyperplane arrangements, Mathematical Surveys and Monographs **226**, American Mathematical Society, Providence, RI, 2017.

[2] H.-J. Bandelt, V. Chepoi , K. Knauer, COMs: complexes of oriented matroids, J. Combin. Theory Ser. A, **156** (2018) 195–237.
[3] A. Björner, Topological methods, in: Handbook of combinatorics, Volume 2, 1819–1872, Elsevier Sci. B. V., Amsterdam, 1995.
[4] A. Björner, M. Las Vergnas, B. Sturmfels, W. White, G.M. Ziegler, Oriented matroids. Second edition. Encyclopedia of Mathematics and its Applications **46**. Cambridge University Press, Cambridge, 1999.
[5] G. Denham, P. Hanlon, Some algebraic properties of the Schechtman-Varchenko bilinear forms, in: New perspectives in algebraic combinatorics (Berkeley, CA, 1996–97), 149–176, Math. Sci. Res. Inst. Publ., **38**, Cambridge University Press, Cambridge, 1999.
[6] R. Gente, The Varchenko Matrix for Cones, PhD-Thesis, Philipps-Universität Marburg, 2013.
[7] W. Hochstättler, S. Keip, K. Knauer. "Kirchberger's Theorem for Complexes of Oriented Matroids." arxiv.2112.03589 (2021).
[8] W. Hochstättler, V. Welker. "The Varchenko determinant for oriented matroids." Mathematische Zeitschrift 293.3 (2019): 1415-1430.
[9] K. Knauer, T. Marc, On tope graphs of complexes of oriented matroids, Discrete Comput. Geom. 63 (2020): 377-417.
[10] H. Randriamaro, The Varchenko Determinant of an Oriented Matroid, Trans. Comb. (10) **4** (2021), 7–18.
[11] V.V. Schechtman, A.N. Varchenko, Arrangements of hyperplanes and Lie algebra homology, Invent. Math. **106** (1991) 139–194.
[12] R.P. Stanley, Enumerative combinatorics. Volume 1. Second edition. Cambridge Studies in Advanced Mathematics **49**,
[13] A. Varchenko, Bilinear form of real configuration of hyperplanes, Adv. Math. **97** (1993) 110–144.
[14] M. L. Wachs, Poset topology: tools and applications. (2006).

FernUniversität in Hagen, Fakultät für Mathematik und Informatik, 58084 Hagen, Germany
Email address: `winfried.hochstaettler@fernuni-hagen.de`

FernUniversität in Hagen, Fakultät für Mathematik und Informatik, 58084 Hagen, Germany
Email address: `sophia.keip@fernuni-hagen.de`

Departament de Matemàtiques i Informàtica, Universitat de Barcelona, Spain
Email address: `kolja.knauer@ub.edu`