The queue-number of posets of bounded width or height

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Abstract. Heath and Pemmaraju [9] conjectured that the queue-number of a poset is bounded by its width and if the poset is planar then also by its height. We show that there are planar posets whose queue-number is larger than their height, refuting the second conjecture. On the other hand, we show that any poset of width 2 has queue-number at most 2, thus confirming the first conjecture in the first non-trivial case. Moreover, we improve the previously best known bounds and show that planar posets of width w have queue-number at most $3w-2$ while any planar poset with 0 and 1 has queue-number at most its width.

1 Introduction

A *queue layout* of a graph consists of a total ordering on its vertices and an assignment of its edges to *queues*, such that no two edges in a single queue are nested. The minimum number of queues needed in a queue layout of a graph G is called its *queue-number* and denoted by $\mathrm{qn}(G)$.

To be more precise, let G be a graph and let L be a linear order on the vertices of G. We say that the edges $uv, u'v' \in E(G)$ are *nested* with respect to L if $u < u' < v' < v$ or $u' < u < v < v'$ in L. Given a linear order L of the vertices of G, the edges $u_1v_1, \ldots, u_kv_k$ of G form a *rainbow* of size k if $u_1 < \cdots < u_k < v_k < \cdots < v_1$ in L. Given G and L, the edges of G can be partitioned into k queues if and only if there is no rainbow of size $k+1$ in L, see [10].

The queue-number was introduced by Heath and Rosenberg in 1992 [10] as an analogy to book embeddings. Queue layouts were implicitly used before and have applications in fault-tolerant processing, sorting with parallel queues, matrix computations, scheduling parallel processes, and communication management in distributed algorithm (see [8,10,12]).

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Perhaps the most intriguing question concerning queue-numbers is whether planar graphs have bounded queue-number.

Conjecture 1 (Heath and Rosenberg [10]).
The queue-number of planar graphs is bounded by a constant.

In this paper we study queue-numbers of posets. The parameter was introduced in 1997 by Heath and Pemmaraju [9] and the main idea is that given a poset one should lay it out respecting its relation. Two elements a, b of a poset are called *comparable* if $a < b$ or $b < a$, and *incomparable*, denoted by $a \parallel b$, otherwise. Posets are visualized by their *diagrams*: Elements are placed as points in the plane and whenever $a < b$ in the poset, and there is no element c with $a < c < b$, there is a curve from a to b going upwards (that is y-monotone). We denote this case as $a \prec b$. The diagram represents those relations which are essential in the sense that they are not implied by transitivity, also known as *cover relations*. The undirected graph implicitly defined by such a diagram is the *cover graph* of the poset. Given a poset P, a *linear extension* L of P is a linear order on the elements of P such that $x <_L y$, whenever $x <_P y$. (Throughout the paper we use a subscript on the symbol $<$, if we want to emphasize which order it represents.) Finally, the *queue-number of a poset* P, denoted by $\mathrm{qn}(P)$, is the smallest k such that there is a linear extension L of P for which the resulting linear layout of G_P contains no $(k+1)$-rainbow. Clearly we have $\mathrm{qn}(G_P) \leqslant \mathrm{qn}(P)$, i.e., the queue-number of a poset is at least the queue-number of its cover graph. It is shown in [9] that even for *planar posets*, that is posets admitting crossing-free diagrams, there is no function f such that $\mathrm{qn}(P) \leqslant f(\mathrm{qn}(G_P))$.

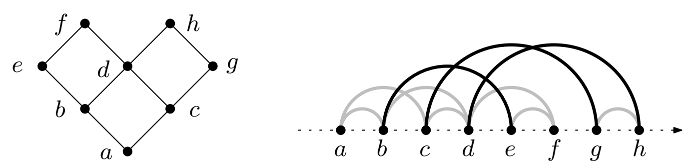

Fig. 1. A poset and a layout with two queues (gray and black). Note that the order of the elements on the spine is a linear extension of the poset.

Heath and Pemmaraju [9] investigated the maximum queue-number of several classes of posets, in particular with respect to bounded width (the maximum number of pairwise incomparable elements) and height (the maximum number of pairwise comparable elements). A set with every two elements being comparable is a *chain*. A set with every two distinct elements being incomparable is an *antichain*. They proved that if $\mathrm{width}(P) \leqslant w$, then $\mathrm{qn}(P) \leqslant w^2$. The lower bound is attained by *weak orders*, i.e., chains of antichains and is conjectured to be the upper bound as well:

Conjecture 2 (Heath and Pemmaraju [9]).
Every poset of width w has queue-number at most w.

Furthermore, they made a step towards this conjecture for planar posets: if a planar poset P has $\text{width}(P) \leqslant w$, then $\text{qn}(P) \leqslant 4w - 1$. For the lower bound side they provided planar posets of width w and queue-number $\lceil \sqrt{w} \rceil$.

We improve the bounds for planar posets and get the following:

Theorem 1. *Every planar poset of width w has queue-number at most $3w - 2$. Moreover, there are planar posets of width w and queue-number w.*

As an ingredient of the proof we show that posets without certain subdivided crowns satisfy Conjecture 2 (c.f. Theorem 5). This implies the conjecture for interval orders and planar posets with (unique minimum) 0 and (unique maximum) 1 (c.f. Corollary 2). Moreover, we confirm Conjecture 2 for the first non-trivial case $w = 2$:

Theorem 2. *Every poset of width 2 has queue-number at most 2.*

An easy corollary of this is that all posets of width w have queue-number at most $w^2 - w + 1$ (c.f. Corollary 1).

Another conjecture of Heath and Pemmaraju concerns planar posets of bounded height:

Conjecture 3 (Heath and Pemmaraju [9]).
Every planar poset of height h has queue-number at most h.

We show that Conjecture 3 is false for the first non-trivial case $h = 2$:

Theorem 3. *There is a planar poset of height 2 with queue-number at least 4.*

Furthermore, we establish a link between a relaxed version of Conjecture 3 and Conjecture 1, namely we show that the latter is equivalent to planar posets of height 2 having bounded queue-number (c.f. Theorem 6). On the other hand, we show that Conjecture 3 holds for planar posets with 0 and 1:

Theorem 4. *Every planar poset of height h with 0 and 1 has queue-number at most $h - 1$.*

Organization of the paper. In Section 2 we consider general (not necessarily planar) posets and give upper bounds on their queue-number in terms of their width, such as Theorem 2. In Section 3 we consider planar posets and bound the queue-number in terms of the width, both from above and below, i.e., we prove Theorem 1. In Section 4 we give a counterexample to Conjecture 3 by constructing a planar poset with height 2 and queue-number at least 4. Here we also argue that proving *any* upper bound on the queue-number of such posets is equivalent to proving Conjecture 1. Finally, we show that Conjecture 3 holds for planar posets with 0 and 1 and that for every h there is a planar poset of height h and queue-number $h - 1$ (c.f. Proposition 3).

2 General Posets of Bounded Width

By Dilworth's Theorem [3], the width of a poset P coincides with the smallest integer w such that P can be decomposed into w chains of P. Let us derive Proposition 1 of Heath and Pemmaraju [9] from such a chain partition.

Proposition 1. *For every poset P, if* $\mathrm{width}(P) \leqslant w$ *then* $\mathrm{qn}(P) \leqslant w^2$.

Proof. Let P be a poset of width w and $C_1, \ldots, C_w$ be a chain partition of P. Let L be any linear extension of P and $a <_L b <_L c <_L d$ with $a \prec d$ and $b \prec c$. Note that we must have either $a \parallel b$ or $c \parallel d$. If follows that if $a \in C_i$, $b \in C_j$, $c \in C_k$, and $d \in C_\ell$, then $(i, \ell) \neq (j, k)$. As there are only w^2 ordered pairs (x, y) with $x, y \in [w]$, we can conclude that every nesting set of covers has cardinality at most w^2. □

Note that in the above proof L is *any* linear extension and that without choosing the linear extension L carefully, upper bound w^2 is best-possible. Namely, if $P = \{a_1, \ldots, a_k, b_1, \ldots, b_k\}$ with comparabilities $a_i < b_j$ for all $1 \leqslant i, j \leqslant k$, then P has width k and the linear extension $a_1 < \ldots < a_k < b_k < \ldots < b_1$ creates a rainbow of size k^2.

We continue by showing that every poset of width 2 has queue-number at most 2, that is, we prove Theorem 2.

Proof (Theorem 2). Let P be a poset of width 2 and minimum element 0 and C_1, C_2 be a chain partition of P. Note that the assumption of the minimum causes no loss of generality, since a 0 can be added without increasing the width nor decreasing the queue-number. Any linear extension L of P partitions the ground set X naturally into inclusion-maximal sets of elements, called *blocks*, from the same chain in $\{C_1, C_2\}$ that appear consecutively along L, see Figure 2. We denote the blocks by $B_1, \ldots, B_k$ according to their appearance along L. We say that L is *lazy* if for each $i = 2, \ldots, k$, each element $x \in B_i$ has a relation to some element $y \in B_{i-1}$. A linear extension L can be obtained by picking any minimal element $m \in P$, put it into L, and recurse on $P \setminus \{m\}$. Lazy linear extensions (with respect to C_1, C_2) can be constructed by the same process where additionally the next element is chosen from the same chain as the element before, if possible. Note that the existence of a 0 is needed in order to ensure the property of laziness with respect to B_2.

Now we shall prove that in a lazy linear extension no three covers are pairwise nesting. So assume that $a \prec b$ is any cover and that $a \in B_i$ and $b \in B_j$. As L is lazy, b is comparable to some element in B_{j-1} (if $j \geqslant 2$) and all elements in $B_1, \ldots, B_{j-2}$ (if $j \geqslant 3$). With $a \prec b$ being a cover, it follows from L being lazy that $i \in \{j-2, j-1, j\}$. If $i = j$, then no cover is nested under $a \prec b$. If $i = j-1$, then no cover $c \prec d$ is nested above $a \prec b$: either $c \in B_i$ and $d \in B_j$ and hence $c \prec d$ is not a cover, or both endpoints would be inside the same chain, i.e., c, d are the last and first element of B_{j-2} and B_j or B_i and B_{i+2}, respectively. This implies $c <_L a <_L d <_L b$ or $a <_L c <_L b <_L r$, respectively, and $c \prec d$ cannot nest above $a \prec b$. If $i = j - 2$, then no cover is nested above $a \prec b$. Thus, either

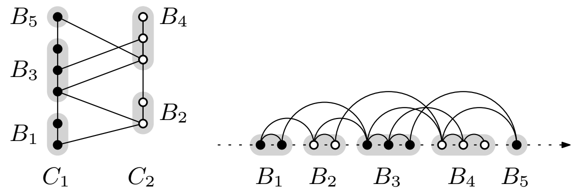

Fig. 2. A poset of width 2 with a 0 and a chain partition C_1, C_2 and the blocks $B_1, \ldots, B_5$ induced by a lazy linear extension with respect to C_1, C_2.

no cover is nested below $a \prec b$, or no cover is nested above $a \prec b$, or both. In particular, there is no three nesting covers and $\mathrm{qn}(P) \leqslant 2$. □

Corollary 1. *Every poset of width w has queue-number at most $w^2 - 2\lfloor w/2 \rfloor$.*

Proof. We take any chain partition of size w and pair up chains to obtain a set S of $\lfloor w/2 \rfloor$ disjoint pairs. Each pair from S induces a poset of width at most 2, which by Theorem 2 admits a linear order with at most two nesting covers. Let L be a linear extension of P respecting all these partial linear extensions.

Now, following the proof of Proposition 1 any cover can be labeled by a pair (i, j) corresponding to the chains containing its endpoint. Thus, in a set of nesting covers any pair appears at most once, but for each i, j such that $(i, j) \in S$ only two of the four possible pairs can appear simultaneously in a nesting. This yields the upper bound. □

For an integer $k \geqslant 2$ we define a *subdivided k-crown* as the poset P_k as follows. The elements of P_k are $\{a_1, \ldots, a_k, b_1, \ldots, b_k, c_1, \ldots, c_k\}$ and the cover relations are given by $a_i \prec b_i$ and $b_i \prec c_i$ for $i = 2, \ldots, k$, $a_i \prec c_{i-1}$ for $i = 1, \ldots, k-1$, and $a_1 \prec c_k$; see the left of Figure 3. We refer to the covers of the form $a_i \prec c_j$ as the *diagonal covers* and we say that a poset P has an *embedded P_k* if P contains $3k$ elements that induce a copy of P_k in P with all diagonal covers of that copy being covers of P.

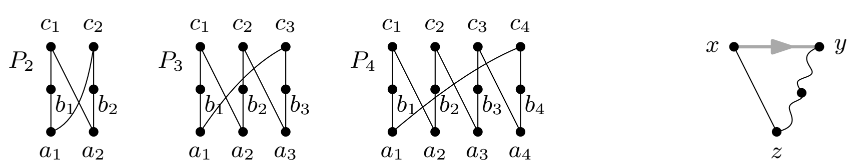

Fig. 3. Left: The posets P_2, P_3, and P_4. Right: The existence of an element z with cover relation $z \prec x$ and non-cover relation $z < y$ gives rise to a gray edge from x to y.

Theorem 5. *If P is a poset that for no $k \geqslant 2$ has an embedded P_k, then the queue-number of P is at most the width of P.*

Proof. Let P be any poset. For this proof we consider the cover graph G_P of P as a directed graph with each edge xy directed from x to y if $x \prec y$ in P. We call these edges the *cover edges*. Now we augment G_P to a directed graph G by introducing for some incomparable pairs $x \parallel y$ a directed edge. Specifically, we add a directed edge from x to y if there exists a z with $z < x, y$ in P where $z \prec x$ is a cover relation and $z < y$ is not a cover relation; see the right of Figure 3. We call these edges the *gray edges* of G.

Now we claim that if G has a directed cycle, then P has an embedded subdivided crown. Clearly, every directed cycle in G has at least one gray edge. We consider the directed cycles with the fewest gray edges and among those let $C = [c_1, \ldots, c_\ell]$ be one with the fewest cover edges.s First assume that C has a cover edge (hence $\ell \geqslant 3$), say c_1c_2 is a gray edge followed by a cover edge c_2c_3. Consider the element z with cover relation $z \prec c_1$ and non-cover relation $z < c_2$ in P. By $z < c_2 \prec c_3$ we have a non-cover relation $z < c_3$ in P. Now if $c_1 \parallel c_3$ in P, then G contains the gray edge c_1c_3 (see Figure 4(a)) and $[c_1, c_3, \ldots, c_\ell]$ is a directed cycle with the same number of gray edges as C but fewer cover edges, a contradiction. On the other hand, if $c_1 < c_3$ in P (note that $c_3 < c_1$ is impossible as $z \prec c_1$ is a cover), then there is a directed path Q of cover edges from c_1 to c_3 (see Figure 4(b)) and $C + Q - \{c_1c_2, c_2c_3\}$ contains a directed cycle with fewer gray edges than C, again a contradiction.

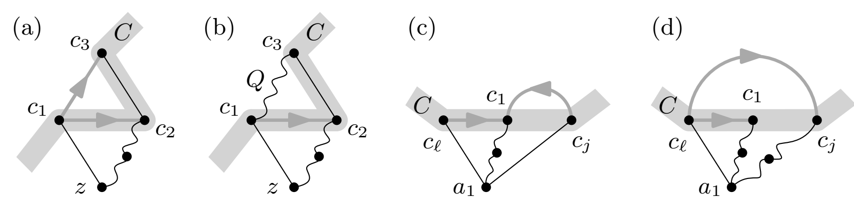

Fig. 4. Illustrations for the proof of Theorem 5.

Hence $C = [c_1, \ldots, c_\ell]$ is a directed cycle consisting solely of gray edges. Note that by the first paragraph $\{c_1, \ldots, c_\ell\}$ is an antichain in P. For $i = 2, \ldots, \ell$ let a_i be the element of P with cover relation $a_i \prec c_{i-1}$ and non-cover relation $a_i < c_i$, as well as a_1 with cover relation $a_1 \prec c_\ell$ and non-cover relation $a_1 < c_1$. As $\{c_1, \ldots, c_\ell\}$ is an antichain and $a_i < c_i$ holds for $i = 1, \ldots, \ell$, we have $\{c_1, \ldots, c_\ell\} \cap \{a_1, \ldots, a_\ell\} = \emptyset$. Let us assume that $a_1 < c_j$ in P for some $j \neq 1, \ell$. If $a_1 \prec c_j$ is a cover relation, then there is a gray edge c_jc_1 in G (see Figure 4(c)) and the cycle $[c_1, \ldots, c_j]$ is shorter than C, a contradiction. If $a_1 < c_j$ is a non-cover relation, then there is a gray edge $c_\ell c_j$ in G (see Figure 4(d)) and the cycle $[c_j, \ldots, c_\ell]$ is shorter than C, again a contradiction.

Hence, the only relations between $a_1, \ldots, a_\ell$ and $c_1, \ldots, c_\ell$ are cover relations $a_1 \prec c_\ell$ and $a_i \prec c_{i-1}$ for $i = 2, \ldots, \ell$ and the non-cover relations $a_i < c_i$ for $i = 1, \ldots, \ell$. Hence $a_1, \ldots, a_\ell$ are pairwise distinct. Moreover, $\{a_1, \ldots, a_\ell\}$ is an antichain in P since the only possible relations among these elements are of the form $a_1 < a_\ell$ or $a_i < a_{i-1}$, which would contradict that $a_1 \prec c_\ell$ and $a_i \prec c_{i-1}$ are cover relations. Finally, we pick for every $i = 1, \ldots, \ell$ an element b_i with $a_i < b_i < c_i$, which exists as $a_i < c_i$ is a non-cover relation. Together with the above relations between $a_1, \ldots, a_\ell$ and $c_1, \ldots, c_\ell$ we conclude that $b_1, \ldots, b_\ell$ are pairwise distinct and these 3ℓ elements induce a copy of P_ℓ in P with all diagonal covers in that copy being covers of P.

Thus, if P has no embedded P_k, then the graph G we constructed has no directed cycles, and we can pick L to be any topological ordering of G. As $G_P \subseteq G$, L is a linear extension of P. For any two nesting covers $x_2 <_L x_1 <_L y_1 <_L y_2$ we have $x_1 \parallel x_2$ or $y_1 \parallel y_2$ or both, since $x_2 \prec y_2$ is a cover. However, if $x_2 < x_1$ in P, then there would be a gray edge from y_2 to y_1 in G, contradicting $y_1 <_L y_2$ and L being a topological ordering of G. We conclude that $x_1 \parallel x_2$ and the left endpoints of any rainbow form an antichain, proving $\mathrm{qn}(P) \leqslant \mathrm{width}(P)$. □

Let us remark that several classes of posets have no embedded subdivided crowns, e.g., graded posets, interval orders (since these are 2+2-free, see [6]), or (quasi-)series-parallel orders (since these are N-free, see [7]). Here, 2+2 and N are the four-element posets defined by $a < b, c < d$ and $a < b, c < d, c < b$, respectively. Also note that while subdivided crowns are planar posets, no planar poset with 0 and 1 has an embedded k-crown. Indeed, already looking at the subposet induced by the k-crown and the 0 and the 1, it is easy to see that there must be a crossing in any diagram. Thus, we obtain:

Corollary 2. *For any interval order, series-parallel order, and planar poset with 0 and 1, P we have* $\mathrm{qn}(P) \leqslant \mathrm{width}(P)$.

3 Planar Posets of Bounded Width

Heath and Pemmaraju [9] show that the largest queue-number among planar posets of width w lies between $\lceil\sqrt{w}\rceil$ and $4w-1$. Here we improve the lower bound to w and the upper bound to $3w-2$.

Proposition 2. *For each w there exists a planar poset Q_w with 0 and 1 of width w and queue-number w.*

Proof. We shall define Q_w recursively, starting with Q_1 being any chain. For $w \geqslant 2$, Q_w consists of a *lower copy* P and a disjoint *upper copy* P' of Q_{w-1}, three additional elements a, b, c, and the following cover relations in between:

- $a \prec x$, where x is the 0 of P
- $y \prec x'$, where y is the 1 of P and x' is the 0 of P'
- $y' \prec c$, where y' is the 1 of P'
- $a \prec b \prec c$

It is easily seen that all cover relations of P and P' remain cover relations in Q_w, and that Q_w is planar, has width w, a is the 0 of Q_w, and c is the 1 of Q_w. See Figure 5 for an illustration.

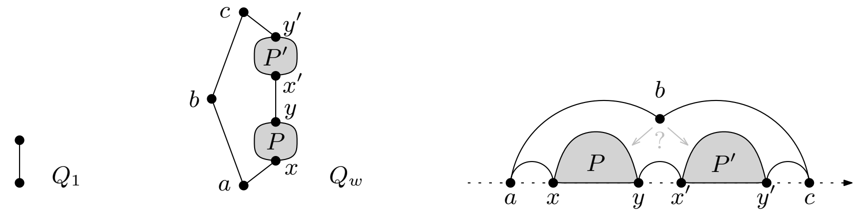

Fig. 5. Recursively constructing planar posets Q_w of width w and queue-number w. Left: Q_1 is a two-element chain. Middle: Q_w is defined from two copies P, P' of Q_{w-1}. Right: The general situation for a linear extension of Q_w.

To prove that $\mathrm{qn}(Q_w) = w$ we argue by induction on w, with the case $w = 1$ being immediate. Let L be any linear extension of Q_w. Then a is the first element in L and c is the last. Since $y \prec x'$, all elements in P come before all elements of P'. Now if in L the element b comes after all elements of P, then P is nested under cover $a \prec b$, and if b comes before all elements of P', then P' is nested under cover $b \prec c$. We obtain w nesting covers by induction on P in the former case, and by induction on P' in the latter case. This concludes the proof. □

Next we prove Theorem 1, namely that the maximum queue-number of planar posets of width w lies between w and $3w - 2$.

Proof (Theorem 1). By Proposition 2 some planar posets of width w have queue-number w. So it remains to consider an arbitrary planar poset P of width w and show that P has queue-number at most $3w - 2$. To this end, we shall add some relations to P, obtaining another planar poset Q of width w that has a 0 and 1, with the property that $\mathrm{qn}(P) \leqslant \mathrm{qn}(Q) + 2w - 2$. Note that this will conclude the proof, as by Corollary 2 we have $\mathrm{qn}(Q) \leqslant w$.

Given a planar poset P of width w, there are at most w minima and at most w maxima. Hence there are at most $2w - 2$ extrema that are not on the outer face. For each such extremum x –say x is a minimum– consider the unique face f with an obtuse angle at x. We introduce a new relation $y < x$, where y is a smallest element at face f, see Figure 6. Note that this way we introduce at most $2w - 2$ new relations, and that these can be drawn y-monotone and crossing-free by carefully choosing the other element in each new relation. Furthermore, every inner face has a unique source and unique sink.

Now consider a cover relation $a \prec_P b$ that is not a cover relation in the new poset Q. For the corresponding edge e from a to b in Q there is one face f with unique source a and unique sink b. Now either way the other edge in f incident to a or to b must be one of the $2w - 2$ newly inserted edges, see again Figure 6. This way we assign $a \prec b$ to one of $2w - 2$ queues, one for each newly inserted edge. Every such queue contains either at most one edge or two incident edges, i.e., a nesting is impossible, no matter what linear ordering is chosen later.

We create at most $2w - 2$ queues to deal with the cover relations of P that are not cover relations of Q and spend another w queues for Q dealing with the remaining cover relations of P. Thus, $\mathrm{qn}(P) \leqslant \mathrm{qn}(Q) + 2w - 2 \leqslant 3w - 2$. □

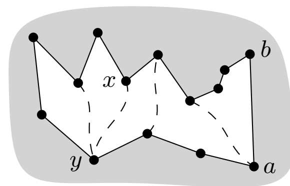

Fig. 6. Inserting new relations (dashed) into a face of a plane diagram. Note that relation $a < b$ is a cover relation in P but not in Q.

4 Planar Posets of Bounded Height

Recall Conjecture 3, which states that every planar poset of height h has queue-number at most h. In the following, we give a counterexample to this conjecture:

Proof (Theorem 3). Consider the graph G that is constructed as follows: Start with $K_{2,10}$ with bipartition classes $\{a_1, a_2\}$ and $\{b_1, \ldots, b_{10}\}$. For every $i = 1, \ldots, 9$ add four new vertices $c_{i,1}, \ldots, c_{i,4}$, each connected to b_i and b_{i+1}. The resulting graph G has 46 vertices, is planar and bipartite with bipartition classes $X = \{b_1, \ldots, b_{10}\}$ and $Y = \{a_1, a_2\} \cup \{c_{i,j} \mid 1 \leqslant i \leqslant 9, 1 \leqslant j \leqslant 4\}$. See Figure 7.

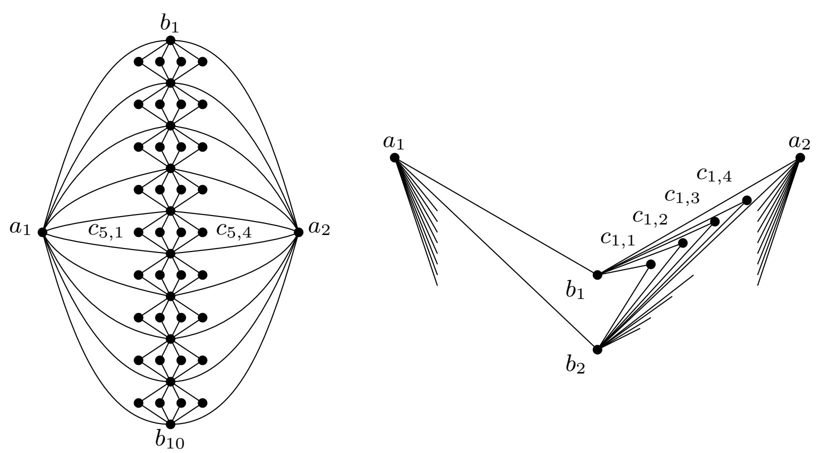

Fig. 7. A planar poset P of height 2 and queue-number at least 4. Left: The cover graph G_P of P. Right: A part of a planar diagram of P.

Let P be the poset arising from G by introducing the relation $x < y$ for every edge xy in G with $x \in X$ and $y \in Y$. Clearly, P has height 2 and hence the cover relations of P are exactly the edges of G. Moreover, by a result of Moore [11] (see also [2]) P is planar because G is planar, also see the right of Figure 7.

We shall argue that $\mathrm{qn}(P) \geqslant 4$. To this end, let L be any linear extension of P. Without loss of generality we have $a_1 <_L a_2$. Note that since in P one bipartition class of G is entirely below the other, any 4-cycle in G gives a 2-rainbow. Let b_{i_1}, b_{i_2} be the first two elements of X in L, b_{j_1}, b_{j_2} be the last two such elements. As $|X| = 10$ there exists $1 \leqslant i \leqslant 9$ such that $\{i, i+1\} \cap \{i_1, i_2, j_1, j_2\} = \emptyset$, i.e., we have $b_{i_1}, b_{i_2} <_L b_i, b_{i+1} <_L b_{j_1}, b_{j_2} <_L a_1 <_L a_2$, where we use that a_1 and a_2 are above all elements of X in P.

Now consider the elements $C = \{c_{i,1}, \ldots, c_{i,4}\}$ that are above b_i and b_{i+1} in P. As $|C| \geqslant 4$, there are two elements c_1, c_2 of C that are both below a_1, a_2 in L, or both between a_1 and a_2 in L, or both above a_1, a_2 in L. Consider the 2-rainbow R in the 4-cycle $[c_1, b_i, c_2, b_{i+1}]$. In the first case R is nested below the 4-cycle $[a_1, b_{i_1}, a_2, b_{i_2}]$, in the second case the cover $b_{j_1} \prec a_1$ is nested below R and R is nested below the cover $b_{i_1} \prec a_2$, and in the third case 4-cycle $[a_1, b_{j_1}, a_2, b_{j_2}]$ is nested below R. As each case results in a 4-rainbow, we have $\mathrm{qn}(P) \geqslant 4$. □

Even though Conjecture 3 has to be refuted in its strongest meaning, it might hold that planar posets of height h have queue-number $O(h)$, or at least bounded by some function $f(h)$ in terms of h, or at least that planar posets of height 2 have bounded queue-number. As it turns out, all these statements are equivalent, and in turn equivalent to Conjecture 1.

Theorem 6. *The following statements are equivalent:*

(i) Planar graphs have queue-number $O(1)$ (Conjecture 1).
(ii) Planar posets of height h have queue-number $O(h)$.
(iii) Planar posets of height h have queue-number at most $f(h)$ for a function f.
(iv) Planar posets of height 2 have queue-number $O(1)$.
(v) Planar bipartite graphs have queue-number $O(1)$.

Proof. **(i)⇒(ii)** Pemmaraju proves in his thesis [13] (see also [4]) that if G is a graph, π is a vertex ordering of G with no $(k+1)$-rainbow, $V_1, \ldots, V_m$ are color classes of any proper m-coloring of G, and π' is the vertex ordering with $V_1 <_{\pi'} \cdots <_{\pi'} V_m$, where within each V_i the ordering of π is inherited, then π' has no $(2(m-1)k+1)$-rainbow. So if P is any poset of height h, its cover graph G_P has $\mathrm{qn}(G_P) \leqslant c$ by (i) for some global constant $c > 0$. Splitting P into h antichains $A_1, \ldots, A_h$ by iteratively removing all minimal elements induces a proper h-coloring of G_P with color classes $A_1, \ldots, A_h$. As every vertex ordering π' of G with $A_1 <_{\pi'} \cdots <_{\pi'} A_h$ is a linear extension of P, it follows by Pemmaraju's result that $\mathrm{qn}(P) \leqslant 2(h-1)\,\mathrm{qn}(G_P) \leqslant 2ch$, i.e., $\mathrm{qn}(P) \in O(h)$.

(ii)⇒(iii)⇒(iv) These implications are immediate.

(iv)⇒(v) Moore proves in his thesis [11] (see also [2]) that if G is a planar and bipartite graph with bipartition classes A and B, and P_G is the poset on element set $A \cup B = V(G)$ where $x < y$ if and only if $x \in A, y \in B, xy \in E(G)$, then P_G is a planar poset of height 2. As G is the cover graph of P_G, we have $\mathrm{qn}(G) \leqslant \mathrm{qn}(P_G) \leqslant c$ for some constant $c > 0$ by (iv), i.e., $\mathrm{qn}(G) \in O(1)$.

(v)⇒(i) This is a result of Dujmović and Wood [5]. □

Finally, we show that Conjecture 3 holds for planar posets with 0 and 1.

Proof (Theorem 4). Let P be a planar poset with 0 and 1. Then P has dimension at most two [1], i.e., it can be written as the intersection of two linear extensions of P. A particular consequence of this is, that there is a well-defined dual poset $P^\star$ in which two distinct elements x, y are comparable in P if and only if they are incomparable in $P^\star$. Poset $P^\star$ reflects a "left of"-relation for each incomparable pair $x \parallel y$ in P in the following sense: Any maximal chain C in P corresponds to a 0-1-path Q in G_P, which splits the elements of $P \setminus C$ into those left of Q and those right of Q. Now $x <_{P^\star} y$ if and only if x is left of the path for every maximal chain containing y (equivalently y is right of the path for every maximal chain containing x). Due to planarity, if $a \prec b$ is a cover in P and C is a maximal chain containing neither a nor b, then a and b are on the same side of the path Q corresponding to C. In particular, if for $x, y \in C$ we have $a <_{P^\star} x$ and $b \parallel y$, then b and y are comparable in $P^\star$, but if $y <_{P^\star} b$ we would get a crossing of C and $a \prec b$. Also see the left of Figure 8. We summarize:

($\star$) If $a \prec b$, $a <_{P^\star} x$ for some $x \in C$ and $b \parallel y$ for some $y \in C$, then $b <_{P^\star} y$.

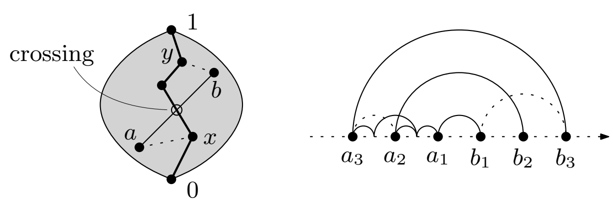

Fig. 8. Left: Illustration of ($\star$): If $a <_{P^\star} x$, $b \parallel y$, $x < y$, and $a \prec b$ is a cover, then $b <_{P^\star} y$ due to planarity. Right: If $a_3 <_L a_2 <_L a_1 <_L b_1 <_L b_2 <_L b_3$ is a 3-rainbow with $a_2, a_3 < a_1$, then $a_3 < a_2$.

Now let L be the *leftmost* linear extension of P, i.e., the unique linear extension L with the property that for any $x \parallel y$ in P we have $x <_L y$ if and only if $x < y$ in $P^\star$. Assume that $a_2 <_L a_1 <_L b_1 <_L b_2$ is a pair of nesting covers $a_1 \prec b_1$ below $a_2 \prec b_2$. Then $a_1 \parallel a_2$ (hence $a_2 <_{P^\star} a_1$) or $b_1 \parallel b_2$ (hence $b_1 <_{P^\star} b_2$) or both. Observe that the latter case is impossible, as for any maximal chain C containing $a_1 \prec b_1$ we would have $a_2 <_{P^\star} a_1$ with $a_1 \in C$ and $b_1 <_{P^\star} b_2$ with $b_1 \in C$, contradicting ($\star$). So the nesting of $a_1 \prec b_1$ below $a_2 \prec b_2$ is either of type A with $a_2 < a_1$, or of type B with $b_1 < b_2$. See Figure 9.

Now consider the case that cover $a_2 \prec b_2$ is nested below another cover $a_3 \prec b_3$, see the right of Figure 8. Then also $a_1 \prec b_1$ is nested below $a_3 \prec b_3$ and we claim that if both, the nesting of $a_1 \prec b_1$ below $a_2 \prec b_2$ as well as the nesting of $a_1 \prec b_1$ below $a_3 \prec b_3$, are of type A (respectively type B), then also the nesting of $a_2 \prec b_2$ below $a_3 \prec b_3$ is of type A (respectively type B). Indeed, assuming type B, we would get $a_3 <_{P^\star} a_2$ and $b_1 <_{P^\star} b_3$, which together with any maximal chain C containing $a_2 < a_1 < b_1$ contradicts ($\star$).

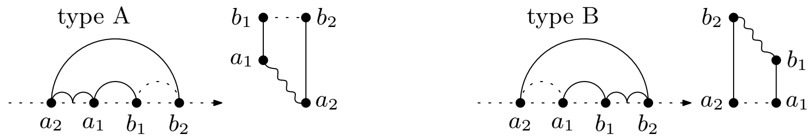

Fig. 9. A nesting of $a_1 \prec b_1$ below $a_2 \prec b_2$ of type A (left) and type B (right).

Finally, let $a_k <_L \cdots <_L a_1 <_L b_1 <_L \cdots <_L b_k$ be any k-rainbow and let $I = \{i \in [k] \mid a_i < a_1\}$, i.e., for each $i \in I$ the nesting of $a_1 \prec b_1$ below $a_i \prec b_i$ is of type A. Then we have just shown that the nesting of $a_j \prec b_j$ below $a_i \prec b_i$ is of type A whenever $i, j \in I$ and of type B whenever $i, j \notin I$. Hence, the set $\{a_i \mid i \in I\} \cup \{a_1, b_1\} \cup \{b_i \mid i \notin I\}$ is a chain in P of size $k+1$, and thus $k \leqslant h-1$. It follows that P has queue-number at most $h-1$, as desired. □

Proposition 3. *For each h there exists a planar poset Q_h of height h and queue-number $h-1$.*

Proof. We shall recursively define a planar poset Q_h of height h and queue-number $h-1$, together with a certain set of marked subposets in Q_h. Each marked subposet consists of three elements x, y, z forming a *V-subposet* in Q_h, i.e., $y < x, z$ but $x \parallel z$, with both relations $y < x$ and $y < z$ being cover relations of Q_h, and y being a minimal element of Q_h. We call such a marked subposet in Q_h a V-poset. Finally, we ensure that the V-posets are pairwise incomparable, namely that any two elements in distinct V-posets are incomparable in Q_h.

For $h = 2$ let Q_2 be the three-element poset as shown in left of Figure 10, which also forms the only V-poset of Q_2. Clearly Q_2 has height 2 and queue-number 1. For $h \geqslant 3$ assume that we already constructed Q_{h-1} with a number of V-posets in it. Then Q_h is obtained from Q_{h-1} by replacing each V-poset by the eight-element poset shown in the right of Figure 10, which introduces (for each V-poset) five new elements. Moreover, two new V-posets are identified in Q_h as illustrated in Figure 10.

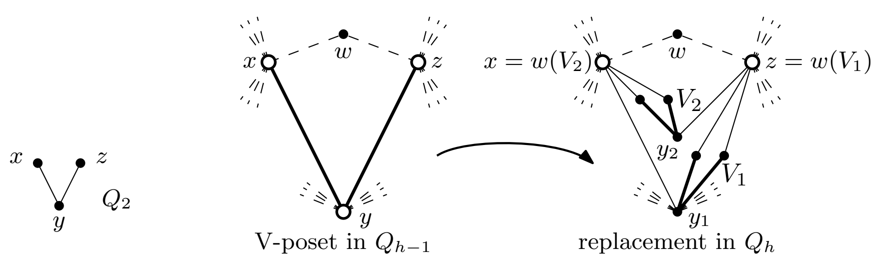

Fig. 10. Constructing planar posets of height h and queue-number $h-1$. Left: Q_2 is a three-element poset and its only V-poset. Right: Q_h is recursively defined from Q_{h-1} by replacing each V-poset by an eight-element poset and identifying two new V-posets.

It is easy to check that Q_h is planar and has height h, since Q_{h-1} has height $h-1$ and the V-posets in Q_{h-1} are pairwise incomparable. Moreover, every V-poset in Q_h contains a minimal element of Q_h and all V-posets in Q_h are pairwise incomparable. Finally, observe that, as long as $h \geqslant 3$, for every V-poset V in Q_h there is a unique smallest element $w = w(V)$ that is larger than all elements in V, see the right of Figure 10.

In order to show that $\mathrm{qn}(Q_h) \geqslant h-1$, we shall show by induction on h that for every linear extension L of Q_h there exists a $(h-1)$-rainbow in Q_h with respect to L whose innermost cover is contained in a V-poset V of Q_h, and, if $h \geqslant 3$, whose second innermost cover has the element $w(V)$ as its upper end. This clearly holds for $h = 2$. For $h \geqslant 3$, consider any linear extension L of Q_h. This induces a linear extension L' of Q_{h-1} as follows: The set X of elements in Q_h not contained in any V-poset is also a subset of the elements in Q_{h-1}. The remaining elements of Q_{h-1} are the minimal elements of the V-posets in Q_{h-1}. For each minimal element y of Q_{h-1} consider the two corresponding V-posets in Q_h with its two corresponding minimal elements y_1, y_2. Let $\hat{y} \in \{y_1, y_2\}$ be the element that comes first in L, i.e., $\hat{y} = y_1$ if and only if $y_1 <_L y_2$. Then we define L' to be the ordering of Q_{h-1} induced by the ordering of $X \cup \{\hat{y} \mid y \in Q_{h-1} - X\}$ in L. Note that L' is a linear extension of Q_{h-1}, even though $X \cup \{\hat{y} \mid y \in Q_{h-1} - X\}$ does not necessarily induce a copy of Q_{h-1} in Q_h.

By induction on Q_{h-1} there exists a $(h-2)$-rainbow R with respect to L' whose innermost cover is contained in a V-poset V and, provided that $h-1 \geqslant 3$, its second innermost cover has $w = w(V)$ as its upper end. Consider the elements x, y, z of V with y being the minimal element, and the two corresponding V-posets V_1, V_2 with minimal elements y_1, y_2 of Q_h, where $y_1 x$ and $y_2 z$ are covers; see Figure 10. By definition of $\hat{y}$ and L', all elements of $\{x, y\} \cup V_1 \cup V_2$ lie between $\hat{y}$ (included) and w (excluded, if $h-1 \geqslant 3$) with respect to L.

Assume without loss of generality that $x <_L z$. If $y_2 <_L y_1$ $(\hat{y} = y_2)$, then the V-poset with y_1 is nested completely under the cover $y_2 z$ and replacing in R the innermost cover by the cover $y_2 z$ and any cover with y_1 gives a $(h-1)$-rainbow with the desired properties. If $y_1 <_L y_2$ $(\hat{y} = y_1)$, then the V-poset with y_2 is nested completely under the cover $y_1 x$ and replacing in R the innermost cover by the cover $y_1 x$ and any cover with y_w gives a $(h-1)$-rainbow with the desired properties, which concludes the proof. □

5 Conclusions

We studied the queue-number of (planar) posets of bounded height and width. Two main problems remain open: bounding the queue-number by the width and bounding it by a function of the height in the planar case, where the latter is equivalent to the central conjecture in the area of queue-numbers of graphs. For the first problem the biggest class known to satisfy it are posets without the embedded the subdivided k-crowns for $k \geqslant 2$ as defined in Section 2. Note, that proving it for $k \geqslant 3$ would imply that Conjecture 2 holds for all 2-dimensional posets, which seems to be a natural next step.

Let us close the paper by recalling another interesting conjecture from [9], which we would like to see progress in:

Conjecture 4 (Heath and Pemmaraju [9]).
Every planar poset on n elements has queue-number at most $\lceil \sqrt{n} \rceil$.

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