LATTICE PATH MATROIDS AND QUOTIENTS

CAROLINA BENEDETTI AND KOLJA KNAUER

Abstract. We characterize the quotients among lattice path matroids (LPMs) in terms of their diagrams. This characterization allows us to show that ordering LPMs by quotients yields a graded poset, whose rank polynomial has the Narayana numbers as coefficients.

Furthermore, we study full lattice path flag matroids and show that – contrary to arbitrary positroid flag matroids – they correspond to points in the nonnegative flag variety. At the basis of this result lies an identification of certain intervals of the strong Bruhat order with lattice path flag matroids.

A recent conjecture of Mcalmon, Oh, and Xiang states a characterization of quotients of positroids. We use our results to prove this conjecture in the case of LPMs.

1. Introduction

Matroids, introduced independently by Whitney [35] and Nakasawa [27], around 1930, are an abstraction of the concept of linear independence from linear algebra, carried to other settings such as graphs, systems of distinct representatives, transcendental extensions of fields, etc. This paper focuses on a class of matroids called *representable* as defined in Section 2.1. The family of representable matroids we are particularly interested in are *positroids*. Positroids appear in the work of da Silva from the perspective of oriented matroids (see [17], [2]), then by Blum [6] in terms of Koszulness of rings associated to a matroid. Finally, Postnikov [29] introduced positroids via a stratification of the totally nonnegative Grassmannian. This latter point of view is the one that has spiked most of the research related to positroids, in particular, since part of the work of Postnikov includes several combinatorial characterizations of them.

A categorical view point on matroids leads to the notion of quotients, see [14, 20, 21]. Matroid quotients are part of standard text books such as [28] and natural appearances can be found in linear algebra and graphic matroids. For instance, out of a graph one can construct a quotient after identifying some vertices. Despite the several ways that there are to define quotients, it can be very difficult to determine the quotients of a general matroid, and even worse, to characterize quotients for a given family of matroids.

The present paper focuses on a family of positroids called *lattice path matroids*, LPMs for short. We provide an answer to the question:

> Given two lattice path matroids M and M' on the same ground set, how can we determine combinatorially if M is a quotient of M'?

Any LPM can be thought of as a diagram in the plane grid as in Figure 1. Such a diagram is bounded above by a (monotone) lattice path U and below by a lattice path L. Any lattice path from the bottom left to the upper right corner inside this grid is identified with a set

B, where $i \in B$ if and only if the ith step of the path is North. We denote the resulting LPM by $M[U, L]$.

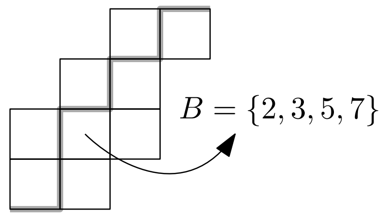

FIGURE 1. A basis in the diagram representing the LPM $M[1246, 2568]$.

LPMs were introduced by Bonin, de Mier, and Noy [10], where fundamental properties were established. Many different aspects of lattice path matroids have been studied: excluded minor characterizations [7], algebraic geometry notions [19, 30, 31], the Tutte polynomial [11, 23, 25], the associated basis polytope in connection with its facial structure [1, 5], specific decompositions in relation with Lafforgue's work [16], as well as its Ehrhart polynomial [4, 5, 22].

The study of LPMs as a subclass of positroids, including analyzing quotients of these, is mostly novel apart from [18], where certain quotients of LPMs related to the tennis ball problem are explored.

One of the main contributions of this paper provides a way to determine all the quotients of a given LPM (Theorem 18). The advantage of this characterisation is that it allows to tell the quotients of an LPM purely based on its diagram. As a consequence of this result, we are able to build a graded poset $\mathcal{P}_n$ whose elements are LPMs ordered by quotients. Some enumerative results regarding $\mathcal{P}_n$ are stated in Corollary 20, where it is shown that the rank function of $\mathcal{P}_n$ has as coefficients the Narayana numbers.

A maximal sequence of distinct matroids on the same ground set, where each matroid is a quotient of the next, is a *full flag matroid*, see [13]. We can view full flag matroids consisting of LPMs as saturated chains in $\mathcal{P}_n$. Our interest in these flags, called lattice path flag matroids (LPFMs), arises from thinking of LPMs as positroids. See Section 2 for the necessary background and motivation.

Positroids can be thought of as cells of the nonnegative Grassmannian. On the other hand, points in the nonnegative flag variety $\mathcal{F}\ell_n^{\geq 0}$ can be thought of as certain full positroid flag matroids (PFMs) [33]. However, not every PFM arises this way (see Example 6). Moreover, in [33] the authors prove that the points in $\mathcal{F}\ell_n^{\geq 0}$ correspond to an intervals in the (strong) Bruhat order. Our second main result shows that every LPFM is, indeed, an interval in the Bruhat order and thus, a point in $\mathcal{F}\ell_n^{\geq 0}$ (Theorem 29 and Corollary 30). Moreover we characterize those intervals in the Bruhat order that come from LPFMs (Theorem 31). In particular, Proposition 33 shows that cubes in the (right) weak Bruhat order are instances of these intervals.

Combining our description of LPM quotients with the fact that LPFMs are points in $\mathcal{F}\ell_n^{\geq 0}$ we achieve our final result Theorem 37: the (realizable) quotient relation among LPMs can be expressed in terms of certain objects called CCW arrows in [24]. This confirms a conjecture of Mcalmon, Oh, Xiang in the case of LPMs.

We finish with diagram representations of LPFMs as suggested by de Mier [18], Higgs lifts and the weak order on LPMs, as well as questions on the structure of LPFMs within $\mathcal{F}\ell_n^{\geq 0}$.

2. Preliminaries

2.1. **Matroids, positroids and the (real) Grassmannian.** There are several equivalent ways to define matroid, see [28]. For our purposes we say that a *matroid* $M = (E, \mathcal{B})$ is a pair consisting of a finite set E and a non-empty collection $\mathcal{B}$ of subsets of E that satisfies:

if $A, B \in \mathcal{B}$ and $a \in A \setminus B$, then there is $b \in B \setminus A$ such that $(A \setminus \{a\}) \cup \{b\} \in \mathcal{B}$.

In this context, we refer to the set E as the ground set of M and the collection $\mathcal{B}$ as the *set of bases* of M. Also, an element $A \in \mathcal{B}$ is said to be a *basis* of M. Since the set E has cardinality n, for some $n \geq 0$, we will identify it with the set $[n] := \{1, \ldots, n\}$. The *uniform matroid* of rank k over $[n]$, denoted $U_{k,n}$, is the matroid whose bases are all the subsets of size k of $[n]$.

Given a matroid $M = ([n], \mathcal{B})$, it is known that elements of $\mathcal{B}$ have all the same cardinality, say $k \geq 0$, just as bases of a finite dimensional vector space have the same size. In this case, we say that *the rank* of M is k, and we denote this as $r(M) = k$. A matroid $M = ([n], \mathcal{B})$ of rank k is said to be *representable* (over $\mathbb{R}$) if there exists a collection of vectors $S = \{u_1, \ldots, u_n\} \subseteq \mathbb{R}^k$ such that $\dim(span(S)) = k$ and $\{i_1, \ldots, i_k\} \in \mathcal{B}$ if and only if $\{u_{i_1}, \ldots, u_{i_k}\}$ is a basis of $span(S)$. In this case, the $k \times n$ matrix whose columns are the set S is said to be a *(matrix) representation of* M. Although almost all matroids are non representable [26], in this paper the matroids we are interested in are the ones that are representable over $\mathbb{R}$. We will in the following elaborate on one of the many reasons why this class is important.

The (real) *Grassmannian* $\mathrm{Gr}_{k,n}$ consists of all the k-dimensional vector subspaces V of $\mathbb{R}^n$. Let $V \in \mathrm{Gr}_{k,n}$ and let $\{v_1, \ldots, v_k\}$ be a basis of V. Then the $k \times n$ matrix A_V whose rows are $\{v_1, \ldots, v_k\}$ gives rise to a representable matroid $M = ([n], \mathcal{B})$ of rank k such that $B = \{i_1, \ldots, i_k\} \in \mathcal{B}$ if and only if $\Delta_B \neq 0$, where Δ_B is the $k \times k$ determinant of A_V obtained from the columns indexed by the set B. Now let us talk about the *nonnegative Grassmannian* $\mathrm{Gr}_{k,n}^{\geq 0}$. As a set, $\mathrm{Gr}_{k,n}^{\geq 0}$ consists of those $V \in \mathrm{Gr}_{k,n}$ for which there exists a full rank $k \times n$ matrix A_V, whose rows span V, such that every maximal minor of A_V is nonnegative. The representable matroid $M = ([n], \mathcal{B})$ arising from such $V \in \mathrm{Gr}_{k,n}^{\geq 0}$, as explained before, is exactly what is called a *positroid.* Note that for all maximal minors to be nonnegative, the ordering of the columns is essential, which is why a positroid is a matroid on $[n]$, where the (natural) ordering of the ground set is part of the input. Let us clarify this with an:

Example 1. The matroid $P = ([4], \mathcal{B})$ where $\mathcal{B} = \{13, 14, 23, 24\}$, is a positroid, since the matrix $A_V = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ is such that each of the maximal minors indexed by the sets $\{13, 14, 23, 24\}$ is positive and the remaining maximal minors are 0. Notice that we are writing ij to denote the subset $\{i, j\}$, as long as there is no confusion. . In particular, this example allows us to conclude that the subspace $V = span\langle(1, 1, 0, 0), (0, 0, 1, 1)\rangle$ is an element of $\mathrm{Gr}_{2,4}$. On the other hand, the matroid $M = ([4], \mathcal{B})$ where $\mathcal{B} = \{12, 14, 23, 34\}$ is representable but is not a positroid. We leave this as an exercise to the reader. Notice that the matroid M corresponds to a relabelling of the elements of P, thus as remarked above being a positroid depends strongly on the ordering of the ground set.

We already mentioned several instances where positroids have appeared. For our purposes, the importance of positroids is that they contain the family of lattice path matroids, as will be defined in Section 3. Although our treatment in the present paper is purely combinatorial, we want to emphasize that our initial interest for developing this project started from the connection between geometry and matroid theory via the Grassmanian (and its relatives), and representable matroids.

Going back to our discussion above, let us scratch the surface on the connection that interests us between geometry and matroid theory. Several decompositions of the Grassmannian have been studied and many of them give rise to different families of representable matroids. In order to mention them we will denote by $\binom{[n]}{k}$ the collection of subsets $A \subset [n]$ such that $|A| = k$.

Definition 2. Let $i \in [n]$. The *i-th Gale order* $\leq_i$ on $[n]$ is the total order given by $i <_i i+1 <_i \cdots <_i n <_i 1 < \cdots <_i i-1$. In particular, $\leq_1$ is the usual order on $[n]$. Let $A, B \in \binom{[n]}{k}$. We say that A *is smaller than* B in the i-th Gale order if, for every r it holds that $a_r \leq_i b_r$, where $A = \{a_1 <_i \cdots <_i a_k\}$ and $B = \{b_1 <_i \cdots <_i b_k\}$. For us the most important will be the 1st Gale order on $\binom{[n]}{k}$, which we will call simply *Gale order* and denote it by $\leq_G$ if no confusion arises.

We have discussed the cells of the Grassmanian and the particular positroid cells. Let us present two further specializations of positroid cells in terms of the 1st Gale order.

- *Schubert cell* Ω_I: Let $I \in \binom{[n]}{k}$. A generic point $U \in \Omega_I$ gives rise to a representable matroid $M_I = ([n], \mathcal{B})$ such that $B \in \mathcal{B}$ if and only if $I \leq_G B$. We call the matroid M_I a Schubert matroid. For example, the matroid $M = ([4], \{13, 14, 23, 24, 34\})$ arises from the generic point $A = \begin{pmatrix} 1 & \star & \star & \star \\ 0 & 0 & 1 & \star \end{pmatrix} \in \Omega_{13}$, where the $\star$'s are generically chosen real numbers. That is, every pair of columns of A, except for 12, is a basis of the column space of A.
- *Richardson cell* Ω_I^J: Let $I, J \in \binom{[n]}{k}$ such that $I \leq_G J$. A generic point $U \in \Omega_I^J$ gives rise to a representable matroid $M_I^J = ([n], \mathcal{B})$ such that $B \in \mathcal{B}$ if and only if $I \leq_G B \leq_G J$. A matroid M_I^J arising this way is known as a *lattice path matroid*. In particular, every Schubert matroid is a lattice path matroid. For example, the Schubert matroid M given above comes from a generic point in Ω_{13}^{34}.

The main goal of this paper is to describe combinatorially *quotients* of lattice path matroids as will be defined shortly. The link with the previous discussion will be made via the *flag variety*.

2.2. **Quotients of matroids.**

As it goes with many concepts in matroid theory, the concept of quotient of matroids has many equivalent definitions. The interested reader is encouraged to consult, for instance, [14, 15, 34]. The definition we provide here is as follows.

Definition 3. [34, Prop. 7.4.7] Consider two matroids M and M' on the ground set $[n]$ with base sets $\mathcal{B}$ and $\mathcal{B}'$, respectively. We say that M is a *quotient* of M' if for all $B' \in \mathcal{B}', p \notin B'$ there is $B \in \mathcal{B}$ such that $B \subseteq B'$ and if $B \cup \{p\} \setminus \{q\} \in \mathcal{B}$ then $B' \cup \{p\} \setminus \{q\} \in \mathcal{B}'$, for all $q \in B$. We denote this by $M \leq_q M'$.

For example, as the reader may check we have that $U_{r,n} \leq_q U_{s,n}$ for all $0 \leq r \leq s \leq n$. Moreover, in [3] the authors give a combinatorial way to determine families of positroids that are a quotient of $U_{k,n}$, for any $0 \leq k \leq n$.

Remark 4. Observe that if $M \leq_q M'$ then $r(M) \leq r(M')$. In particular, $r(M) = r(M')$ implies $M = M'$. On the other hand, Definition 3 can be restated as follows. Given $p \in [n] \setminus B$ we will set $B_p := \{q \in B \mid B + p - q \in \mathcal{B}\}$. Then we obtain that $M \leq_q M'$ if and only if for all $B' \in \mathcal{B}'$ and $p \in [n] \setminus B'$ there is $B \in \mathcal{B}$ such that $B \subseteq B'$ and $B_p \subseteq B'_p$.

Although Definition 3 seems tricky to work with, as one may suspect, the notion of matroid quotient is better understood for certain families of matroids. For example, for $k \leq n$, given a full rank $k \times n$ matrix A let M_A be the realizable matroid on $[n]$ of rank k that A gives rise to. Now let A' be the $i \times n$ submatrix obtained from A by deleting its bottom $k - i$ rows, for some $i \in [k-1]$. Then the representable matroid $M_{A'}$ that A' gives rise to, is a quotient of the matroid M_A. What we are interested in is a handy and combinatorial way to determine when two lattice path matroids M and M' on $[n]$ are such that $M \leq_q M'$. In fact, we care about giving a characterisation of *flags* of LPMs.

Definition 5 ([13])**.** A *flag matroid* is a sequence $\mathcal{F} = (M_0, M_1, \ldots, M_k)$ of distinct matroids on the ground set $[n]$ such that M_i is a quotient of M_{i+1} for $i \in \{0, 1, \ldots, k-1\}$. If $k = n$, then we say that $\mathcal{F}$ is a *full* flag matroid. Each of the M_i's is said to be a *constituent of* $\mathcal{F}$. If $B_0 \subseteq \cdots \subseteq B_k$ is a sequence where B_i is a basis of M_i, we refer to it as a *flag of bases in* $\mathcal{F}$. If every M_i is a positroid we say that $\mathcal{F}$ is a *positroid flag matroid* (PFM). If every M_i is an LPM we say that $\mathcal{F}$ is a *lattice path flag matroid* (LPFM).

From Definition 5 we remark that if $\mathcal{F} = (M_0, M_1, \ldots, M_k)$ is a flag matroid then $r(M_0) < r(M_1) < \cdots < r(M_k)$. In particular if the flag $\mathcal{F}$ is a full flag, then M_0 is the matroid $U_{0,n}$ and $M_n = U_{n,n}$. For our purposes, we will only focus on full flag matroids either if in the PFM or the LPFM case.

Now we want to extend the dictionary between $\mathrm{Gr}_{k,n}$ and representable matroids, given so far. The *(real) full flag variety* $\mathcal{F}\ell_n$ consists of sequences (flags) of vector spaces $\mathcal{F} : \{\mathbf{0}\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{R}_n$ such that $V_i \in \mathrm{Gr}_{i,n}$ for $i = 1, \ldots, n$. Thus each such $\mathcal{F}$ can be thought of as a full rank $A_{n\times n}$ matrix whose top j rows give rise to a representable matroid M_j of rank j. Therefore, the point $\mathcal{F} \in \mathcal{F}\ell_n$ gives rise to the full flag matroid $\mathcal{F} = (M_0, M_1, \ldots, M_n)$. In this case $\mathcal{F}$ is said to be a *representable flag matroid* (over $\mathbb{R}$), and A *represents* the flag matroid $\mathcal{F}$. However, even if two representable matroids M and M' are such that $M <_q M'$, they do not necessarily form (part of) a representable flag matroid. This is, there may be no matrix A that gives rise to both of them, simultaneously (see [13, Section 1.7.5] or [15, Example 6.9]).

Finally, the *nonnegative full flag variety* $\mathcal{F}\ell_n^{\geq 0}$ consists of sequences $\mathcal{F} : V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{R}_n$ of vector spaces such that $\mathcal{F}$ can be given by a full rank $A_{n\times n}$ matrix whose top j

rows span V_j as a point in $\mathrm{Gr}_{j,n}^{\geq 0}$ for each $j \in [n]$. That is, A is such that each submatrix A_j has nonnegative maximal minors and its row-space spans V_j, for each $j \in [n]$. In this case we say that $\mathcal{F}$ is *nonnegatively representable*. The following problems are in order:

(P1) Does every full positroid flag matroid $\mathcal{F}$ come from a point in $\mathcal{F}\ell_n^{\geq 0}$?
(P2) Does every full lattice path flag matroid $\mathcal{F}$ come from a point in $\mathcal{F}\ell_n^{\geq 0}$?

From now on when we refer to a flag matroid (of any kind) we mean a full flag matroid. Thus, LPFMs refer to full flags of LPMs, and similarly for PFM. Now, if the answer to problem P1 were affirmative, then P2 would be as well, since the family of LPFMs is a subset of the family of PFMs. The discussion we have conveyed here is summarized in Table 1.

In this paper, we will see that the answer to problem P2 is yes. Now let us illustrate why the answer to P1 is negative.

Geometry	**Matroids**
Point V in $\mathrm{Gr}_{k,n}$	Representable matroid $M = ([n], \mathcal{B})$ of rank k
Richardson cell Ω_I^J	Lattice path matroid $M[I,J]$
Point V in $\mathrm{Gr}_{k,n}^{\geq 0}$	Positroid $M = ([n], \mathcal{B})$ of rank k
Flag $F : V_1 \subset \cdots \subset V_n$ in $\mathcal{F}\ell_n$	Representable flag matroid $M_1 \leq_q \cdots \leq_q M_n$
Flag $F : V_1 \subset \cdots \subset V_n$ in $\mathcal{F}\ell_n^{\geq 0}$	(P1)
(P2)	lattice path flag matroid $M_0 \leq_q M_1 \leq_q \cdots \leq_q M_n$
(P1)	positroid flag matroid $M_0 \leq_q M_1 \leq_q \cdots \leq_q M_n$

Table 1. Bridge between geometry and realizable matroids.

Example 6. Let M_1 be the positroid on [3] whose set of bases is $\mathcal{B}_1 = \{1, 3\}$ and let $M_2 = U_{2,3}$ be the uniform matroid of rank 2 on [3]. That is, the bases of M_2 are $\mathcal{B}_2 = \{12, 13, 23\}$. We leave to the reader the task to check that M_1 and M_2 are positroids[1] and that $M_1 \leq_q M_2$. Thus the flag $\mathcal{F} : U_{0,3} \leq_q M_1 \leq_q M_2 \leq_q U_{3,3}$ is a PFM. If $\mathcal{F}$ where to come from an element in $\mathcal{F}\ell_n^{\geq 0}$ then there would be a 3×3 matrix

$$A = \begin{pmatrix} a & 0 & b \\ c & d & e \\ f & g & h. \end{pmatrix}$$

such that $\det A > 0$ and also the submatrices $\begin{pmatrix} a & 0 & b \end{pmatrix}$ and $\begin{pmatrix} a & 0 & b \\ c & d & e \end{pmatrix}$ would be a representation of the positroids M_1 and M_2, respectively. This forces $a > 0$ and $b > 0$ since $\mathcal{B}_1 = \{1, 3\}$. On the other hand, since $12 \in \mathcal{B}_2$ then $ad > 0$ and thus $d > 0$. Similarly, since $23 \in \mathcal{B}_2$ then $-bd > 0$ and thus $d < 0$ which is a contradiction. Thus, we are not able to obtain the PFM $\mathcal{F} : U_{0,3} \leq_q M_1 \leq_q M_2 \leq_q U_{3,3}$ as coming from a point in $\mathcal{F}\ell_n^{\geq 0}$.

It is known that every uniform matroid $U_{k,n}$ is an LPM and, as mentioned above, in [3] the authors give a partial characterization of positroids M such that $M \leq_q U_{k,n}$. Here, we

[1]In fact every uniform matroid is a positroid.

are interested in particular in a description of those LPMs M' such that given an LPM M it follows that $M' \leq_q M$. Thus, a complete answer to this question, which we will give, does not imply the aforementioned result in [3] since some quotients of $U_{k,n}$ are not LPMs.

To our knowledge it is open whether every positroid flag matroid corresponds to a point in the flag-variety. Furthermore, we do not know if every LPFM corresponds to a point in the Richardson flag variety

3. Quotients of LPMs

We have defined the i-th Gale order in Definition 2. For the rest of the paper however, we will restrict ourselves to the 1st Gale order and will simply call it Gale order, which we will recall now. Let $B, B' \in \binom{[n]}{k}$. We say that B *is smaller than* B' in the Gale order if $b_i \leq b'_i$, for all $i \in [k]$, where $B = b_1 < \ldots < b_k$ and $B' = b'_1 < \ldots < b'_k$, for some $k \leq n$. We denote this by $B \leq_G B'$.

In view of this, let us recall the definition of lattice path matroid.

Definition 7. Let $0 \leq k \leq n$ and let $U, L \in \binom{[n]}{k}$ be such that $U \leq_G L$. The *lattice path matroid* $M[U, L]$ is the matroid over the set $[n]$ whose collection of bases is given by $\mathcal{B} = \{B \in \binom{[n]}{k} \mid U \leq_G B \leq_G L\}$.

Setting $M = M[U, L]$ in Definition 7 it follows that M has rank k. In particular U and L are bases of M. Moreover, if $U = \{u_1 < \cdots < u_k\}$ and $L = \{\ell_1 < \cdots < \ell_k\}$ then U corresponds to the lattice path from $(0, 0)$ to $(k, n-k)$ whose north steps are labelled by U, and similarly for L. Thus, if B is any basis of M then B corresponds to a lattice path from $(0, 0)$ to $(k, n-k)$ whose labels are in B and B lies between U and L since $U \leq_G B \leq_G L$. See Figure 1, where the basis $\{2, 3, 5, 7\}$ of $M[1246, 2568]$ is represented in the diagram.

The following is derived from the very definition of $\leq_G$.

Observation 8. *Let $0 \leq k \leq n$ and let $M[U, L]$ be an LPM of rank k over $[n]$. The Gale order endows the set of bases of $M[U, L]$ with the poset structure of the interval $[U, L]_G$ of the bases of $U_{k,n}$ ordered by $\leq_G$.*

Observation 8 in particular yields that ordering the bases of an LPM by $\leq_G$ endows the set $\mathcal{B}$ of bases with a distributive lattice structure, that has been characterized in [22]. See Figure 2 for an example.

In what remains for this section we intend to describe combinatorially quotients of LPMs. In particular, we will determine when $M[U \setminus \{u\}, L \setminus \{\ell\}]$ is a quotient of $M[U, L]$, where $u \in U$ and $\ell \in L$. Let us start by gathering some more intuition. Given $A \in \binom{[n]}{k}$ we denote its elements using lower case as $A = \{a_1 < \cdots < a_k\}$.

If $M = M[U, L]$, it is not true in general that $M[U \setminus \{u_j\}, L \setminus \{\ell_i\}]$ is a quotient of M, for any choice of $i, j \in [k]$. As an example let $M = M[1357, 3578]$ and take the basis $B = 1467$, also take $j = 4$ and $i = 1$. Using the notation from Remark 4 we have that $B_5 = \{q \in B \mid B + 5 - q \in M\} = \{4, 6\}$. Moreover, for any $i \leq r \leq j$ one can check that in the matroid $M[135, 578]$ we obtain $(B \setminus \{b_r\})_5 = B \setminus \{b_r\} \not\subseteq B_5$. We also point out that not every quotient of an LPM is an LPM. Indeed, every matroid on $[n]$ (whether positroid or not) is a quotient of the LPM $U_{n,n}$. Maybe a little bit more surprising is that not even

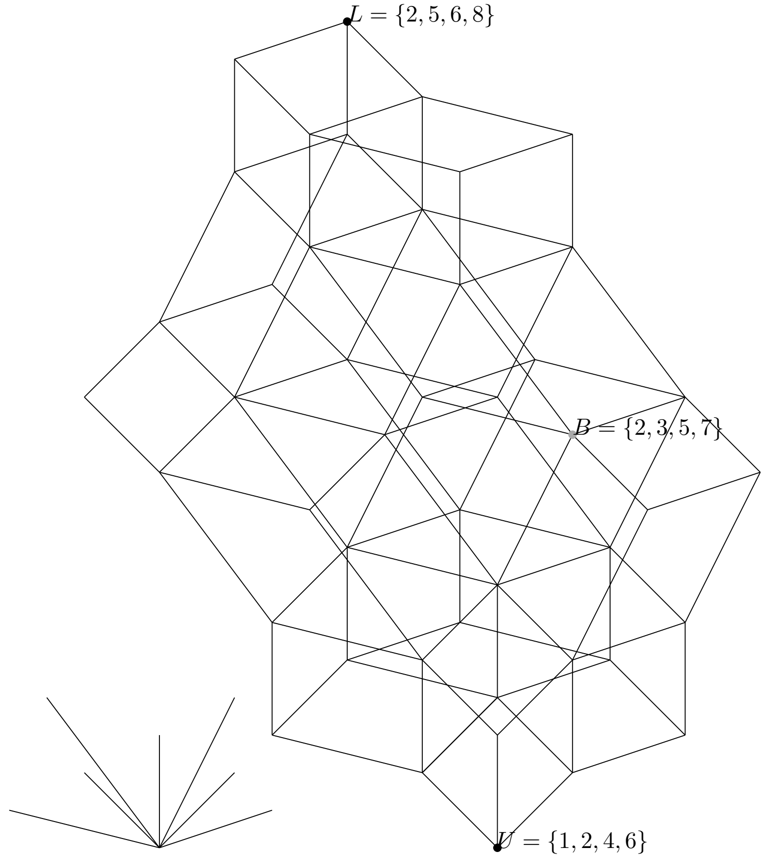

FIGURE 2. The lattice of bases of $M[1246, 2568]$.

truncations (which are special quotients) of LPMs are LPMs. An example of this situation comes from taking the LPM given by th direct sum $M = U_{1,2} \oplus U_{1,2} \oplus U_{1,2}$. Its rank-two truncation is not an LPM, although it is a positroid, see [9].

The following will be essential for our results and is illustrated in Figure 3 as a visual aid.

Lemma 9. *Let $M = M[U, L]$ of rank k and let $B = \{b_1 < \cdots < b_k\}$ be a basis of M. Let $p \in [n]$ such that $p \notin B$ and choose x such that $b_{x-1} < p < b_x$. Then $B_p = \{b_s < \ldots < b_t\}$ where*

(a) $1 \leq s \leq x$ *and* $b_{s+1} \leq \ell_s, \ldots, b_{x-1} \leq \ell_{x-2}, p \leq \ell_{x-1}$,
(b) $x - 1 \leq t \leq r$ *and* $p \geq u_x, b_x \geq u_{x+1}, \ldots, b_{t-1} \geq u_t$.

Proof. For the proof consider Figure 3, where the basis B is a monotone path P in the LPM diagram. Since $p \notin B$, it corresponds to a horizontal segment of P. Now, B_p consists of those vertical segments q of P that can be made horizontal such that after making p vertical, the path Q corresponding to $B \setminus \{q\} \cup \{p\}$ remains within the boundaries of the diagram. These segments are (a) between the last time B touched L before arriving at p and p itself or (b) after p and the next time B touches U. This is what is expressed through indices in the statement of the lemma. ∎

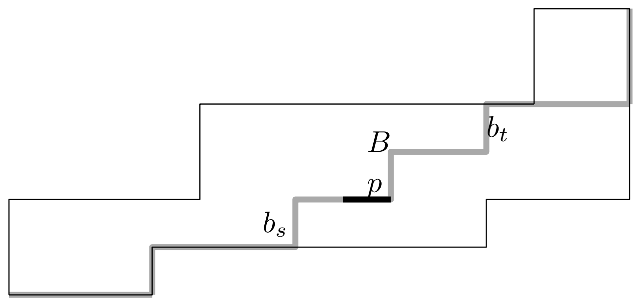

FIGURE 3. An illustration of Lemma 9: the steps in B that can be replaced by p in order to produce a new basis are those between b_s and b_t.

Definition 10. Let $M = M[U, L]$ be an LPM where $U = \{u_1, \cdots, u_k\}$, $L = \{\ell_1, \cdots, \ell_k\}$. Let $1 \leq i, j \leq k$. We say that (ℓ_i, u_j) is a *good pair of M* if

(1) $i \leq j$,
(2) $u_j - \ell_i \leq j - i$.

If $\{u_1, \ldots, u_z\} \subseteq U$ and $\{\ell_1, \ldots, \ell_z\} \subseteq L$ then we call the sequence $((\ell_{i_1}, u_{j_1}), \ldots, (\ell_{i_z}, u_{j_z}))$ a *pairing of M*. We say that a pairing of M is *good* if (ℓ_{i_r}, u_{j_r}) is a good pair of $M[U', L']$ where $U' = U \setminus \{u_{j_1}, \ldots, u_{j_{r-1}}\}$ and $L' = L \setminus \{\ell_{j_1}, \ldots, \ell_{j_{r-1}}\}$, for $1 \leq r \leq z - 1$.

Graphically, being a good pair can be visualized as follows. The step u_j is such that its northern vertex (a, b) determines the closed region R_{u_j} bounded below by L, and lies in the halfspaces $x \geq a$ and $y \leq b$. Then the pair (ℓ_i, u_j) is a good pair if ℓ_i lies in R_{u_j}. Figure 4 depicts a bad pair (ℓ_i, u_j). Every good pair (ℓ_i, u_j) allows us to characterize LPMs of rank $k-1$ on that are a quotient of a given LPM $M = M[U, L]$ of rank k as the upcoming result shows.

Proposition 11. *Let $M = M[U, L]$ be such that $r(M) = k$ and let (ℓ_i, u_j) be a good pair of M. Then the matroid $M' = M[U \setminus \{u_j\}, L \setminus \{\ell_i\}]$ of rank $k-1$ is a quotient of M.*

Proof. Let $B \in \mathcal{B}(M)$ and let (ℓ_i, u_j) be a good pair of M for some $1 \leq i \leq j \leq k$. For any $r \in [k]$ it holds that $B \setminus \{b_r\} \subseteq B$. Furthermore, since $U \leq_G B \leq_G L$ and if $i \leq r \leq j$ we have $U \setminus \{u_j\} \leq_G U \setminus \{u_r\} \leq_G B \setminus \{b_r\} \leq_G L \setminus \{\ell_r\} \leq_G L \setminus \{\ell_i\}$. Therefore $B \setminus \{b_r\}$ is a basis of M'.

Now let $p \notin B$. We will show that if $\max(0, u_j - \ell_i) \leq j - i$, then r can be chosen such that $(B \setminus \{b_r\})_p \subseteq B_p$. We use the description of B_p provided by Lemma 9. We want to choose $i \leq r \leq j$ such that for $L' = L \setminus \{\ell_i\}$, $B' = B \setminus \{b_r\}$, $U' = U \setminus \{u_j\}$ and the correspondingly defined s', t' we have that $b_s \leq b'_{s'}$ and $b_t \geq b'_{t'}$.

Case 1: Let $i < s$ and $t < j$. In this situation it holds that $\ell_i \leq \ell_{s-1} < b_s < \ldots < p < \ldots < b_t < u_{t+1} \leq u_j$. Thus, $u_j - \ell_i > t - s + 2 \geq j - i$, which contradicts our assumption on (ℓ_i, u_j) being a good pair. Hence, we cannot have $i < s$ and $t < j$ simultaneously.

Case 2: If $s > i$, then we set $r = i$. We get either $\ell'_{s-2} \leq \ell_{s-1} < b_s = b'_{s-1} \leq b'_{s'}$ or $s - 1 = 1 \leq s'$. Since $r \leq j \leq t$, one can see that either $u'_t = u_{t+1} > b_t \geq b'_{t-1} \geq b'_{t'}$ or $t = r \geq t'$.

Case 3: Similarly, if $t < j$, then we set $r = j$ and we obtain that either $u'_{t+1} \geq u_{t+1} > b_t = b'_t \geq b'_{t'}$ or $t = r \geq t'$. By the above we have $s \leq i \geq r$, we compute either $\ell'_{s-1} = \ell_{s-1} < b_s \leq b'_s \leq b'_{s'}$ or $s = 1 \leq s'$.

Case 4: If $s \leq i$ and $t \geq j$ any choice of $i \leq r \leq j$ yields a good B'. Indeed, as above we will get $u'_t = u_{t+1} > b_t \geq b'_{t-1} \geq b'_{t'}$ and $\ell'_{s-1} = \ell_{s-1} < b_s \leq b'_s \leq b'_{s'}$. ☐

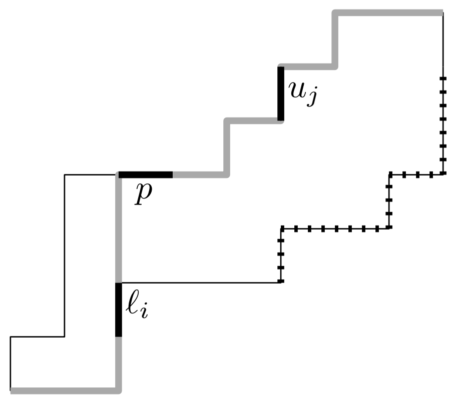

FIGURE 4. An LPM with a bad pair (ℓ_i, u_j) . The gray basis B has $(B \setminus \{b_r\})_p \not\subseteq B_p$ for all r. Exactly those $\ell \in L$ on the dotted path yield good pairs with u_j.

Lemma 12. *Let $M = M[U, L]$ and $M' = M[U', L']$. If $M \leq_q M'$, then $U \subseteq U'$ and $L \subseteq L'$.*

Proof. We only show $U \subseteq U'$, the proof that $L \subseteq L'$ is analogous. Suppose, by contradiction, that $U \not\subseteq U'$ and choose the smallest $p \in U \setminus U'$. A quick computation using Lemma 9 yields that $u' < p$, where $u' = \max\{y : y \in U'_p\}$. Let now $B \in M$ such that $B \subseteq U'$. Such B exists as $M \leq_q M'$. By the choice of p we have $b_i = u_i = u'_i$ for all $i < x$, where as in Lemma 9 $u'_{x-1} < p < u'_x$. Since $p \in U$ and $B \subseteq U'$ we have $b_x > p = u_x$. Thus, $b_x \notin U'_p$. However, Lemma 9 yields $b_x \in B_p$, leading to a contradiction. ☐

Lemma 13. *Let $M = M[U, L]$ and $M' = M[U', L']$ such that $M \leq_q M'$ and $r(M') = k$. Let $U' \setminus U = \{u'_{i_1} < \ldots < u'_{i_z}\}$ and $L' \setminus L = \{\ell'_{j_1} < \ldots < \ell'_{j_z}\}$ so that $k - z = r(M)$. Then $\{i_1, \ldots, i_z\} \geq_G \{j_1, \ldots, j_z\}$.*

Proof. We argue by contradiction. Suppose that $\{i_1, \ldots, i_z\} \not\geq_G \{j_1, \ldots, j_z\}$ and let w be the smallest index such that $i_w < j_w$. In particular, the minimality of the choice of w yields $i_w > i_{w-1} \geq j_{w-1} > \ldots > j_1$ and $\ell'_{i_w} = \ell_{i_w - w+1}$. Moreover, we have $u'_{i_w} < u'_{i_w+1} = u_{i_w-w+1}$. Consider now the set $B' = \{u'_1, \ldots, u'_{i_w}, \ell'_{i_w+1}, \ldots, \ell'_k\}$, which is a basis of M'. By the quotient relation there is a set Z of size z such that $U \leq_G B \leq_G L$ where $B := B' \setminus Z$.

Now, since $U \leq_G B$ we have $u'_{i_w} < u_{i_w-w+1} \leq b_{i_w-w+1}$ which by the shape of B' implies $\ell'_{i_w+1} \leq b_{i_w-w+1}$. But by $\ell'_{i_w+1} > \ell'_{i_w} = \ell_{i_w-w+1}$, this yields $b_{i_w-w+1} > \ell_{i_w-w+1}$ and contradicts $B \leq_G L$. ☐

If $M = M[U, L]$ is an LPM on the ground set $[n]$ then its dual matroid M^* is such that $M^* = M[\overline{L}, \overline{U}]$ where $\overline{A} := [n] \setminus A$ for $A \subseteq [n]$. Then, Lemma 13 can be stated in terms of M^* and M'^* since $M \leq_q M'$ if and only if $M'^* \leq_q M^*$, see [14, Proposition 7.4.7] and $U' \setminus U = \overline{U} \setminus \overline{U'}$. Thus we obtain the following result whose proof we leave to the reader.

Lemma 14. *Let M^* and M'^* as above. If $\overline{U} \setminus \overline{U'} = \{\overline{u}_{i_1}, \ldots, \overline{u}_{i_z}\}$ and $\overline{L} \setminus \overline{L'} = \{\overline{\ell}_{j_1}, \ldots, \overline{\ell}_{j_z}\}$ then $\{i_1, \ldots, i_z\} \leq_G \{j_1, \ldots, j_z\}$.*

Let M and M' as in Lemma 13. The following definition is an extension of Definition 10 and will allow us to provide a sequence $M_1, \ldots, M_{z-1}$ of LPMs of ranks $k-z+1, \ldots, k-1$, respectively, such that $M \leq_q M_1 \leq_q \cdots \leq_q M_{z-1} \leq_q M'$.

Definition 15 (Greedy pairing)**.** Let $M = M[U, L]$ and $M' = M[U', L']$ be LPMs over $[n]$ such that $L \subseteq L', U \subseteq U'$ and set $U' \setminus U = \{u'_{i_1} < \ldots < u'_{i_z}\}$, $L' \setminus L = \{\ell'_{j_1} < \ldots < \ell'_{j_z}\}$. We refer to the sequence $((\ell'_{j_1}, u'_{i_1}), \ldots, (\ell'_{j_z}, u'_{i_z}))$ as *the greedy pairing of* $(L' \setminus L, U' \setminus U)$.

Lemma 16. *Let $M = M[U, L]$ and $M' = M[U', L']$ be such that $M \leq_q M'$ and $r(M') = k$. Let $U' \setminus U = \{u'_{i_1} < \ldots < u'_{i_z}\}$ and $L' \setminus L = \{\ell'_{j_1} < \ldots < \ell'_{j_z}\}$ so that $k - z = r(M)$. Then the greedy pairing $((\ell'_{j_1}, u'_{i_1}), \ldots, (\ell'_{j_z}, u'_{i_z}))$ is good.*

Proof. Let (ℓ, u) be an element of the greedy pairing. We want to show that (ℓ, u) satisfies Definition 10. By Lemma 13 it follows that $(\ell, u) = (\ell'_{j_y}, u'_{i_y})$ for some $\ell'_{j_y} \in L', u'_{i_y} \in U'$ where $i_y \geq j_y$.

Now, using Lemma 14 we have that $(\ell, u) = (\overline{\ell}_{j_r}, \overline{u}_{i_r})$ for some $\overline{\ell}_{j_r} \in \overline{L}$, $\overline{u}_{i_r} \in \overline{U}$ with $i_r \leq j_r$. Thinking of L as a lattice path, this means, that starting from $(0, 0)$, there are as many east steps in L before ℓ as there are east steps before u in U. Then by the choice of the greedy pairing, we have that ℓ is (weakly) to the right of u in M'. We conclude that (ℓ, u) is good. □

The next result will be the remaining ingredient towards the proof of the main theorem in this section.

Lemma 17. *Let $M = M[U, L]$, $\ell_i < \ell_{i'} \in L$ and $u_j < u_{j'} \in U$. If (ℓ_i, u_j) and $(\ell_{i'}, u_{j'})$ are good then (ℓ_i, u_j) is good in $M[U \setminus \{u_{j'}\}, L \setminus \{\ell_{i'}\}]$ and $(\ell_{i'}, u_{j'})$ is good in $M[U \setminus \{u_j\}, L \setminus \{\ell_i\}]$.*

Proof. The first statement follows since removal of $(\ell_{i'}, u_{j'})$ does not change the positions of (ℓ_i, u_j). The second statement follows because the removal of (ℓ_i, u_j) shifts both segments $(\ell_{i'}, u_{j'})$ one unit to the right and downwards, so if they were good before they are still good afterwards. □

Note that the condition of the comparability of the pairs is necessary (see Figure 7). Now we are ready to state the main result of this section.

Theorem 18. *[Characterizing quotients of LPMs] Let $M = M[U, L]$ and $M' = M[U', L']$ be LPMs on the ground set $[n]$. We have that $M \leq_q M'$ if and only if $U \subseteq U'$, $L \subseteq L'$ and the greedy pairing of $(L' \setminus L, U' \setminus U)$ is good.*

Proof. "$\Rightarrow$" this follows as a consequence of Lemmas 12 and 16.
"$\Leftarrow$" We can induct on the size of $U' \setminus U$. We take a first good pair and get a quotient M'' of M' by Proposition 11. Now, since we had a greedy pairing by Lemma 17 all previously good pairs remain good. Moreover, the pairing remains greedy. So we can apply induction and get $M \leq_q M''$. By transitivity of the quotient relation we get $M \leq_q M'$. □

3.1. **The quotient poset of LPMs.** Theorem 18 allows us to construct a graded poset $\mathcal{P}_n$ whose elements are LPMs on $[n]$ and whose ordering relation is $\leq_q$. The left side of Figure 5 displays $\mathcal{P}_3$. It is worth mentioning that the set of matroids $\mathcal{M}_n$ over the set $[n]$ is endowed with a graded poset structure using the order $\leq_q$ (see [34, Prop. 7.4.7]). However, this construction does not guarantee that the matroids obtained as quotients of a given ones remain LPMs. Thus, the proerties of the poset $\mathcal{P}_n$ that we analyze now is not obtained for free.

Proposition 19. *The poset $\mathcal{P}_n$ is graded with minimum $U_{0,n}$ and maximum $U_{n,n}$.*

Proof. Let $M \leq_q M'$ and consider a chain $C = (M = M_0 <_q \ldots < M_z = M')$. If two consecutive elements $M_i <_q M_{i+1}$ have non-consecutive ranks, i.e., $r(M_{i+1}) - r(M_i) > 1$, then by Theorem 18, the greedy pairing given by M_i and M_{i+1} allows us to enlarge the chain C. Hence, each saturated chain in the interval $[M, M']_q$ in $\mathcal{P}_n$ has size $r(M') - r(M) = |U' \backslash U|$. The statement about maximum and minimum is clear. $\square$

The curious reader might wonder whether $\mathcal{P}_n$ is a lattice. This, however is not the case. For instance in $\mathcal{P}_3$ matroids $M[12, 23]$ and $M[12, 23]$ are both coverings of the matroids $M[1, 3]$ and $M[1, 2]$, i.e., they do not have a unique meet (see Figure 5). We also point out that the poset $\mathcal{P}_3$ considered here is a subposet from the one considered in [3, Section 3] where all positroids on [3], not only LPMs, are considered.

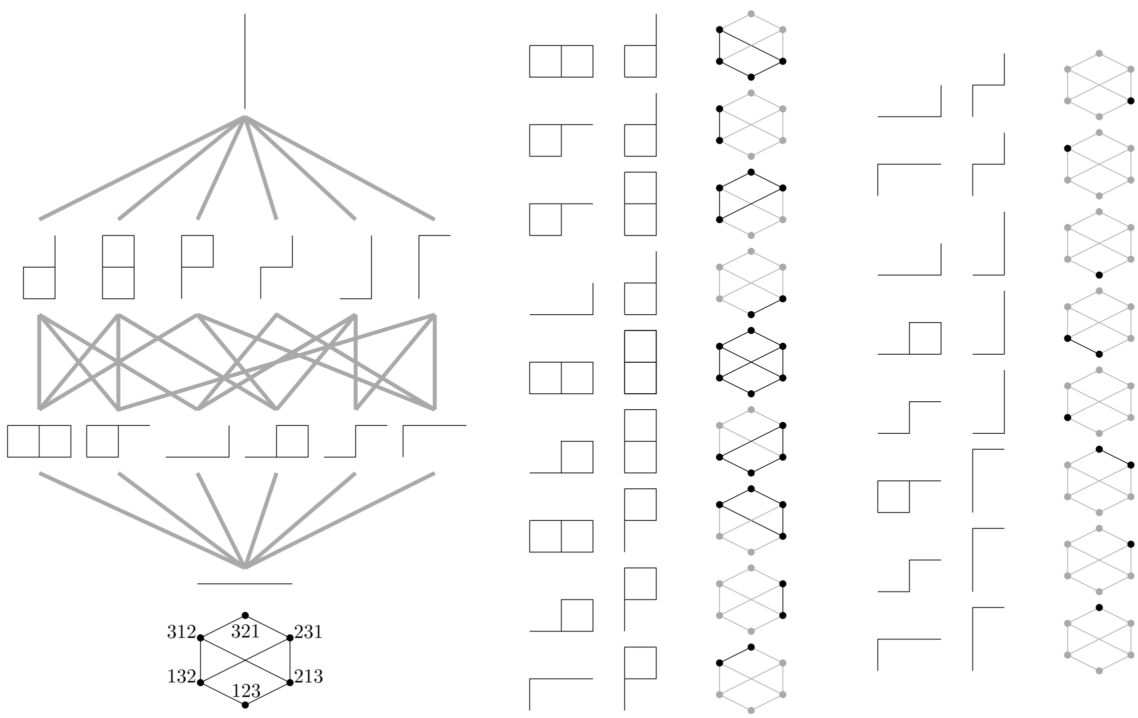

FIGURE 5. On the upper left the poset $\mathcal{P}_3$. On the lower left the strong Bruhat order $(S_3, \leq_B)$. On the right the corresponding intervals in $(S_3, \leq_B)$ given by each maximal chain, as explained in Section 4.

Let us explore a bit more the poset $\mathcal{P}_n$. For $k \in \{0, \dots, n\}$ denote by $a(n,k) := \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$. The numbers $a(n,k)$ are known as *Narayana numbers*, and count the number of Dyck paths from 0 to $2n$ with k peaks (see [32, Exercise 6.36] and the right of Figure 6).

Corollary 20. *The poset $\mathcal{P}_n$ has $a(n+1, k+1)$ elements of rank $n-k$, for each $k \in \{0, \dots, n\}$.*

Proof. Our proof will be based on two observations:

(a) Every LPM of rank $n-k$ corresponds to a greedy pairing $((\ell'_{j_1}, u'_{i_1}), \dots, (\ell'_{j_k}, u'_{i_k}))$ of length k obtained from $M' = M[\{1, \dots, n\}, \{1, \dots, n\}] = U_{n,n}$.

(b) There is a bijection between such greedy pairings $((\ell'_{j_1}, u'_{i_1}), \dots, (\ell'_{j_k}, u'_{i_k}))$ and the Dyck paths from 0 to $2(n+1)$ with $k+1$ peaks.

For part (a), if $M = M[U, L] \in \mathcal{P}_n$ then $M \leq_q M'$ by Theorem 18, M corresponds to the greedy pairing on $([n] \setminus L, [n] \setminus U)$.

For part (b) given a greedy pairing $((\ell'_{j_1}, u'_{i_1}), \dots, (\ell'_{j_k}, u'_{i_k}))$, consider the sequence of points $(j_1, i_1), \dots, (j_k, i_k) \in [n] \times [n]$. Since this is a greedy pairing we have $j_1 < \dots < j_k$, $i_1 < \dots < i_k$ and $i_r \geq j_r$ for all $1 \leq r \leq k$. This is, the points sit weakly above the skew diagonal in the grid $[n] \times [n]$ and the upper left quadrant of each point is empty. Note that the properties $i_r \geq j_r$ characterizes all good pairs since we are in $U_{n,n}$. Now, adding points $(0,0)$ and $(n+1, n+1)$ allows to associate M with a Dyck path from 0 to $2(n+1)$ with $k+1$ peaks. See Figure 6. □

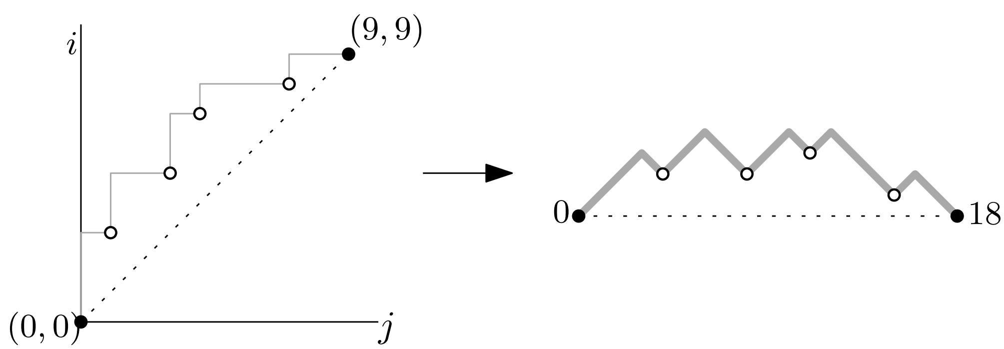

FIGURE 6. The LPM $M[1246, 2568]$ as quotient of $U_{8,8}$ with greedy pairing $(1,3), (3,5), (4,7), (7,8)$ and the corresponding Dyck path.

Corollary 20 gives an idea of how to analyze the ranks of general intervals in the quotient poset of LPMs, but we leave this for further investigation.

Question 21. *Are rank functions of intervals of $\mathcal{P}_n$ unimodal?*

Theorem 18 together with Lemma 17 sheds some light on the structure of the order complex of an interval $[M, M']_q$ in the poset $\mathcal{P}_n$. The idea is that if $((\ell'_{i_1}, u'_{j_1}), \dots, (\ell'_{i_z}, u'_{j_z}))$ is the greedy pairing on $(U' \setminus U, L' \setminus L)$ then any permutation of the set of pairs $\{(\ell'_{i_1}, u'_{j_1}), \dots, (\ell'_{i_z}, u'_{j_z})\}$ gives rise to a sequence that is a good pairing. That is, every such permutation corresponds to a saturated chain in the interval $[M, M']_q$. However, not all saturated chains arise this way.

For instance the interval $[U_{1,3}, U_{3,3}]_q$ in $\mathcal{P}_3$ has 3 saturated chains, two of which come as permutations of the set $\{(1,2),(2,3)\}$, The third chain corresponds to the sequence $((1,3),(2,2))$. Notice that $((2,2),(1,3))$ is not a good pairing on $(12,23)$ as $(1,3)$ is not a good pair of $M = M[13,13]$. See Figure 7.

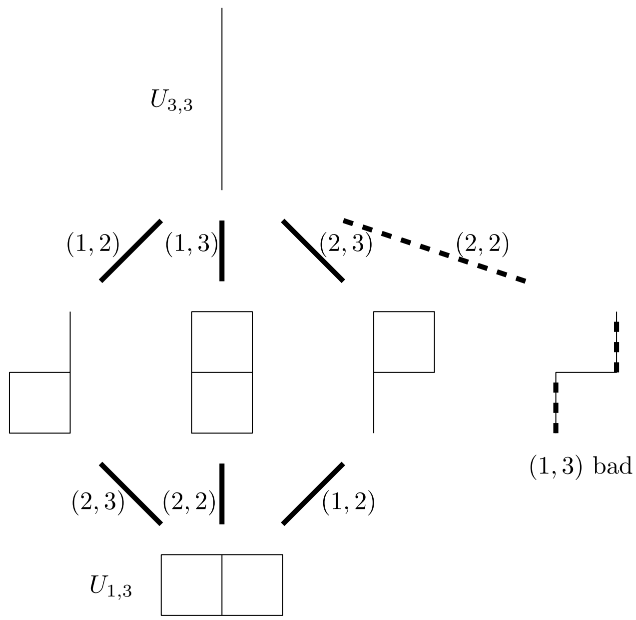

FIGURE 7. The interval $[U_{1,3}, U_{3,3}]_q$ in $\mathcal{P}_3$.

4. LPMs and the nonnegative flag variety

In this section we will study saturated chains in the interval $[U_{0,n}, U_{n,n}]_q$ of the poset $\mathcal{P}_n$. That is, we study (full) lattice path flag matroids, LPFMs. Recall that following Definition 5 an LPFM is a sequence $\mathcal{F} : (M_0, M_1, \dots, M_n)$ of LPMs where $M_0 \leq_q M_1 \leq_q \cdots \leq_q M_n$ is a saturated chain in $\mathcal{P}_n$. That is, each M_i is an LPM on $[n]$ and for $i = 0, \dots, n-1$:

(a) M_i is a quotient of M_{i+1}
(b) $r(M_i) + 1 = r(M_{i+1})$.

One of the main results of our paper will show us that the family of LPFMs is included into $\mathcal{F}\ell_n^{\geq 0}$. That is, every LPFM can be represented by a point in $\mathcal{F}\ell_n^{\geq 0}$ and thus we can think of the family of LPFMs as properly contained inside $\mathcal{F}\ell_n^{\geq 0}$. In order to achieve this, we will make use of *matroid polytopes*, defined next.

Definition 22. Let $\{e_1, \dots, e_n\}$ be the canonical basis of $\mathbb{R}^n$.

(1) Let M be a matroid on $[n]$ of rank k and let $\mathcal{B}$ its set of bases. The *matroid polytope of* M is the polytope Δ_M in $\mathbb{R}^n$ given as the convex hull $\Delta_M := conv\{e_B \,|\, B \in \mathcal{B}\}$ where $e_B = \sum_{i \in B} e_i$.

(2) Let $\mathcal{F} : (M_0, \ldots, M_r)$ be a flag matroid whose constituents M_i are matroids on $[n]$. The *flag matroid polytope* $\Delta_{\mathcal{F}}$ is the polytope in $\mathbb{R}^n$ given by

$$\Delta_{\mathcal{F}} := conv\{e_{B_0} + \cdots + e_{B_r} \mid \mathcal{B} = (B_0, B_1, \ldots, B_r) \text{ is a flag of bases of } \mathcal{F}\}.$$

For those familiar with polytopes, if Δ_i denotes the matroid polytope of M_i for each M_i as in (2) of Definition 22, then the polytope $\Delta_{\mathcal{F}}$ is the Minkowski sum $\Delta_1 + \cdots + \Delta_n$ (see [13, Cor. 1.13.5]). Also, notice that Definition 22(2) does not assume the flag is full, as $r \leq n$. When $r = n$ then $\Delta_{\mathcal{F}}$ is such that each of its vertices is a permutation of the point $(1, 2 \ldots, n)$. In particular if $\mathcal{F}$ is the *uniform flag matroid* $\mathfrak{U}_n = (U_{0,n}, U_{1,n}, \ldots, U_{n,n})$ then $\Delta_{\mathcal{F}}$ has $n!$ vertices, given by all the permutations of $(1, 2 \ldots, n)$. That is, the polytope $\Delta_{\mathcal{F}}$ is such that its 1-skeleton is the permutahedron. Now, notice that since $U_{n,n}$ has only one basis $B = \{12 \ldots n\}$ then $e_B = (1, 1, \ldots, 1)$. Thus, any full flag matroid $\mathcal{F} = (M_0, M_1, \ldots, M_{n-1}, U_{n,n})$ is such that its polytope $\Delta_{\mathcal{F}}$ is a translation of the polytope $\Delta_{\mathcal{F}'}$, by $(1, \ldots, 1)$, where $\mathcal{F}' = (M_0, M_1, \ldots, M_{n-1})$, and the latter polytope has vertices which are permutations of $(0, 1, \ldots, n-1)$.

Example 23. Consider the LPFM given by $\mathcal{F} : M_1 \leq_q M_2 \leq_q M_3$ where $M_1 = U_{1,3}$, $M_2 = M[13, 23]$ and $M_3 = U_{3,3}$. Then the flags of bases of $\mathcal{F}$ are

$$\begin{array}{ll} 1 \subset 13 \subset 123 & 2 \subset 23 \subset 123 \\ 3 \subset 13 \subset 123 & 3 \subset 23 \subset 123. \end{array}$$

Each of these flags gives rise, respectively, to the points $(3, 1, 2), (1, 3, 2), (2, 1, 3), (1, 2, 3)$ in $\mathbb{R}^3$. Thus the polytope $\Delta_{\mathcal{F}}$ is the convex hull of these four points and it is depicted in Figure 5 along with all the polytopes arising from full flags of LPMs over the set [3].

Definition 24. Let $u, v \in S_n$. We say that v *covers* u in the (strong) *Bruhat order*, denoted $u \prec_B v$ if $v = u(i, j)$ for some $i < j$ such that if $i < k < j$ then $u(k) < u(i)$ or $u(k) > u(j)$. The Bruhat order of S_n is the transitive closure of this covering relation.

The next main result in this paper shows that every flag matroid polytope $\Delta_{\mathcal{F}}$ over $[n]$, where $\mathcal{F} : (M_0, M_1, \ldots, M_n)$ is a LPFM, is such that (its 1-skeleton) is an interval in the strong Bruhat order $\leq_B$ of S_n. The importance of this result is that, as shown in [33, Proposition 2.7], every flag matroid $\mathcal{F}$ arising from a point in $\mathcal{F}\ell_n^{\geq 0}$ is such that its flag matroid polytope is (its 1-skeleton) an interval in the (strong) Bruhat order S_n. Conversely, every interval in the Bruhat order can be thought of as the 1-skeleton of a flag matroid that arises as a point of $\mathcal{F}\ell_n^{\geq 0}$. In Example 23 the 1-skeleton of $\Delta_{\mathcal{F}}$ corresponds to the interval $[123, 312]_B$ in S_3.

Let $\mathcal{F} : (M_0, M_1, \ldots, M_n)$ be an LPFM. Given two flags of bases of $\mathcal{F}$, namely $\mathfrak{B} : (B_0, B_1, \ldots, B_n)$ and $\mathfrak{B}' : (B_0', B_1', \ldots, B_n')$, we say that $\mathfrak{B}$ is *smaller* than $\mathfrak{B}'$ if and only if $B_i \leq_G B_i'$ for all $i \in [n]$. We denote this as $\mathfrak{B} \leq_G \mathfrak{B}'$. With this notation in mind, we say that the permutation $\pi = \pi_{\mathfrak{B}}$ associated to the flag of bases $\mathfrak{B}$ is the permutation in S_n such that $\pi(i) = B_i \setminus B_{i-1}$, for $i = 1, \ldots, n$. We refer to π as the *Gale permutation of* $\mathfrak{B}$. On the other hand, the *Bruhat permutation of* $\mathfrak{B}$ is the permutation $\tau = \tau_{\mathfrak{B}}$ in S_n such that $\tau(i) = \overline{\pi}^{-1}(i)$ where $\overline{\pi}(i) = \pi(n - i + 1)$. It is worth pointing out that such $\mathfrak{B}$ determines π (and thus τ) uniquely. Thus we will say that $\pi = \pi_{\mathfrak{B}} \leq_G \pi_{\mathfrak{B}'} = \pi'$ if and only if $\mathfrak{B} \leq_G \mathfrak{B}'$, where $\mathfrak{B}$ and $\mathfrak{B}'$ are flags of bases of the uniform flag matroid $\mathfrak{U}_n = (U_{0,n}, U_{1,n}, \ldots, U_{n,n})$.

Example 25. Consider the LPFM $\mathcal{F} : M_1 \leq_q M_2 \leq_q M_3 \leq_q U_{4,4}$ where $M_1 = M[1,3]$, $M_2 = M[14,34]$ and $M_1 = M[124,134]$. The polytope $\Delta_{\mathcal{F}}$ is the convex hull of 6 points in $\mathbb{R}^4$. Each point arises from each flag of bases of $\mathcal{F}$, which the reader can compute.[2] In Figure 8 we depict on the left the constituents of $\mathcal{F}$. Below each of them appear their bases set. Also, each covering relation $\prec_q$ is labelled by the corresponding good pair. On the right hand side appears the interval $[1243, 4213]_B$ in S_4 whose permutations correspond, bijectively, to the vertices of $\Delta_{\mathcal{F}}$, i.e. to the collection of flags of bases of $\mathcal{F}$. For instance the flag of bases $\mathfrak{B} : (3, 34, 134, 1234)$ is such that its Bruhat permutation $\tau = 2143$ corresponds to the vertex $(2,1,4,3)$. On the other hand, its Gale permutation is $\pi = 3412$. If $\mathfrak{B}' : (3, 34, 234, 1234)$ then its corresponding Gale and Bruhat permutations are $\pi' = 3421$, $\tau' = 1243$. Moreover, $\pi' \geq_G \pi$ and $\tau' \leq_B \tau$.

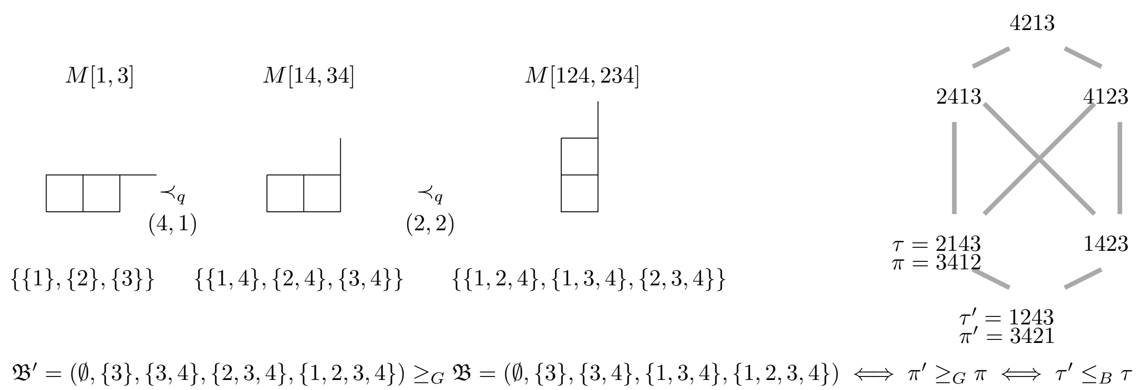

FIGURE 8. An LPFM and its flag matroid polytope. Its vertices constitute the interval $[1243, 4213]_B$ in the Bruhat order.

Lemma 26. *Let $\tau', \tau \in S_n$ the Bruhat permutations of flags $\mathfrak{B}', \mathfrak{B}$, respectively. If $\tau' \prec_B \tau$ then $\mathfrak{B}' \geq_G \mathfrak{B}$.*

Proof. Let $\tau' = i_1 \cdots i_n$. Now, $\tau' \prec_B \tau$ if and only if $\tau = \tau'(r,s)$ for some $r < s$ such that $i_r < i_s$ and if $r < t < s$ then $i_t \notin [i_r, i_s]$. Thus τ is obtained from τ' by exchanging positions r and s. In view of this, we get that B_j, the j-th component of $\mathfrak{B}$, satisfies

$$\begin{cases} B_j = B'_j \setminus \{s\} \cup \{r\} & \text{if } j \in \{n - i_s + 1, \dots, n - i_r\} \\ B_j = B'_j & \text{otherwise} \end{cases}.$$

The result follows. □

Lemma 27. *Let $\mathfrak{B}, \mathfrak{B}'$ flags of bases of $\mathfrak{U}_n$ and $\pi := \pi_{\mathfrak{B}} \prec_G \pi_{\mathfrak{B}'} =: \pi'$. Then there are $i < j \in [n]$ such that $\pi(k) = \pi'(k)$ for all $k \in [n] \setminus \{i,j\}$, $\pi(i) = a < a' = \pi'(i)$, $\pi(i,j) = \pi'$ and $\pi'(k) \notin [a, a']$ for all $i < k < j$.*

Proof. We prove the contrapositive. To this end, we assume that π, π' are two different permutations in S_n such that they differ on more than two positions, or, $\pi(i,j) = \pi'$ and

[2]We have omitted $U_{4,4}$ as it provides no further information.

there is $i < \ell < j$ such that $\pi(\ell) \in [a, a']$. Let $i < j$ the first two indices such that $\pi(i) \neq \pi'(i)$ and $\pi(j) \neq \pi'(j)$. Without loss of generality we may assume $\pi(i) < \pi'(i)$. If $\pi(j) > \pi'(j)$ then $\pi_{\mathfrak{B}} \not\leq_G \pi_{\mathfrak{B}'}$. Thus suppose also that $\pi(j) < \pi'(j)$. Let us consider the permutation $\sigma = \pi(i,k)(j,\ell)$ where $k = \pi'(i), \ell = \pi'(j)$. We have $\pi <_G \sigma \leq_G \pi'$. If $\sigma \neq \pi'$ then we are done as this shows that π' does not cover π. Hence we only need to care about the case where $\sigma = \pi'$, which by construction of σ is equivalent to saying $\pi(i,j) = \pi'$. Therefore, $\pi = a_1 \cdots a_r a a_{r+2} \cdots a' a_{j+1} \cdots a_n$ and $\pi' = a_1 \cdots a_r a' a_{r+2} \cdots a a_{j+1} \cdots a_n$ and by assumption we let $a_\ell \in [a, a']$ where $i < \ell < j$. We thus obtain $\pi <_G \pi(i,\ell) <_G \pi'$, and thus π' does not cover π. □

Lemma 28. *Let $\mathfrak{B}, \mathfrak{B}'$ be flags of bases of $\mathfrak{U}_n$ and let $\pi := \pi_{\mathfrak{B}}, \pi' := \pi_{\mathfrak{B}'}$ their respective Gale permutation, and $\tau := \tau_{\mathfrak{B}}, \tau' := \tau_{\mathfrak{B}'}$ their corresponding Bruhat permutation. Suppose that there are $i < j \in [n]$ such that $\pi(k) = \pi'(k)$ for all $k \in [n] \setminus \{i,j\}$, $\pi(i) = a < a' = \pi'(i)$, $\pi(i,j) = \pi'$ and $\pi'(k) \notin [a, a']$ for all $i < k < j$. Then $\tau' \prec_B \tau$.*

Proof. Set $r = i - 1$ and $s = j - 1$. By assumption $\pi = a_1 \cdots a_r a b_1 \cdots b_s a' c_1 \cdots c_t$ and $\pi' = a_1 \cdots a_r a' b_1 \cdots b_s a c_1 \cdots c_t$, in one-line notation, where $b_\ell \notin [a, a']$. Therefore τ and τ' coincide for every $k \in [n] \setminus \{a, a'\}$. It holds that $\tau(a) = n - r$, $\tau(a') = n - (r + s)$ and $\tau'(a, a') = \tau$. Thus we only need to show that $\tau(k) \notin [n - r, n - (r + s)]$ for $a < k < a'$. There are two cases to consider.

Case 1: $b_\ell < i$. The values belonging to the interval $[n - r, n - (r + s)]$ in τ correspond precisely to the positions b_ℓ, as τ records the order of appearance of each element from π. Hence, the values in $[n - r, n - (r + s)]$ are assigned to positions to the left of a in τ. We conclude that $\tau' \prec_B \tau$.

Case 2: $b_\ell > j$. This is analogous to Case 1. In this situation the values in $[n - r, n - (r + s)]$ are assigned to positions to the right of a' in τ. □

The following result asserts that there is an order-reversing (or antitone) map between the Bruhat order $(S_n, \leq_B)$ and the Gale order $(\mathfrak{U}_n, \leq_G)$.

Theorem 29. *Let $\mathfrak{B}, \mathfrak{B}'$ be flags of bases of $\mathfrak{U}_n$ and $\pi_{\mathfrak{B}}, \pi_{\mathfrak{B}'}$ and $\tau_{\mathfrak{B}}, \tau_{\mathfrak{B}'}$ their Gale and Bruhat permutations as above. The following are equivalent:*

(i) $\mathfrak{B} \leq_G \mathfrak{B}'$,
(ii) $\pi_{\mathfrak{B}} \leq_G \pi_{\mathfrak{B}'}$,
(iii) $\tau_{\mathfrak{B}} \geq_B \tau_{\mathfrak{B}'}$.

Proof. The equivalence of (i) and (ii) is just by definition. Lemma 26 shows $(iii) \implies (i)$. Finally, $(ii) \implies (iii)$ follows by first applying Lemma 27 and then Lemma 28. □

Corollary 30. *Every lattice path flag matroid polytope corresponds to an interval in the Bruhat order.*

Proof. Let $\mathcal{F} : (M_0, M_1, \ldots, M_n)$ be an LPFM, with $M_i = M[U_i, L_i]$ for all $0 \leq i \leq n$. By Theorem 29 we can argue directly in the order $\leq_G$ on the flags. We show that the set of flags of bases $\mathcal{F}$ coincides with the interval $[(U_0, \ldots, U_n), (L_0, \ldots, L_n)]_G$. The inclusion "$\subseteq$", follows since by definition every flag $(B_0, B_1, \ldots, B_n)$ of bases of $\mathcal{F}$ must be such that $U_i \leq_G B_i \leq_g L_i$, for all $i = 0, 1, \ldots, n$.

To see the reverse inclusion " $\supseteq''$, let $\mathfrak{B} = (B_0, \ldots, B_n) \in [(U_0, \ldots, U_n), (L_0, \ldots, L_n)]_G$. Thus, $B_i \in [U_i, L_i]_G$ for all $0 \leq i \leq n$. Now, by Observation 8 this simply means that B_i is a base of $M_i = M[U_i, L_i]$. Hence, $\mathfrak{B}$ is a flag of bases of $\mathcal{F}$. $\square$

We can rephrase Corollary 30 by saying that if $\mathcal{F}$ is an LPFM then its matroid polytope $\Delta_{\mathcal{F}}$ is such that its 1-skeleton corresponds to an interval in $(S_n, \leq_B)$. It is however not that easy to decide which intervals arise from LPFMs.

The following Theorem establishes in terms of Gale permutations and Bruhat permutations, the condition for a sequence of LPMs to be a flag matroid. To this end we will make use of Definition 10 and translate it in terms of the aforementioned permutations. We will make use of the *standardization map* $\mathrm{st}_S : S \to [\ell]$ where S is a ℓ-subset of positive integers. The map st_S is the unique bijection from S to $[\ell]$ that preserves order. We also denote by $\pi([k]) = \{\pi(1), \ldots, \pi(k)\}$ whenever $\pi \in S_n$ and $1 \leq k \leq n$.

Theorem 31. *Let $\mathfrak{B} \leq_G \mathfrak{B}'$ be flags of $\mathfrak{U}_n$ and $\pi_{\mathfrak{B}} \leq_G \pi_{\mathfrak{B}'}$ and $\tau_{\mathfrak{B}} \geq_B \tau_{\mathfrak{B}'}$ the permutations associated as above. The following are equivalent:*

(i) the order-interval $[\mathfrak{B}, \mathfrak{B}']_G$ constitutes the set of flags of bases of an LPFM,
(ii) for all $1 \leq k \leq n$ the maps $st_{\pi_{\mathfrak{B}}([k])} : \pi_{\mathfrak{B}}([k]) \to [k]$ and $st_{\pi_{\mathfrak{B}'}([k])} : \pi_{\mathfrak{B}'}([k]) \to [k]$ are such that $\max\{0, \pi_{\mathfrak{B}}(k) - \pi_{\mathfrak{B}'}(k)\} \leq st_{\pi_{\mathfrak{B},k}}(\pi_{\mathfrak{B}}(k)) - st_{\pi_{\mathfrak{B}'}([k])}(\pi_{\mathfrak{B}'}(k))$,
(iii) for every $1 \leq k \leq n$ let $a_k = \tau_{\mathfrak{B}}^{-1}(n-k+1)$, $a'_k = \tau_{\mathfrak{B}'}^{-1}(n-k+1)$. Then $a_k - a'_k \leq st_{\{a_1,\ldots,a_k\}}(a_k) - st_{\{a'_1,\ldots,a'_k\}}(a'_k)$.

Proof. "(i) $\iff$ (ii)": Let $\mathfrak{B} = (B_0, \ldots, B_n)$ and $\mathfrak{B}' = (B'_0, \ldots, B'_n)$. Clearly, $[\mathfrak{B}, \mathfrak{B}']_G$ constitutes the set of flags of an LPFM $\mathcal{F}$ if and only if $\mathcal{F} = (M_0, \ldots, M_n)$ with $M_k := M[B_k, B'_k]$ for $1 \leq k \leq n$. Now, by Theorem 18, this is equivalent to $(\pi_{\mathfrak{B}'}(k), \pi_{\mathfrak{B}}(k))$ being a good pair in $M_k =: M[U, L]$ for each $1 \leq k \leq n$. Indeed, if $\pi_{\mathfrak{B}'}(k)$ has position i, then it corresponds do ℓ_i and if $\pi_{\mathfrak{B}}(k)$ has position j, then it corresponds do u_j. We denote $n_k := j - i$. Now, the first condition of good pair ($j \geq i$) just means that the position of $\pi_{\mathfrak{B}}(k)$ in $\{\pi_{\mathfrak{B}}(1), \ldots, \pi_{\mathfrak{B}}(k)\}$ is at least n_k higher than the position of $\pi_{\mathfrak{B}'}(k)$ in $\{\pi_{\mathfrak{B}'}(1), \ldots, \pi_{\mathfrak{B}'}(k)\}$. The second condition for being a good pair ($u_j - \ell_i \leq j - i$) is however just says $\pi_{\mathfrak{B}}(k) - \pi_{\mathfrak{B}'}(k) \leq n_k$. Thus, $\pi_{\mathfrak{B}}(k) - \pi_{\mathfrak{B}'}(k) \leq n_k = \mathrm{st}_{\pi_{\mathfrak{B}}([k])}(\pi_{\mathfrak{B}}(k)) - \mathrm{st}_{\pi_{\mathfrak{B}'}([k])}(\pi_{\mathfrak{B}'}(k))$, for all $1 \leq k \leq n$.

"(ii) $\iff$ (iii)": We just translate (ii) in terms of Bruhat permutations. For every $1 \leq k \leq n$ let $a_k = \tau_{\mathfrak{B}}^{-1}(n-k+1)$, $a'_k = \tau_{\mathfrak{B}'}^{-1}(n-k+1)$. Thus we rewrite (ii) as $a_k - a'_k \leq n_k$ where $n_k = \mathrm{st}_{\{a_1,\ldots,a_k\}}(a_k) - \mathrm{st}_{\{a'_1,\ldots,a'_k\}}(a'_k)$. $\square$

Example 32. We illustrate Theorem 31 with two flags $\mathfrak{B}$ and $\mathfrak{B}'$ in $\mathfrak{U}_4$ whose Gale permutations are, respectively, $\pi_{\mathfrak{B}} = 2413$, $\pi_{\mathfrak{B}'} = 4321$. Thus, the Bruhat permutations are, respectively, $\tau_{\mathfrak{B}} = 2413$ and $\tau_{\mathfrak{B}'} = 1234$. Notice that $\pi_{\mathfrak{B}} \leq_G \pi_{\mathfrak{B}'}$ and $\tau_{\mathfrak{B}} \geq_B \tau_{\mathfrak{B}'}$. Following the notation in the proof of the theorem, setting $k = 3$ we summarize as follows the calculations needed to verify the condition to be a good pair. However notice that in order to verify $\mathfrak{B} \leq_G \mathfrak{B}'$ one needs to do the corresponding calculations for every $k \in [n]$.

Our next result establishes that cubes in the weak Bruhat order come from LPFMs.

$\pi_{\mathfrak{B}}([3])$	$\pi_{\mathfrak{B}}(3)$	$\mathrm{st}_{\pi_{\mathfrak{B}}([3])}(\pi_{\mathfrak{B}}(3))$	$\pi_{\mathfrak{B}'}([3])$	$\pi_{\mathfrak{B}'}(3)$	$\mathrm{st}_{\pi_{\mathfrak{B}'}([3])}(\pi_{\mathfrak{B}}(3))$	$u_j - \ell_i \leq j - i$
$\{1,2,4\}$	1	1	$\{2,3,4\}$	2	1	$1-2 \leq 1-1$

$\tau_{\mathfrak{B}}^{-1}(\{4,3,2\})$	$\tau_{\mathfrak{B}}^{-1}(2)$	$\mathrm{st}_{\tau_{\mathfrak{B}}^{-1}([2])}(\tau_{\mathfrak{B}}^{-1}(3))$	$\tau_{\mathfrak{B}'}^{-1}([\{4,3,2\})$	$\tau_{\mathfrak{B}'}^{-1}(2)$	$\mathrm{st}_{\tau_{\mathfrak{B}'}^{-1}([3])}(\tau_{\mathfrak{B}}^{-1}(2))$	$u_j - \ell_i \leq j - i$
$\{1,2,4\}$	1	1	$\{2,3,4\}$	2	1	$1-2 \leq 1-1$

Proposition 33. *Let $s_i = (i, i+1)$ be a simple transposition in S_n. Let τ, τ' be permutations of S_n such that $\tau \leq_B \tau'$ where $\tau' = \tau s_{i_1} \cdots s_{i_m}$ for some $i_1, \ldots, i_m \in [n-1]$. If the s_{i_j} commute pairwise then $[\mathfrak{B}', \mathfrak{B}]_G$ constitute the set of flags of bases of an LPFM on $[n]$.*

Proof. Set $k_{i_j} := n - \tau'(i_j) + 1$ for all $j \in [m]$, $I_1 = \{k_{i_j} \mid j \in [m]\}$ and $I_2 = \{k_{i_j+1} \mid j \in [m]\}$. Notice that the permutations τ and τ' are such that $\tau^{-1}([n] \setminus (I_1 \cup I_2)) = \tau'^{-1}([n] \setminus (I_1 \cup I_2))$. Now, following notation of Theorem 31 part (iii) we get Table 2, which concludes the proof.

$a_k - a'_k$	n_k	
0	0	if $k \in [n] \setminus (I_1 \cup I_2)$
-1	0	if $k \in I_1$
1	1	if $k \in I_2$

Table 2. Proof of Proposition 33.

∎

5. On a conjecture of Mcalmon, Oh, and Xiang

In this section we will provide proof of a conjecture made by Mcalmon, Oh, and Xiang [24] which aims to characterize quotients of positroids (with no loops or coloops) combinatorially. As we already know, LPMs are a subfamily of positroids and thus, our purpose now is to state and prove this conjecture for LPMs using the results we have developed already. Recall that if $A \subseteq [n]$ then $\overline{A}$ denotes the set $[n] \setminus A$.

Definition 34. Let $M = M[U, L]$ be an LPM over $[n]$ where $U = \{u_1, \ldots, u_k\}$ and $L = \{\ell_1, \ldots, \ell_k\}$. Let $\overline{L} = \{\overline{\ell}_1, \ldots, \overline{\ell}_{n-k}\}$ and $\overline{U} = \{\overline{u}_1, \ldots, \overline{u}_{n-k}\}$ and assume that M has no loops nor coloops.

- (RI) A *row-interval* of M is a cyclic interval of the form $\{\ell_i, \ell_i + 1, \ldots, n, 1, \ldots, u_i\}$, for every $i \in \{1, \ldots, k\}$. We denote such an interval by $[\ell_i, u_i]$.
- (CI) A *column-interval* of M is an interval of the form $\{\overline{\ell_i}, \overline{\ell_i} + 1, \ldots, \overline{u_i}\}$, for every $i \in \{1, \ldots, n-k\}$. We denote such an interval by $[\overline{\ell_i}, \overline{u_i}]$.

An *interval* of M is either a row or a column interval of M.

Let M be as in Definition 34. The *decorated permutation* π_M, or simply π, associated to the positroid M is the permutation on the set $[n]$ given by

$$\begin{cases} \pi(u_i) = \ell_i & \text{for } i \in \{1, \ldots, k\} \\ \pi(\overline{u_i}) = \overline{\ell_i} & \text{for } i \in \{1, \ldots, n-k\}. \end{cases}$$

If $a \in [n]$ is a loop of M, then $\pi(a) = \underline{a}$. If $a \in [n]$ is a coloop of M then $\pi(a) = a$. That is, loops and coloops are the only fixed points of π and they are either decorated with an underline or not decorated, respectively. However since we are considering M to be loop-free and coloop-free, then no fixed points will arise in the corresponding permutation π.

Before we illustrate these concepts with an example, we point out that there is a bijective correspondence between positroids on $[n]$ and decorated permutation on $[n]$ (see [29]). However, for our purposes the definition we are providing here for such permutations, has been adapted to LPMs. Also, sometimes in the literature the definition given for decorated permutation would differ from ours by taking the inverse π^{-1}, of the one we provided here.

As an example, consider the LPM given by $M = [13, 25]$ over the set [5]. Then its decorated permutation in one-line notation is $\pi = 21534$. Also, the row-intervals of M are $[2, 1] = \{2, 3, 4, 5, 1\}$ and $[5, 3] = \{5, 1, 2, 3\}$. On the other hand, the column-intervals of M are $[1, 2] = \{1, 2\}$, $[3, 4] = \{3, 4\}$ and $[4, 5] = \{4, 5\}$. In general, given any M as in Definition 34, it follows that each of its row-intervals contains the set $\{1, n\}$. On the other hand, the only column-interval that contains n is the column-interval $[\overline{\ell_{n-k}}, n]$. This discussion leads us to the following observation which will be used throughout in the proof of Theorem 37.

Observation 35. *Let M, M' be LPMs on $[n]$ that are loop and coloop free. Also assume that $M' \neq U_{n-1,n}$. If a column-interval of M' is expressed as union of intervals of M, then these are all column-intervals.*

It is worth pointing out that Observation 35 does not hold in general for row-intervals. For instance consider $M' = [123, 245]$ and $M = [13, 25]$. Then the row-interval $[4, 2] = \{1, 2, 4, 5\}$ of M' can only be represented as union of column-intervals $[1, 2] \cup [4, 5]$. Now, in [24], what the authors call *CCW*-arrows of an arbitrary positroid, correspond in the case of an LPM to its intervals as given in Definition 34. Moreover, in [24] the authors consider *realizable quotients*, which we denote with $\trianglelefteq_q$. Namely, if M and M' are positroids over $[n]$ of ranks $k < \ell$, respectively, then $M \trianglelefteq_q M'$ if there exists a point $A' \in Gr^{\geq 0}_{\ell,n}$ such that A' represents M' and the submatrix A obtained from A' by keeping its top k rows is such that A represents M and $A \in Gr^{\geq 0}_{k,n}$.

Conjecture 36 (Mcalmon, Oh, Xiang '19)**.** For positroids M, M' we have $M \trianglelefteq_q M'$ if and only if every CCW-arrow of M' is the union of CCW-arrows of M.

Now we are ready to prove and strengthen this conjecture for LPMs.

Theorem 37. *Let M and M' be LPMs on $[n]$ without loops or coloops. The following are equivalent:*

(i) $M \leq_q M'$,
(ii) $M \trianglelefteq_q M'$,
(iii) every interval of M' can be expressed as union of intervals of M.

Proof. (i) $\Rightarrow$ (ii): If M and M' are LPMs on $[n]$ and $M \leq_q M'$, then by Theorem 18 there is an LPFM $\mathcal{F} : (M_0 \leq_q \ldots \leq_q M_n)$ with $M = M_i$ and $M' = M_j$ for some $0 \leq i < j \leq n$. By Corollary 30, $\mathcal{F}$ corresponds to an interval of the (strong) Bruhat order S_n. Now, by [33, Proposition 2.7], $\mathcal{F}$ can be thought of as a point of $\mathcal{F}\ell^{\geq 0}_n$. In particular, $M \trianglelefteq_q M'$.
(ii) $\Rightarrow$ (i): it is clear by definition.
(iii) $\Rightarrow$ (i): If $M' = U_{n-1,n}$, then all coloop-free matroids on $[n]$ are quotients of M and we are done. Now let $M = M[U, L]$, $M' = M[U', L']$ with $M' \neq U_{n-1,n}$ and assume that (iii)

holds. In order to prove (i) we will show that $U \subseteq U'$, $L \subseteq L'$ and that the greedy pairing, as given in Definition 15, is good.

By hypothesis, and using Observation 35, every column-interval $[\overline{\ell'}, \overline{u'}]$ of M' can expressed as union of intervals $\bigcup_{j=1}^{k} [\overline{\ell}_{i_j}, \overline{u}_{i_j}]$ in M, where each of the intervals $[\overline{\ell}_{i_j}, \overline{u}_{i_j}]$ are column-intervals of M. This implies that $\overline{\ell'}_{i_j} \in \overline{L}$ and $\overline{u'}_{i_j} \in \overline{U}$ for every $j \in \{1, \ldots, k\}$. Since $\overline{\ell'} \in \overline{L}'$ and $\overline{u'} \in \overline{U}'$ then, again by Observation 35, it follows that $\overline{\ell'} \in \overline{L}$ and $\overline{u'} \in \overline{U}$. Therefore, $U \subseteq U'$ and $L \subseteq L'$. In particular, we have $\operatorname{rank}(M) \leq \operatorname{rank}(M')$ and $M = M'$ if $\operatorname{rank}(M) = \operatorname{rank}(M')$.

Now, letting $U' \setminus U = \{u'_{i_1}, \ldots, u'_{i_z}\}$ and $L' \setminus L = \{\ell'_{j_1}, \ldots, \ell'_{j_z}\}$, take the greedy pairing $((\ell'_{j_1}, u'_{i_1}), \ldots, (\ell'_{j_z}, u'_{i_z}))$. In order to prove (i), suffices to show that the greedy pairing is good, by Theorem 18. Suppose then that this is not the case, and assume that (ℓ'_{i_s}, u'_{j_s}) is the first bad pair in this list. There are then, two cases to consider that make (ℓ'_{i_s}, u'_{j_s}) bad.

Case 1: If the step ℓ'_{i_s} is above the step u'_{j_s}, i.e., $j_s < i_s$, consider the row-interval $[\ell'_{j_s}, u'_{j_s}]$ in M'. By the choice of (ℓ'_{i_s}, u'_{j_s}) it follows that $\ell'_{j_s} \in L$. Hence, in M there is no column-interval beginning with ℓ'_{j_s}. Thus, in order to represent $[\ell'_{j_s}, u'_{j_s}]$ as union of intervals in M, the row-interval of M starting with $\ell'_{j_s} =: \ell_j \in L$ has to be used. This interval is the interval $[\ell_j, u_j]$. But then, since $u'_{j_s} \notin U$, we have that $u_j > u'_{j_s}$ and thus $[\ell_j, u_j]$ contains properly the interval $[\ell'_{j_s}, u'_{j_s}]$. This contradicts the fact that $[\ell'_{j_s}, u'_{j_s}]$ is union of intervals in M.

Case 2: Assume the step ℓ'_{i_s} is to the left of the step u'_{j_s}. Let $\overline{\ell'}$ be the smallest element in $\overline{L'}$ larger than ℓ'_{i_s}. Graphically, $\overline{\ell'}$ is the first east step after ℓ'_{i_s} in the southern boundary of the diagram of M'. Thus $\overline{\ell'}$ determines the column-interval $[\overline{\ell'}, \overline{u'}]$ in M'. In order to express this interval as union of intervals in M, again to Observation 35 tells us that only column-intervals in M can be used. In particular, since $\overline{\ell} := \overline{\ell'} \in \overline{L}$, the column-interval $[\overline{\ell}, \overline{u}]$ of M has to be used, where $\overline{u} \in \overline{U}$. However, $\overline{u} > \overline{u}'$ as $\overline{\ell} > \ell'_{i_s}$ and the step ℓ'_{i_s} becomes horizontal in M making the containment $[\overline{\ell'}, \overline{u'}] \subsetneq [\overline{\ell}, \overline{u}]$ proper. As in Case 1, this contradicts the fact that $[\overline{\ell'}, \overline{u'}]$ is union of intervals in M. Thus we conclude that M is a quotient of M'.

(i) $\Rightarrow$ (iii): Let $M <_q M'$. It is sufficient to assume that $\operatorname{rank}(M) = \operatorname{rank}(M') - 1$. Hence, by Theorem 18 there is a good pair (ℓ', u') such that $U = U' \setminus \{u'\}$ and $L = L' \setminus \{\ell'\}$. Let $[\overline{\ell'}, \overline{u'}]$ be a column-interval of M' and let us prove that it can be written as union of intervals in M. Since the pair (ℓ', u') is good, in the diagram of M, steps ℓ' and u' become horizontal and thus the horizontal step $\overline{u'}$ appears weakly to the right of $\overline{\ell'}$ in M. Hence, we can write the interval $[\overline{\ell'}, \overline{u'}]$ as union of column-intervals $\bigcup_{j=1}^{k} [\overline{\ell}_{i_j}, \overline{u}_{i_j}]$ in M in such a way that $\overline{\ell}_{i_1} = \overline{\ell'}$ and $\overline{u}_{i_k} = \overline{u'}$. In this way, every column-interval of M' can be written as required in M.

Now, consider a row-interval $[\ell, u]$ in M'. If $\ell = \ell'$ then $\ell \in \overline{L}$ and we take the column-intervals in M of the form $[\overline{\ell}, \overline{u}]$ where $\overline{\ell} \in \overline{L}$ and $\overline{\ell} \geq \ell$. The union of these intervals together with the unique row-interval $[\_, u]$ of M with end-point u gives us $[\ell, u]$ if such interval exists, which happens if $u \neq u'$. On the other hand, reasoning in a very analogous way, when $u = u'$ we obtain $[\ell, u]$ as union of the column-intervals in M of the form $[\overline{\ell}, \overline{u}]$ where $\overline{u} \in \overline{U}$ and $\overline{u} \leq u$, along with the unique row-interval $[\ell, \_]$ of M whose initial point is ℓ. If $\ell \neq \ell'$ and $u \neq u'$, then we take in M the union of the row-intervals $[\ell, \_] \cup [\_, u]$. The result follows. $\square$

6. Further remarks

6.1. **Towards LPM flag diagrams.** In [18] a certain class of (partial) LPFMs was studied, i.e., $\mathcal{F} : (M_0, M_1, \ldots, M_k)$ such that $U_{0,n} = M_0 <_q \ldots <_q M_k = U_{n,n}$ where all components are LPMs and $k \leq n$. Given a flag of bases $\mathfrak{B} = (B_0, B_1, \ldots, B_k)$ in $\mathcal{F}$, one can associate a monotone path P of length n in $\mathbb{Z}^k$ by setting the ith step to e_j if $i \in B_j \setminus B_{j-1}$ for all $1 \leq i \leq k$. Note that if $\mathcal{F} = (U_{0,n}, M, U_{n,n})$ where $M = M[U, L]$, then the set of paths obtained this way just corresponds to the paths in the diagram of $M[U, L]$. It is thus natural to define the *diagram* $D_{\mathcal{F}}$ of $\mathcal{F}$ as the set of points in $\mathbb{Z}^k$ that are on paths associated to flags of bases of $\mathcal{F}$. See Figure 9 for an example.

Problem 38. (a) Characterize the set of diagrams of LPFMs. (b) Characterize those paths in a diagram that correspond to flags. Are these all the monotone ones?

This question is already present in [18, Figure 6], where an example shows that already pretty reasonable sets in $\mathbb{Z}^3$ are not the diagram of an LPFM. We hope that the results of the present paper allow to shed new light on this problem.

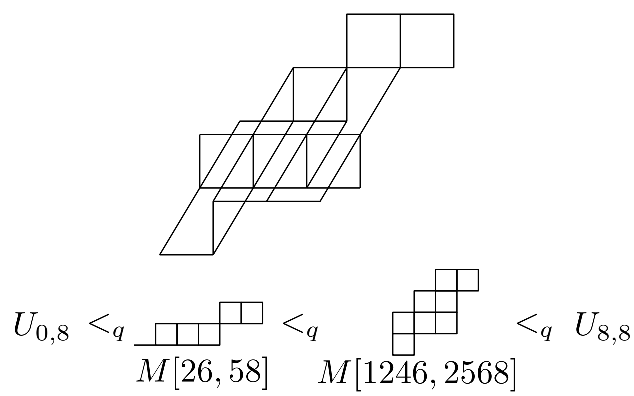

Figure 9. An LPFM and its diagram.

6.2. **Weak order and Higgs lift.** Let $\mathcal{M}_{k,n}$ be the collection of matroids over the set $[n]$ of fixed rank k. This collection is endowed with a partial ordering $\leq_w$, known as the *(rank-preserving) weak order* given as follows: if $M, M' \in \mathcal{M}_k, n$ then $M \leq_w M'$ if and only if every basis of M is a basis of M'. See [34, Prop. 7.4.7] for several cryptomorphic descriptions of the rank-preserving weak order relation.

In the case of LPMs, the weak order corresponds to diagram containment. That is, if $\mathcal{L}_{k,n}$ denotes the set of LPMs of rank k over $[n]$ and $M = M[U, L], M' = M[U', L'] \in \mathcal{L}_{k,n}$ then $M \leq_w M'$ if and only if $U \geq_G U'$ and $L \leq_G L'$.

Since $(\binom{[n]}{k}, \leq_G)$ has a lattice structure, by Observation 8 we have that $(\mathcal{L}_{k,n}, \leq_w)$ becomes an upper semilattice by setting the join $M[U, L] \vee M[U', L'] := M[U \wedge_G U', L \vee_G L']$. In particular, since $\leq_G$ is a distributive lattice, intervals in $(\mathcal{L}_{r,n}, \leq_w)$ are distributive lattices, as well. Also maxima and minima are easily determined as we now state.

Observation 39. *The poset $(\mathcal{L}_{k,n}, \leq_w)$ is isomorphic to the upper semilattice of intervals of the Gale order $(\binom{[n]}{k}, \leq_G)$ ordered by inclusion. Its unique maximum is $U_{k,n}$. It has $\binom{n}{k}$ minima corresponding to the elements of $\binom{[n]}{k}$.*

One can wonder how the weak order and the quotient relation interact. In Figure 10 we illustrate all the LPMs that belong to the interval $[U_{0,8}, M]_q$, where $M = M[1246, 2568]$. That is, $N \in [U_{0,8}, M]_q$ if and only if N is an LPM and $N \leq_q M$. Matroids in this interval that have the same rank have been ordered using $\leq_w$. Notice also that although $M[12, 58] <_w M[12, 68]$ and $M[12, 68] <_q M[124, 268]$, it does not follow that $M[12, 58]$ is a quotient of $M[124, 268]$. Thus, the union of quotient relation and rank preserving weak order is not an order relation.

Given two matroids M and M' such that $M \leq_q M'$, we say that a matroid M'' is the ith *Higgs lift* of M towards M' if M'' is the maximal matroid (with respect to $\leq_w$) such that $r(M'') = r(M) + i$ and $M \leq_q M'' \leq_q M'$. See [12, Propositions 2.2, 2.6] and [8] for proof that the Higgs lift always exists. Notice that the ith Higgs lift of $U_{0,n}$ towards M' is simply the *i-truncation of* M', i.e., its base set is given by $\mathcal{B}'_i := \{X \in \binom{[n]}{i} \mid \exists B' \in \mathcal{B}' : X \subseteq B'\}$.

With the above notation one can now wonder if a given class of matroids $\mathcal{C}$ is closed under taking Higgs lifts. That is, if $M \leq_q M'$ are in $\mathcal{C}$ and $i \leq r(M') - r(M)$, then we ask if there is a unique maximal (with respect to $\leq_w$) $M'' \in \mathcal{C}$ such that $r(M'') = r(M) + i$ and $M \leq_q M'' \leq_q M'$. In general there exists no Higgs lift within the class of LPMs: going back to Figure 10, we see that there is no unique maximum with respect to $\leq_w$ among the rank 3 LPMs in the interval $[U_{0,8}, M[1246, 2568]]$.

Observation 40. *The class of LPMs is not closed under Higgs lifts.*

Recall that another question that remains open about the different ranks of the quotient order of LPMs is whether they are unimodal on any interval $[M, M']_q$ (see Question 21). Note that we have answered this in the positive for the entire poset $\mathcal{P}_n$ itself.

6.3. **LPFMs and the flag variety.** We want to point out that the reach of Theorem 18 goes beyond characterizing combinatorially quotients of LPMs. Indeed, this characterization also proves that given two LPMs M and M' there exists a representable flag matroid that has them as constituents. This is not true for realizable matroids in general, see [13, 1.7.5 Example 7]. More precisely, our results show that there exits a point $A \in \mathcal{F}\ell_n^{\geq 0}$ that realizes simultaneously M and M'. This is not true for positroids in general, as pointed out in Example 6. There are however several questions about the class of LPFMs from the point of view of the flag variety that we are interested in.

Problem 41. Does the set of LPFMs on $[n]$ correspond to (a subset of) a proper subvariety of $F\ell_n^{\geq 0}$?

Our results completely characterize quotients of LPMs from a combinatorial point of view. Wanting to go further, we would like to understand the intervals in the (strong) Bruhat order corresponding to full flags of LPMs. Proposition 33 tells us that intervals corresponding to cubes in the (right) weak Bruhat order come from flags of LPMs, although not every flag of LPMs give rise to a weak interval (see Figure 5). Notice, however, that in Theorem 31

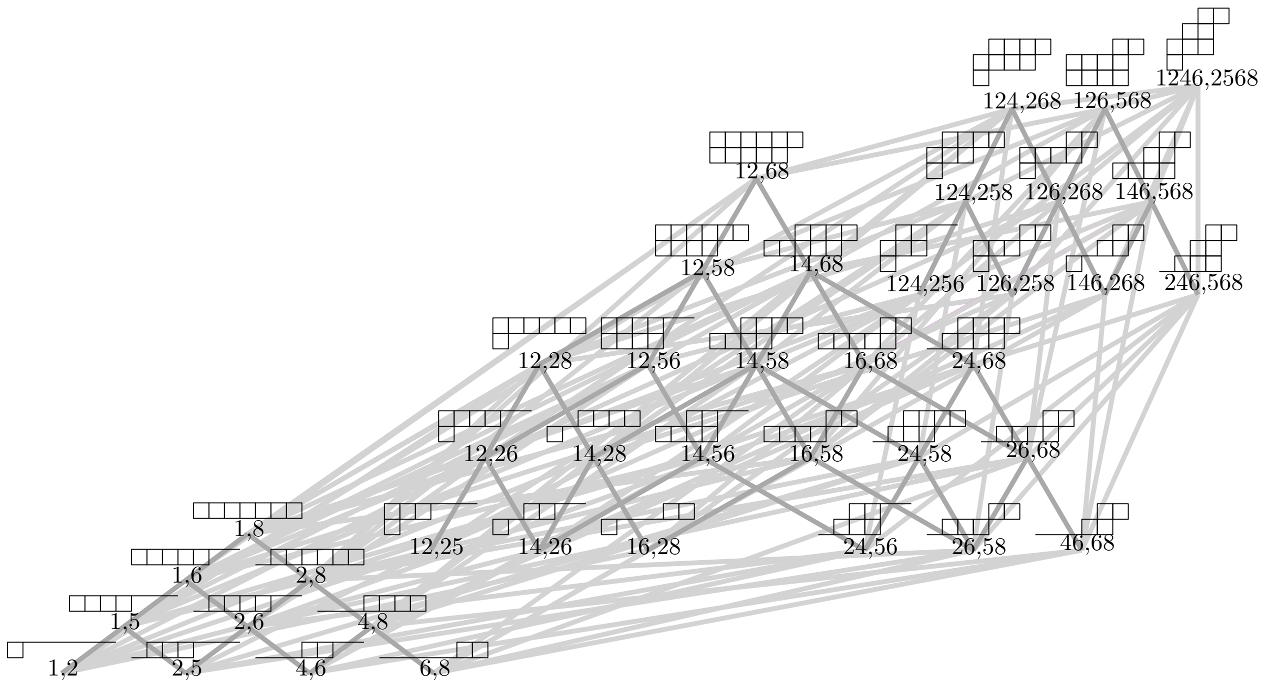

FIGURE 10. The LPM $M[1246, 2568]$ and its (non-trivial) quotients. Each rank is ordered with respect to the weak order, the corresponding cover relations are in dark gray. Quotient cover relations are in light gray.

we give a characterization of intervals in $(S_n, \leq_B)$ that arise from LPFMs. However, this characterization may be tricky to handle and a simpler one might be of use.

Finally, from the polytopal perspective, one may wonder if the intersection of LPFM polytopes is an LPFM polytope of. However this is not the case (see Figure 5). On the other hand the union of LPM intervals does not have a "natural" join. More precisely, take two LPFMs (M_1, M_2, M_3) and (M_1, M_2', M_3) such that $M_2 \cup M_2'$ is not contained in any quotient of M_3. An example would be $M_1 = M[12, 68]$, $M_2 = M[124, 268]$, $M_2' = M[126, 568]$, $M_3 = M[1246, 2568]$, see Figure 10. Thus, an LPFM containing both must also increase M_3.

Acknowledgments. The first author was supported by grant FAPA of the Faculty of Science at Universidad de los Andes. The second author was partially supported by the French *Agence nationale de la recherche* through project ANR-17-CE40-0015, by the Spanish *Ministerio de Economía, Industria y Competitividad* through grant RYC-2017-22701 and grant PID2019-104844GB-I00. We also thank Lauren Williams for insightful conversations and Suho Oh for sharing their conjecture with us.

REFERENCES

[1] S. AN, J. JUNG, AND S. KIM, *Facial structures of lattice path matroid polytopes*, Discrete Math., 343 (2020), p. 11. Id/No 111628.

[2] F. ARDILA, F. RINCÓN, AND L. WILLIAMS, *Positively oriented matroids are realizable*, J. Eur. Math. Soc. (JEMS), 19 (2017), pp. 815–833.

[3] C. Benedetti, A. Chavez, and D. Tamayo, *Quotients of uniform matroids*, Elec. J. Comb., 29 (2022).

[4] C. Benedetti, K. Knauer, and J. Valencia, *Ehrhart theory of lattice path matroids*, 2022.

[5] H. Bidkhori, *Lattice Path Matroid Polytopes*, ArXiv e-prints, (2012).

[6] S. Blum, *Base-sortable matroids and Koszulness of semigroup rings*, Eur. J. Comb., 22 (2001), pp. 937–951.

[7] J. E. Bonin, *Lattice path matroids: the excluded minors*, J. Combin. Theory Ser. B, 100 (2010), pp. 585–599.

[8] J. E. Bonin, C. Chun, and S. D. Noble, *Delta-matroids as subsystems of sequences of Higgs lifts*, Adv. Appl. Math., 126 (2021), p. 27. Id/No 101910.

[9] J. E. Bonin and A. de Mier, *Lattice path matroids: structural properties*, Eur. J. Comb., 27 (2006), pp. 701–738.

[10] J. E. Bonin, A. de Mier, and M. Noy, *Lattice path matroids: enumerative aspects and Tutte polynomials*, J. Combin. Theory Ser. A, 104 (2003), pp. 63–94.

[11] J. E. Bonin and O. Giménez, *Multi-path matroids*, Combin. Probab. Comput., 16 (2007), pp. 193–217.

[12] J. E. Bonin and W. R. Schmitt, *Splicing matroids*, Eur. J. Comb., 32 (2011), pp. 722–744.

[13] A. V. Borovik, I. M. Gel'fand, and N. White, *Coxeter Matroids*, vol. 216, London: Birkhäuser, 2003.

[14] T. Brylawski, *Constructions*. Theory of matroids, Encycl. Math. Appl. 26, 127-223 (1986)., 1986.

[15] A. Cameron, R. Dinu, M. Michałek, and T. Seynnaeve, *Flag matroids: algebra and geometry*, 2018.

[16] V. Chatelain and J. L. Ramírez Alfonsín, *Matroid base polytope decomposition*, Adv. in Appl. Math., 47 (2011), pp. 158–172.

[17] I. P. da Silva, ed., *Quelques propriétés des matroides orientés*, Ph.D. Dissertation, Université Paris VI, 1987.

[18] A. De Mier, *A natural family of flag matroids*, SIAM J. Discrete Math., 21 (2007), pp. 130–140.

[19] E. Delucchi and M. Dlugosch, *Bergman complexes of lattice path matroids.*, SIAM J. Discrete Math., 29 (2015), pp. 1916–1930.

[20] C. Heunen and V. Patta, *The category of matroids*, Appl. Categ. Struct., 26 (2018), pp. 205–237.

[21] D. A. Higgs, *Strong maps of geometries*, J. Comb. Theory, 5 (1968), pp. 185–191.

[22] K. Knauer, L. Martínez-Sandoval, and J. L. Ramírez Alfonsín, *On lattice path matroid polytopes: integer points and Ehrhart polynomial*, Discrete Comput. Geom., 60 (2018), pp. 698–719.

[23] K. Knauer, L. Martínez-Sandoval, and J. L. R. Alfonsín, *A Tutte polynomial inequality for lattice path matroids*, Advances in Applied Mathematics, 94 (2018), pp. 23 – 38. Special issue on the Tutte polynomial.

[24] R. Mcalmon, O. Suho, and X. David, *Flats of a positroid from its decorated permutation*, Sémin. Lothar. Comb., 82B (2019), pp. 82b.43, 12.

[25] J. Morton and J. Turner, *Computing the Tutte polynomial of lattice path matroids using determinantal circuits.*, Theor. Comput. Sci., 598 (2015), pp. 150–156.

[26] P. Nelson, *Almost all matroids are nonrepresentable*, Bull. Lond. Math. Soc., 50 (2018), pp. 245–248.

[27] H. Nishimura and S. Kuroda, eds., *A lost mathematician, Takeo Nakasawa. The forgotten father of matroid theory*, Basel: Birkhäuser, 2009.

[28] J. G. Oxley, *Matroid theory*, vol. 21, Oxford: Oxford University Press, 2011.

[29] A. Postnikov, *Total positivity, Grassmannians, and networks*, arXiv), (2006), p. arXiv:math/0609764.

[30] J. Schweig, *On the h-vector of a lattice path matroid*, Electron. J. Combin., 17 (2010), pp. Note 3, 6.

[31] J. Schweig, *Toric ideals of lattice path matroids and polymatroids*, J. Pure Appl. Algebra, 215 (2011), pp. 2660–2665.

[32] R. Stanley, *Enumerative Combinatorics Vol. 2*, vol. 62, Cambridge: Cambridge University Press, paperback ed. ed., 2001.

[33] E. Tsukerman and L. Williams, *Bruhat interval polytopes*, Adv. Math., 285 (2015), pp. 766–810.

[34] N. White, *Theory of matroids*, vol. 26, Cambridge: Cambridge University Press, paperback ed. ed., 1986.

[35] H. Whitney, *On the abstract properties of linear dependence.*, Bull. Am. Math. Soc., 40 (1934), p. 663.