ON k-NEIGHBORLY REORIENTATIONS OF ORIENTED MATROIDS

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Abstract. We study the existence and the number of k-neighborly reorientations of an oriented matroid. This leads to k-variants of McMullen's problem and Roudneff's conjecture, the case $k = 1$ being the original statements on complete cells in arrangements. Adding to results of Larman and García-Colín, we provide new bounds on the k-McMullen's problem and prove the conjecture for several ranks and k by computer. Further, we show that k-Roudneff's conjecture for fixed rank and k reduces to a finite case analyse. As a consequence we prove the conjecture for odd rank r and $k = \frac{r-1}{2}$ as well as for rank 6 and $k = 2$ with the aid of the computer.

1. Introduction

An *oriented matroid* is a pair $\mathcal{M} = (E, \mathcal{C})$ of a finite *ground set* E and a set of *sign-vectors* $\mathcal{C} \subseteq \{+, -, 0\}^E$ called *circuits* satisfying a certain set of axioms, see Definition 2.1. The *size* of a member $X \in \{+, -, 0\}^E$ is the size of its *support* $\underline{X} = \{e \in E \mid X_e \neq 0\}$. Throughout the paper all oriented matroids are considered *simple*, i.e., all circuits have size at least 3. The *rank* r of $\mathcal{M}$ is one less than the size of the largest circuit of $\mathcal{M}$. An oriented matroid of rank r is called *uniform* if all its circuits are of size $r + 1$. Most of the problems we study in this paper reduce to uniform oriented matroids.

Oriented matroids generalize the oriented linear algebra of Euclidean space in the following way: let $M \in \mathbb{E}^{m\times n}$ a real $m \times n$ matrix, then $\mathcal{M}_M = ([n], \mathcal{C}_M)$ is an oriented matroid where $X \in \mathcal{C}_M$ if there is a minimal linear combination $\lambda_{i_1} r_{i_1} + \ldots + \lambda_{i_k} r_{i_k} = \mathbf{0}$ of rows of M such that $\underline{X} = \{i_1, \ldots, i_k\}$ and X_{i_j} is the sign of λ_{i_j} for all $1 \leq j \leq k$. The rank of $\mathcal{M}_M$ is the rank of M. If $\mathcal{M}$ arises this way from a matrix M, it is called *realizable*. Realizable oriented matroid form a small subclass of all oriented matroids [3, Corollary 7.4.3], but capture hyperplane arrangements, point configurations, linear programming, and directed graphs. See the book [3] for a broad overview.

Let us move to generalization of hyperplane arrangements yielding a complete picture of oriented matroids. For a subset $R \subseteq E$ the *reorientation of* R is the oriented matroid ${}_{-R}\mathcal{M}$ obtained from $\mathcal{M}$ by reversing the sign of X_e for every $e \in R$ and $X \in \mathcal{C}$. The set of all oriented matroids that can be obtained this way from $\mathcal{M}$ is the *reorientation class* $[\mathcal{M}]$ of $\mathcal{M}$. A corner stone of the theory is the Topological Representation Theorem [10], which identifies the reorientation class $[\mathcal{M}]$ of a simple rank r oriented matroid $\mathcal{M}$ with an arrangement $\mathcal{A}^{\mathbb{E}}_{\mathcal{M}}$ of pseudospheres in Euclidean space $\mathbb{E}^r$ or equivalently with an arrangement of pseudohyperplanes $\mathcal{A}^{\mathbb{P}}_{\mathcal{M}}$ in projective space $\mathbb{P}^{r-1}$. See Section 2.1 for more precise definitions of these topological objects.

An oriented matroid $\mathcal{M}$ is *acyclic* if every circuit has positive and negative signs.

Definition 1.1. An oriented matroid $\mathcal{M} = (E, \mathcal{C})$ is *k-neighborly* if for every subset $R \subseteq E$ of size at most k the reorientation ${}_{-R}\mathcal{M}$ is acyclic.

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It follows from the definition that a k-neighborly oriented matroid is k'-neighborly for all $0 \leq k' \leq k$. Note that $\mathcal{M}$ is 0-neighborly if and only if $\mathcal{M}$ is acyclic. The acyclic reorientations of an oriended matroid $\mathcal{M}$ correspond to the maximal cells of $\mathcal{A}^{\mathbb{E}}_{\mathcal{M}}$. If $\mathcal{M}$ is 1-neighborly, then $\mathcal{M}$ is called *matroid polytope.* The maximal cells of $\mathcal{A}^{\mathbb{E}}_{\mathcal{M}}$ corresponding to the 1-neighborly reorientations of an oriended matroid $\mathcal{M}$ are called *complete cells*, since they are bounded by all the pseudospheres of $\mathcal{A}^{\mathbb{E}}_{\mathcal{M}}$. In general, a k-neighborly reorientation of $\mathcal{M}$ corresponds to a maximal cell T of $\mathcal{A}^{\mathbb{E}}_{\mathcal{M}}$ such that for any set F of at most k pseudospheres there is another cell ${}_FT$ that is separated from T by exactly the pseudospheres in F. We will give several cryptomorhpic descriptions of k-neighborliness in Section 3. If $\mathcal{M}$ has rank r, then it can be at most $\lfloor \frac{r-1}{2} \rfloor$-neighborly and in this case $\mathcal{M}$ is often just called *neighborly.* There is quite some work about neighborly oriented matroids, starting with Sturmfels [30] and [3, Section 9.4] but also more recent works such as [23, 25]. In the realizable setting k-neighborliness is related to polytopes in the following ways. A hyperplane arrangement corresponds to the reorientation class of a realizable k-neighborly oriented matroid if and only if it can be obtained by taking the facet defining hyperplanes of a k-neighborly polytope. Equivalently, a point configuration corresponds to an acyclic reorientation of a realizable k-neighborly oriented matroid if there is a projective transformation such that the points become the vertices of a k-neighborly polytope. Another nice instance comes from the co-graphic oriented matroid $\mathcal{M}^*(G)$ of a graph G, where the k-neighborly reorientations of $\mathcal{M}^*(G)$ corresond to the $(k+1)$-arc-connected orientations of G.
In this paper we study k-neighborly reorientations of oriented matroids. In particular we study two well-known conjectures on 1-neighborly oriented matroids, i.e., complete cells, and their generalization to arbitrary k.

1.1. **k-McMullen problem.** This problem is about the existence of k-neighborly reorientations of uniform oriented matroids. Denote by $\nu(r,k)$ (respectively $\nu_{\mathrm{R}}(r,k)$) the largest n such that any (realizable) uniform oriented matroid $\mathcal{M}$ of rank r and n elements has a k-neighborly reorientation. Clearly, $\nu(r,k) \leq \nu_{\mathrm{R}}(r,k)$. Note that since every uniform oriented matroid has an acyclic orientation $\nu(r,0) = +\infty$. This parameter was originally only studied for $k = 1$. First, it was defined for realizable uniform oriented matroids [18] and then for general uniform oriented matroids [6]. The following conjecture is known as the *McMullen problem*:

Conjecture 1.2 (McMullen 1972)**.** *For any $r \geq 3$ it holds $\nu(r,1) = 2r-1$.*

The inequality $2r-1 \leq \nu_{\mathrm{R}}(r,1)$ has been shown in [18] for realizable uniform oriented matroids and in [6] for general uniform oriented matroids, i.e., $2r-1 \leq \nu(r,1)$. This conjecture has been verified for $r \leq 5$ [11], but remains open otherwise. After a series of results [18, 20], the currently best-known upper bound $\nu(r,1) < 2(r-1) + \lceil \frac{r}{2} \rceil$ is due to Ramírez Alfonsín [27]. For general positive k we propose the following strengthening of Conjecture 1.2:

Question 1.3. *Does $\nu(r,k) = r + \lfloor \frac{r-1}{k} \rfloor$ hold for all $k = 1, \dots, \lfloor \frac{r-1}{2} \rfloor$ and $r \geq 3$?*

Since $\nu(r,0) = +\infty$ the case $k = 0$ holds in a sense. If $k = 1$ then $r \geq 3$ since $k \leq \lfloor \frac{r-1}{2} \rfloor$ and so, it is just Conjecture 1.2 (McMullen problem). In [13] the authors show that every realizable uniform oriented matroid of rank r on $r + \lfloor \frac{r-1}{k} \rfloor$ elements has a k-neighborly reorientation, i.e., $r + \lfloor \frac{r-1}{k} \rfloor \leq \nu_{\mathrm{R}}(r,k)$. On the other hand they present examples of realizable uniform oriented matroids of rank r and order $2r-k+1$ without k-neighborly reorientations, i.e., $\nu_{\mathrm{R}}(r,k) < 2r-k+1$. Note that yet another variant of McMullen's problem has been studied recently in [14].

Remark 1.4. *Indeed in [13, Theorem 1] it is claimed that* $r+\lceil\frac{r-1}{k}\rceil \leq \nu_{\mathrm{R}}(r,k) < 2r-k-1$, *but what they actually prove are the above mentioned bounds. Regarding the lower bound, the authors introduce a parameter* $\lambda(r,k)$ *and prove that* $\lambda(r,k) \leq (k+1)(r-1)+(k+2)$ *(see [13, Lemma 9]) and that* $\nu_{\mathrm{R}}(r,k) = max\{w \in \mathbb{N} : w \geq \lambda(w-r-1,k)\}$ *(see [13, Equation (1)]). Hence,* $\lambda(w-r-1,k) \leq (k+1)(w-r-1)+(k+2)$ *and so, for the positive integer* w *such that* $(k+1)(w-r-1)+(k+2) \leq w$, *we obtain by [13, Equation (1)] that* $w \leq \nu_{\mathrm{R}}(r,k)$. *On the other hand, from the inequality* $(k+1)(w-r-1)+(k+2) \leq w$ *it follows that* $w \leq r+\frac{r-1}{k}$ *and since* w *is an integer,* $w \leq r+\lfloor\frac{r-1}{k}\rfloor$. *Therefore, as* $\nu_{\mathrm{R}}(r,k) = max\{w \in \mathbb{N} : w \geq \lambda(w-r-1,k)\}$, *we conclude that* $r+\lfloor\frac{r-1}{k}\rfloor \leq \nu_{\mathrm{R}}(r,k)$. *In fact, we show in Theorem 6.3 that* $\nu_{\mathrm{R}}(r,k) < r+\lceil\frac{r-1}{k}\rceil$ *for* $r=6$ *and* $k=2$, *thus their claimed lower bound cannot hold.*
With respect the upper bound, they construct in Definitions 8 *and* 9 *of [13] (for the cases* $k=2$ *and* $k \geq 3$, *respectively) examples of realizable uniform oriented matroids, called Lawrence oriented matroids, of rank* r *and order* $2r-k+1$ *(and not* $2r-k-1$ *as mentioned in Theorem* 1 *of [13]) without* k*-neighborly reorientations. Then,* $\nu_{\mathrm{R}}(r,k) < 2r-k+1$. *In fact, notice that* $r+\lfloor\frac{r-1}{k}\rfloor < 2r-k-1$ *is not true for* $r=5$ *and* $k=2$, *thus their claimed upper bound cannot hold.*

In Section 4 we prove that $r-1+\lfloor\frac{r-1}{k}\rfloor \leq \nu(r,k)$ for $r \geq 3$ and every $k = 1,\ldots,\lfloor\frac{r-1}{2}\rfloor$ (Theorem 4.2). Hence, we are only off by 1 of the lower bound claimed in Question 1.3. Moreover, we prove the lower bound $r+\lfloor\frac{r-1}{k}\rfloor \leq \nu(r,k)$ for some cases (Theorem 4.3). Further, with the help of the computer, we improve the upper bound $\nu_{\mathrm{R}}(r,k) < 2r-k+1$ for some small values of r and k and as a consequence, we answer Question 1.3 affirmatively showing that $\nu(5,2) = \nu_{\mathrm{R}}(5,2) = 7$, $\nu(6,2) = \nu_{\mathrm{R}}(6,2) = 8$ and $\nu(7,3) = \nu_{\mathrm{R}}(7,3) = 9$ (Theorem 6.3).

1.2. **k-Roudneff's conjecture.** This problem is about the number of k-neighborly reorientations of a rank r oriented matroid $\mathcal{M}$ on n elements, denoted by $m(\mathcal{M},k)$. The *cyclic polytope* of dimension d with n vertices, $C_d(t_1,\ldots,t_n)$, discovered by Carathéodory [4], is the convex hull in $\mathbb{R}^d$ of $n \geq d+1 \geq 3$ different points $x(t_1),\ldots,x(t_n)$ of the moment curve $\mu : \mathbb{R} \longrightarrow \mathbb{R}^d,\ t \mapsto (t,t^2,\ldots,t^d)$. Cyclic polytopes play an important role in the combinatorial convex geometry due to their connection with certain extremal problems. For example, the Upper Bound theorem due to McMullen [22]. *Cyclic arrangements* are defined as the dual of the point set given by the vertices of the cyclic polytope. As for cyclic polytopes, cyclic arrangements also have extremal properties. For instance, Shannon [29] has introduced cyclic arrangements on dimension d as examples of projective arrangements with a minimum number of cells with $(d+1)$ facets. The oriented matroid associated to a cyclic arrangement of dimension $d = r-1$ is the *alternating oriented matroid* $C_r(n)$. It is the uniform oriented matroid of rank r and ground set $E = [n] := \{1,\ldots,n\}$ such that every circuit $X \in \mathcal{C}$ is *alternating*, i.e., $X_{i_j} = -X_{i_{j+1}}$ for all $1 \leq j \leq r$ if $\underline{X} = \{i_1,\ldots,i_{r+1}\}$ and $i_1 < \ldots < i_{r+1}$. The matroid $C_r(n)$ is realizable and intimately related to the cyclic polytope and cyclic arrangements. More precisely, $C_r(n) = \mathcal{M}_M$ where given different points $x(t_1),\ldots,x(t_n) \subseteq \mathbb{R}^{r-1}$, the ith row of the matrix M is $(t^0,t,t^2,\ldots,t^{r-1})$, see [3, Proposition 9.4.1].
Denote by $c_r(n,k) = m(C_r(n),k)$ the number of k-neighborly reorientations of $C_r(n)$. Since $C_r(n)$ is uniform and 0-neighborly reorientations are just the acyclic reorientations, we have $c_r(n,0) = 2\sum_{i=0}^{r-1}\binom{n-1}{i}$, see e.g. [5]. Roudneff [28] proved that $c_r(n,1) \geq 2\sum_{i=0}^{r-3}\binom{n-1}{i}$ and that is an equality for all $n \geq 2r-1$. In [12] it is shown that $c_r(n,1) = 2(\binom{r-1}{n-r+1}+\binom{r}{n-r}+\sum_{i=0}^{r-3}\binom{n-1}{i})$ for $n \geq r+1$.
The following has been conjectured by Roudneff (originally in terms of projective pseudohyperplane arrangements [28, Conjecture 2.2]).

Conjecture 1.5 (Roudneff 1991)**.** For any $r \geq 3$, $n \geq 2r-1$ it holds $m(\mathcal{M}, 1) \leq c_r(n, 1)$.

The above conjecture is stated for $r \geq 3$ since for $r = 1, 2$ there is only one reorientation class and clearly $m(\mathcal{M}, 1) = c_r(n, 1)$ (see [3, Section 6.1]). The case $r = 3$ is not difficult to prove, the case $r = 4$ has been shown in [26] and recently also for $r = 5$ [16] as well as for Lawrence oriented matroids [24]. Furthermore, in [2] it is shown that for realizable oriented matroids of rank r on n elements, the number of 1-neighborly reorientations is $2\binom{n}{r-3} + O(n^{r-4})$, i.e., Roudneff's conjecture holds asymptotically in the realizable setting. In [24, Question 2] the authors asked if the conjecture holds for $n \geq r+1$ and again, it turns that it is true for $r \leq 5$ [16] and for Lawrence oiented matroids [24]. In the same manner as for McMullen's conjecture we propose the k-variant of the above question.

Question 1.6. *Is it true that* $m(\mathcal{M}, k) \leq c_r(n, k)$*, for all* $n > r \geq 3$*,* $0 \leq k \leq \lfloor \frac{r-1}{2} \rfloor$*?*

The above question holds for $k = 0$, since all oriented matroids of given rank r and number n of elements have at most the number of acyclic reorientations of (any) uniform oriented matroid [5], moreover it holds trivially for $n \leq r+2$ (Remark 2.3). For $k = 1$ Question 1.6 combines Roudneff's conjecture and [24, Question 2]. Hence, the answer is positive for $k = 1$ if $r \leq 5$. We prove in Theorem 5.13 that in order to answer Question 1.6 affirmatively for a fixed r and k, it is enough to prove it for uniform oriented matoids with $r+1 \leq n \leq 2(r-k)+1$ and all rank $r' \leq r$ uniform oriented matroid $\mathcal{M}'$ on $n' = 2(r'-k)+1$ elements. Thus, the question reduces to a finite number of cases. As a consequence we answer Question 1.6 in the positive for odd r and $k = \lfloor \frac{r-1}{2} \rfloor$ (Corollary 5.14). Moreover, in Theorem 6.4 we answer Question 1.6 in the affirmative for $r = 6$ and $k = 2$. One might think that as in the Upper Bound Theorem [22] the reorientation class of $\mathcal{C}_r(n)$ is unique in attaining the maximum in Question 1.6. However, in Theorem 6.1 we show that different reorientation classes attain $c_r(n, k)$, for $r = 5, n = 8, 9$ and $k = 2$ and for $r = 7, n = 10$ and $k = 3$.

As a tool in the study of Question 1.6 we make use of a refinement of $m(\mathcal{M}, k)$, namely we define the *o-vector* of $\mathcal{M}$, as the vector with entries $o(M, k)$, for every $k = 0, 1, \ldots, \lfloor \frac{r-1}{2} \rfloor$, where $o(\mathcal{M}, k)$ is the number of reorientations of $\mathcal{M}$ that are k-neighborly but not $(k+1)$-neighborly. In Theorem 5.8 we compute this parameter for the alternating oriented matroid, which lies at the heart of the proof of Theorem 5.13. We then proceed to study $o(\mathcal{M}, k)$ as a parameter of independent interest, and note that here the role of the alternating oriented matroid is more complicated than $m(\mathcal{M}, k)$.
On the one hand, the alternating oriented matroid maximizes $o(\mathcal{M}, k)$ for $n \leq r+2$ (Remark 2.3), for odd r and $k = \lfloor \frac{r-1}{2} \rfloor$ (Corollary 5.14) and for $r = 6$ and $k = 2$ (Theorem 6.4). In Theorem 6.4 we show that for $r = 5$, $k = 1$ and $n = 8, 9$, for $r = 6$, $k = 2$ and $n = 9$, and for $r = 7$, $k = 2$ and $n = 10$, the alternating oriented matroid is even unique (up to reorientation) with this property. On the other hand, for $r = 6$, $k = 1$ and $n = 9$ and for $r = 7$, $k = 1$ and $n = 10$, there are (up to reorientation) 91 and 312336 uniform oriented matroids $\mathcal{M}$ of rank r on n elements such that $o(\mathcal{M}, 1) > o(\mathcal{C}_6(9), 1)$ and $o(\mathcal{M}, 1) > o(\mathcal{C}_7(10), 1)$, respectively (Theorem 6.4 (c) and (d)).

1.3. **Organization of the paper.** The structure of the paper is as follows.

In Section 2, we introduce further basic notions of oriented matroid theory and explain the correspondence between arrangements and oriented matroids.

In Section 3 we study several cryptomorphic descriptions of k-neighborliness, most importantly the notion of k'-orthogonality which generalizes usual orthogonality of sign-vectors, as well as metric descriptions in terms of the tope graph (Proposition 3.2). Several of our results are stated in terms of k'-orthogonality which is equivalent to k-neighborliness, with $k = k' - 1$.

While in Section 4 we study k-McMullen's conjecture and give lower bounds on the parameter $\nu(r, k)$, in Section 5 we study k-Roudneff's conjecture mostly through the function $o(\mathcal{M}, k)$. In Subection 5.1 we study the tope graph of the alternating polytope and obtain $o(C_r(n), k)$ for large enough n (Theorem 5.8). In Subsection 5.2 we show that we may restrict Question 1.6 to uniform oriented matroids of finite order (Theorem 5.13) and solve Question 1.6 for odd r and $k = \frac{r-1}{2}$ (Corollary 5.14).

In Section 6 we present computational results for low rank. We explain our computer program that obtains $o(\mathcal{M}, k)$ for uniform oriented matroids $\mathcal{M}$ via the chirotope. We then obtain the maximal $o(\mathcal{M}, k)$ among all $\mathcal{M}$ of rank r and n elements for several values of r, n, and k. We give lower and upper bounds for $\nu(r, k)$ and answer Question 1.3 for several parameters. Using Theorem 5.13, we answer Question 1.6 in the affirmative for $r = 6, k = 2$ and $n \geq 9$ (Theorem 6.4).

2. Oriented matroids: notation and terminology

Let us give some basic notions and definitions in oriented matroid theory apart from those given in the introduction. We assume some knowledge and standard notation of the theory of oriented matroids, for further reference the reader can consult the textbook [3]. For a *sign vector* $X \in \{+, -, 0\}^E$ on ground set E we denote by $X^+ := \{e \in E \mid X_e = +\}$, the set of *positive elements* of X, and $X^- := \{e \in E \mid X_e = -\}$, its set of *negative elements*. Hence, the set $\underline{X} = X^+ \cup X^-$ is the support of X. We denote by $-X$ the sign-vector ${}_{-E}X$ where all signs are revered, i.e., such that $-X^+ = X^-$ and $-X^- = X^+$. We say that X is *positive* if $X^- = \emptyset$.

Definition 2.1. An oriented matroid $\mathcal{M} = (E, \mathcal{C})$ is a pair of a finite ground set E and a collection of signed sets on E called *circuits*, satisfying the following axioms:

(C0) $\emptyset \notin \mathcal{C}$,
(C1) $X \in \mathcal{C}$ if and only if $-X \in \mathcal{C}$,
(C2) if $X, Y \in \mathcal{C}$ and $\underline{X} \subseteq \underline{Y}$, then $X = \pm Y$,
(C3) if $X, Y \in \mathcal{C}$, $X \neq -Y$, and $e \in X^+ \cap Y^-$, then there exists a $Z \in \mathcal{C}$, such that $Z^+ \subseteq X^+ \cap Y^+ \setminus e$ and $Z^- \subseteq X^- \cap Y^- \setminus e$.

Given an oriented matroid $\mathcal{M} = (E, \mathcal{C})$ the *contraction* of $F \subset E$ is the oriented matroid $\mathcal{M}/F = (E \setminus F, \mathcal{C}/F)$, where $\mathcal{C}/F$ is the set of support-minimal sign-vectors from $\{X \setminus F \mid X \in \mathcal{C}\} \setminus \{\mathbf{0}\}$. If $\mathcal{M}$ is a uniform oriented matroid of rank r, then $\mathcal{M}/F$ is uniform of rank $\max(0, r - |F|)$. The *deletion* of F from $\mathcal{M}$ is the oriented matroid $\mathcal{M} \setminus F = ((E \setminus F, \mathcal{C} \setminus F)$, where $\mathcal{C} \setminus F = \{X \setminus F \mid X \in \mathcal{C}, \underline{X} \cap F = \emptyset\} \setminus \{\mathbf{0}\}$. If $\mathcal{M}$ is a uniform oriented matroid of rank r, then $\mathcal{M}/F$ is uniform of rank $\min(|E \setminus F|, r)$.

An equivalent characterization of oriented matroids was given by Lawrence [21].

Definition 2.2. Let $r \geq 1$ be an integer be a set. An oriented matroid of rank r is a pair $\mathcal{M} = (E, \chi)$ of a finite ground set E and a *chirotope* $\chi : E^r \to \{+, 0, -\}$ satisfying:

(B0) χ is no identically zero,
(B1) χ is alternating,
(B2) if $x_1, \ldots, x_r, y_1, \ldots, y_r \in E$ and $\chi(y_i, x_2, \ldots, x_r)\chi(y_1, \ldots, y_{i-1}, x_1, y_{i+1}, \ldots, y_r) \geq 0$ for all $i \in [r]$, then $\chi(x_1, \ldots, x_r)\chi(y_1, \ldots, y_r) \geq 0$.

If $\chi : E^r \to \{-, +\}$ is a chirotope, then $\mathcal{M} = (E, \chi)$ is uniform. Moreover, if $E = [n]$ and $\chi(B) = +$ for any ordered tuple $B = (b_1 < \ldots < b_r)$, then the uniform matroid

$\mathcal{M} = (E, \chi)$ is the alternating oriented matroid of rank r on n elements: $C_r(n)$. In this model the *reorientation* ${}_{-R}\mathcal{M}$ is obtained by defining

$$_{-R}\chi(x_1, \ldots, x_r) = (-\chi(x_1, \ldots, x_r))^{|\{x_1,\ldots,x_r)\} \cap R|}.$$

Given an oriented matroid $\mathcal{M} = (E, \chi)$ of rank r, its *dual* is the oriented matroid $\mathcal{M}^* = (E, \chi^*)$ of rank $n - r$ defined by setting

$$\chi^*(x_1, \ldots, x_{n-r}) = \chi(x'_1, \ldots, x'_r)\mathrm{sign}(x_1, \ldots, x_{n-r}, x'_1, \ldots, x'_r),$$

where $(x'_1, \ldots, x'_r)$ is any permutation of $E \setminus \{x_1, \ldots, x_{n-r}\}$. In particular, $\mathcal{M}$ is uniform of rank r if and only if $\mathcal{M}^*$ is uniform of rank $n - r$. The following observation will be useful throughout this paper.

Remark 2.3. *For every $r \leq n \leq r+2$, there is exactly one reorientation class of uniform rank r oriented matroids on n elements.*

Proof. It follows from the above definitions that $\mathcal{M}_1 \in [\mathcal{M}_2]$ if and only if $\mathcal{M}_1^* \in [\mathcal{M}_2^*]$. Hence, the number of reorientation classes of uniform oriented matroids of rank r on n elements is equal to the number of reorientation classes of uniform oriented matroids of rank $n - r$ on n elements. Thus, the number of reorientation classes of uniform oriented matroids of rank r on $r \leq n \leq n + 2$ elements is equal to the number of reorientation classes of uniform oriented matroids of rank at most 2 on n elements. For the latter it is well-known that there is only one reorientation class, see e.g. [3, Section 6.1]. □

Two sign vectors $X, Y \in \{+, -, 0\}^E$ are called *orthogonal* if either there are $e, f \in E$ such that $X_eY_e = +$ and $X_fY_f = -$ or $\underline{X} \cap \underline{Y} = \emptyset$. The set $\mathcal{L}$ of *covectors* of $\mathcal{M}$ consists of all sign vectors $X \in \{+, -, 0\}^E$ such that X is orthogonal to every circuit $Y \in \mathcal{C}$ of $\mathcal{M}$. The set $\mathcal{C}^*$ of *cocircuits* of $\mathcal{M}$ consists of the support-minimal elements of $\mathcal{L} \setminus \{\mathbf{0}\}$. The *topes* of $\mathcal{M}$ are defined as $\mathcal{L} \cap \{+, -\}^E$, i.e., as the maximal covectors of $\mathcal{M}$. All these three sets uniquely determine $\mathcal{M}$ and oriented matroids can be axiomatized in terms of them as well, see [3] for covectors, cocircuits, and [7, 15] for topes. Finally, the *tope graph* $\mathcal{G}(\mathcal{M})$ is the graph whose vertex set are the tope of $\mathcal{M}$, where two vertices are adjacent if they differ in a single coordinate. The (unlabelled) $\mathcal{G}(\mathcal{M})$ determines the reorientation class of $\mathcal{M}$ and purely graph theoretical polynomial time verifiable characterizations of tope graphs of oriented matroids have been obtained recently, see [17].

2.1. **Topological Representation Theorem.** In this subsection we briefly explain the Topological Representation Theorem and present the formulation of McMullen's problem and Roudneff's conjecture in their original versions. This is done merely out of illustrative reasons. None of our proofs or arguments outside of this subsection relies on topological representations.

The combinatorial properties of arrangements of pseudohyperplanes can be studied in the language of oriented matroids. The Topological Representation Theorem [10] states that the reorientation classes of simple oriented matroids on n elements and rank r are in one-to-one correspondence with the classes of isomorphism of arrangements of n pseudospheres in S^{r-1}. See [3, Theorem 1.4.1] for the precise definitions of such arrangements. There is a natural identification between pseudospheres and pseudohyperplanes as follows: Recall that $\mathbb{P}^{r-1}$ is the topological space obtained from S^{r-1} by identifying all pairs of antipodal points. The double covering map $\pi : S^{r-1} \to \mathbb{P}^{r-1}$, given by $\pi(x) = \{x, -x\}$, gives an identification of centrally symmetric subsets of S^{r-1} and general subsets of $\mathbb{P}^{r-1}$. This way centrally symmetric pseudospheres in S^{r-1} correspond to pseudohyperplanes in $\mathbb{P}^{r-1}$. Since the pseudoshperes in the Topological Representation Theorem can be assumed to be centrally symmetric, we get a statement in terms of pseudohyperplanes in $\mathbb{P}^{r-1}$, i.e., the reorientation classes of simple oriented matroids on n elements and rank r are in one-to-one

correspondence with the classes of isomorphism of arrangements of n pseudohyperplanes in $\mathbb{P}^{r-1}$. See [3, Exercise 5.8]. In this model one usual uses the *dimension* $d = r - 1$ of the rank. Given an arrangement $\mathcal{H}(d, n)$ of n pseudohyperplanes in $\mathbb{P}^d$ representing an reorientation class of $[\mathcal{M}]$, any given element of class can be obtained by choosing for each pseudohyperplane $H_e \in \mathcal{H}(d, n)$ in which of its two sides is positive and which is negative. Now, every point in $x \in \mathbb{P}^d$ yields a sign-vector X, where X_e is $0, +, -$ depending on whether x lies on H_e, on its positive side, or its negative side.
An arrangement $\mathcal{H}(d, n)$ of n pseudohyperplanes in $\mathbb{P}^d$ is called *simple* if every intersection of d pseudohyperplanes is a unique distinct point. Simple arrangements correspond to reorientation classes of uniform oriented matroids. The maximal cells of the arrangement $\mathcal{H}(d, n)$ correspond to one half of the the topes of the oriented matroid $\mathcal{M}$ (obtained by factoring the antipodal map). The topes then corresponds to the acyclic reorientation of $\mathcal{M}$, by orienting the pseudohyperplanes such that points inside the corresponding maximal cell are all-positive. A complete cell of $\mathcal{H}(d, n)$ is a maximal that is bounded by every hyperplane of the arrangement. In the corresponding reorientation of $\mathcal{M}$, reorienting any element of E results in another acyclic reorientation, i.e., the one corresponding to the adjacent maximal cell. This is, complete cells correspond to 1-neighborly reorientations.
Alternating oriented matroids of rank $r = d + 1$ on n elements are equivalent to cyclic arrangements of n hyperplanes in $\mathbb{P}^d$, which hence have $\frac{1}{2}c_{r-1}(n, 1)$ complete cells . Since the latter number for $n \geq 2r - 1$ can be expressed as a sum of binomial coefficients (see [26, Theorem 2.1]), we get Roudneff's original conjecture [26, Conjecture 2.2]:

Conjecture (Roudneff 1991)**.** Every arrangement of $n \geq 2d+1 \geq 5$ (pseudo)hyperplanes in $\mathbb{P}^d$ has at most $\sum_{i=0}^{d-2} \binom{n-1}{i}$ complete cells.

McMullen's problem (Conjecture 1.2) was originally formulated for realizable uniform oriented matroids in the language of projective tranformations [18] and then, more generally in terms of uniform oriented matroids [6]. A *projective transformation* $T : \mathbb{R}^d \to \mathbb{R}^d$ is a function such that $T(x) = \frac{Ax+b}{\langle c,x\rangle+\delta}$, where A is a linear transformation of $\mathbb{R}^d$, $b, c \in \mathbb{R}^d$ and $\delta \in \mathbb{R}$, is such that at least one of $c \neq 0$ or $\delta \neq 0$. Further, T is said to be *permissible* for a set $X \subset \mathbb{R}^d$ if $\langle c, x\rangle + \delta \neq 0$ for all $x \in X$ (see [31, Appendix 2.6]). It turns out that permissible projective transformations on n points in $\mathbb{R}^d$ corresponds to acyclic reorientations of the corresponding rank $r = d + 1$ oriented matroid $\mathcal{M}$ on n elements, see [6, Theorem 1.2]. On the other hand, it is well-known that a set X of n points in general position in $\mathbb{R}^d$ corresponds to a realizable uniform oriented matroid $\mathcal{M}$ of rank $r = d + 1$ on n elements and a permissible projective transformation leading X to the vertices of a convex polytope corresponds to a 1-neighborly reorientation of $\mathcal{M}$. So, we get McMullen's problem in its original version [18, P.1.]:

Conjecture (McMullen 1972)**.** Determine the largest number $\nu(d)$ such that any set of $\nu(d)$ points, lying in general position in $\mathbb{R}^d$, may be mapped by a permissible projective transformation onto the set of vertices of a convex polytope.

3. Orthogonality and neighborliness

Following Las Vergnas [19] a subset $F \subseteq E$ is a *face* of $\mathcal{M} = (E, \mathcal{L})$ if there is a covector $Y \in \mathcal{L}$ such that $F = E \setminus Y^+$ and $Y^- = \emptyset$.

Lemma 3.1. *An oriented matroid $\mathcal{M} = (E, \mathcal{C})$ is k-neighborly if and only if every subset $F \subseteq E$ of size at most k is a face.*

Proof. Let $F \subseteq E$ such that every subset $F' \subseteq F$ the reorientation ${}_{-F}\mathcal{M}$ is acyclic. Thus, we have that F' properly intersects X^+ or X^- for every circuit X of $\mathcal{M}$ or is disjoint if

$X^+, X^- \neq \emptyset$. This implies that the sign-vector Y that is positive on $F \setminus E$ and 0 on F is orthogonal to all circuits of $\mathcal{M}$. Hence all subsets of F are faces. Conversely if all subsets if $F' \subseteq F$ are faces, then $_{-F}\mathcal{M}$ is acyclic. The claim follows from Definition 1.1. □

Given two sign-vector $X, Y \in \{+, -, 0\}^E$, their *separation* is the set $S(X, Y) = \{e \in E \mid X_e \cdot Y_e = -\}$. For convenience, we denote by $H(X, Y) = \{e \in E \mid X_e \cdot Y_e = +\}$. We define the *orthogonality* of X and Y by

$$X \perp Y = \min\{|H(X, Y)|, |S(X, Y)|\}.$$

We say that X, Y are *k-orthogonal* if $X \perp Y \geq k$. We call a sign-vector $T \in \{+, -\}^E$, *k-orthogonal* (to $\mathcal{M}$) if $X \perp T \geq k$ for all $X \in \mathcal{C}$. Note that for $X \in \{0, +, -\}^E$ and $T \in \{+, -\}^E$ of $\mathcal{M}$ we have $X \perp T \geq 1$ if and only if X and T are orthogonal. Hence, the sign-vectors $T \in \{+, -\}^E$ that are 1-orthogonal to $\mathcal{M}$ constitute the set of topes $\mathcal{T}$ of $\mathcal{M}$.
The following establishes a correspondence between $k-1$-neighborly reorientations and k-orthogonal topes.

Proposition 3.2. *Let $\mathcal{M}$ be a rank r oriented matroid on $n = |E|$ elements, $R \subseteq E$, and T the sign-vector that is negative on R and positive on $E \setminus R$, and $k = 1, \ldots, \lfloor \frac{r+1}{2} \rfloor$. Then, T is a k-orthogonal tope if and only if $_{-R}\mathcal{M}$ is $k-1$-neighborly.*

Proof. "⇒". Let T be a k-orthogonal tope of $\mathcal{M}$, i.e., $H(X, T) \geq k$ and $S(X, T) \geq k$ for every circuit X of $\mathcal{M}$ and denote $R = T^-$. Now, let us consider the oriented matroid $_{-R}\mathcal{M}$. Then, we notice that T', the resulting sign-vector obtained from reorienting each element of R in T, is a sign vector with only $+$ entries, i.e., $T' = \{+\}^E$. Let Y be the resulting circuit of $_{-R}\mathcal{M}$ obtaining from reorienting a circuit X of $\mathcal{M}$. Hence, notice that $H(X, T) = H(Y, T') = |Y^+|$ and $S(X, T) = S(Y, T') = |Y^-|$, obtaining that $|Y^+| > k-1$ and $|Y^-| > k-1$ for every circuit Y of $_{-R}\mathcal{M}$. Hence, $_{-R}\mathcal{M}$ is $(k-1)$-neighborly.
"⇐". Let $R \subseteq E$ be such that $_{-R}\mathcal{M}$ is $k-1$-neighborly and $T \in \{+, -\}^E$ be such that $T^- = R$. By definition $|Y^+| > k-1$ and $|Y^-| > k-1$ for every circuit Y of $_{-R}\mathcal{M}$. Let T' be the tope of $_{-R}\mathcal{M}$ such that $T' \in \{+\}^E$, which exists because $_{-R}\mathcal{M}$ is acyclic such that T is obtained by reoriented each element of R in T'. For every circuit Y of $_{-R}\mathcal{M}$, let X be the resulting circuit of $\mathcal{M}$ obtaining from reorienting each element of R' in Y. Hence, notice that $H(Y, T') = |Y^+| = H(X, T)$ and $S(Y, T') = |Y^-| = S(X, T)$, obtaining that $H(X, T) \geq k$ and $S(X, T) \geq k$ for every circuit X of $\mathcal{M}$. Therefore T is a k-orthogonal tope of $\mathcal{M}$. □

We say that a tope T of $\mathcal{M}$ is *k-neighborly* if the oriented matroid $_{-T^-}\mathcal{M}$, obtained from reorienting such that T becomes all positive, is k-neighborly . By Proposition 3.2, k-neighborly topes are in correspondence with the $(k+1)$-orthogonal topes. Given an oriented matroid $\mathcal{M}$ with tope set $\mathcal{T}$ and $T \in \mathcal{T}$ denote $\mathcal{M}^{\perp_k} = \{T \in \mathcal{T} \mid T \text{ is } k\text{-orthogonal}\}$. Further denote $\text{ort}(T) = \max\{k \mid T \text{ is } k\text{-orthogonal}\}$ and $\mathcal{O}_k(\mathcal{M}) = \{T \in \mathcal{T} \mid \text{ort}(T) = k\}$, i.e., $\mathcal{O}_k(\mathcal{M}) = \mathcal{M}^{\perp_k} \setminus \mathcal{M}^{\perp_{k+1}}$, the set of k-orthogonal but not $k+1$-orthogonal topes in $\mathcal{M}$. By Proposition 3.2 we get:

Corollary 3.3. *For every $k = 1, \ldots, \lfloor \frac{r+1}{2} \rfloor$, $m(\mathcal{M}, k-1) = |\mathcal{M}^{\perp_k}|$. Moreover, $o(\mathcal{M}, k-1) = |\mathcal{O}_k(\mathcal{M})|$.*

Recall that tope graph $\mathcal{G}(\mathcal{M})$ of an oriented matroid $\mathcal{M}$ on n elements naturally embedded as an induced subgraphs into the hypercube $\{+, -\}^E \cong Q_n$. The edges of $\mathcal{G}(\mathcal{M})$ are partitioned into classes, where two edges are equivalent if their corresponding adjacent topes differ in exactly the same entry $e \in E$ (see Figure 1). In order to present a graph theoretical description of k-orthogonal topes for any graph G with vertex v and k a non-negative integer, define the *ball* of radius k and center v in G, denoted by $B_k^G(v)$, as the induced subgraph of G with set of vertices $V(B_k^G(v)) = \{u \in V(G) \mid d_G(u,v) \leq k\}$. Here $d_G(u,v)$ denotes the distance between u and v in the graph G.

Proposition 3.4. *Let $\mathcal{M}$ be a rank r oriented matroid on n elements, $k = 0, \ldots, \lfloor \frac{r-1}{2} \rfloor$, and $T \in \mathcal{T}$ a tope. Then T is k-neighborly if and only if $B_k^{\mathcal{G}(\mathcal{M})}(T) \cong B_k^{Q_n}(T)$.*

Proof. By Definition 1.1, T is k-neighborly if and only if reorienting any $F \subseteq E$ of size at most k in ${}_{-T}\text{-}\mathcal{M}$ is acyclic. Direct computation yields that this is equivalent to any reorientation of ${}_{-F}T$ being at least 1-orthogonal. Hence this changing any set of at most k coordinates in T yields another vertex of $\mathcal{G}(\mathcal{M})$. Since $\mathcal{G}(\mathcal{M})$ is an induced subgraph of Q_n this is equivalent to $B_k^{\mathcal{G}(\mathcal{M})}(T) \cong B_k^{Q_n}(T)$. $\square$

4. Beyond McMullen

The inequality $2r - 1 \leq \nu(r, 1)$ has been shown in [6]. In this section we give some lower bounds for $\nu(r,k)$ and $k \geq 2$.

In order to prove that $n \leq \nu(r,k)$, one has to prove that any uniform rank r oriented matroid $\mathcal{M}$ on n elements has a reorientation that is k-neighborly. In particular, if $n \leq r+2$ then $\mathcal{M}$ is in the same reorientation class as the alternating oriented matroid by Remark 2.3. So, we have the following observation.

Remark 4.1. *If $\lfloor \frac{r-1}{k} \rfloor \leq 2$, then $r + \lfloor \frac{r-1}{k} \rfloor \leq \nu(r,k)$.*

Theorem 4.2. *For every $k = 2, \ldots, \lfloor \frac{r-1}{2} \rfloor$, we have $r - 1 + \lfloor \frac{r-1}{k} \rfloor \leq \nu(r,k)$.*

Proof. Let $\mathcal{M} = (E, \mathcal{C}^*)$ be a uniform rank r oriented matroid on $n = r-1+\lfloor \frac{r-1}{k} \rfloor$ elements with set of cocircuits $\mathcal{C}^*$. First notice that $|\underline{C}| = \lfloor \frac{r-1}{k} \rfloor$ for every cocircuit $C \in \mathcal{C}^*$ and since $n \geq (k+1)\lfloor \dfrac{r-1}{k} \rfloor$, there exist $k+1$ cocircuits $C_1, \ldots, C_{k+1} \in \mathcal{C}^*$ mutually disjoint. Let R be the set of elements $x \in \bigcup_{i=1}^{k+1} \underline{C_i}$ such that $x \in \underline{C_i^-}$ for some $i \in \{1, \ldots, k+1\}$ and consider ${}_{-R}\mathcal{M}$, the oriented matroid resulting from reorienting the set R. Now, let $S \subseteq E$ be any k-set and observe that $S \cap \underline{C_i} = \emptyset$ for some $i \in \{1, \ldots, k+1\}$. Then, $S \subseteq E \setminus \underline{C_i}$, where $\underline{C_i}$ is the support of a positive cocircuit in ${}_{-R}\mathcal{M}$, obtaining that S is a face of ${}_{-R}\mathcal{M}$ and so, ${}_{-R}\mathcal{M}$ is a k-neighborly matroid polytope, concluding the proof. $\square$

The next result improves Theorem 4.2 in some cases.

Theorem 4.3. *For every $k = 2, \ldots, \lfloor \frac{r-1}{2} \rfloor$, we have $r + \lfloor \frac{r-1}{k} \rfloor \leq \nu(r,k)$ if $r-1 \equiv \beta \bmod k$, where $\beta = \lceil \frac{k-1}{2} \rceil, \ldots, k-1$,*

Proof. Let $\mathcal{M} = (E, \mathcal{C}^*)$ be a uniform rank r oriented matroid on $n = |E| = r + \lfloor \frac{r-1}{k} \rfloor$ elements with set of cocircuits $\mathcal{C}^*$ and notice that $|\underline{C}| = \lfloor \frac{r-1}{k} \rfloor + 1$ for every cocircuit $C \in \mathcal{C}^*$. As $r - 1 = \alpha k + \beta$ for some positive integer α, then $\alpha = \lfloor \frac{r-1}{k} \rfloor$ and so, $n = (k+1)\alpha + \beta + 1$.

Next, we will consider a partition of E into two sets, A and B, as follows. Consider any set $B = \{b_1, \ldots, b_{\beta+1}\}$ of cardinality $\beta + 1$ and let $A = \bigcup_{i=1}^{k+1} A_i = E \setminus B$, where $|A_i| = \alpha$ and $A_i \cap A_j = \emptyset$ for every distinct $i, j \in \{1, \ldots, k+1\}$. Notice that $|B| \geq \lceil \frac{k+1}{2} \rceil$ since by hypothesis $\beta \geq \lceil \frac{k-1}{2} \rceil$. Thus, for every $i = 1, \ldots, \lfloor \frac{k+1}{2} \rfloor$ we will consider the sets

$$D_i = A_{2i-1} \cup A_{2i} \cup b_i$$

of cardinality $2\alpha + 1$ and if k is even, we will also consider the set $D_{\frac{k+2}{2}} = A_{k+1} \cup b_{\frac{k+2}{2}}$, of cardinality $\alpha + 1$.

First observe that $D_i \cap D_j = \emptyset$ for every distinct $i, j \in \{1, \ldots, \lceil \frac{k+1}{2} \rceil\}$. Now, for each $i \in \{1, \ldots, \lfloor \frac{k+1}{2} \rfloor\}$ consider $\mathcal{M}/(E \setminus D_i)$, the contraction of $E \setminus D_i$ from $\mathcal{M}$, which we

will denote by $\mathcal{M}_{D_i}$. So, $\mathcal{M}_{D_i}$ is a uniform oriented matroid with ground set D_i and rank $r_i = r - |E \setminus D_i| = \alpha + 1$. Hence, the cocircuits of $\mathcal{M}_{D_i}$ have cardinality $|D_i| - (\alpha+1) + 1 = \alpha + 1$ and so, the cocircuits of $\mathcal{M}_{D_i}$ are also cocircuits of $\mathcal{M}$. On the other hand, $\mathcal{M}_{D_i}$ has order $2\alpha + 1 = 2r_i - 1$ and since $2r_i - 1 \leq \nu(r_i, 1)$ (see [6]), there exists a reorientation of $\mathcal{M}_{D_i}$ on $R_i \subseteq D_i$ such that ${}_{-R_i}\mathcal{M}_{D_i}$ is a 1-neighborly matroid polytope.
If k is even, denote by F the cocircuit of $\mathcal{M}$ with support $\underline{F} = D_{\frac{k+2}{2}}$. Now, consider the set

$$R = \begin{cases} \bigcup\limits_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} R_i & \textit{if } k \textit{ is odd}, \\ \bigcup\limits_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} R_i \cup \underline{F}^- & \textit{if } k \textit{ is even}, \end{cases}$$

and notice that $R_i \cap R_j = \emptyset$ for every distinct $i, j = 1, \ldots, \lfloor \frac{k+1}{2} \rfloor$. Moreover, if k is even then also $R_i \cap \underline{F}^- = \emptyset$ for every $i = 1, \ldots, \lfloor \frac{k+1}{2} \rfloor$. We will see that ${}_{-R}\mathcal{M}$ is a k-neighborly matroid polytope. Let $S \subset E$ be any k-set and first suppose that $|S \cap D_i| \leq 1$ for some $i \in \{1, \ldots, \lfloor \frac{k+1}{2} \rfloor\}$. As ${}_{-R_i}\mathcal{M}_{D_i}$ is a 1-neighborly matroid polytope, then there exists a positive cocircuit C of ${}_{-R_i}\mathcal{M}_{D_i}$ such that $\underline{C} \cap S = \emptyset$. Then, S is a face of ${}_{-R}\mathcal{M}$ since C is also a positive cocircuit of ${}_{-R}\mathcal{M}$, concluding that ${}_{-R}\mathcal{M}$ is a k-neighborly matroid polytope.
Now suppose that $|S \cap D_i| \geq 2$ for every $i = 1, \ldots, \lfloor \frac{k+1}{2} \rfloor$ and let denote $D = \cup_{i=1}^{\lfloor \frac{k+1}{2} \rfloor} D_i$. Then, $|S \cap D| \geq 2\lfloor \frac{k+1}{2} \rfloor$ concluding that k is even since $|S| = k$. Moreover, as $|S \cap D| = k$, then $|S \cap \underline{F}| = 0$, concluding that S is a face of ${}_{-R}\mathcal{M}$ since $\underline{F}$ is the support of a positive cocircuit of ${}_{-R}\mathcal{M}$. Therefore, the theorem holds. □

5. Beyond Roudneff

5.1. **The o-vector of $\mathcal{C}_r(n)$ and its tope graph.** The main result of this subsection is to obtain the o-vector of $\mathcal{C}_r(n)$, for $n \geq 2(r-k) + 1$ (Theorem 5.8).

Let n be a positive integer. Given a sign-vector $T \in \{+, -\}^n$, notice that T partitions $\{1, \ldots, n\}$ into signed *blocks* $B_1, \ldots, B_m$, $1 \leq m \leq n$, where B_j denotes the j-th maximal set of consecutive elements having the same sign. Sign-vectors with m blocks can be easily counted as there are $m - 1$ possibilities to choose the change of sign in $n - 1$ places. So, the number of sign-vectors with m blocks is $2\binom{n-1}{m-1}$ since for every sign-vector T there exists also $-T$.

Remark 5.1. *The number of sign-vectors of n elements with m blocks is $2\binom{n-1}{m-1}$.*

Throughout this subsection, we will denote by $\mathcal{T}$ and $\mathcal{C}$ the set of topes and circuits of $\mathcal{C}_r(n)$, respectively. Given a tope $T \in \mathcal{T}$ with m blocks, we denote $O(m) = \lceil \frac{r+1-m}{2} \rceil$. Next, we prove that $O(m)$ is the minimum orthogonality that T can have.

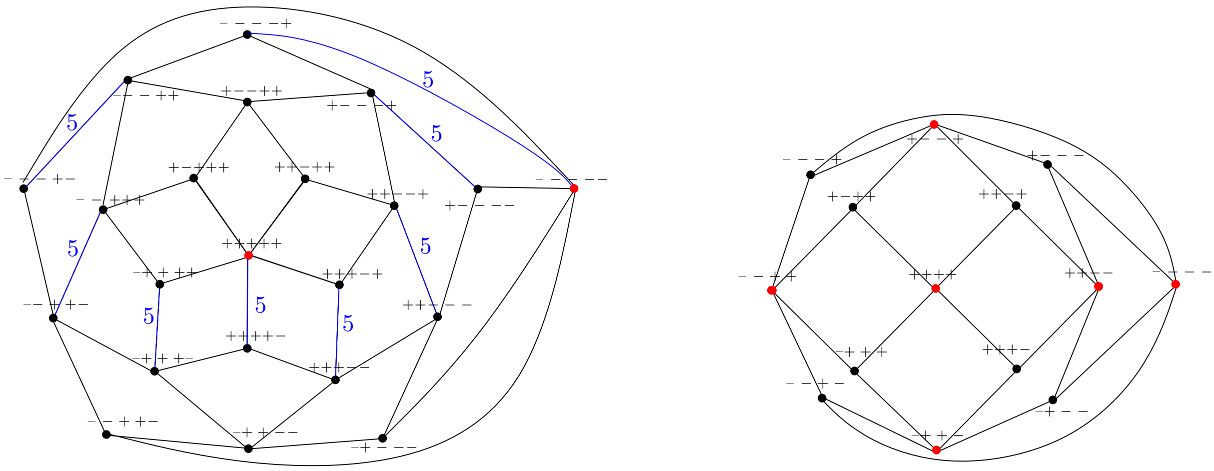
Lemma 5.2. *Let $T \in \mathcal{T}$ be with m blocks, then* $\operatorname{ort}(T) \geq O(m)$.

Proof. Let $B_1, B_2, \ldots, B_m$ be the blocks of T and let $X \in \mathcal{C}$. As X is alternating, we have that $\min\{|S(X,T) \cap B_i|, |H(X,T) \cap B_i|\} \geq \lceil \frac{|X \cap B_i| - 1}{2} \rceil$ for every $i = 1, 2, \ldots, m$. Thus, $X \perp T = \min\{|H(X,T)|, |S(X,T)|\} \geq \sum\limits_{i=1}^{m} \lceil \frac{|X \cap B_i| - 1}{2} \rceil \geq \lceil \frac{r+1-m}{2} \rceil = O(m)$ for every $X \in \mathcal{C}$, concluding that T is $O(m)$-orthogonal. Therefore, as $\operatorname{ort}(T) = \max\{k \mid T \text{ is } k-\text{orthogonal}\}$, it follows that $\operatorname{ort}(T) \geq O(m)$, concluding the proof. □

Lemma 5.3. *Let $n \geq r + 1 \geq 2$, then $T \in \mathcal{T}$ if and only if T has at most r blocks.*

Proof. Recall that $T \in \mathcal{T}$ if and only if T is 1-orthogonal. On the other hand, as every circuit $X \in \mathcal{C}$ is alternating, it follows that T is 1-orthogonal if and only if T has at most r blocks. □

For $n \geq r+1 \geq 3$ we define the *graph of blocks*, denoted by $B(r,n)$, as the graph whose vertices are the sign-vectors $T \in \{+,-\}^n$ with at most r blocks and two sign-vectors T, T' are adjacent if $|S(T,T')| = 1$ (see Figure 1). Thus, the adjacency of the vertices in the graph $B(r,n)$ corresponds to the adjacency of topes in $\mathcal{G}(\mathcal{C}_r(n))$ and so, it follows from Lemma 5.3 that $B(r,n)$ is just the tope graph of $\mathcal{C}_r(n)$.



$B(3,5)$ $B(3,4)$

FIGURE 1. The graphs $B(3,5)$ and $B(3,4)$ corresponding to the tope graphs of $\mathcal{C}_3(5)$ and $\mathcal{C}_3(4)$, respectively. Red vertices correspond to 2-orthogonal topes (i.e., 1-neighborly topes which in turn are complete cells) and adjacent topes between a blue edge, differ exactly in the fifth entry.

We say that a block is an *even block* (*odd block*) if it has an even (odd) number of elements. From now, we will denote by $B_1, B_2, \dots, B_m$ the blocks of a tope $T \in \mathcal{T}$. Denote by $\mathcal{B}_e$ and $\mathcal{B}_o$ the set of even and odd blocks of T, respectively. We will also denote $\mathbf{B}_e = |\mathcal{B}_e|$ and $\mathbf{B}_o = |\mathcal{B}_o|$. Given a tope T and a circuit X, let denote $S^X = \{i \mid \min\{|S(X,T) \cap B_i|, |H(X,T) \cap B_i|\} = |S(X,T) \cap B_i|\}$ and $H^X = \{i \mid \min\{|S(X,T) \cap B_i|, |H(X,T) \cap B_i|\} = |H(X,T) \cap B_i|\}$, see Figure 2.

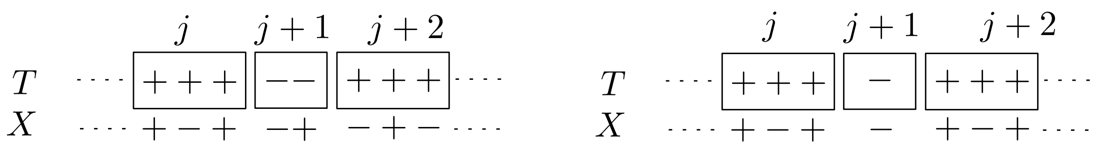


FIGURE 2. In the first example $j \in S^X \setminus H^X$, $j+1 \in S^X \cap H^X$ and $j+2 \in H^X \setminus S^X$. In the second example, $j, j+1, j+2 \in S^X \setminus H^X$.

The following observation can be deduced by a simple parity argument and will be very useful for the next lemmas.

Remark 5.4. *Let $T \in \mathcal{T}$ and $X \in \mathcal{C}$. Let $j < j'$ and suppose that $|\underline{X} \cap B_j|$ and $|\underline{X} \cap B_{j'}|$ are odd, and that $|\underline{X} \cap B_i|$ is even for every $i \in \{j+1, \dots, j'-1\}$. Then,*

$$|\{j+1, \dots, j'-1\}| = j' - j - 1 \text{ is even if and only if } j, j' \in S^X \text{ or } j, j' \in H^X.$$

The above remark holds since the blocks and the circuits are alternating. In the first example of Figure 2 we observe that if $j' = j+2$, then $j' - j - 1 = 1$ is odd and so, $j \in S^X \setminus H^X$ and $j' \in H^X \setminus S^X$ by Remark 5.4. In the second example, we notice that $j' - j - 1 = 0$ is even if $j' = j+1$ and so, $j, j' \in S^X$. Moreover, applying Remark 5.4 now to the blocks $j+1$ and $j+2$ of the second example, it follows that $j+1, j+2 \in S^X$.

The following lemmas provide a connection between blocks and orthogonality that will allow us to prove Theorem 5.8.

Lemma 5.5. *Let $T \in \mathcal{T}$ be with m blocks. If $r+1 \leq n - \mathbf{B}_e$, then* $\operatorname{ort}(T) = O(m)$.

Proof. We will find a circuit $X \in \mathcal{C}$ such that $X \perp T = O(m)$, where $O(m) = \lceil \frac{r+1-m}{2} \rceil$. First suppose that $r+1-m$ is even. Then, there exists $X \in \mathcal{C}$ such that $|\underline{X} \cap B_i|$ is odd for every $1 \leq i \leq m$, since $|\underline{X}| > m$ (Lemma 5.3) and $|\underline{X}| \leq n - \mathbf{B}_e$. Thus, by Remark 5.4 and by possibly exchanging X with $-X$, we can assume that $i \in S^X$ for every $1 \leq i \leq m$ and hence,

$$X \perp T = \min\{|H(X,T)|, |S(X,T)|\} = \sum_{i=1}^{m} |S(X,T) \cap B_i|$$

$$= \sum_{i=1}^{m} \frac{|\underline{X} \cap B_i| - 1}{2} = \frac{(r+1-m)}{2} = O(m).$$

Now suppose that $r+1-m$ is odd and consider $X \in \mathcal{C}$ such that $|\underline{X} \cap B_i|$ is odd for every $i \leq m-1$ and $|(\bigcup_{i=1}^{m-1} B_i) \cap \underline{X}|$ is maximum possible. If there are no more elements in $\underline{X}$ to place, i.e., $|\underline{X} \cap B_i|$ is odd for every $1 \leq i \leq m-1$ and $\underline{X} \cap B_m = \emptyset$, we obtain by Remark 5.4 and by possibly exchanging X with $-X$, that $i \in S^X$ for every $1 \leq i \leq m-1$. Therefore,

$$X \perp T = \sum_{i=1}^{m-1} |S(X,T) \cap B_i| = \sum_{i=1}^{m-1} \frac{|\underline{X} \cap B_i| - 1}{2}$$

$$= \frac{r+1-(m-1)}{2} = \frac{r+2-m}{2} = O(m),$$

as desired. If there are still elements in $\underline{X}$ to be placed, by the maximality of $|(\bigcup_{i=1}^{m-1} B_i) \cap \underline{X}|$ and since $|\underline{X}| \leq n - \mathbf{B}_e$, we may place such elements in B_m (see Figure 3). Notice that $|\underline{X} \cap B_m|$ is even since $r+1-m$ is odd, obtaining that $m \in S^X \cap H^X$. Hence, $i \in S^X$ for every $1 \leq i \leq m$ and then

$$X \perp T = \sum_{i=1}^{m} |S(X,T) \cap B_i| = \sum_{i=1}^{m-1} \frac{|\underline{X} \cap B_i| - 1}{2} + \frac{|\underline{X} \cap B_m|}{2}$$

$$= \frac{r+1-(m-1)}{2} = \frac{r+2-m}{2} = O(m).$$

Therefore, there exists a circuit X such that $X \perp T = O(m)$ and so, $\operatorname{ort}(T) \leq O(m)$. Thus, $\operatorname{ort}(T) = O(m)$ by Lemma 5.2, concluding the proof. □

The following example, for $n = 13, r+1 = 9, m = 4$ and $\mathbf{B}_e = 1$, illustrates the case of the above lemma when $r+1-m$ is odd. We choose a circuit X such that $|\underline{X} \cap B_i|$ is odd for every $i \leq m-1 = 3$ and maximum possible. In this case, we notice that there are still elements in $\underline{X}$ to be placed, so, we may placed such elements in $B_m = B_4$. Finally, we observe that $X \perp T = \min\{|H(X,T)|, |S(X,T)|\} = \min\{6, 3\} = 3 = O(m)$, as required.

T	$+++$	$--$	$+++$	$-----$
X	$+-+$	-0	$+-+$	$-+000$

FIGURE 3. An example of the case $r+1-m$ odd in Lemma 5.5.

Given a tope $T \in \mathcal{T}$ and a circuit $X \in \mathcal{C}$ of $\mathcal{C}_r(n)$, let us denote the sets

$$S^X_{<j} = \{i \mid i < j,\ B_i \in \mathcal{B}_o \text{ and } \min\{|S(X,T) \cap B_i|, |H(X,T) \cap B_i|\} = |S(X,T) \cap B_i|\}$$

and

$$H^X_{<j} = \{i \mid i < j,\ B_i \in \mathcal{B}_o \text{ and } \min\{|S(X,T) \cap B_i|, |H(X,T) \cap B_i|\} = |H(X,T) \cap B_i|\},$$

see Figures 4 and 5. Notice that these sets contain integers i such that $B_i \in \mathcal{B}_o$ is an odd block. The following lemma provides an upper bound for $\mathrm{ort}(T)$ in terms of its blocks.

Lemma 5.6. *Let $T \in \mathcal{T}$ be with m blocks and $n \geq r+2$. If $r+1 > n - \mathbf{B}_e$, then*

$$\mathrm{ort}(T) \leq \frac{n - \mathbf{B}_e - m}{2} + r + 1 - (n - \mathbf{B}_e) + \lfloor \frac{\mathbf{B}_o}{2} \rfloor.$$

Proof. Let $\alpha = r + 1 - (n - \mathbf{B}_e)$. In order to prove the lemma, it is enough to find $X \in \mathcal{C}$ such that $X \perp T \leq \frac{n-\mathbf{B}_e-m}{2} + \alpha + \lfloor \frac{\mathbf{B}_o}{2} \rfloor$. Let us denote by $B^e_1, \ldots, B^e_\alpha$, the first α blocks of $\mathcal{B}_e$ (from left to right) and notice that $\alpha > 0$ since by hypothesis $r+1 > n - \mathbf{B}_e$. Then, $B^e_\alpha = B_j$ for some $j \in \{1, \ldots, m\}$. Consider $X \in \mathcal{C}$ as follows:

- $|\underline{X} \cap B_i| = |B_i|$ for every $B_i \in \mathcal{B}_o$,
- $|\underline{X} \cap B^e_i| = |B^e_i|$ for every $i = 1, \ldots, \alpha$ and
- $|\underline{X} \cap B_i| = |B_i| - 1$ otherwise.

Such a circuit exists, since $|\underline{X}| = n - \mathbf{B}_e + \alpha = r + 1$. Thus, as $|\underline{X} \cap B_i|$ is odd for every $i > j$, by Remark 5.4 and by possibly exchanging X with $-X$, we can assume that $i \in S^X$ for every $i > j$. Hence,

$$\begin{aligned} (1) \qquad \sum_{i>j} |S(X,T) \cap B_i| &= \sum_{i>j, B_i \in \mathcal{B}_e} |S(X,T) \cap B_i| + \sum_{i>j, B_i \in \mathcal{B}_o} |S(X,T) \cap B_i| \\ &= \sum_{i>j, B_i \in \mathcal{B}_e} \frac{|B_i| - 2}{2} + \sum_{i>j, B_i \in \mathcal{B}_o} \frac{|B_i| - 1}{2}. \end{aligned}$$

Consider the following cases.

Case 1. $|H^X_{<j}| \leq \lfloor \frac{\mathbf{B}_o}{2} \rfloor$ (see Figure 4).

Then,

$$\sum_{i\leq j}|S(X,T)\cap B_i| = \sum_{i\in S^X_{<j}\cup H^X_{<j}}|S(X,T)\cap B_i| + \sum_{i\leq j, B_i\in\mathcal{B}_e}|S(X,T)\cap B_i| \tag{2}$$

$$= \sum_{i\in S^X_{<j}}\frac{|B_i|-1}{2} + \sum_{i\in H^X_{<j}}\frac{|B_i|+1}{2} + \sum_{i\leq j, B_i\in\mathcal{B}_e}\frac{|B_i|}{2}$$

$$= \sum_{i\in S^X_{<j}}\frac{|B_i|-1}{2} + \sum_{i\in H^X_{<j}}\frac{|B_i|-1}{2} + |H^X_{<j}| + \sum_{i\leq j, B_i\in\mathcal{B}_e}\frac{|B_i|-2}{2} + \alpha$$

$$\leq \sum_{i\in S^X_{<j}\cup H^X_{<j}}\frac{|B_i|-1}{2} + \sum_{i\leq j, B_i\in\mathcal{B}_e}\frac{|B_i|-2}{2} + \alpha + \lfloor\frac{\mathbf{B}_o}{2}\rfloor.$$

Thus, by Equations (1) and (2) we obtain that

$$X\perp T = \min\{|H(X,T)|,|S(X,T)|\} \leq \sum_{i\leq j}|S(X,T)\cap B_i| + \sum_{i>j}|S(X,T)\cap B_i|$$

$$\leq \sum_{i\in S^X_{<j}\cup H^X_{<j}}\frac{|B_i|-1}{2} + \sum_{i\leq j, B_i\in\mathcal{B}_e}\frac{|B_i|-2}{2} + \alpha + \lfloor\frac{\mathbf{B}_o}{2}\rfloor$$

$$+ \sum_{i>j, B_i\in\mathcal{B}_e}\frac{|B_i|-2}{2} + \sum_{i>j, B_i\in\mathcal{B}_o}\frac{|B_i|-1}{2} = \frac{n-\mathbf{B}_e-m}{2} + \alpha + \lfloor\frac{\mathbf{B}_o}{2}\rfloor$$

concluding that $\mathrm{ort}(T) \leq \frac{n-\mathbf{B}_e-m}{2} + \alpha + \lfloor\frac{\mathbf{B}_o}{2}\rfloor$ and so, the lemma holds in this case.

Case 2. $|H^X_{<j}| \geq \lceil\frac{\mathbf{B}_o}{2}\rceil$ (see Figure 5).

We will slightly modify X in order to find another circuit $X'\in\mathcal{C}$ such that $X'\perp T \leq \frac{n-\mathbf{B}_e-m}{2} + \alpha + \lfloor\frac{\mathbf{B}_o}{2}\rfloor$. First, observe that $\lceil\frac{\mathbf{B}_o}{2}\rceil \leq |S^X_{<j}| + |H^X_{<j}| \leq \mathbf{B}_o$. So, $|S^X_{<j}| + |H^X_{<j}| = \mathbf{B}_o - p$ for some $p\in\{0,\ldots,\lfloor\frac{\mathbf{B}_o}{2}\rfloor\}$. Then, $|S^X_{<j}| = \mathbf{B}_o - p - |H^X_{<j}| \leq \mathbf{B}_o - p - \lceil\frac{\mathbf{B}_o}{2}\rceil$, concluding that

$$|S^X_{<j}| \leq \lfloor\frac{\mathbf{B}_o}{2}\rfloor - p \tag{3}$$

As $|S^X_{<j}| + |H^X_{<j}| + |\{i \mid i>j, B_i\in\mathcal{B}_o\}| = \mathbf{B}_o$, we also conclude that

$$|\{i \mid i>j, B_i\in\mathcal{B}_o\}| = p \tag{4}$$

Let $B_{j'}$ be the last block of $\mathcal{B}_e$ (from left to right). As $n\geq r+2$, we have that $\alpha<\mathbf{B}_e$ and then $j'>j$. Choose $e\in B_j$ and $e'\in B_{j'}\setminus\underline{X}$ (such e' exists since $|\underline{X}\cap B_{j'}| = |B_{j'}|-1$) and consider the circuit $\underline{X'} = \underline{X} - e\cup e'$. Hence, notice that

$$S^X_{<j} = S^{X'}_{<j} \tag{5}$$

Now, we claim that $j\in H^{X'}$. Let $i' = \max\{i \mid i<j \text{ and } B_i\in\mathcal{B}_o\}$ and first suppose that $i'\in S^X$. As we have assumed that $j+1\in S^X$, applying Remark 5.4 to the blocks $B_{i'}$ and B_{j+1}, respect to the circuit X, we obtain that $(j+1)-i'-1$ is even. Hence, as $i'\in S^{X'}$ by Equation (5) and $j-i'-1$ is odd, applying Remark 5.4 now to the blocks $B_{i'}$ and B_j, respect to the circuit X', we obtain that $j\in H^{X'}$, as desired. The case when $i'\in H^X$ can

be treated analogously, so the claim holds. Thus, by the above claim and by Remark 5.4 it can be deduced that

$$i \in H^{X'} \text{ if } j \leq i < j' \text{ and } i \in S^{X'} \text{ if } i > j' \tag{6}$$

Therefore, we conclude the following:

(a) $\sum\limits_{i \in S^{X'}_{<j}} |H(X',T) \cap B_i| = \sum\limits_{i \in S^{X'}_{<j}} \frac{|B_i|+1}{2} = \sum\limits_{i \in S^{X'}_{<j}} \frac{|B_i|-1}{2} + |S^{X'}_{<j}| \leq \sum\limits_{i \in S^{X'}_{<j}} \frac{|B_i|-1}{2} + \lfloor \frac{\mathbf{B}_o}{2} \rfloor - p,$

by Equations (3) and (5).

(b) $\sum\limits_{i \in H^{X'}_{<j}} |H(X',T) \cap B_i| = \sum\limits_{i \in H^{X'}_{<j}} \frac{|B_i|-1}{2}.$

(c) $|H(X',T) \cap B_{j'}| + \sum\limits_{i<j \text{ and } B_i \in \mathcal{B}_e} |H(X',T) \cap B_i| = \frac{|B_{j'}|}{2} + \sum\limits_{i<j \text{ and } B_i \in \mathcal{B}_e} \frac{|B_i|}{2}$

$$= \sum_{i<j, i=j',\, B_i \in \mathcal{B}_e} \frac{|B_i|-2}{2} + \alpha,$$

since such blocks are just $B^e_1, \ldots, B^e_{\alpha-1}$ and $B_{j'}$.

(d) $\sum\limits_{i \geq j,\, i \neq j'} |H(X',T) \cap B_i| = \sum\limits_{j \leq i < j',\, B_i \in \mathcal{B}_e} \frac{|B_i|-2}{2} + \sum\limits_{j < i < j',\, B_i \in \mathcal{B}_o} \frac{|B_i|-1}{2} + \sum\limits_{i > j'} \frac{|B_i|+1}{2}$

$$\leq \sum_{j \leq i < j',\, B_i \in \mathcal{B}_e} \frac{|B_i|-2}{2} + \sum_{j < i < j',\, B_i \in \mathcal{B}_o} \frac{|B_i|-1}{2} + \sum_{i > j'} \frac{|B_i|-1}{2} + p,$$

by Equations (4) and (6).

Finally, by (a), (b), (c) and (d), we obtain that,

$$\begin{aligned} X' \perp T &= \sum_{i<j} |H(X',T) \cap B_i| + \sum_{i \geq j} |H(X',T) \cap B_i| \\ &\leq \sum_{i \in S^{X'}_{<j} \cup H^{X'}_{<j}} \frac{|B_i|-1}{2} + \lfloor \frac{\mathbf{B}_o}{2} \rfloor - p + \alpha + \sum_{i<j, i=j', B_i \in \mathcal{B}_e} \frac{|B_i|-2}{2} \\ &+ \sum_{j \leq i < j', B_i \in \mathcal{B}_e} \frac{|B_i|-2}{2} + \sum_{j < i < j', B_i \in \mathcal{B}_o} \frac{|B_i|-1}{2} + \sum_{i > j'} \frac{|B_i|-1}{2} + p \\ &= \frac{n - \mathbf{B}_e - m}{2} + \alpha + \lfloor \frac{\mathbf{B}_o}{2} \rfloor, \end{aligned}$$

concluding that $\text{ort}(T) \leq \frac{n-\mathbf{B}_e-m}{2} + \alpha + \lfloor \frac{\mathbf{B}_o}{2} \rfloor$ and so, the lemma holds. □

The example of Figure 4, for $n = 13, r+1 = 10, m = 9, \mathbf{B}_e = 4, \mathbf{B}_o = 5, \alpha = 1$ and $j = 3$, illustrate Case 1 of the above lemma. We consider the circuit X as in the proof and we notice that $X \perp T = 3 = \frac{n-\mathbf{B}_e-m}{2} + \alpha + \lfloor \frac{\mathbf{B}_o}{2} \rfloor$, as required.
The example of Figure 5, for $n = 13, r+1 = 11, m = 9, \mathbf{B}_e = 4, \mathbf{B}_o = 5, \alpha = 2$ and $j = 6$, illustrates the Case 2 of Lemma 5.6. We consider the circuit X as in the proof and we notice that $X \perp T = 5 > 4 = \frac{n-\mathbf{B}_e-m}{2} + \alpha + \lfloor \frac{\mathbf{B}_o}{2} \rfloor$. Then, we consider the circuit $\underline{X'} = \underline{X} - e \cup e'$ and we observe that $X' \perp T = 3 < 4 = \frac{n-\mathbf{B}_e-m}{2} + \alpha + \lfloor \frac{\mathbf{B}_o}{2} \rfloor$, as required.

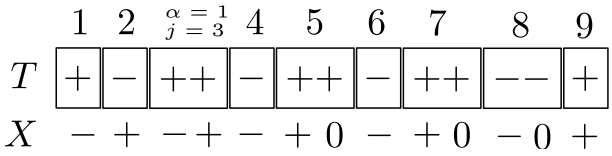


FIGURE 4. An example of the case $|H^X_{<j}| \leq \lfloor\frac{\mathbf{B}_o}{2}\rfloor$ of Lemma 5.6. Notice that $2 = |H^X_{<j}| = |H^X_{<3}| = \lfloor\frac{\mathbf{B}_o}{2}\rfloor$, $H^X_{<j} = \{1, 2\}$, $S^X = \{3, 4, 5, 6, 7, 8, 9\}$ and $H^X = \{1, 2, 3\}$.

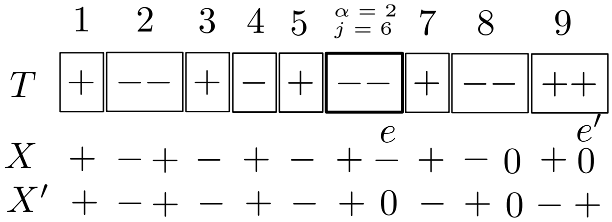


FIGURE 5. An example of the case $|H^X_{<j}| > \lfloor\frac{\mathbf{B}_o}{2}\rfloor$ of Lemma 5.6. Notice that $|H^X_{<j}| = |H^X_{<6}| = |\{3, 4, 5\}| = 3$, $\lfloor\frac{\mathbf{B}_o}{2}\rfloor = 2$, $S^X = \{1, 2, 6, 7, 8, 9\}$, $H^X = \{2, 3, 4, 5, 6\}$, $S^{X'} = \{1, 2, 9\}$, $H^{X'} = \{2, 3, 4, 5, 6, 7, 8, 9\}$ and $S^X_{<j} = S^{X'}_{<j} = \{1\}$.

Proposition 5.7. *Let $n \geq 2(r-k)+3 \geq r+2$ and let $T \in \mathcal{T}$ be with m blocks. Then the following holds.*

(i) *If $O(m) \geq k$, then $\text{ort}(T) = O(m)$;*
(ii) *If $O(m) \leq k-1$ then $\text{ort}(T) \leq k-1$.*

Proof. Recall that $O(m) = \lceil\frac{r+1-m}{2}\rceil$.
(i) As $k \leq O(m) \leq \frac{r+2-m}{2}$, then $m \leq r+2-2k$. Hence, as $\mathbf{B}_e \leq m$ and $2(r-k)+3 \leq n$ we obtain that $r+1+\mathbf{B}_e \leq r+1+m \leq 2(r-k)+3 \leq n$, concluding that $r+1 \leq n-\mathbf{B}_e$. Thus, $\text{ort}(T) = O(m)$ by Lemma 5.5 and the result holds in this case.
(ii) Let $n = 2(r-k)+3+l$, where $l \geq 0$ is an integer. As each even block have at least two elements, we have that $\mathbf{B}_e \leq \lfloor\frac{n}{2}\rfloor$. If n is odd, then l is even and $\lfloor\frac{n}{2}\rfloor = r-k+1+\frac{l}{2}$. Similarly, $\lfloor\frac{n}{2}\rfloor = r-k+1+\frac{l+1}{2}$ if n is even. Hence, $\mathbf{B}_e \leq \lfloor\frac{n}{2}\rfloor = r-k+1+\lceil\frac{l}{2}\rceil$ and so, $\mathbf{B}_e = r-k+1+\lceil\frac{l}{2}\rceil - j$ for some integer $j \geq 0$. On the other hand, let $\alpha = r+1+\mathbf{B}_e-n$ and suppose first that $\alpha \leq 0$. Then $r+1 \leq n-\mathbf{B}_e$ and by Lemma 5.5, $\text{ort}(T) = O(m) \leq k-1$, as desired. Now suppose that $\alpha > 0$, then $\text{ort}(T) \leq \frac{(n-\mathbf{B}_e)-m}{2} + \alpha + \lfloor\frac{\mathbf{B}_o}{2}\rfloor$ by Lemma 5.6. As $\alpha = r+1+(r-k+1+\lceil\frac{l}{2}\rceil-j)-(2(r-k)+3+l) = k-1-j-\lfloor\frac{l}{2}\rfloor$ and $m = \mathbf{B}_e+\mathbf{B}_o$, we obtain that $\text{ort}(T) \leq \frac{(2(r-k)+3+l)-(r-k+1+\lceil\frac{l}{2}\rceil)-(r-k+1+\lceil\frac{l}{2}\rceil+\mathbf{B}_o)}{2} + (k-1-j-\lfloor\frac{l}{2}\rfloor) + \lfloor\frac{\mathbf{B}_o}{2}\rfloor$ and thus, $\text{ort}(T) \leq \frac{1+\lfloor\frac{l}{2}\rfloor-\lceil\frac{l}{2}\rceil-\mathbf{B}_o}{2} + \frac{2k-2-2j-2\lfloor\frac{l}{2}\rfloor}{2} + \lfloor\frac{\mathbf{B}_o}{2}\rfloor$, concluding that

$$\text{ort}(T) \leq \frac{2k-1-l-\mathbf{B}_o}{2} + \lfloor\frac{\mathbf{B}_o}{2}\rfloor.$$

If $\mathbf{B}_o$ is odd, then $\text{ort}(T) \leq \frac{2k-1-l-\mathbf{B}_o}{2} + \frac{\mathbf{B}_o-1}{2} \leq k-1$. If $\mathbf{B}_o$ is even, then $l \geq 1$ is odd. Therefore, $\text{ort}(T) \leq \frac{2k-1-l-\mathbf{B}_o}{2} + \frac{\mathbf{B}_o}{2} \leq \frac{2k-2-\mathbf{B}_o}{2} + \frac{\mathbf{B}_o}{2} = k-1$, concluding the proof. □

Recall that $\mathcal{O}_k(\mathcal{M}) = \{T \in \mathcal{T} \mid \text{ort}(T) = k\}$. For short, let us denote $\mathcal{O}_k(\mathcal{C}_r(n))$ by $\mathcal{O}_k$. Let denote by $\mathcal{T}_m$ the set of $T \in \mathcal{T}$ with exactly m blocks. Next, we will obtain the o-vector of $\mathcal{C}_r(n)$, for n large enough.

Theorem 5.8. *If $n \geq 2(r-k)+1 \geq r+2$, then*

$$o(\mathcal{C}_r(n), i) = 2\binom{n}{r-1-2i}$$

for every $i = k, \dots, \lfloor \frac{r-1}{2} \rfloor$.

Proof. For every $k = 0, \dots, \lfloor \frac{r-1}{2} \rfloor$ let $k' = k+1$ and consider $m \in \{1, \dots, r\}$ such that $\dfrac{r+2-m}{2} \in \mathbb{N}$ and $O(m) \geq k'$, where $O(m) = \lceil \frac{r+1-m}{2} \rceil$.

We first claim that $\mathcal{T}_m \cup \mathcal{T}_{m-1} = \mathcal{O}_{O(m)}$. Let $T \in \mathcal{T}_m \cup \mathcal{T}_{m-1}$. By Proposition 5.7 (i), $\mathrm{ort}(T) = O(m)$ since $O(m) \geq k'$ and $n \geq 2(r-k')+3$, concluding that $T \in \mathcal{O}_{O(m)}$. Now, let T' be a tope of $\mathcal{C}_r(n)$ with m' blocks and such that $T' \notin \mathcal{T}_m \cup \mathcal{T}_{m-1}$. We will prove that $T' \notin \mathcal{O}_{O(m)}$. As $O(m) = O(m-1)$ by the election of m and $m' \notin \{m, m-1\}$, we obtain that $O(m') \neq O(m)$. If $O(m') \geq k'$, then $\mathrm{ort}(T') = O(m')$ by Proposition 5.7 (i), concluding that $T \notin \mathcal{O}_{O(m)}$ since $\mathrm{ort}(T') = O(m') \neq O(m)$. If $O(m') \leq k-1$, then $\mathrm{ort}(T') \leq k'-1$ by Proposition 5.7 (ii). As $O(m) \geq k'$, we obtain that $\mathrm{ort}(T') \neq O(m)$ and so, $T' \notin \mathcal{O}_{O(m)}$. Hence, $\mathcal{T}_m \cup \mathcal{T}_{m-1} = \mathcal{O}_{O(m)}$ and the claim holds.

As $|\mathcal{T}_m| = 2\binom{n-1}{m-1}$ and $|\mathcal{T}_{m-1}| = 2\binom{n-1}{m-2}$ by Remark 5.1, we obtain by the above claim that $|\mathcal{O}_{O(m)}| = 2(\binom{n-1}{m-1} + \binom{n-1}{m-2}) = 2\binom{n}{m-1}$. On the other hand, as $O(m) = \frac{r+2-m}{2}$ by the election of m, then $m = r+2-2O(m)$ and so,

$$|\mathcal{O}_{O(m)}| = 2\binom{n}{r+1-2O(m)}$$

for every $O(m) \geq k'$. Finally, as $o(\mathcal{C}_r(n), k) = |\mathcal{O}_{k+1}|$ for every $k = 0, \dots, \lfloor \frac{r-1}{2} \rfloor$ (Corollary 3.3), the theorem holds. □

The above theorem is best possible. For instance, Proposition 5.9 shows that $o(\mathcal{C}_r(n), k) \neq 2$ if $n = r+1$ and $k = \frac{r-1}{2}$ (i.e., for $n < 2(r-k)+1$), while the formula given in Theorem 5.8 for these values give us $2\binom{n}{r-1-2k} = 2$. Further, Example 6.2 shows several values of r, k and $n < 2(r-k)+1$, where $o(\mathcal{C}_r(n), k) \neq 2\binom{n}{r-1-2k}$.

Below, we obtain the o-vector of $\mathcal{C}_r(r+1)$. Given two sign-vectors $X, Y \in \{+,-\}^n$, we denote by $X \cdot Y$ the vector whose i-th entry is $-$ if $i \in S(X,T)$ and $+$ if $i \in H(X,T)$.

Proposition 5.9. *Let $r \geq 3$ and $0 \leq k \leq \lfloor \frac{r-1}{2} \rfloor$, then*

$$o(\mathcal{C}_r(r+1), k) = \begin{cases} \binom{r+1}{k+1} & \text{if } k = \frac{r-1}{2}; \\ 2\binom{r+1}{k+1} & \text{otherwise.} \end{cases}$$

Proof. Let X be the unique circuit of $\mathcal{C}_r(r+1)$ that starts with the sign $+$. Consider the set $\mathcal{Y} = \{Y \in \{+,-\}^{r+1}\} \setminus \{Z, -Z\}$, where $Z \in \{+\}^{r+1}$. Now, let define the function $f : \mathcal{T} \to \mathcal{Y}$ as $f(T) = X \cdot T$. We will see that f is bijective. Consider any tope $T \in \mathcal{T}$ and notice that $X \cdot T \notin \{Z, -Z\}$ since by Lemma 5.3, T has at most r blocks. Thus, f is injective since for every $T, T' \in \mathcal{T}$, clearly $X \cdot T \neq X \cdot T'$ and $X \cdot T, X \cdot T' \in \mathcal{Y}$. Now, let $T = (T_1, \dots, T_{r+1})$ be such that

$$T_i = \begin{cases} + & \text{if } i \text{ is odd and } Y_i = +; \\ - & \text{if } i \text{ is odd and } Y_i = -; \\ + & \text{if } i \text{ is even and } Y_i = -; \\ - & \text{if } i \text{ is even and } Y_i = +, \end{cases}$$

for every $i = 1, \ldots, r+1$. Then, we notice that $X \cdot T = Y$ since X is alternating. Moreover, T has at most r blocks since $Y \notin \{Z, -Z\}$. Hence, $T \in \mathcal{T}$ by Lemma 5.3 and so, f is bijective.
Therefore, as $\text{ort}(T) = X \perp T = \min\{(X \cdot T)^+, (X \cdot T)^-\}$, computing $|\mathcal{O}_{k+1}(\mathcal{C}_r(r+1))|$ is equivalent to counting the number of sign-vectors $Y \in \mathcal{Y}$ with $|Y^+| = k+1$ and the number of sign-vectors $Y \in \mathcal{Y}$ with $|Y^-| = k+1$, for every $k = 0, \ldots, \lfloor \frac{r-1}{2} \rfloor$, which is $\binom{r+1}{k+1}$ if $k = \frac{r-1}{2}$ and $2\binom{r+1}{k+1}$ otherwise. Finally, as $|\mathcal{O}_{k+1}(\mathcal{C}_r(r+1))| = o(\mathcal{C}_r(r+1), k) =$ by Corollary 3.3, the result holds. ☐

5.2. **The o-vector of a general oriented matroid.** In this subsection we study the o-vector of general oriented matroids $\mathcal{M}$. The main results show that Question 1.6 can be reduced to a finite case analysis for fixed r and its affirmative answer for $k = \frac{r-1}{2}$ (Theorem 5.13 and Corollary 5.14).
Given a tope T of $\mathcal{M}$ and $e \in E$, denote by ${}_{-e}T$ the sign-vector obtained from reorienting the element e in T.

Lemma 5.10. *Let $\mathcal{M} = (E, \mathcal{C})$ be an oriented matroid of odd rank $r \geq 3$ on $|E| \geq r+1$ elements and suppose that $\mathcal{M}$ has a $(\frac{r+1}{2})$-orthogonal tope T. Then, $\text{ort}({}_{-e}T) = \frac{r-1}{2}$ for every $e \in E$.*

Proof. Let $e \in E$ and first consider a circuit $X \in \mathcal{C}$ with $e \notin \underline{X}$ (if exists). Then, $X \perp {}_{-e}T = \frac{r+1}{2}$ since $|S(X,T)| = |S(X, {}_{-e}T)| = |H(X,T)| = |H(X, {}_{-e}T)| = \frac{r+1}{2}$. Now, consider $X \in \mathcal{C}$ such that $e \in \underline{X}$ and notice that $X \perp {}_{-e}T = \frac{r-1}{2}$. Therefore, $X \perp {}_{-e}T \geq \frac{r-1}{2}$ for every circuit $X \in \mathcal{C}$, concluding that ${}_{-e}T$ is $(\frac{r-1}{2})$-orthogonal. Moreover, since there exists $X \in \mathcal{C}$ such that $X \perp T' = \frac{r-1}{2}$, we conclude that $\text{ort}({}_{-e}T) = \frac{r-1}{2}$. ☐

For any uniform oriented matroid $\mathcal{M}$ on E elements and $e \in E$, the tope T of $\mathcal{M}$ will be mapped to some topes in the deletion $\mathcal{M} \setminus e$ and in the contraction $\mathcal{M}/e$, which will be denoted by $T_{\setminus e}$ and $T_{/e}$, respectively.

Lemma 5.11. *Let $\mathcal{M}$ be a uniform rank $r \geq 3$ oriented matroid on $n = |E|$ elements and $e \in E$. If T is k-orthogonal, $1 \leq k \leq \lfloor \frac{r+1}{2} \rfloor$, then*

(a) *$T_{\setminus e}$ is k-orthogonal in $\mathcal{M} \setminus e$;*
(b) *$T_{/e}$ is k-orthogonal in $\mathcal{M}/e$ if ${}_{-e}T$ is k-orthogonal and $k < \frac{r+1}{2}$.*

Proof. (a) By Proposition 3.4, $B_{k-1}^{\mathcal{G}(\mathcal{M})}(T) \cong B_{k-1}^{Q_n}(T)$. The tope graph of $\mathcal{M} \setminus e$ can be obtained by contracting all the edges of $\mathcal{G}(\mathcal{M})$, corresponding to e. Hence, $B_{k-1}^{\mathcal{G}(\mathcal{M}\setminus e)}(T_{\setminus e}) \cong B_{k-1}^{Q_{n-1}}(T_{\setminus e})$ and so, $T_{\setminus e}$ is k-orthogonal in $\mathcal{M} \setminus e$.
(b) Notice that $\mathcal{M}/e$ has rank $r' = r-1$ and $k \leq \lfloor \frac{r'+1}{2} \rfloor$. If $T_{/e}$ is not k-orthogonal in $\mathcal{M}/e$, then there exists a circuit Y of $\mathcal{M}/e$ such that $T_{/e} \perp Y \leq k-1$, but then taking the circuit $\underline{X} = \underline{Y} \cup e$ of $\mathcal{M}$, we obtain that $T \perp X \leq k-1$ or ${}_{-e}T \perp X \leq k-1$, since T and ${}_{-e}T$ differ in exactly the entry e, contradicting then the fact that T and ${}_{-e}T$ were k-orthogonal. Therefore, $T_{/e}$ is k-orthogonal and the result holds. ☐

Figure 1 shows the tope graph of $\mathcal{M} = \mathcal{C}_3(5)$ and $\mathcal{M} \setminus e = \mathcal{C}_3(4)$, where $e = 5$. Observe that k-orthogonal topes in $\mathcal{M}$ are mapped to k-orthogonal topes in $\mathcal{M} \setminus e$, for $k = 1, 2$. Next, we will obtain a bound for $m(\mathcal{M}, k)$ in terms of $\mathcal{M} \setminus e$ and $\mathcal{M}/e$.

Lemma 5.12. *Let $\mathcal{M}$ be a uniform rank $r \geq 3$ oriented matroid and let $e \in E$. Then, $m(\mathcal{M}, k) \leq m(\mathcal{M} \setminus e, k) + m(\mathcal{M}/e, k)$ for every $k = 0, \ldots, \lfloor \frac{r-1}{2} \rfloor$.*

Proof. Let $k' = k+1$ and let T be a k'-orthogonal tope of $\mathcal{M}$. First suppose that $k' = \frac{r+1}{2}$. Then by Lemma 5.10, every tope adjacent to T in $\mathcal{G}(\mathcal{M})$, in particular ${}_{-e}T$ is

not k'-orthogonal. Hence, the mapping $\mathcal{M}^{\perp_{k'}} \to (\mathcal{M} \setminus e)^{\perp_{k'}}$ sending T to $T_{\setminus e}$ which by Lemma 5.11 (a) is well-defined furthermore is injective. Hence, $|\mathcal{M}^{\perp_{k'}}| \leq |(\mathcal{M} \setminus e)^{\perp_{k'}}|$ and by Corollary 3.3 we get $m(\mathcal{M}, k) \leq m(\mathcal{M} \setminus e, k)$.
If $k' < \frac{r+1}{2}$ notice that $\mathcal{M}/e$ has rank $r' = r-1$ and so, $k' \leq \lfloor \frac{r'+1}{2} \rfloor$. If however, the neighbor ${}_{-e}T$ of T with respect to e was also k'-orthogonal, then both will be mapped to the same k'-orthogonal tope $T_{\setminus e}$ of $\mathcal{M} \setminus e$, but then the tope $T_{/e}$ of $\mathcal{M}/e$ is k'-orthogonal by Lemma 5.11 (b). Thus, we conclude that $|\mathcal{M}^{\perp_{k'}}| \leq |(\mathcal{M} \setminus e)^{\perp_{k'}}| + |(\mathcal{M}/e)^{\perp_{k'}}|$. By Corollary 3.3, this yields $m(\mathcal{M}, k) \leq m(\mathcal{M} \setminus e, k) + m(\mathcal{M}/e, k)$. □

An analogue of the above result in terms of $o(\mathcal{M}, k)$ is not true since topes T with $\operatorname{ort}(T) = k$ do not necessarily satisfy that $\operatorname{ort}(T_{\setminus e}) = k$ and $\operatorname{ort}(T_{/e}) = k$, even if its neighbor T' has $\operatorname{ort}(T') = k$ as in Lemma 5.11. For instance, Figure 1 shows examples of topes T with $\operatorname{ort}(T) = 1$ and $\operatorname{ort}(T_{\setminus e}) = 2$. In fact, for $r = 4$ and $n = 6$ we have that $36 = o(\mathcal{C}_4(6), 0) > o(\mathcal{C}_4(6) \setminus e, 0) + o(\mathcal{C}_4(6)/e, 0) = o(\mathcal{C}_4(5), 0) + o(\mathcal{C}_3(5), 0) = 10 + 20 = 30$, where these values are obtained from Example 6.2.

The following reduces Question 1.6 for fixed r to a finite number of cases.

Theorem 5.13. *Let $r \geq 3$ and $0 \leq k \leq \lfloor \frac{r-1}{2} \rfloor$. If $m(\mathcal{M}', k) \leq c_{r'}(n', k)$ for every uniform oriented matroid $\mathcal{M}'$ of rank $r' \leq r$ on $n' = 2(r'-k)+1$ elements, then $m(\mathcal{M}, k) \leq c_r(n, k)$ for every oriented matroid $\mathcal{M}$ of rank r on $n \geq 2(r-k)+1$ elements.*

Proof. Every rank r oriented matroid $\mathcal{M}$ on n elements can be perturbed to become a uniform rank r oriented matroid $\mathcal{M}'$ on n elements, see [3, Corollary 7.7.9]. In particular, we have that the tope graph $\mathcal{G}(\mathcal{M})$ is a subgraph of the tope graph $\mathcal{G}(\mathcal{M}')$. Hence, $m(\mathcal{M}, k) \leq m(\mathcal{M}', k)$ for every $k = 0, \ldots, \lfloor \frac{r-1}{2} \rfloor$. So, let us consider a uniform rank r oriented matroid $\mathcal{M}$ on n elements and let $e \in E$.
Let us show the claim by induction on r and n. By Lemma 5.12, $m(\mathcal{M}, k) \leq m(\mathcal{M} \setminus e, k) + m(\mathcal{M}/e, k)$. Now, fix r and let $n > 2(r-k)+1$. Notice that the inequality $m(\mathcal{M} \setminus e, k) \leq c_r(n-1, k)$ then follows by induction on n since we know that it is verified for all uniform rank r oriented matroids on $n-1$ elements. On the other hand, the inequality $m(\mathcal{M}/e, k) \leq c_{r-1}(n-1, k)$ follows since by assumption all uniform rank $r' = r-1$ oriented matroid $\mathcal{M}'$ on $n' = 2(r'-k)+1$ elements satisfy $m(\mathcal{M}', k) \leq c_r(n', k)$. Thus by induction this also holds for $n-1 \geq 2(r-k)+1 \geq 2(r'-k)+1$. Now, a straightforward computation using Theorem 5.8 and the fact that $\sum_{i=k}^{\lfloor \frac{r-1}{2} \rfloor} o(\mathcal{C}_r(n), i) = c_r(n, k)$, yields

$$c_r(n-1, k) + c_{r-1}(n-1, k) = c_r(n, k).$$

Thus, we obtain that $m(\mathcal{M}, k) \leq c_r(n, k)$. □

The following result answers Question 1.6 in the affirmative for odd r and $k = \frac{r+1}{2}$.

Corollary 5.14. *Let $\mathcal{M}$ be an oriented matroid of odd rank $r \geq 3$ on $n \geq r+2$ elements and $k = \frac{r-1}{2}$. Then, $m(\mathcal{M}, k) \leq c_r(n, k) = 2$.*

Proof. As $k = \frac{r-1}{2}$, we notice that $o(\mathcal{M}, k) = m(\mathcal{M}, k)$ and $o(\mathcal{C}_r(n), k) = c_r(n, k)$. Moreover, $c_r(n, k) = 2$ by Theorem 5.8. In order to prove that $m(\mathcal{M}, k) \leq c_r(n, k)$ for $n \geq r+2$, it is sufficient by Theorem 5.13 to verify it for all uniform rank r oriented matroid $\mathcal{M}$ on $n = r+2 = 2(r-k)+1$ elements, since for smaller rank $r' < r$ we will have $k > \frac{r'-1}{2}$ and hence the k-entries of all o-vectors are 0. As in that case there is only one reorientation class (see Remark 2.3), we obtain that $o(\mathcal{M}, k) = o(\mathcal{C}_r(n), k)$, concluding the proof. □

6. Low rank

Question 1.3 and by Theorem 5.13 also Question 1.6 can be reduced to uniform oriented matroids and reduce to a finite problem for fixed rank. Finschi and Fukuda [8,9] generated (up to isomorphism) all the chirotopes of uniform rank r oriented matroids on n elements, for $4 \le r \le 7$ and $n = r+3$, and moreover classified them by realizability and for $r = 5$ and $n = 9$ (where some of the data and also their source code for the enumeration is available only upon request from Lukas Finschi). So, we will now quickly explain how we can obtain the o-vector of a uniform oriented matroid from its chirotope:

chirotope → circuits:
Note that in a uniform oriented matroid the chirotope $\chi(B) \neq 0$ for every ordered set of size r. Further, the supports of its circutis correspond to all sets of size $r+1$. It is well-known, see [3, Section 3.5], that from the chirotope of a unifrom oriented matroid, we obtain the signs of a circuit X with $\underline{X} = \{b_1, ..., b_{r+1}\}$ via:

$$\chi(B) = -X_{b_i} \cdot X_{b_{i+1}} \cdot \chi(B'),$$

where $B = \underline{X} \setminus b_i$ and $B' = \underline{X} \setminus b_{i+1}$. This allows us to compute the set $\mathcal{C}$ from χ.

circuits → o-vector:
For any sign-vector $T \in \{+,-\}^n$, we obtain $\mathrm{ort}(T) = \min\{X \perp T \mid X \in \mathcal{C}\}$ and so, using the correspondence $o(\mathcal{M}, k) = |\mathcal{O}_{k+1}(\mathcal{M})|$ given in Corollary 3.3, the o-vector of $\mathcal{M}$.

We implemented this procedure in Python (available at [1]) giving us the o-vector of all the reorientation classes from the database. Recall that $[C_r(n)]$ denotes the class of reorientations of the alternating oriented matroid $\mathcal{C}_r(n)$. We resume the results in the following theorem:

Theorem 6.1. *Let $\mathcal{M} \notin [C_r(n)]$ be a uniform rank r oriented matroid on n elements, then the following hold:*

(a) *if $r = 5$ and $n = 8$, then $o(\mathcal{M}, 1) < o(\mathcal{C}_5(8), 1)$, $m(\mathcal{M}, 1) < c_5(8, 1)$, $m(\mathcal{M}, 2) \le c_5(8, 2)$ and there are exactly 3 reorientation classes with $m(\mathcal{M}, 2) = c_5(8, 2)$. Moreover, there exists $\mathcal{M}$ realizable such that $m(\mathcal{M}, 2) = 0$;*
(b) *if $r = 5$ and $n = 9$, then $o(\mathcal{M}, 1) < o(\mathcal{C}_5(9), 1)$, $m(\mathcal{M}, 1) < c_5(9, 1)$, $m(\mathcal{M}, 2) \le c_5(9, 2)$ and there are exactly 23 reorientation classes with $m(\mathcal{M}, 2) = c_5(9, 2)$;*
(c) *if $r = 6$ and $n = 9$, then $m(\mathcal{M}, 1) < c_6(9, 1)$ and $m(\mathcal{M}, 2) < c_6(9, 2)$. Moreover, there are exactly 91 reorientation classes having $o(\mathcal{M}, 1) > o(\mathcal{C}_6(9), 1)$ and there exists $\mathcal{M}$ realizable such that $m(\mathcal{M}, 2) = 0$;*
(d) *if $r = 7$ and $n = 10$, then $m(\mathcal{M}, 1) < c_7(10, 1)$, $0 < o(\mathcal{M}, 2) < o(\mathcal{C}_7(10), 2)$, $m(\mathcal{M}, 2) < c_7(10, 2)$, $m(\mathcal{M}, 3) \le c_7(10, 3)$ and there are exactly 37 reorientation classes with $m(\mathcal{M}, 3) = c_7(10, 3)$. Moreover, there are exactly 312336 reorientation classes having $o(\mathcal{M}, 1) > o(\mathcal{C}_7(10), 1)$ and there exists $\mathcal{M}$ realizable such that $m(\mathcal{M}, 3) = 0$.*

Example 6.2. Since the chirotope of the alternating oriented matroid is always +, using our computer program we compute the o-vector of $\mathcal{C}_r(n)$ for some values of r and n in Figure 6. For all these values and k such that $n \le 2(r-k)$, we notice that $o(\mathcal{C}_r(n), k) \neq 2\binom{n}{r-1-2k}$, showing that Theorem 5.8 is best possible.

Beyond McMullen. Next, we answer Question 1.3 affirmatively for $(r, k) \in (5, 2), (6, 2), (7, 3)$. Further, we show that the lower bound in Question 1.3 is tight in one more case, i.e., $10 \le \nu(7, 2)$.

	$k=0$	$k=1$	$k=2$
$o(\mathcal{C}_3(5),k)$	20	2	
$o(\mathcal{C}_4(5),k)$	10	20	
$o(\mathcal{C}_4(6),k)$	36	16	
$o(\mathcal{C}_4(7),k)$	70	14	
$o(\mathcal{C}_5(7),k)$	56	56	2
$o(\mathcal{C}_5(8),k)$	136	60	2
$o(\mathcal{C}_5(9),k)$	252	72	2

	$k=0$	$k=1$	$k=2$	$k=3$
$o(\mathcal{C}_6(8),k)$	80	128	32	
$o(\mathcal{C}_6(9),k)$	234	186	18	
$o(\mathcal{C}_6(10),k)$	500	244	20	
$o(\mathcal{C}_6(11),k)$	924	330	22	
$o(\mathcal{C}_7(9),k)$	108	234	150	2
$o(\mathcal{C}_7(10),k)$	370	450	110	2
$o(\mathcal{C}_7(11),k)$	902	682	110	2

	$k=0$	$k=1$	$k=2$	$k=3$
$o(\mathcal{C}_7(12),k)$	1844	994	132	2
$o(\mathcal{C}_7(13),k)$	3432	1430	156	2
$o(\mathcal{C}_8(10),k)$	140	380	420	64
$o(\mathcal{C}_8(11),k)$	550	924	440	22
$o(\mathcal{C}_8(12),k)$	1512	1632	464	24
$o(\mathcal{C}_8(13),k)$	3406	2600	572	26
$o(\mathcal{C}_8(14),k)$	6860	4008	728	28

FIGURE 6.

Theorem 6.3. *We have* $\nu(5,2) = \nu_{\mathrm{R}}(5,2) = 7$, $\nu(6,2) = \nu_{\mathrm{R}}(6,2) = 8$, $\nu(7,3) = \nu_{\mathrm{R}}(7,3) = 9$, *and* $10 \leq \nu_{\mathrm{R}}(7,2)$.

Proof. By Remark 4.1, $7 \leq \nu(5,2)$, $8 \leq \nu(6,2)$ and $9 \leq \nu(7,3)$ (and so, also $7 \leq \nu_{\mathrm{R}}(5,2)$, $8 \leq \nu_{\mathrm{R}}(6,2)$ and $9 \leq \nu_{\mathrm{R}}(7,3)$). The lower bound $10 \leq \nu(7,2)$ holds since by Theorem 6.1 (d), $0 < o(\mathcal{M},2)$ for any uniform rank 7 oriented matroid $\mathcal{M}$ on 10 elements. On the other hand, the upper bounds $\nu_{\mathrm{R}}(5,2) < 8$, $\nu_{\mathrm{R}}(6,2) < 9$ and $\nu_{\mathrm{R}}(7,3) < 10$ hold by Theorem 6.1 (a), (c) and (d), respectively, since there exists a rank r realizable uniform oriented matroid $\mathcal{M}$ on n elements such that $m(\mathcal{M},k) = 0$, for $(r,k,n) \in (5,2,8), (6,2,9), (7,3,10)$. Then, $\nu(5,2) < 8$, $\nu(6,2) < 9$ and $\nu(7,3) < 10$ and the result follows. □

Beyond Roudneff. Next, we answer Question 1.6 affirmatively for $r = 6$ and $k = 2$.

Theorem 6.4. *Let* $\mathcal{M}$ *be a rank* 6 *oriented matroid on* $n \geq 9$ *elements. Then,* $m(\mathcal{M},2) \leq c_6(n,2)$.

Proof. By Theorem 6.1 (c), $m(\mathcal{M},2) \leq c_6(9,2)$ for all rank 6 uniform oriented matroid $\mathcal{M}$ on 9 elements. On the other hand, it is known that $m(\mathcal{M}',2) \leq c_5(n,2)$ for any rank 5 oriented matroid $\mathcal{M}'$ on $n \geq 7$ elements by Corollary 5.14. Then, the result follows for $n \geq 9$ by Theorem 5.13. □

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