Geometry of convex geometries

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Abstract

We prove that any convex geometry $\mathcal{A} = (U, \mathcal{C})$ on n points and any ideal $\mathcal{I} = (U', \mathcal{C}')$ of $\mathcal{A}$ can be realized as the intersection pattern of an open convex polyhedral cone $K \subseteq \mathbb{R}^n$ with the orthants of $\mathbb{R}^n$. Furthermore, we show that K can be chosen to have at most m facets, where m is the number of critical rooted circuits of $\mathcal{A}$. We also show that any convex geometry of convex dimension d is realizable in $\mathbb{R}^d$ and that any multisimplicial complex (a basic example of an ideal of a convex geometry) of dimension d is realizable in $\mathbb{R}^{2d}$ and that this is best possible. From our results it also follows that distributive lattices of dimension d are realizable in $\mathbb{R}^d$ and that median systems are realizable. We leave open whether each median system of dimension d is realizable in $\mathbb{R}^{O(d)}$.

1 Introduction

Several fundamental combinatorial structures constitute abstract generalizations of geometric settings. Matroids generalize the linear independence in vector spaces, oriented matroids (OMs) capture the combinatorics of regions in a central hyperplane arrangement in $\mathbb{R}^d$, ample/lopsided sets (AMPs) encode the regions of the arrangement of coordinate hyperplanes intersected with a convex set, and convex geometries/antimatroids represent an abstraction of Euclidean convexity restricted to a finite set. Finally, complexes of oriented matroids (COMs) are a common generalization of oriented matroids and ample sets and capture the combinatorics of regions in an arbitrary hyperplane arrangement in $\mathbb{R}^d$ restricted to a convex set K. Although this geometric model is a desirable property for a respective combinatorial structure, the realizability question is hard. For example, the problem of characterizing which oriented matroids come from hyperplane arrangements is intractable [26, 29, 31] and the realizability of a convex geometry in $\mathbb{R}^2$ is $\exists\mathbb{R}$-complete [2, 18]. All the above structures can be seen as set systems, see Remark 3. This allows to view convex geometries as ample sets and ample sets as COMs.

We investigate the realizability question for convex geometries and generalize them to ideals of convex geometries. Convex geometries (alias antimatroids) have been introduced and investigated by Edelman and Jamison [13] in the context of abstract convexity and by Korte and Lovasz [23, 24] in combinatorics. Kashiwabara, Nakamura, and Okamoto [19] proved that any convex geometry $\mathcal{A} = (U, \mathcal{C})$ on n points can be realized in $\mathbb{R}^n$ using a generalized convex shelling. Richter and Rogers [28] proved that $\mathcal{A}$ can be realized in this way in $\mathbb{R}^d$, where d is the convex dimension of $\mathcal{A}$. In this paper, we consider a simpler (and dual) version of realizability via hyperplane arrangements and convex sets, as in the case of OMs and COMs. We prove that any convex geometry $\mathcal{A}$ is realizable in this way in $\mathbb{R}^n$. Furthermore, we show that the convex set realizing $\mathcal{A}$ can be chosen to be a polyhedral cone with at most m facets, where m is the number of critical rooted circuits of $\mathcal{A}$. We also establish that any ideal $\mathcal{I} = (U, \mathcal{C}')$ of $\mathcal{A}$ is realizable in $\mathbb{R}^n$ by a convex polyhedron with $m + k$ facets where k is the number of positive circuits. We also show that any convex geometry of convex dimension d is realizable in $\mathbb{R}^d$. As an application of our results on ideals, we show that any multisimplicial complex of dimension d is realizable in $\mathbb{R}^{2d}$ and this is optimal. It follows that distributive lattices of dimension d are realizable in $\mathbb{R}^d$ and that median systems are realizable. We leave open whether each median system of dimension d is realizable in $\mathbb{R}^{O(d)}$ but show that any tree (median system of dimension 1) is realizable in $\mathbb{R}^2$.

2 Preliminaries

In this section, we define the main combinatorial structures investigated in this paper and their realizability.

2.1 Set families and systems of sign vectors

Let U be a set of size n. A *set family* $\mathcal{S}$ is any collection of subsets of U. We denote by $\mathcal{S}^*$ the complement $2^U \setminus \mathcal{S}$ of the family $\mathcal{S}$. Any set family $\mathcal{S} \subseteq 2^U$ can be viewed as a subset of vertices of the n-dimensional hypercube $\mathrm{Q}_n = \mathrm{Q}(U)$. Denote by $G(\mathcal{S})$ the subgraph of Q_n induced by the vertices of Q_n corresponding to the sets of $\mathcal{S}$; $G(\mathcal{S})$ is called the *1-inclusion graph* of $\mathcal{S}$. A set family $\mathcal{S}$ is called *isometric* if $G(\mathcal{S})$ is an *isometric subgraph* of the hypercube Q_n, i.e., the distances in $G(\mathcal{S})$ and in Q_n between any two vertices of $G(\mathcal{S})$ are equal. An X*-cube* Q of Q_n is the 1-inclusion graph of the set family $\{Y \cup X' : X' \subseteq X\}$, where Y is a subset of $U \setminus X$, called the *support* of Q. If $|X| = n'$, then any X-cube is a n'-dimensional subcube of Q_n and Q_n contains $2^{n-n'}$ X-cubes.

Let $\mathcal{L}$ be a *system of sign vectors* on U, i.e., maps from U to $\{-1, 0, +1\}$. The elements of $\mathcal{L}$ are referred to as *covectors* and denoted by capital letters X, Y, Z. For $X \in \mathcal{L}$, the subset $\underline{X} = \{e \in U : X_e \neq 0\}$ is the *support* of X and its complement $X^0 = U \setminus \underline{X} = \{e \in U : X_e = 0\}$ is the *zero set* of X.

The systems of sign vectors generalize set families. Indeed, any set family $(U, \mathcal{S})$ can be encoded as a system of sign vectors by setting for each set $X \in \mathcal{S}$, $X_e = -1$ if $e \notin X$ and $X_e = +1$ if $e \in X$. In this representation, each set X is encoded by a $\{-1, +1\}$-vector. The encoding with $\{-1, 0, +1\}$-vectors is useful when encoding cubes of Q_n. Indeed, each X-cube Q with support Y can be encoded by $\{-1, 0, +1\}$-vector $X(Q)$, where $X(Q)_e = +1$ if $e \in Y$, $X(Q)_e = -1$ if $e \in (U \setminus X) \setminus Y$, and $X(Q)_e = 0$ if $e \in X$.

For each subset $X \subset U$, the (open) X*-orthant* of $\mathbb{R}^n$ is the set $\mathcal{O}(Y)$ of all points $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ such that $x_e > 0$ if $e \in X$ and $x_e < 0$ if $e \notin X$. More generally, for a covector $X \in \{-1, 0, +1\}^U$, the X*-generalized orthant* is the set of all points $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ such that $x_e > 0$ if $X_e = +1$ and $x_e < 0$ if $X_e = -1$. Notice that the X-generalized orthant is the union of all Y-orthants $\mathcal{O}(Y)$ such that $Y = \underline{X} \cup Y'$ with $Y' \subseteq X^0$.

2.2 OMs and COMs

We recall the basic theory of OMs and COMs from [7] and [5], respectively. Co-invented by Bland and Las Vergnas [8] and Folkman and Lawrence [16], and further investigated by Edmonds and Mandel [15] and many other authors, oriented matroids (OMs) represent a unified combinatorial theory of orientations of ordinary matroids, which simultaneously captures the basic properties of sign vectors representing the regions in a hyperplane arrangement in $\mathbb{R}^d$ and of sign vectors of the circuits in a directed graph. OMs provide a framework for the analysis of combinatorial properties of geometric configurations occurring in discrete geometry and in machine learning. Point and vector configurations, order types, hyperplane and pseudo-line arrangements, convex polytopes, directed graphs, and linear programming find a common generalization in this language. The Topological Representation Theorem of [16] connects the theory of OMs on a deep level to arrangements of pseudohyperplanes and distinguishes it from the theory of ordinary matroids.

Let $\mathcal{L} \subseteq \{-1, 0, +1\}^U$ be a system of sign vectors. For the sake of this paper we assume that $\mathcal{L}$ is *simple*, i.e., $\{X_e : X \in \mathcal{L}\} = \{-1, 0, +1\}$. For $X, Y \in \mathcal{L}$, $\mathrm{Sep}(X, Y) = \{e \in U : X_e Y_e = -1\}$ is the *separator* of X and Y. The *composition* of X and Y is the sign vector $X \circ Y$, where for all $e \in U$, $(X \circ Y)_e = X_e$ if $X_e \neq 0$ and $(X \circ Y)_e = Y_e$ if $X_e = 0$.

Definition 1. An *oriented matroid* (OM) [7] is a system $\mathcal{M} = (U, \mathcal{L})$ of sign vectors satisfying

(C) (Composition) $X \circ Y \in \mathcal{L}$ for all $X, Y \in \mathcal{L}$.

(SE) (Strong elimination) for each pair $X, Y \in \mathcal{L}$ and for each $e \in \mathrm{Sep}(X, Y)$, there exists $Z \in \mathcal{L}$ such that $Z_e = 0$ and $Z_f = (X \circ Y)_f$ for all $f \in U \setminus \mathrm{Sep}(X, Y)$.

(Sym) (Symmetry) $-\mathcal{L} = \{-X : X \in \mathcal{L}\} = \mathcal{L}$, that is, $\mathcal{L}$ is closed under sign reversal.

Complexes of Oriented Matroids (COMs) were introduced by Bandelt, Chepoi, and Knauer [5] as a natural common generalization of ample sets and OMs. They are defined by replacing the global axiom (Sym) with a weaker local axiom:

Definition 2. A *complex of oriented matroids* (COM) [5] is a system of sign vectors $\mathcal{M} = (U, \mathcal{L})$ satisfying (SE) and the following axiom:

(FS) (Face symmetry) $X \circ -Y \in \mathcal{L}$ for all $X, Y \in \mathcal{L}$.

One can see that OMs are exactly the COMs containing the zero vector $\mathbf{0}$, see [5]. Let $\leq$ be the product ordering on $\{-1,0,+1\}^U$ relative to $0 \leq -1,+1$. The poset $(\mathcal{L},\leq)$ of a COM $\mathcal{M}=(U,\mathcal{L})$ with an artificial maximum $\hat{1}$ forms the (graded) *big face (semi)lattice* $\mathcal{F}_{\text{big}}(\mathcal{M})$. For $X \in \mathcal{L}$ a covector of a COM $\mathcal{M}=(U,\mathcal{L})$, the *face* of X is $\uparrow X := \{Y \in \mathcal{L} : X \leq Y\}$, see [5,7]. By [5, Lemma 4], each face $\uparrow X$ of a COM $\mathcal{M}$ is an OM.

The *topes* $\mathcal{T}$ of $\mathcal{L}$ are the co-atoms of $\mathcal{F}_{\text{big}}(\mathcal{M})$. By simplicity the topes are $\{-1,+1\}$-vectors and $\mathcal{T}$ can be seen as a family of subsets of U. For each $T \in \mathcal{T}$, an element $e \in U$ belongs to the corresponding set if and only if $T_e = +1$. Since by [5,21] the topes determine $\mathcal{M}$ uniquely, we make frequent use of the following:

Remark 3. Every simple COM $\mathcal{M}$ can be seen as the set family, defined by its set of topes $\mathcal{T}$.

The *tope graph* $G(\mathcal{M})$ of a COM $\mathcal{M}=(U,\mathcal{L})$ is the 1-inclusion graph of the set $\mathcal{T}$ of topes of $\mathcal{L}$, i.e., the subgraph of the hypercube induced by the vertices corresponding to $\mathcal{T}$. The COM $\mathcal{M}=(U,\mathcal{L})$ can be recovered up to relabelling and reorientation from $G(\mathcal{M})$ and $G(\mathcal{M})$ is an isometric subgraph of Q_n, see [5,21].

2.3 Ample sets

Ample sets [4] (originally introduced as lopsided sets by Lawrence [25]) are combinatorial structures somewhat opposed to oriented matroids. They capture an important variety of combinatorial objects, e.g., diagrams of (upper locally) distributive lattices, median graphs or CAT(0) cube complexes, convex geometries and conditional antimatroids, see [4]. *Ample sets* are exactly the COMs, in which all faces are cubes. Ample sets can be defined and characterized in a multitude of combinatorial ways in the language of set families, i.e., in terms of topes; see [4,9,25]. One of them is via shattering and strong shattering.

Let $\mathcal{S}$ be a family of subsets of an n-element set U. For a set $Y \subset U$, the *trace* of $\mathcal{S}$ to Y is defined as $\mathcal{S}|_Y = \{X \cap Y : X \in \mathcal{S}\}$. A subset X of U is *shattered* by $\mathcal{S}$ if for all $Y \subseteq X$ there exists $S \in \mathcal{S}$ such that $S \cap X = Y$, i.e. $\mathcal{S}_X = 2^X$. The *Vapnik-Chervonenkis dimension* (the *VC-dimension* for short) VC-dim$(\mathcal{S})$ of $\mathcal{S}$ is the cardinality of the largest subset of U shattered by $\mathcal{S}$. A subset X of U is *strongly shattered* by $\mathcal{S}$ if the 1-inclusion graph $G(\mathcal{S})$ of $\mathcal{S}$ contains an X-cube. Denote by $\overline{\mathcal{X}}(\mathcal{S})$ and $\underline{\mathcal{X}}(\mathcal{S})$ the families consisting of all shattered and of all strongly shattered sets of $\mathcal{S}$, respectively. Clearly, $\underline{\mathcal{X}}(\mathcal{S}) \subseteq \overline{\mathcal{X}}(\mathcal{S})$ and both $\overline{\mathcal{X}}(\mathcal{S})$ and $\underline{\mathcal{X}}(\mathcal{S})$ are closed under taking subsets, i.e., $\overline{\mathcal{X}}(\mathcal{S})$ and $\underline{\mathcal{X}}(\mathcal{S})$ are *abstract simplicial complexes*. The VC-dimension VC-dim$(\mathcal{S})$ of $\mathcal{S}$ is thus the size of a largest set shattered by $\mathcal{S}$, i.e., the dimension of the simplicial complex $\overline{\mathcal{X}}(\mathcal{S})$. A family $\mathcal{S}$ of subsets of U is *ample* whenever the simplicial complexes $\overline{\mathcal{X}}(\mathcal{S})$ and $\underline{\mathcal{X}}(\mathcal{S})$ coincide, i.e., each shattered set is strongly shattered. From this definition it follows that the VC-dimension of an ample set $\mathcal{S}$ coincides with the dimension of a largest cube included in $\mathcal{S}$. The complement $\mathcal{S}^* = 2^U \setminus \mathcal{S}$ of an ample set $\mathcal{S}$ is also ample. Finally, ample sets are isometric.

2.4 Convex geometries

Convex geometries, introduced and investigated by Edelman and Jamison [13], are the abstract convexity spaces satisfying one of the most important properties of Euclidean convexity: each convex set is the convex hull of its extremal points. Convex geometries are a particular case of ample sets [4]. A *convex geometry* [13] is a pair $\mathcal{A}=(U,\mathcal{C})$ of a finite *universe* U and a collection of *convex sets* $\mathcal{C} \subseteq 2^U$ satisfying the following three conditions:

(C1) $\varnothing \in \mathcal{C}$ and $U \in \mathcal{C}$;

(C2) $X \cap Y \in \mathcal{C}$ for all $X,Y \in \mathcal{C}$; (intersection-closed)

(C3) if $X \in \mathcal{C} \setminus \{U\}$, then there exists $e \in U \setminus X$ such that $X \cup e \in \mathcal{C}$. (extendable)

See the left side of Figure 1 for an example. The *convex hull* conv(A) of a set $A \subset U$ is the intersection of all convex sets of $\mathcal{A}$ containing A; by (C2), conv(A) is the smallest element of $\mathcal{C}$ containing A. A point x of a convex set X is called an *extreme point* of X if $X \setminus \{x\}$ is also convex. Denote by ex(X) the set of all extreme points of X. Convex geometries can be also characterized by (C1), (C2), and the following axiom:

(C4) $X = \text{conv}(\text{ex}(X))$ for any $X \in \mathcal{C}$.

Convex geometries can be also characterized by (C1), (C2) and the *anti-exchange* axiom:

(C5) if $X \in \mathcal{C}$ and p,q are two different points of $U \setminus X$, then $q \in \text{conv}(X \cup p)$ implies that $p \notin \text{conv}(X \cup q)$.

For other characterizations of convex geometries, their properties, and examples, see the foundational paper by Edelman and Jamison [13]. Convex geometries are ample sets [4]. If $\mathcal{A} = (U, \mathcal{C})$ is a convex geometry, then the set family $\mathcal{A}^*$ on U consisting of the complements $U \setminus C$ of sets C of $\mathcal{C}$ is union closed and is called an *antimatroid*. For a nice exposition of the theory of convex geometries/antimatroids, see the book by Korte, Lovász, and Schrader [24]. Below we will present a characterization of convex geometries via rooted circuits and the notion of convex dimension.

2.5 Bouquets and ideals of convex geometries

We now consider a generalization of ideals of convex geometries, originally introduced under the name of conditional antimatroids in [4]. Analogously to convex geometries bouquets of convex geometries are also ample sets [4]. A pair $\mathcal{A} = (U, \mathcal{C})$ of a finite *universe* U and a collection of *convex sets* $\mathcal{C} \subseteq 2^U$ is called *bouquet of convex geometries* if it satisfies the following two conditions:

(C1′) $\varnothing \in \mathcal{C}$;

(C2) $X \cap Y \in \mathcal{C}$ for all $X, Y \in \mathcal{C}$;

(C3′) for all $X, Y \in \mathcal{C}$ with $Y \subset X$, there is $e \in X \setminus Y$ such that $Y \cup e \in \mathcal{C}$. (locally extendable)

A pair $\mathcal{I} = (U', \mathcal{C}')$ is called an *ideal of a convex geometry* if there is a convex geometry $\mathcal{A} = (U, \mathcal{C})$ such that $U' \subseteq U$, $\mathcal{C}' \subseteq \mathcal{C}$ and if $X \in \mathcal{C}, Y \in \mathcal{C}'$ and $X \subseteq Y$, then $X \in \mathcal{C}'$. Equivalently, for any $X \in \mathcal{C}'$, its principal ideals in $\mathcal{C}'$ and $\mathcal{C}$ coincide, where the *principal ideal* $\downarrow X$ of X in $\mathcal{C}$ (respectively, in $\mathcal{C}'$) is the set of all subsets $Y \subseteq X$ such that $Y \in \mathcal{C}$ (respectively, $Y \in \mathcal{C}'$). It is easy to see that intervals of bouquets of convex geometries are convex geometries and that ideals of convex geometries are bouquets of convex geometries. However, there are bouquets of convex geometries that are not ideals.

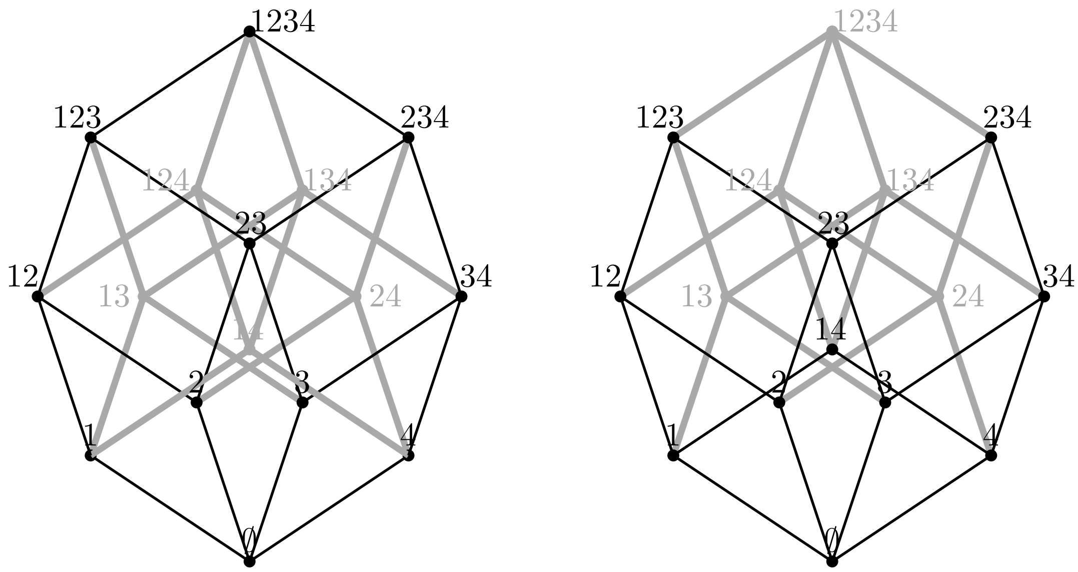

Figure 1: Left: A convex geometry. Right: A bouquet of convex geometries that is not an ideal of a convex geometry. Both are represented in black as a subgraph of the 4-dimensional hypercube (in gray).

Proposition 4. *There exist bouquets of convex geometries that are not ideals of convex geometries.*

Proof. Consider the bouquet of convex geometries in Figure 1, right, i.e., $U' = \{1, 2, 3, 4\}$ and

$$\mathcal{C}' = \{\varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3\}, \{2, 3, 4\}\}.$$

Suppose that $\mathcal{A}' = (U', \mathcal{C}')$ is an ideal of a convex geometry $\mathcal{A} = (U, \mathcal{C})$. Starting with $\{1, 4\}$ and applying iteratively (C3), there exists a set $Y \subseteq U \setminus U'$ such that one of the sets $Y \cup \{1, 2, 4\}$ or $Y \cup \{1, 3, 4\}$ must be in $\mathcal{C}$. Now applying (C2) with respect to $Y \cup \{1, 2, 4\}$ and $\{2, 3, 4\}$ or $Y \cup \{1, 3, 4\}$ and $\{1, 2, 3\}$, we get that $\{2, 4\}$ or $\{1, 3\}$ are in $\mathcal{C}$. Hence $\mathcal{A}' = (E', \mathcal{C}')$ is not an ideal of $\mathcal{A}$. $\square$

3 Realizability

In this section, we recall the notion of realizability of COMs and investigate some general properties of realizable COMs. We also compare the notion of realizability with the notion of generalized convex shelling, which can be seen as a dual realizability.

3.1 Realizability of COMs

We continue with the definition of realizable COMs given in the paper [5], which generalizes realizability of oriented and affine oriented matroids [7] and of ample sets [25]. An *affine arrangement of hyperplanes* $\mathcal{H}$ is a finite set of affine hyperplanes of $\mathbb{R}^d$. We will denote by $\mathcal{H}_d$ the set of d coordinate hyperplanes of $\mathbb{R}^d$. Let K be a relatively open convex set of $\mathbb{R}^d$. Each (oriented) hyperplane $H = \{x \in \mathbb{R}^d : ax = b\}$ in an arrangement of hyperplanes $\mathcal{H}$ splits $\mathbb{R}^d$ into the positive part $H^+ = \{x \in \mathbb{R}^d : ax > b\}$, the negative part $H^- = \{x \in \mathbb{R}^d : ax < b\}$, and the zero part H. The arrangement $\mathcal{H}$ partitions $\mathbb{R}^d$ into relatively open convex regions, called *cells*. All points belonging to the same cell have the same sign vector with respect to the hyperplanes of $\mathcal{H}$. Restrict the arrangement pattern to K, that is, remove all sign vectors which represent the regions of the partition disjoint from K. The resulting set of sign vectors of the cells of K is denoted $\mathcal{L}(\mathcal{H}, K)$ and constitutes a COM $\mathcal{M}(\mathcal{H}, K) = (H, \mathcal{L}(\mathcal{H}, K))$, see [5].

If $\mathcal{H}$ is a central arrangement with K being any relatively open convex set containing the origin, then $\mathcal{M}(\mathcal{H}, K)$ coincides with the notion of *realizable oriented matroid* [7]. If the arrangement $\mathcal{H}$ is affine and K is the entire space, then $\mathcal{M}(\mathcal{H}, K)$ coincides with the *realizable affine oriented matroid*. Finally, the *realizable ample sets* arise by taking the central arrangement $\mathcal{H}_d$ of coordinate hyperplanes restricted to a full-dimensional open convex set K of $\mathbb{R}^d$ (this model was first considered in [25]). A COM $\mathcal{M}$ is called *realizable* if there exists an affine arrangement of hyperplanes $\mathcal{H}$ and a relatively open convex set K of $\mathbb{R}^d$ such that $\mathcal{M} = \mathcal{M}(\mathcal{H}, K)$. Then the pair $(\mathcal{H}, K)$ is called a *realization* of $\mathcal{M}$. As noticed in [5], for a realizable COM the set K always can be selected to be a relatively open polyhedron. For a realizable COM $\mathcal{M}$, we denote by $\dim_{\mathbb{E}}(\mathcal{M})$ the smallest d such that $\mathcal{M}$ has a realization in $\mathbb{R}^d$ and call it the *Euclidean dimension* of $\mathcal{M}$.

In realizable COMs, $X \leq Y$ for two covectors X, Y if and only if the cell corresponding to Y is contained in the cell corresponding to X. Consequently, the topes of realizable COMs are the covectors of the inclusion maximal cells (which all have dimension d), called *regions*. Therefore, the tope graph of a realizable COM with realization $(\mathcal{H}, K)$ can be viewed as the adjacency graph of regions: the vertices of this graph are the regions of K defined by the hyperplane arrangement $\mathcal{H}$ and two regions are adjacent in this graph if they are separated by a unique hyperplane of the arrangement (recall that a hyperplane H *separates* two disjoint relatively open convex sets A and B if A and B belong to distinct halfspaces defined by H).

To realize a COM $\mathcal{M}$, it is necessary to find an arrangement of hyperplanes $\mathcal{H}$, a relatively open convex set K, and a sign-preserving bijection between the topes of $\mathcal{M}$ and the sign vectors of the regions (maximal cells) defined by $\mathcal{H}$ and included in K. Now, we will show that $\mathcal{H}$ can be selected to be the set of coordinate hyperplanes $\mathcal{H}_d$ of $\mathbb{R}^d$ and K to be a relatively open polyhedron of $\mathbb{R}^d$. This immediately follows from the following lemma:

Lemma 5. *Let $\mathcal{M} = (U, \mathcal{L})$ be a realizable COM. Then $\mathcal{M}$ has a realization $(\mathcal{H}'', K'')$, where K'' is a relatively open polyhedral cone, and $\mathcal{H}''$ is a central arrangement of hyperplanes, whose normal vectors generate the ambient space.*

Proof. Let $\mathcal{M} = (U, \mathcal{L})$ be a realizable COM with a realization $\mathcal{M} = \mathcal{M}(\mathcal{H}, K)$, where $\mathcal{H}$ is an arrangement of affine hyperplanes and K is a relatively open polyhedron of $\mathbb{R}^d$, as provided by [5]. First, we consider $\mathbb{R}^d$ as the hyperplane $x_{d+1} = 1$ of $\mathbb{R}^{d+1}$. Second, we transform each hyperplane $H_e \in \mathcal{H}, e \in U$ to a central hyperplane H'_e of $\mathbb{R}^{d+1}$ passing via H_e and the origin of coordinates. Denote the resulting arrangement by $\mathcal{H}'$. The supporting hyperplanes S_j of the closure of K define a second arrangement of hyperplanes $\mathcal{S}$ of $\mathbb{R}^d$. Analogously to $\mathcal{H}$, the arrangement $\mathcal{S}$ can be transformed into an arrangement $\mathcal{S}'$ of central hyperplanes of $\mathbb{R}^{d+1}$. Then K belongs to one halfspace defined by each hyperplane $S'_j \in \mathcal{S}'$. The intersection of those positive open halfspaces is an open polyhedron K' containing K. Then $\mathcal{L} = \mathcal{L}(\mathcal{H}', K')$ holds. Each cell $C \subset K$ of the realization $\mathcal{M}(\mathcal{H}, K)$ gives rise to an open polyhedral cone $C' \subset K'$ (containing C) of the realization $\mathcal{M}(\mathcal{H}', K')$. The cone C' is the intersection of positive or negative halfspaces defined by the hyperplanes of $\mathcal{H}'$ and by the halfspaces of the hyperplanes of $\mathcal{S}'$ containing K'.

If the normal vectors of $\mathcal{H}'$ generate a proper subspace $W \subset \mathbb{R}^{d+1}$ of dimension d', then we consider $\mathcal{H}'$ as an arrangement in W, intersect K' with W, and affinely transform W to $\mathbb{R}^{d'}$. This yields a realization $(\mathcal{H}'', K'')$ of $\mathcal{M}$ with the desired properties. □

Proposition 6. *Let $\mathcal{M} = (U, \mathcal{L})$ be a realizable COM. Then $\mathcal{M}$ has a realization $(\mathcal{H}_n, K)$, where K is a relatively open polyhedral cone, and $\mathcal{H}_n$ is the set of the coordinates hyperplanes.*

Proof. Let $(\mathcal{H}'', K'')$ be a realization of $\mathcal{M}$ in $\mathbb{R}^d$ provided by Lemma 5. The vectors normal to the central hyperplanes of $\mathcal{H}''$ generate $\mathbb{R}^d$. Now, we can extend $\mathcal{H}''$ to a hyperplane arrangement in a larger $\mathbb{R}^n$, so that the normal vectors of $\mathcal{H}'''$ are a basis of $\mathbb{R}^n$, while defining K''' just as the restriction of K'' to $\mathbb{R}^d$. Now, there exists a linear transformation f of $\mathbb{R}^n$ mapping these vectors to the coordinate vectors. Then f maps the hyperplanes of $\mathcal{H}'''$ to the coordinate hyperplanes $\mathcal{H}_n$ and the relatively open polyhedron K''' to a relatively open polyhedron K. Furthermore, each relatively open cell C''' of K''' bounded by a set $\mathcal{H}_0'''$ of hyperplanes of $\mathcal{H}'''$ is mapped to a nonempty cell C of K bounded by the set $\mathcal{H}_0$ of hyperplanes of $\mathcal{H}_n$, which are images of the hyperplanes of $\mathcal{H}_0'''$. Consequently, $(\mathcal{H}_n, K)$ realizes $\mathcal{M}$. $\square$

3.2 General properties of realizability

We continue with some properties of realizable COMs and OMs. For this we continue with the notions of restriction, contraction, and minors for COMs. Let $\mathcal{M} = (U, \mathcal{L})$ be a COM and $A \subseteq U$. Given a sign vector $X \in \{\pm 1, 0\}^U$ by $X \setminus A$ we refer to the *restriction* of X to $U \setminus A$, that is $X \setminus A \in \{\pm 1, 0\}^{U \setminus A}$ with $(X \setminus A)_e = X_e$ for all $e \in U \setminus A$. Note that in the realizable setting $\mathcal{M}(\mathcal{H}, K)$ this operation corresponds to adding some hyperplanes from $\mathcal{H}$ as halfspaces to K. The *deletion* of A is defined as $(U \setminus A, \mathcal{L} \setminus A)$, where $\mathcal{L} \setminus A := \{X \setminus A : X \in \mathcal{L}\}$. Note that in the realizable setting $\mathcal{M}(\mathcal{H}, K)$ this operation corresponds to removing some hyperplanes from $\mathcal{H}$. The *contraction* of A is defined as $(U \setminus A, \mathcal{L}/A)$, where $\mathcal{L}/A := \{X \setminus A : X \in \mathcal{L} \text{ and } \underline{X} \cap A = \varnothing\}$. Note that in the realizable setting $\mathcal{M}(\mathcal{H}, K)$ this operation corresponds to intersecting K with some hyperplanes from $\mathcal{H}$. If $\mathcal{L}'$ arises by deletions and contractions from $\mathcal{L}$, $\mathcal{L}'$ is said to be *minor* of $\mathcal{L}$. With the arguments about realizability together with [5] we get.

Lemma 7 ([5, Lemma 1])**.** *The class of realizable COMs is closed under taking minors and restrictions. Moreover, if $\mathcal{M}'$ is obtained by such operations from $\mathcal{M}$, then $\dim_{\mathbb{E}}(\mathcal{M}') \leq \dim_{\mathbb{E}}(\mathcal{M})$.*

A COM $\mathcal{M} = (U, \mathcal{L})$ is called *free* of dimension d if its tope graph is isomorphic to Q_d. A COM $\mathcal{M}$ has VC-dimension d if the topes of $\mathcal{M}$ define a set-family of VC-dimension d.

Lemma 8 ([10, Lemma 1])**.** *A COM has VC-dimension $\leq d$ if and only if it does not have a free COM of dimension $d+1$ as a minor.*

Lemma 9. *The Euclidean dimension of a free COM of dimension d is at least d.*

Proof. A hyperplane arrangement of d hyperplanes in $\mathbb{R}^{d-1}$ has less than 2^d maximal cells, see e.g., [7, Exercise 4.3.5]. Thus, the free COM (which has 2^d topes) cannot be realized in $\mathbb{R}^{d-1}$. $\square$

The previous lemmas together imply:

Theorem 10. *For every realizable COM $\mathcal{M}$, we have $\dim_{\mathrm{VC}}(\mathcal{M}) \leq \dim_{\mathbb{E}}(\mathcal{M}) \leq |U|$.*

Proof. For the lower bound, if $\mathcal{M}$ has $\dim_{\mathrm{VC}}(\mathcal{M}) = r$, then by Lemma 8 we can remove hyperplanes from its realization and obtain a realization of Q_r, which is an OM of VC-dimension r by Lemma 9, hence by Lemma 7 it has $r \leq \dim_{\mathbb{E}}(\mathrm{Q}_r) \leq \dim_{\mathbb{E}}(\mathcal{M})$. The upper bound is Proposition 6. $\square$

As a side remark we provide a quick answer to a question communicated to us privately by Kunin, Lienkaemper, and Rosen, i.e., that a non-realizable OM cannot be realized as a COM either:

Remark 11. If $\mathcal{M}$ is an OM realizable as a COM, then $\mathcal{M}$ is realizable by a central hyperplane arrangement.

Proof. Let $\mathcal{M} = (U, \mathcal{L})$ be an OM and suppose that it can be realized as a (realizable) COM $\mathcal{M} = \mathcal{M}(\mathcal{H}, K)$ in $\mathbb{R}^d$. Suppose without loss of generality that K is full-dimensional, otherwise we project into the affine hull of K. Since $\mathcal{M}$ is an OM, we have that $\mathbf{0} \in \mathcal{L}$ (recall that $\mathbf{0}$ is the all-zeros vector). Hence, there is a point $x \in K$ in which all the hyperplanes of $\mathcal{H}$ intersect. After translating x to the origin of $\mathbb{R}^d$ we see that $\mathcal{H}$ is a central hyperplane arrangement. Hence, its combinatorics is determined by any arbitrary small sphere S around x, see e.g. [7, Section 1.2.(c)]. Therefore, we can set $K = \mathbb{R}^d$ and get a a representation of $\mathcal{M}$ by a central hyperplane arrangement. $\square$

3.3 Generalized convex shellings

Convex geometries are considered as generalizations of Euclidean convexity in the following sense. Let $\mathcal{A} = (U, \mathcal{C})$ be a convex geometry. We say that $\mathcal{A}$ is *CG-realizable* if U can be viewed as a finite subset of points of $\mathbb{R}^d$ such that $X \subseteq U$ belongs to $\mathcal{C}$ if and only if the Euclidean convex hull of X does not contain other points of U: $\mathrm{conv}_{\mathbb{R}^d}(X) \cap U = X$. Not every convex geometry can be realized in this way. Kashiwabara et al. [19] gave a realization theorem for all convex geometries using the notion of generalized convex shellings. Let P and Q be two finite sets of $\mathbb{R}^d$ such that the Euclidean convex hull of Q does not intersect P. Then the family of subsets $\mathcal{C}(P, Q) = \{X \subseteq P : \mathrm{conv}(X \cup Q) \cap P = X\}$ of P is called the *generalized convex shelling on P with respect to Q*. It is easy to see that any generalized convex shelling is a convex geometry. Conversely, Kashiwabara et al. [19] proved that any convex geometry arises as a generalized convex shelling. If $Q = \varnothing$, then this leads to the notion of CG-realizability. If $\mathcal{C} = \mathcal{C}(P, Q)$, then we will say that $\mathcal{C}(P, Q)$ is a *generalized convex shelling* of $\mathcal{A} = (P, \mathcal{C})$. Richter and Rogers [28] proved that any convex geometry of convex dimension d has a generalized convex shelling in $\mathbb{R}^d$.

Extending the notation for generalized convex shellings, given two finite point sets P and Q of $\mathbb{R}^d$, let

$$\mathcal{M}(P, Q) = \{X \subseteq P : \exists \text{ halfspace } H^+ \text{ s.t. } Q \subset H^+ \text{ and } P \cap H^+ = X\}.$$

We say that (P, Q) is a *point representation* of a COM $\mathcal{M}$ if $\mathcal{M} = \mathcal{M}(P, Q)$, i.e., the set $\mathcal{T}$ of topes of $\mathcal{M}$ consists precisely of the $\{-1, +1\}$-vectors of all sets of $\mathcal{M}(P, Q)$ as in Remark 3. Trivially, $P \in \mathcal{M}(P, Q)$. Now, we establish a correspondence between (hyperplane) realizations and point realizations of COMs and a link with generalized convex shellings. For this purpose we define a COM to be *acyclic* if it has the all plus-vector $(+1, \ldots, +1)$ as a tope.

Proposition 12 (Point representations of COMs)**.** *An acyclic COM $\mathcal{M} = (U, \mathcal{L})$ is (hyperplane) realizable if and only if $\mathcal{M}$ is point realizable.*

Proof. Let $\mathcal{M} = (U, \mathcal{L})$ be an acyclic realizable COM. We have $\mathcal{M} = \mathcal{M}(\mathcal{H}', K')$ where $\mathcal{H}'$ is a central hyperplane arrangement $\mathcal{H}'$ and K' an open polyhedral cone in $\mathbb{R}^{d+1}$ as in Lemma 5. Now, the vectors v_e, normal to the hyperplanes $H'_e, e \in U$ of the arrangement $\mathcal{H}'$, gives rise to a set $\mathcal{V}$ of vectors of $\mathbb{R}^{d+1}$. Analogously, the vectors u_j, normal to the hyperplanes S'_j of $\mathcal{S}'$ defining the polyhedron K' and such that their product with the vectors with ends inside K' is positive, gives rise to a set $\mathcal{U}$ of vectors of $\mathbb{R}^{d+1}$. For a cell C of the initial realization, each point p of C defines a vector v_p belonging to the cone C'. Let H_{v_p} be the central hyperplane of $\mathbb{R}^{d+1}$ having v_p as the normal vector. Then all vectors of $\mathcal{U}$ belong to a halfspace defined by H_{v_p}. Analogously, each vector $v_e \in \mathcal{V}$ belongs to the positive side of the hyperplane H_{v_p} if and only if the cell C is located on the positive side of the hyperplane $H_e \in \mathcal{H}$ (and thus the cone C' is on the positive side of the central hyperplane $H'_e \in \mathcal{H}$). Therefore, the topes of $\mathcal{M}$ can be identified (in the set-theoretical language) with the subsets X of U such that there exists a central hyperplane H' of $\mathbb{R}^{d+1}$ such that all U belong to one halfspace defined by H' and a vector $v_e, e \in U$ belongs to the same halfspace defined by H' if and only if $e \in X$.

Now, we can return back to $\mathbb{R}^d$ by considering a hyperplane H_0 of $\mathbb{R}^{d+1}$ parallel to $\mathbb{R}^d$, which intersects all vectors of $\mathcal{V} \cup \mathcal{U}$. This hyperplane exists, since by acyclicity of $\mathcal{M}$ all elements of $\mathcal{V}$ have a positive last coordinate and the same holds for the elements of $\mathcal{U}$ by the choice of normal vectors for the defining hyperplanes of K'. Let $p_e \in H_0$ denote the point defined by the vector $v_e \in \mathcal{V}$ and let $q_j \in H_0$ denote the point defined by the vector $u_j \in \mathcal{U}$. Denote by P and Q the resulting point configurations of $\mathbb{R}^d$. The correspondence between points in the model $(\mathcal{H}', K')$ and hyperplanes in the model (P, Q) establishes that for any point p in a cell C of $(\mathcal{H}', K')$, and thus for any tope X of $\mathcal{M}(\mathcal{H}', K')$, there is a hyperplane H_p such that H_p induces the sign-pattern of X in the model (P, Q).

Conversely, the preceding transformation can be reversed. Namely, given sets P, Q in $H_0 \subseteq \mathbb{R}^{d+1}$, we can define the hyperplane arrangements $\mathcal{H}$ and $\mathcal{S}$ having P and Q as normal vectors, respectively. Letting K be the polyhedral cone defined by $\mathcal{S}$, we get a realization $\mathcal{M} = \mathcal{M}(\mathcal{H}, K)$. □

Remark 13. The essential difference between point representation of Proposition 12 provided by $\mathcal{M}(P, Q)$ and generalized convex shelling of [19] provided by $\mathcal{C}(P, Q)$ is that instead of a single halfspace H^+ containing Q and intersecting P in X, the generalized convex shelling requires that $\mathrm{conv}(X \cup Q) \cap P = X$. This is equivalent to the existence of a set of halfspaces $H_1^+, \ldots, H_k^+$, all containing Q and such that $X = P \cap (\cap_{i=1}^k H_i^+)$. (As a set of halfspaces $H_1^+, \ldots, H_k^+$ one can take the halfspaces defined by the support hyperplanes of the facets of the polytope $\mathrm{conv}(X \cup Q)$ and containing this polytope). This is a natural requirement for intersection-closed set families, because any set family admitting a generalized convex shelling is closed by intersections: if $H_1^+, \ldots, H_k^+$ and $S_1^+, \ldots, S_m^+$

are two sets of halfspaces such that $Q \subset (\cap_{i=1}^k H_i^+) \cap (\cap_{j=1}^m S_j^+)$ and $P \cap (\cap_{i=1}^k H_i^+) = X$, $P \cap (\cap_{i=1}^m S_j^+) = Y$, then $P \cap (\cap_{i=1}^k H_i^+) \cap (\cap_{i=1}^m S_j^+) = X \cap Y$.

Therefore, the hyperplane realizations and point representations of COMs are much simpler than generalized convex shellings and correspond to classical OM-realizations. The main results of our paper (Theorem 26 and Theorem 35) must be considered from this point of view, while compared to the results of [19] and [28] in the case of convex geometries. However, in contrast to our results generalized convex shellings characterize convex geometries [19].

In the following we establish a connection between point representations and generalized convex shellings. A COM $\mathcal{M}$ is called *intersection-closed* if the set of topes of $\mathcal{M}$ defines an intersection-closed family of sets. Note that by simplicity this implies that $(-1,\dots,-1)$ is a tope of $\mathcal{M}$. Obviously, convex geometries are acyclic and their ideals are intersection closed COMs. Acyclicity corresponds to the universe being convex, i.e., $(+1,\dots,+1)$ is a tope of $\mathcal{M}$.

Proposition 14. *Let $P, Q \subset \mathbb{R}^d$ be finite. If $\mathcal{M} = \mathcal{M}(P,Q)$ is a simple acyclic intersection-closed COM, then $\mathcal{C}(P,Q)$ is a generalized convex shelling. In particular, $\mathcal{M}$ is a convex geometry.*

Proof. Since $\mathcal{M} = \mathcal{M}(P,Q)$ contains the tope $(-1,\dots,-1)$, necessarily $\mathrm{conv}(Q) \cap P = \varnothing$. From the definition of the point representations and generalized convex shellings, we conclude that $\mathcal{M} \subseteq \mathcal{C}(P,Q)$. To prove the converse inclusion $\mathcal{C}(P,Q) \subseteq \mathcal{M}$, pick any set X from $\mathcal{C}(P,Q)$. Then there exists a set $H_1^+,\dots,H_k^+$ of halfspaces, all containing Q and such that $X = P \cap (\cap H_i^+)$. Let $X_i = P \cap H_i^+, i = 1,\dots,k$. Then each X_i belongs to $\mathcal{M}(P,Q)$. Since $\mathcal{M}(P,Q) = \mathcal{M}$, each X_i belongs to $\mathcal{M}$. Since $\mathcal{M}$ is intersection-closed and $\cap_{i=1}^k X_i = X$, we conclude that $X \in \mathcal{M}$, establishing that $\mathcal{C}(P,Q) \subseteq \mathcal{M}$. $\square$

4 Convex geometries and their ideals

In this section, we present further results about convex geometries and their ideals that will be used in our proofs. We also present examples of ideals of convex geometries.

4.1 Rooted circuits

In the proof of realizability of convex geometries, we will use the characterization of convex geometries via rooted circuits and critical rooted circuits. Let $\mathcal{A} = (U,\mathcal{C})$ be a convex geometry. A *rooted set* is a pair (C,r) consisting of a subset C of U and an element r of C. A convex geometry $\mathcal{C}$ is *reconstructed* from a collection of rooted sets $\mathcal{F}$ if $\mathcal{C} = \{X \subseteq U : (C,r) \in \mathcal{F} \Rightarrow X \cap C \neq C \setminus \{r\}\}$, i.e., $X \subseteq U$ belongs to $\mathcal{C}$ if and only if no rooted set $(C,r) \in \mathcal{F}$ meets X in $C \setminus \{r\}$. A rooted set (C,r) is a *rooted circuit* [23] of $\mathcal{C}$ if $\mathcal{C}|_C = 2^C \setminus \{C \setminus \{r\}\}$. Denote by $\mathcal{R}(\mathcal{A})$ the set of all rooted circuits of $\mathcal{A}$. $\mathcal{R}(\mathcal{A})$ has the following properties established in [23] (for proofs, see Lemma 2 and Proposition 4 of [11]):

Theorem 15 ([23,24])**.** *Let $\mathcal{A} = (U,\mathcal{C})$ be a convex geometry. Then:*

(i) If (C,r) is a rooted circuit of $\mathcal{C}$, then $r \in \mathrm{conv}(C \setminus \{r\})$;

(ii) $\mathcal{A}$ can be reconstructed from the family $\mathcal{R}(\mathcal{A})$ of its rooted circuits.

Dietrich [11] provided the following axiomatization of convex geometries via the rooted circuits:

Theorem 16 ([11])**.** *Let $\mathcal{R}$ be a set of rooted subsets of a finite set U. Then $\mathcal{R}$ is the set of rooted circuits of a convex geometry if and only if $\mathcal{R}$ satisfies the following two properties:*

(1) $(C_1,r_1),(C_2,r_2) \in \mathcal{R}$ and $C_1 \subseteq C_2$ implies $C_1 = C_2$ and $r_1 = r_2$;

(2) $(C_1,r_1),(C_2,r_2) \in \mathcal{R}$ and $r_1 \in C_2 \setminus \{r_2\}$ implies that there exists $(C_3,r_2) \in \mathcal{R}$ such that $C_3 \subseteq (C_1 \cup C_2) \setminus \{r_1\}$.

Korte and Lovász [23] (see also the book [24]) identified a canonical subset of rooted circuits, which they call critical rooted circuits. In terms of convex geometries, a rooted circuit (C,a) of a convex geometry $\mathcal{A} = (U,\mathcal{C})$ is a *critical rooted circuit* if $\mathrm{conv}(C) \setminus \{a\}$ does not belong to $\mathcal{C}$ but $\mathrm{conv}(C) \setminus \{a,b\}$ belongs to $\mathcal{C}$ for any $b \in C \setminus \{a\}$. Denote by $\mathcal{R}_0(\mathcal{A})$ the set of all critical rooted circuits of $\mathcal{A}$. The importance of the family $\mathcal{R}_0(\mathcal{A})$ of critical rooted circuits stems from the fact that $\mathcal{A}$ can be uniquely determined by $\mathcal{R}_0(\mathcal{A})$ and $\mathcal{R}_0(\mathcal{A})$ is minimal with respect to this property. This is: if $\mathcal{A}$ can be determined by $\mathcal{R}' \subseteq \mathcal{R}(\mathcal{A})$, then $\mathcal{R}_0(\mathcal{A}) \subseteq \mathcal{R}'$ [23, 24] (for a precise definition of "determined by", see [24]).

To each rooted set (C, r) of a convex geometry $\mathcal{A} = (U, \mathcal{C})$ we can associate the cube $Q(C, r)$: $Q(C, r)$ consists of all subsets of U of the form $(C \setminus r) \cup X'$ with $X' \subseteq U \setminus C$. The cube $Q(C, r)$ can be encoded by the sign vector $X(C, r)$, where $X_e(C, r) = +1$ if $e \in C \setminus \{r\}$, $X_r(C, r) = -1$, and $X_e(C, r) = 0$ if $e \in U \setminus C$. If (C, r) is a rooted circuit, then the trace of $\mathcal{C}$ on C coincides with $2^C \setminus \{C \setminus \{r\}\}$, thus the cube $Q(C, r)$ belongs to the complement $\mathcal{A}^*$ of $\mathcal{A}$ and, furthermore, $Q(C, r)$ is a maximal by inclusion cube of $\mathcal{A}^*$. Consequently, $Q(C, r)$ is an $(U \setminus C)$-cube of $\mathcal{A}^*$. The converse also holds:

Lemma 17. *Let $\mathcal{A} = (U, \mathcal{C})$ be a convex geometry. If Q is a maximal cube of $\mathcal{A}^*$, then there exists $(C, r) \in \mathcal{R}(\mathcal{A})$ such that $Q = Q(C, r)$.*

Proof. Suppose that Q consists of the sets $\{Y \cup X' : X' \subseteq X\}$ for $Y \subseteq U \setminus X$. Since $Y \notin \mathcal{C}$, by Theorem 15(ii), there exists a rooted circuit (C, r) such that $Y \cap C = C \setminus \{r\}$. But then Q is contained in the cube $Q(C, r)$ of $\mathcal{A}^*$ whose vertices are the sets $\{(C \setminus \{r\}) \cup X' : X' \subseteq U \setminus C\}$. Since Q is maximal, $Q = Q(C, r)$. □

4.2 Positive circuits

In case of ideals of convex geometries, along with rooted circuits we also have to define positive circuits. Let $\mathcal{I} = (U', \mathcal{C}')$ be an ideal of a convex geometry $\mathcal{A} = (U, \mathcal{C})$. A *positive circuit* of $\mathcal{I}$ *with respect to* $\mathcal{A}$ is a subset $P \subseteq U$ such that $\mathcal{C}'_{|P} = 2^P \setminus \{P\}$. In particular, for any $e \in U \setminus U'$, $\{e\}$ is a positive circuit of $\mathcal{I}$ with respect to $\mathcal{A}$. Given an ideal $\mathcal{I}$ of a convex geometry $\mathcal{A}$, we denote by $\mathcal{P}(\mathcal{I})$ the set of positive circuits of $\mathcal{I}$ with respect to $\mathcal{A}$. We now show that $\mathcal{C}'$ is precisely the set of elements of $\mathcal{C}$ not containing positive circuits.

Lemma 18. *For any ideal $\mathcal{I} = (U', \mathcal{C}')$ of a convex geometry $\mathcal{A} = (U, \mathcal{C})$, we have*

$$\mathcal{C}' = \{X \in \mathcal{C} : \forall P \in \mathcal{P}(\mathcal{I}), P \not\subseteq X\}.$$

Proof. Let $\mathcal{C}'' = \{X \in \mathcal{C} : \forall P \in \mathcal{P}(\mathcal{I}), P \not\subseteq X\}$. Pick any $X \in \mathcal{C}'$ and any $P \subseteq X$. Then $P \cap X = P \in \mathcal{C}'_{|P}$ and thus $\mathcal{C}'_{|P} \neq 2^P \setminus \{P\}$. This establishes that $P \notin \mathcal{P}(\mathcal{I})$ for any $P \subseteq X$ and consequently $X \in \mathcal{C}''$. Therefore $\mathcal{C}' \subseteq \mathcal{C}''$.

Suppose now that there exists $Y \in \mathcal{C}'' \setminus \mathcal{C}'$ and assume that Y is such a set of minimal size. By minimality of Y, for any $e \in Y$, $Y \setminus \{e\} \in \mathcal{C}''$ if and only if $Y \setminus \{e\} \in \mathcal{C}'$. Let $P_0 = \{e \in Y : Y \setminus \{e\} \in \mathcal{C}'\}$. Since $\mathcal{C}'$ is intersection-closed, for any $\varnothing \subsetneq P' \subseteq P_0$, we have $Y \setminus P' \in \mathcal{C}'$. This shows that $2^{P_0} \setminus \{P_0\} \subseteq \mathcal{C}'_{|P_0} \subseteq 2^{P_0}$. Since $P_0 \subseteq Y$ and since $Y \in \mathcal{C}''$, by the definition of $\mathcal{C}''$, we get $P_0 \notin \mathcal{P}(\mathcal{I})$ and thus $\mathcal{C}'_{|P_0} = 2^{P_0}$ by the definition of $\mathcal{P}(\mathcal{I})$. Consequently, there exists $X \in \mathcal{C}'$ such that $P_0 \subseteq X$. Let $Z = X \cap Y$ and observe that $P_0 \subseteq Z$. Since $\mathcal{C}$ is intersection-closed, $Z \in \mathcal{C}$. If $Z = Y$, then $Y \subseteq X \in \mathcal{C}'$, contradicting the fact that $\mathcal{C}'$ is an ideal of $\mathcal{C}$. Thus we have $Z \subsetneq Y$. Since $Z \in \mathcal{C}'$ and $Y \notin \mathcal{C}'$, there exists a set $Z \subseteq A \subseteq Y$ and an element $e \in Y \setminus A$ such that $A \in \mathcal{C}'$ and $B = A \cup \{e\} \in \mathcal{C} \setminus \mathcal{C}'$ (A and e exist since $\mathcal{C}$ is ample and thus isometric). By minimality of Y, $B = Y$. Consequently $e \in P_0$ by the definition of P_0, but this is impossible since $P_0 \subseteq Z \subseteq A$. This establishes that $\mathcal{C}'' \subseteq \mathcal{C}'$. □

4.3 Convex dimension

Edelman and Jamison [13] provided a nice characterization of convex geometries via order convexity. Given a universe $U = \{e_1, \ldots, e_n\}$, a total order $<$ on U, and the reflexive closure $\leq$ of $<$, call an *ending interval* of $\leq$ any set of the form $\{e \in U : e_i \leq e\}$ for some $e_i \in U$. Given a set $\Upsilon = \{\leq_1, \ldots, \leq_d\}$ of d total orders on U, we say that a set family $\mathcal{C} \subseteq 2^U$ is *generated* by the set Υ if $\varnothing \in \mathcal{C}$ and a nonempty set C belongs to $\mathcal{C}$ if and only if C is the intersection of d ending intervals, one from each order $\leq_i$ of Υ, i.e., if there exists $(e_i)_{1 \leq i \leq d} \in U^d$ such that $C = \{e \in U : e_i \leq_i e, \forall 1 \leq i \leq d\}$. Edelman and Jamison proved the following result:

Theorem 19 ([13, Theorem 5.2])**.** *Any set family $\mathcal{C} \subseteq 2^U$ generated by a set of total orders on U is a convex geometry and, conversely, any convex geometry $\mathcal{A} = (U, \mathcal{C})$ can be generated by a set of total orders on U.*

The *convex dimension* $\mathrm{cdim}(\mathcal{A})$ of a convex geometry $\mathcal{A} = (U, \mathcal{C})$ is the least number of total orders generating $\mathcal{C}$.

4.4 (Ideals of) convex geometries as meet-(semi)lattices

A poset $L = (X, \leq)$ is a *meet-semilattice* if for every $x, y \in L$ there is unique largest element $x \wedge y \in L$ such that $x \wedge y \leq x, y$; $x \wedge y$ is called the *meet* of x and y. A poset $L = (X, \leq)$ is a *join-semilattice* if for every $x, y \in L$ there is unique smallest element $x \vee y \in L$ such that $x \vee y \geq x, y$; $x \vee y$ is called the *join* of x and y. A poset is a *lattice* if it is both a join- and a meet-semilattice. Note that both join and meet are associative and commutative operations, so in order to take the join or meet over all the elements of a set Y we sometimes just write $\bigvee Y$ and $\bigwedge Y$, respectively. Note that if a poset has a global minimum $\hat{0}$ or a global maximum $\hat{1}$, then we set $\bigvee \emptyset = \hat{0}$ and $\bigwedge \emptyset = \hat{1}$, respectively. An element $x \in L$ is called *join-irreducible* if $x = \bigvee Y$ implies $x \in Y$ for all $Y \subseteq L$. The set of join-irreducible elements is denoted by $\mathcal{J}(L)$. Similarly, an element $x \in L$ is called *meet-irreducible* if $x = \bigwedge Y$ implies $x \in Y$ for all $Y \subseteq L$. The set of meet-irreducible elements is denoted $\mathcal{M}(L)$. Following Dilworth a *lower locally distributive lattice* (LLD) is a lattice L such that for all $x \in L$ there is a unique inclusion-minimal $J_x \subseteq \mathcal{J}(L)$ such that $x = \bigvee J_x$ (in fact, Dilworth introduced the dual lattices, the *upper locally distributive lattices (ULD)*). This definition is equivalent to Edelman's notion of semi-distributive lattice, but since there are several notions of semidistributivity in the literature, we prefer the name LLD. We refer to [20, 27, 32] for different equivalent characterizations, different names, as well as many different instances of LLDs and ULDs. We continue with the characterization of lattices of convex geometries provided by Edelman [12]:

Theorem 20 ([12, Theorem 3.3])**.** *A lattice is lower locally distributive if and only if it is isomorphic to the inclusion-order of convex sets of a convex geometry.*

The meet-irreducible elements of $L = (\mathcal{C}, \subseteq)$ correspond to the copoints of the convex geometry $\mathcal{A} = (U, \mathcal{C})$; a *copoint* attached at point p is a maximal by inclusion convex set of $\mathcal{C}$ not containing p. Edelman and Jamison [13] characterized convex geometries as the convexity spaces for which each copoint has a unique attaching point. Furthermore, Edelman and Saks [14] characterized the convex dimension $\mathrm{cdim}(\mathcal{C})$ of a convex geometry $\mathcal{A}$ in term of the poset of all meet-irreducibles in the following nice way:

Theorem 21 ([14])**.** *For a convex geometry $\mathcal{A} = (U, \mathcal{C})$, $\mathrm{cdim}(\mathcal{C})$ is equal to the size of the largest antichain of the poset $(\mathcal{M}(L), \subseteq)$, where $L = (\mathcal{C}, \subseteq)$.*

An *ideal* I of a poset is a subset that is downwards-closed, i.e., if $x \leq y$ and $y \in I$, then $x \in I$. From Theorem 20 by definition we get a justification of the name *ideals* of convex geometries:

Remark 22. A meet-semilattice is an ideal of a lower locally distributive lattice if and only if it is isomorphic to the inclusion-order of convex sets of an ideal of a convex geometry.

4.5 Examples of ideals of convex geometries

Edelman and Jamison [13] and Korte et al. [24] presented numerous examples of convex geometries and antimatroids, arising from geometry, language theory, and chip firing games. Bandelt et al. [4] presented simplicial complexes and median set systems as examples of bouquets of convex geometries. We show that multisimplicial complexes and median set systems are ideals of particular convex geometries, called downset alignments.

A *simplicial complex* L on a set U is a family of subsets of U, called *simplices* or *faces* of L, such that if $\sigma \in L$ and $\sigma' \subseteq \sigma$, then $\sigma' \in L$. The *facets* of L are the maximal (by inclusion) faces of L. The *dimension* d of L is the size of its largest face minus one. A multi-subset σ of U is a subset of U such that each element $e \in U$ is given with its multiplicity $n_\sigma(e)$ (the number of times, e occurs in σ). A *multisimplicial complex* L on a set U is a family of multi-subsets of U, such that if $\sigma \in L$ and σ' is a multi-subset of U such that $n_{\sigma'}(e) \leq n_\sigma(e)$ for each $e \in U$, then $\sigma' \in L$. The *size* of a face σ of L is the size of $\{e \in U : n_\sigma(e) > 0\}$. The *dimension* d of L is the size of its largest face minus one.

A *median set system* is a set family $(U, \mathcal{C})$ satisfying (C1′), (C2), and

(C6) for any $x \neq y$ in U, there exists some $K \in \mathcal{C}$ with $|\{x, y\} \cap K| = 1$,

(C7) $K_i, M_i \in \mathcal{C}$ $(i = 1, 2, 3)$ with $K_i \cup K_j \subseteq M_k$ for $\{i, j, k\} = \{1, 2, 3\}$ implies $K_1 \cup K_2 \cup K_3 \in \mathcal{C}$.

The name is justified by the fact that by virtue of (C2) and (C7), $\mathcal{C}$ is closed under the median operation m of 2^U defined by

$$m(L_1, L_2, L_3) := (L_1 \cap L_2) \cup (L_1 \cap L_3) \cup (L_2 \cap L_3).$$

The resulting meet-semilattice is called a *median semilattice* [3,6,30]. Median semilattices are locally lower distributive. Indeed, by a result of Birkhoff and Kiss [6] they can be characterized by the property that all principal ideals $\downarrow x$ are distributive lattices and three elements have an upper bound whenever each pair of them does.

Every abstract finite median algebra (for which the former set-theoretic ternary operation is axiomatized) or, equivalently, any median graph can be represented by a median set system via the Sholander embedding [30] into some power set 2^U. An inherent feature of median algebras/graphs is that they may be oriented so that any element can serve as the empty set in the associated set representation: a median set system $\mathcal{C}$ is mapped onto another one $\mathcal{C} \triangle Z := \{A \triangle Z : K \in \mathcal{C}\}$, by the automorphism of 2^U taking the symmetric difference with a fixed set $Z \in \mathcal{C}$.

Let $\leq$ be a partial order on U. The *downset alignment* $\mathcal{D}_P$ [13] of the poset $P = (U, \leq)$ consists of all ideals of P. It was noticed in [13] that the downset alignments are convex geometries and it was shown in [13, Theorem 3.2] that a convex geometry $\mathcal{A} = (U, \mathcal{C})$ is a downset alignment if and only if $\mathcal{C}$ is union-closed. From the lattice point of view, downset alignments simply correspond to distributive lattices.

We now define bouquets of downset alignments in the same vein as bouquets of convex geometries were defined. We say that a pair $\mathcal{A} = (U, \mathcal{C})$ is a *bouquet of downset alignments* if $\mathcal{C}$ satisfies (C1′), (C2), (C3′), and

(C8) for all $X, Y \in \mathcal{C}$, if there exists $Z \in \mathcal{C}$ such that $X \cup Y \subseteq Z$, then $X \cup Y \in \mathcal{C}$. (locally union-closed)

Note that bouquets of downset alignments are precisely bouquets of convex geometries with the additional property of being locally union-closed. Note also that principal ideals of bouquets of downset alignments are downset alignments and that ideals of downset alignments are bouquets of downset alignments. Differently from bouquets of convex geometries, that are not always ideals of convex geometries by Proposition 4, the next result shows that bouquets of downset alignments and ideals of downset alignments coincide:

Theorem 23. *Every bouquet of downset alignments $\mathcal{A} = (U, \mathcal{C})$ with ℓ maximal convex sets is an ideal of a downset alignment $\mathcal{A}^+$ such that* $\text{VC-dim}(\mathcal{A}^+) \leq \ell \cdot \text{VC-dim}(\mathcal{C})$.

Proof. Consider a bouquet of downset alignments $\mathcal{A} = (U, \mathcal{C})$. Let $X_1, \ldots, X_\ell$ be the maximal convex sets of $\mathcal{A}$ and let $\downarrow X_1, \ldots, \downarrow X_\ell$ be their principal ideals. For each $1 \leq i \leq \ell$, let $d_i = \text{VC-dim}(\downarrow X_i)$. We show by induction on ℓ that $\mathcal{A}$ is an ideal of a downset alignment $\mathcal{A}^+$ such that $\text{VC-dim}(\mathcal{A}^+) \leq \sum_{i=1}^{\ell} d_i$.

If $\ell = 1$, then $\mathcal{A}$ is a downset alignment and we are done. Otherwise, take an element $M \in \mathcal{C}$ that can be written as the intersection of at least two maximal convex sets of $\mathcal{A}$, and that is inclusion maximal for this property. Let $\mathcal{S}$ be the set of all maximal convex sets of $\mathcal{A}$ containing M, and assume without loss of generality, that $\mathcal{S} = \{X_1, \ldots, X_k\}$. Consequently, $M = \bigcap_{i=1}^{k} X_i$. Let $\mathcal{C}'_0$ be the set of all sets of the form $Y_1 \cup \ldots \cup Y_k$ where for each $0 \leq i \leq k$, $Y_i \in \mathcal{C}$ and $M \subseteq Y_i \subseteq X_i$. Now define $\mathcal{C}'$ as $\mathcal{C} \cup \mathcal{C}'_0$ and set $\mathcal{A}' = (U, \mathcal{C}')$. Observe that the maximal convex sets of $\mathcal{A}'$ are the sets $X_{k+1}, \ldots, X_\ell$, and $X' = \bigcup_{i=1}^{k} X_i$.

We first show that $\mathcal{C}'$ is still a bouquet of downset alignments, i.e., that $\mathcal{C}'$ satisfies (C1′), (C2), (C3′) and (C8). Observe that $\varnothing \in \mathcal{C} \subseteq \mathcal{C}'$, establishing (C1′). To prove that $\mathcal{C}'$ is intersection-closed, let $A, B \in \mathcal{C}'$. If $A, B \in \mathcal{C}$, then there is nothing to show, so suppose first $A \in \mathcal{C}$ and $B \in \mathcal{C}' \setminus \mathcal{C}$. Hence we can represent $B = Y_1 \cup \ldots \cup Y_k$ and we consider $A \cap (Y_1 \cup \ldots \cup Y_k) = (A \cap Y_1) \cup \ldots \cup (A \cap Y_k)$. Since all of these sets are subsets of A, since $\mathcal{C}$ is locally union-closed, their union is in $\mathcal{C}$. If $A, B \in \mathcal{C}' \setminus \mathcal{C}$, then $A = Y_1 \cup \ldots \cup Y_k$ and $B = Y'_1 \cup \ldots \cup Y'_k$ and $(Y_1 \cup \ldots \cup Y_k) \cap (Y'_1 \cup \ldots \cup Y'_k) = Y_1 \cap (Y'_1 \cup \ldots \cup Y'_k) \cup \ldots Y_k \cap (Y'_1 \cup \ldots \cup Y'_k)$ but for each i here we have $M \subseteq Y_i \cap (Y'_1 \cup \ldots \cup Y'_k) \subseteq X_i$, hence their union is in $\mathcal{C}'$ by definition of $\mathcal{C}'$.

In order to show that $\mathcal{C}'$ is locally union-closed let $A, B \in \mathcal{C}'$ and let $A, B \subseteq Z$. Since we can assume that $A \notin \mathcal{C}$, without loss of generality we can set $Z = \bigcup_{i=1}^{k} X_i$ and $A = Y_1 \cup \ldots \cup Y_k$ as before. Now, $B = (B \cap X_1) \cup \ldots \cup (B \cap X_k)$, where each term is in $\mathcal{C}$. Then, $A \cup B = (Y_1 \cup (B \cap X_1) \cup \ldots \cup Y_k \cup (B \cap X_k))$, where each term is in $\mathcal{C}$ by local union-closedness and each term contains M as a subset. Hence by definition of $\mathcal{C}'$, $A \cup B \in \mathcal{C}'$.

Now, let us show that $\mathcal{C}'$ is locally extendable. For this purpose let $A, B \in \mathcal{C}'$ be distinct sets such that B is a maximal convex set and $A \subseteq B$. If $B \in \mathcal{C}$, then since $\downarrow B$ does not contains any set of $\mathcal{C}'_0$, also $A \in \mathcal{C}$ and there is nothing to show. So suppose that $B \in \mathcal{C}' \setminus \mathcal{C}$, i.e., that $B = X' = \bigcup_{i=1}^{k} X_i$ (since B is a maximal convex set of $\mathcal{C}'$). If $A \in \mathcal{C}$, then $A \cap X_i \subsetneq X_i$ for some $1 \leq i \leq k$, and thus, there is an element $e \in X_i \setminus A \subseteq B \setminus A$ such that $(A \cap X_i) \cup e \in \mathcal{C}$ since $\mathcal{C}$ is locally extendable. Now, $\mathcal{C}'$ is locally union-closed also $A \cup (A \cap X_i) \cup e = A \cup e \in \mathcal{C}'$. If $A \in \mathcal{C}' \setminus \mathcal{C}$, then $A = Y_1 \cup \ldots \cup Y_k$ with $Y_i \subseteq X_i$ for $1 \leq i \leq k$ and $Y_j \subsetneq X_j$ for some $1 \leq j \leq k$ (since $A \subsetneq B = X'$). Since $\mathcal{C}$ is locally extendable, there is $e \in X_j \setminus Y_j \subseteq B \setminus A$ such that $Y_j \cup e \in \mathcal{C}$. But then by definition of $\mathcal{C}'$, we have that $A \cup e \in \mathcal{C}'$.

Let us now show that $\mathcal{C}$ is an ideal of $\mathcal{C}'$. For this, suppose there exist $X \in \mathcal{C}$ and $R \in \mathcal{C}'_0 = \mathcal{C}' \setminus \mathcal{C}$ such that $R \subseteq X$. We can assume that X is a maximal convex set and since $M \subseteq R$ by definition of $\mathcal{C}'_0$, X necessarily belongs to $\mathcal{S}$

and we can assume that $X = X_1$. Let $R = Y_1 \cup \ldots \cup Y_k \subseteq X_1$ where for all $1 \le i \le k$, $M \subseteq Y_i \subseteq X_i$ and $Y_i \in C$. Since $R \notin \mathcal{C}$, necessarily $Y_1 \subsetneq R$ and thus we can assume that $M \subsetneq Y_2$. Since, $Y_2 \subseteq R \subseteq X_1$ and $Y_2 \subseteq X_2$, we have $M \subsetneq Y_2 \subseteq X_1 \cap X_2$, contradicting the maximality of M.

We now show that $\text{VC-dim}(\downarrow X') \le \sum_{i=1}^{k} \text{VC-dim}(\downarrow X_i)$. Consider a subset $Z \subseteq X'$ with $|Z| = \text{VC-dim}(\downarrow X')$ that is shattered by $\downarrow X'$. For each $1 \le i \le k$, let $Z_i = X \cap X_i$. We claim that Z_i is shattered by $\downarrow X_i$. Indeed, for any $Z'' \subseteq Z_i \subseteq Z$, there exists $X'' \in \downarrow X'$ such that $X'' \cap Z = Z''$. Observe that $(X'' \cap X_i) \cap Z_i = X'' \cap X_i \cap Z = Z''$ since $X'' \cap Z = Z'' \subseteq Z_i \subseteq X_i$. Note also that $X'' \cap X_i \in \mathcal{C}'$ and thus $X'' \cap X_i \in \downarrow X_i$. Consequently, Z_i is shattered by $\downarrow X_i$. Therefore, since $Z = \cup_{i=1}^{k} Z_i$, $\text{VC-dim}(\downarrow X') = |Z| \le \sum_{i=1}^{k} |Z_i| \le \sum_{i=1}^{k} \text{VC-dim}(\downarrow X_i)$, and we are done.

Consequently, $\mathcal{A}'$ is a bouquet of downset alignments, $\mathcal{A}$ is an ideal of $\mathcal{A}'$, and the maximal convex sets of $\mathcal{A}'$ are the sets $X_{k+1}, \ldots, X_\ell$, and $X' = \bigcup_{i=1}^{k} X_i$. Since $k \ge 2$, $\mathcal{C}'$ has less inclusion-maximal convex sets than $\mathcal{C}$. By induction hypothesis, $\mathcal{A}'$ is an ideal of a downset alignment $\mathcal{A}^+$ such that $\text{VC-dim}(\mathcal{A}^+) \le \text{VC-dim}(\downarrow X') + \sum_{i=k+1}^{\ell} \text{VC-dim}(\downarrow X_i) \le \sum_{i=1}^{k} \text{VC-dim}(\downarrow X_i) + \sum_{i=k+1}^{\ell} \text{VC-dim}(\downarrow X_i) = \sum_{i=1}^{\ell} \text{VC-dim}(\downarrow X_i)$. By transitivity, $\mathcal{A}$ is an ideal of $\mathcal{A}^+$. This concludes the proof. □

The grid $\mathbb{N}^n$ can be viewed as the covering graph of the poset $(\mathbb{N}^n, \le)$, where for $x, y \in \mathbb{N}^n$ with $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, we set $x \le y$ if and only $x_i \le y_i$ for $i = 1, \ldots, n$. Since ideals of the grid $(\mathbb{N}^n, \le)$ are ample, the VC-dimension $\dim_{\text{VC}}(L)$ of any such ideal L is the largest subcube of L. Notice that $\dim_{\text{VC}}(L)$ can be smaller than the dimension n of the grid. It is obvious that there exists a bijection between the ideals of the grid $(\mathbb{N}^n, \le)$ and the multisimplicial complexes on a set X of size n. The simplicial complexes are in bijection with the ideals of the n-cube $\{0,1\}^n \subset \mathbb{N}^n$. The VC-dimension of any (multi)simplicial complex coincides with its dimension. Hence, we can apply Theorem 23 to obtain the following result:

Corollary 24. *Every multisimplicial complex of dimension d and ℓ facets is an ideal of a downset alignment of VC-dimension at most ℓd.*

We get an analogous bound for median set systems (since they principal ideals are distributive lattices, median set systems are bouquets of downset alignments):

Corollary 25. *Every median set system of VC-dimension d and having ℓ maximal elements is an ideal of a downset alignment of VC-dimension at most ℓd.*

5 Realization of convex geometries and of their ideals

In this section we prove the main results of the paper: any ideal $\mathcal{I}$ of $\mathcal{A}$ of a convex geometry $\mathcal{A} = (U, \mathcal{C})$ admits a realization $(\mathcal{H}_n, K(\mathcal{R}_0))$ in $\mathbb{R}^n$, where $n = |U|$, $\mathcal{H}_n$ are the coordinate hyperplanes of $\mathbb{R}^n$, and the number of facets of $K(\mathcal{R}_0)$ is at most the number of critical rooted circuits of $\mathcal{A}$ and positive circuits of $\mathcal{I}$. In particular, this yields a representation for convex geometries.

Furthermore, we prove that each convex geometry is realizable in dimension equal to its convex dimension. Finally, we prove that trees and multisimplicial complexes have realizations in dimension bounded linearly in their VC-dimension.

5.1 Realization of ideals of convex geometries

Let $U = \{1, \ldots, n\}$ and let $\mathcal{A} = (U, \mathcal{C})$ be a convex geometry on U having $\mathcal{R} := \mathcal{R}(\mathcal{A})$ as the set of rooted circuits and $\mathcal{R}_0 := \mathcal{R}_0(\mathcal{A})$ as the set of critical rooted circuits. For each rooted circuit (C, r) of $\mathcal{A}$ consider the hyperplane $H(C, r)$ defined by the equation $nx_r = \sum_{e \in C \setminus \{r\}} x_e$. Denote by $K(C, r)$ the open halfspace of $\mathbb{R}^n$ determined by $H(C, r)$ as

$$K(C, r) = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : nx_r > \sum_{e \in C \setminus \{r\}} x_e \right\}.$$

Let $K(\mathcal{R}) = \bigcap_{(C,r) \in \mathcal{R}} K(C, r)$ and $K(\mathcal{R}_0) = \bigcap_{(C,r) \in \mathcal{R}_0} K(C, r)$ be the open polyhedra that are respectively the intersection of all $K(C, r)$ taken over all rooted circuits (C, r) of $\mathcal{A}$ and the intersection of all $K(C, r)$ taken over all critical rooted circuits (C, r) of $\mathcal{A}$. Obviously, $K(\mathcal{R}) \subseteq K(\mathcal{R}_0)$.

Consider now an ideal $\mathcal{I} = (U', \mathcal{C}')$ of the convex geometry $\mathcal{A} = (U, \mathcal{C})$. Let $\mathcal{P} := \mathcal{P}(\mathcal{I})$ be the set of positive circuits of $\mathcal{I}$ with respect to $\mathcal{A}$. For each $P \in \mathcal{P}$, the hyperplane $H(P)$ is defined by the equation $\sum_{e\in P} x_e = 0$. Denote by $K(P)$ the open halfspace of $\mathbb{R}^n$ determined by $H(P)$ as

$$K(P) = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{e\in P} x_e < 0 \right\}.$$

Finally, consider the open polyhedra $K(\mathcal{P}) = \bigcap_{P\in\mathcal{P}} K(P)$, $K = K(\mathcal{R}) \cap K(\mathcal{P})$, and $K_0 = K(\mathcal{R}_0) \cap K(\mathcal{P})$. Clearly, $K \subseteq K_0 \subseteq K(\mathcal{P})$.

Theorem 26. *For a convex geometry $\mathcal{A} = (U, \mathcal{C})$, the pairs $(\mathcal{H}_n, K(\mathcal{R}))$ and $(\mathcal{H}_n, K(\mathcal{R}_0))$ are realizations of $\mathcal{A}$. More generally, for any ideal $\mathcal{I} = (U', \mathcal{C}')$ of $\mathcal{A}$, the pairs $(\mathcal{H}_n, K)$ and $(\mathcal{H}_n, K_0)$ are realizations of $\mathcal{I}$.*

The proof of the first assertion is a direct consequence of the second assertion. The proof that $(\mathcal{H}_n, K(\mathcal{R}))$ and $(\mathcal{H}_n, K)$ are realizations of $\mathcal{A}$ and $\mathcal{I}$ is inspired by (but is simpler than) the proof of Kashiwabara et al. [19] that convex geometries can be represented via generalized convex shellings. The proof of the second assertion follows from four lemmas, which we prove first. We first show that for any $X \in \mathcal{C}'$, the convex set K intersects the X-orthant $\mathcal{O}(X)$ of $\mathbb{R}^n$.

Lemma 27. *For any $X \in \mathcal{C}'$, the X-orthant $\mathcal{O}(X)$ of $\mathbb{R}^n$ intersects the convex polyhedron K.*

Proof. We prove that the X-orthant $\mathcal{O}(X)$ intersects the convex set K, i.e., that there exists a point $x^* = (x^*_1, \ldots, x^*_n) \in \mathcal{O}(X) \cap K$. Let $|X| = k$ and suppose without loss of generality that $X = \{e_1, \ldots, e_k\}$. First, we set $x^*_{e_1} = x^*_{e_2} = \cdots = x^*_{e_k} = 1$. In the following, we set a negative value for each $x^*_e, e \notin X$. Observe that for any rooted circuit (C, r) such that $C \subseteq X$, we have $nx^*_r = n = |U| > (|C| - 1) = \sum_{e\in C\setminus\{r\}} x_e$ and thus $x^* \in K(C, r)$, independently of the values of the remaining $n - k$ coordinates of x^*. Notice also that, since X belongs to $\mathcal{C}'$, X does not contain any positive circuit of $\mathcal{C}'$. Since $X \in \mathcal{C}$, by axiom (C3) of convex geometries, the elements of $U \setminus X$ can be ordered $e_{k+1}, \ldots, e_n$, such that all the sets $X_i = X \cup \{x_{k+1}, \ldots, x_{k+i}\} = X_{i-1} \cup \{x_{k+i}\}, i = 1, \ldots, n - k$ are convex sets of $\mathcal{C}$. We define the remaining $n - k$ coordinates of x^* (which have to be negative) following the order $e_{k+1}, \ldots, e_n$. Suppose that after $i - 1$ steps the coordinates $x^*_{e_{k+1}}, \ldots, x^*_{e_{k+i-1}}$ have been defined. At stage i we need to define the value of $x^*_{e_{k+i}}$ in order to satisfy

- all constraints of the form $nx^*_r > \sum_{e\in C\setminus\{r\}} x^*_e$, where $(C, r) \in \mathcal{R}(\mathcal{A})$, $e_{k+i} \in C$, and $C \subset X_{k+i}$;
- all constraints of the form $\sum_{e\in P} x^*_e < 0$, where $P \in \mathcal{P}$, $e_{k+i} \in P$, and $P \subset X_{k+i}$.

For any rooted circuit (C, r) such that $e_{k+i} \in C \subseteq X_{k+i}$, the values of x^*_e for all elements $e \in C \setminus \{e_{k+i}\}$ have been already defined. Since $X_{k+i-1} \in \mathcal{C}$ and since (C, r) is a rooted circuit of $\mathcal{A}$, we know that $r \neq e_{k+i}$. Consequently, if we set $x^*_{e_{k+i}} < nx^*_r - \sum_{e\in C\setminus\{r,e_{k+i}\}} x^*_e$, the point x^* is in the halfspace $K(C, r)$.

Similarly, for any positive circuit P of $\mathcal{I}$ such that $e_{k+i} \in P \subseteq X_{k+i}$, the values of x^*_e for all elements $e \in P \setminus \{e_{k+i}\}$ have been already defined. Setting $x^*_{e_{k+i}} < -\sum_{e\in P\setminus\{e_{k+i}\}} x^*_e$ ensures that the point x^* is in the halfspace $K(P)$. By choosing a negative value for $x^*_{e_{k+i}}$ that is small enough, we can satisfy all the constraints over the rooted circuits (C, r) and the positive circuits P contained in the set X_{k+i}.

Consequently, we have constructed a point x^* belonging to the X-orthant $\mathcal{O}(X)$ and to the convex polyhedron K. □

We now show that if a set Y is not in $\mathcal{C}$, then we can find a rooted circuit (C, r) whose hyperplane $H(C, r)$ separates the Y-orthant $\mathcal{O}(Y)$ from the polyhedron $K(\mathcal{R})$ (and thus from K and K_0).

Lemma 28. *For any $Y \in \mathcal{C}^*$, there exists a rooted circuit (C, r) of $\mathcal{A}$ such that $\mathcal{O}(Y) \cap K(C, r) = \varnothing$. Consequently, the hyperplane $H(C, r)$ separates $\mathcal{O}(Y)$ from $K(\mathcal{R})$.*

Proof. Since $Y \notin \mathcal{C}$, by Theorem 15 there exists a rooted circuit (C, r) such that $C \cap Y = C \setminus \{r\}$. Pick any point $y = (y_1, \ldots, y_n)$ from the Y-orthant $\mathcal{O}(Y)$ of $\mathbb{R}^n$ (i.e., $y_e > 0$ for $e \in Y$ and $y_e < 0$ for $e \notin Y$). Since $C \cap Y = C \setminus \{r\}$, we have $y_r < 0$ and $y_e > 0$ for any $e \in C \setminus \{r\}$. Consequently, y does not satisfy the inequality $ny_r > \sum_{e\in C\setminus\{r\}} y_e$, and thus $y \notin K$. As a result, each orthant $\mathcal{O}(Y)$ with $Y \in \mathcal{C}^*$ is separated from $K(\mathcal{R})$ by a hyperplane of the form $H(C, r)$ with $(C, r) \in \mathcal{R}$. □

Similarly, if $Y \in \mathcal{C} \setminus \mathcal{C}'$, then we can find a positive circuit $P \in \mathcal{P}$ such that the hyperplane $H(P)$ separates the Y-orthant $\mathcal{O}(Y)$ from the polyhedron $K(\mathcal{P})$ (and thus from K and K_0).

Lemma 29. *For any $Y \in \mathcal{C} \setminus \mathcal{C}'$, there exists a positive circuit $P \in \mathcal{P}$ such that $\mathcal{O}(Y) \cap K(P) = \varnothing$. Consequently, the hyperplane $H(P)$ separates $\mathcal{O}(Y)$ from $K(\mathcal{P})$.*

Proof. By Lemma 18, there exists a positive circuit $P \in \mathcal{P}$ such that $P \subseteq Y$. Consider a point $y = (y_1, \dots, y_n)$ from the Y-orthant $\mathcal{O}(Y)$ of $\mathbb{R}^n$. Since $P \subseteq Y$, we have $y_e > 0$ for any $e \in P$. Consequently, y does not satisfy the inequality $\sum_{e \in P} y_e < 0$, and thus $y \notin K(\mathcal{P})$. As a result, each orthant $\mathcal{O}(Y)$ with $Y \in \mathcal{C} \setminus \mathcal{C}'$ is separated from $K(\mathcal{P})$ by a hyperplane of the form $H(P)$ with $P \in \mathcal{P}$. $\square$

Finally, we show that if $Y \in \mathcal{C}^*$, then the orthant $\mathcal{O}(Y)$ does not intersect the polyhedron $K(\mathcal{R}_0)$ defined by the critical rooted circuits of $\mathcal{A}$. This follows from the following result.

Lemma 30. *For any $Y \in \mathcal{C}^*$, there exists a rooted set (A, r) and real numbers $(a_g)_{g \in A \setminus \{e\}}$ such that*

(1) $Y \cap A = A \setminus \{r\}$,

(2) $0 < a_g \leq \frac{1}{|Y|}$ for each $g \in A \setminus \{r\}$,

(3) for any $x \in K(\mathcal{R}_0)$, $x_r > \sum_{g \in A \setminus \{r\}} a_g x_g$.

Proof. In order to prove the lemma, we employ an *elimination procedure* similar to Gaussian elimination. Note that the rooted sets (A, r) we consider are not necessarily circuits. We show the lemma by reverse induction on $|Y|$.

The following claim ensures that the property holds for any $Y \in \mathcal{C}^*$ such that $Y \cup \{r\} \in \mathcal{C}$ for some $r \in U \setminus Y$.

Claim 31. *For any $Y \in \mathcal{C}^*$, if there exists $r \in U \setminus Y$ such that $Y \cup \{r\} \in \mathcal{C}$, then there exists a critical rooted circuit (C, r) such that $C \subseteq Y$, and thus for each $x \in K(\mathcal{R}_0)$, we have $x_r > \sum_{g \in C \setminus \{r\}} \frac{1}{n} x_g$.*

Proof. Consider a minimal subset $Z \subseteq Y$ such that $Z \in \mathcal{C}^*$ and $Z \cup \{r\} \in \mathcal{C}$. Let $C = \mathrm{ex}(Z \cup \{r\})$ and note that $Z \cup \{r\} = \mathrm{conv}(C)$ by (C4). By the definition of C, for any $z \in C$, we have $Z \cup \{r\} \setminus \{z\} \in \mathcal{C}$, and by minimality of Z, we have $Z \setminus \{z\} \in \mathcal{C}$. Consequently, (C, r) is a critical rooted circuit of $\mathcal{C}$ and thus by the definition of $K(\mathcal{R}_0)$, for each $x \in K(\mathcal{R}_0)$, we have $x_r > \sum_{g \in C \setminus \{r\}} \frac{1}{n} x_g$. $\Diamond$

Suppose now that $Y \in \mathcal{C}^*$ and that for any $e \in U \setminus Y$, we also have $Y \cup \{e\} \in \mathcal{C}^*$. By induction hypothesis, for any such set $Y \cup \{e\}$, there exists a rooted set (A_e, r_e) and coefficients $(a_g)_{g \in A_e \setminus \{r_e\}}$ such that

- $(Y \cup \{e\}) \cap A_e = A_e \setminus \{r_e\}$,
- $0 < a_g \leq \frac{1}{|Y|+1}$ for each $g \in A_e \setminus \{r_e\}$,
- $x_{r_e} > \sum_{g \in A_e \setminus \{r_e\}} a_g x_g$ for any $x \in K(\mathcal{R}_0)$.

If there exists $e \in U \setminus Y$ such that $e \notin A_e$, then $Y \cap A_e = A_e \setminus \{r_e\}$ and we are done. Consequently, we now assume that for any $e \in U \setminus Y$, we have $e \in A_e$, i.e., for any $e \in U \setminus Y$, we have $Y \cap A_e = Y \setminus \{e, r_e\}$. Consider the digraph D_Y having $U \setminus Y$ as the vertex-set and where there is an arc (e, f) precisely when $f = r_e$. Note that in D_Y every vertex has an out-neighbor and consequently, D_Y contains a cycle. Consider a shortest cycle $(e_0, e_1, \dots, e_k, e_{k+1} = e_0)$ in D_Y. By induction hypothesis, for each $0 \leq i \leq k$, there exists a rooted set (A_{e_i}, e_{i+1}) and coefficients $(a_{i,g})_{g \in A_{e_i} \setminus \{e_{i+1}\}}$ such that

- $(Y \cup \{e_i\}) \cap A_{e_i} = A_{e_i} \setminus \{e_{i+1}\}$,
- $0 < a_{i,g} \leq \frac{1}{|Y|+1}$ for each $g \in A_e \setminus \{e_{i+1}\}$,
- for any $x \in K(\mathcal{R}_0)$, $x_{e_{i+1}} > \sum_{g \in A_{e_i} \setminus \{e_{i+1}\}} a_{i,g} x_g \quad = \quad a_{i,e_i} x_{e_i} + \sum_{g \in A_{e_i} \setminus \{e_i, e_{i+1}\}} a_{i,g} x_g$.

For each $0 \leq i \leq k$, let $b_i = a_{i,e_i}$ and observe that $0 < b_i \leq \frac{1}{|Y|+1} < 1$. Consequently, $0 < \prod_{i=0}^{k} b_i < 1$. For each $0 \leq i \leq k$ and each $g \in Y \setminus A_{e_i}$, let $a_{i,g} = 0$. Observe that for any $0 \leq i \leq k$, for every $x \in K(\mathcal{R}_0)$, we thus obtain $x_{e_{i+1}} > b_i x_{e_i} + \sum_{g \in Y} a_{i,g} x_g$. We have the following inequality:

Claim 32. *For every $0 \le i \le k$ and every point $x \in K(\mathcal{R}_0)$, we have:*

$$x_{e_{i+1}} > \left(\prod_{j=0}^{i} b_j\right) x_{e_0} + \sum_{g \in Y} \left(\sum_{j=0}^{i} \left(\prod_{\ell=j+1}^{i} b_\ell\right) a_{j,g}\right) x_g.$$

Proof. We prove the claim by induction on i. The claim trivially holds for $i = 0$. Assume that the claim holds for $i < k$ and note that for any point $x \in K(\mathcal{R}_0)$, we have:

$$\begin{aligned} x_{e_{i+2}} &> b_{i+1} x_{e_{i+1}} + \sum_{g \in Y} a_{i+1,g} x_g \\ &> b_{i+1} \left(\left(\prod_{j=0}^{i} b_j\right) x_{e_0} + \sum_{g \in Y} \left(\sum_{j=0}^{i} \left(\prod_{\ell=j+1}^{i} b_\ell\right) a_{j,g}\right) x_g \right) + \sum_{g \in Y} a_{i+1,g} x_g \\ &= \left(\prod_{j=0}^{i+1} b_j\right) x_{e_0} + \sum_{g \in Y} \left(\sum_{j=0}^{i} \left(\prod_{\ell=j+1}^{i+1} b_\ell\right) a_{j,g}\right) x_g + \sum_{g \in Y} a_{i+1,g} x_g \\ &= \left(\prod_{j=0}^{i+1} b_j\right) x_{e_0} + \sum_{g \in Y} \left(\sum_{j=0}^{i+1} \left(\prod_{\ell=j+1}^{i+1} b_\ell\right) a_{j,g}\right) x_g. \end{aligned}$$

$\diamondsuit$

Consequently, for every point $x \in K(\mathcal{R}_0)$, we have

$$x_{e_0} = x_{e_{k+1}} > \left(\prod_{j=0}^{k} b_j\right) x_{e_0} + \sum_{g \in Y} \left(\sum_{j=0}^{k} \left(\prod_{\ell=j+1}^{k} b_\ell\right) a_{j,g}\right) x_g.$$

This implies the inequality

$$\left(1 - \prod_{j=0}^{k} b_j\right) x_{e_0} > \sum_{g \in Y} \left(\sum_{j=0}^{k} \left(\prod_{\ell=j+1}^{k} b_\ell\right) a_{j,g}\right) x_g,$$

yielding

$$x_{e_0} > \frac{1}{1 - \prod_{j=0}^{k} b_j} \sum_{g \in Y} \left(\sum_{j=0}^{k} \left(\prod_{\ell=j+1}^{k} b_\ell\right) a_{j,g}\right) x_g.$$

For any $g \in Y$, let $a'_g = \frac{1}{1-\prod_{j=0}^{k} b_j} \sum_{j=0}^{k} \left(\prod_{\ell=j+1}^{k} b_\ell\right) a_{j,g}$. Let $C' \subseteq Y$ be the support of $(a'_g)_{g \in Y}$, i.e., the set $\{g \in Y : a'_g \neq 0\}$. Observe that the rooted set $(C' \cup \{e_0\}, e_0)$ and the coefficients $(a'_g)_{g \in C'}$ satisfy Conditions (1) and (3) of the lemma. Note that for each $0 \le i \le k$ and each $g \in Y$, we have $0 < b_i \le \frac{1}{|Y|+1}$ and $0 < a_{i,g} \le \frac{1}{|Y|+1}$. It is obvious that $a'_g > 0$ for any $g \in C'$. In order to establish Condition (2), it is then enough to show that $a'_g \le \frac{1}{|Y|}$ for any $g \in Y$. Let $\Delta = \frac{1}{|Y|+1}$ and note that $\frac{1}{1-\prod_{j=0}^{k} b_j} \le \frac{1}{1-\Delta^{k+1}}$. Consequently, for every $g \in Y$, we have

$$\begin{aligned} a'_g &= \frac{1}{1 - \prod_{j=0}^{k} b_j} \sum_{j=0}^{k} \left(\prod_{\ell=j+1}^{k} b_\ell\right) a_{j,g} \le \frac{1}{1 - \Delta^{k+1}} \sum_{j=0}^{k} \left(\prod_{\ell=j+1}^{k} \Delta\right) \Delta = \frac{\Delta}{1 - \Delta^{k+1}} \sum_{j=0}^{k} \Delta^{k-j} \\ &= \frac{\Delta}{1 - \Delta^{k+1}} \sum_{j=0}^{k} \Delta^{j} = \frac{\Delta}{1 - \Delta^{k+1}} \cdot \frac{1 - \Delta^{k+1}}{1 - \Delta} = \frac{\Delta}{1 - \Delta} = \frac{\frac{1}{|Y|+1}}{1 - \frac{1}{|Y|+1}} = \frac{1}{|Y|+1} \cdot \frac{|Y|+1}{|Y|+1-1} \\ &= \frac{1}{|Y|}, \end{aligned}$$

which concludes the proof of the lemma. $\square$

Proof of Theorem 26. The proof that $(\mathcal{H}_n, K)$ is a realization of $\mathcal{I}$ is obtained by combining Lemmas 27, 28, and 29 and the equality $K = K(\mathcal{R}) \cap K(\mathcal{P})$. The proof that $(\mathcal{H}_n, K_0)$ is a realization of $\mathcal{I}$ is obtained by combining Lemmas 27, 30, and 29 and the equality $K_0 = K(\mathcal{R}_0) \cap K(\mathcal{P})$. Finally, the proof of the first assertion of the theorem that the pairs $(\mathcal{H}_n, K(\mathcal{R}))$ and $(\mathcal{H}_n, K(\mathcal{R}_0))$ are realizations of $\mathcal{A}$ is a direct consequence of the second assertion because a convex geometry $\mathcal{A}$ is an ideal of itself not containing positive circuits (and thus $K = K(\mathcal{R})$ and $K_0 = K(\mathcal{R}_0)$). $\square$

Example 33. Consider the convex geometry $\mathcal{A} = (U, \mathcal{C})$ on the left of Figure 1. Here, $U = \{1,2,3,4\}$ and $\mathcal{C} = \{\varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{2,3\}, \{3,4\}, \{1,2,3\}, \{2,3,4\}, \{1,2,3,4\}\}$. The rooted circuits of $\mathcal{A}$ are $(\{1,2,3\},2)$, $(\{2,3,4\},3)$, $(\{1,2,4\},2)$, and $(\{1,3,4\},3)$. Then in $\mathbb{R}^4$, $\mathcal{A}$ is realized by the convex set defined by the following inequalities:

$$x_1 + x_3 < 4x_2 \qquad x_2 + x_4 < 4x_3 \qquad x_1 + x_4 < 4x_2 \qquad x_1 + x_4 < 4x_3$$

On the other hand, among the rooted circuits of $\mathcal{A}$, only $(\{1,2,3\},2)$ and $(\{2,3,4\},3)$ are critical. The two first inequalities above correspond to these two critical rooted circuits. These two inequalities are sufficient to have a realization of $\mathcal{A}$ in $\mathbb{R}^4$. Indeed, from the two first inequalities, we obtain $4x_1 + x_4 < 15x_2$ and $x_1 + 4x_4 < 15x_3$. This implies that there is no point in the realization corresponding to the sets $\{1,4\}, \{1,2,4\}$, and $\{1,3,4\}$, that are precisely the subsets of U that are forbidden by the circuits $(\{1,2,4\},2)$, and $(\{1,3,4\},3)$.

From Proposition 12 and Proposition 14, we obtain the following corollary.

Corollary 34. *Any convex geometry $\mathcal{A} = (U, \mathcal{C})$ admits a generalized convex shelling $\mathcal{C}(U, Q_0)$ where $|Q_0|$ is the number of critical rooted circuits of $\mathcal{A}$.*

5.2 Realization of convex geometries and convex dimension

In the spirit of Kashiwabara et al. [19], Richter and Rogers [28] established that every convex geometry of convex dimension $d := \mathrm{cdim}(\mathcal{C})$ can be represented as a generalized convex shelling in $\mathbb{R}^d$. The proof of the following result can be seen as a dualization of the proof of [28].

Theorem 35. *A convex geometry $\mathcal{A} = (U, \mathcal{C})$ of convex dimension $d := \mathrm{cdim}(\mathcal{C})$ is realizable in $\mathbb{R}^d$.*

Proof of Theorem 35. Let $\mathcal{A} = (U, \mathcal{C})$ be defined by a set of d total orders $(\leq_i)_{1 \leq i \leq d}$. For each $e \in C$, let $j_i(e)$ be the index of e in $\leq_i$, (i.e., there are precisely $j_i(e)$ elements $e' \in U$ such that $e' \leq_i e$). Consider the sequence of integers $(a_j)_{0 \leq j \leq |U|}$ such that $a_j = \frac{d+1}{d-1}(d^j - 1)$ for each $0 \leq j \leq |U|$. Observe that $a_{j+1} = da_j + d + 1$ for any $0 \leq j < |U|$. For each $e \in U$, let $b_i(e) = a_{j_i}(e)$ for each $1 \leq i \leq d$ and consider the hyperplane $H_e = \{x : \sum_{i=1}^d \frac{x_i}{b_i(e)} = 1\}$ and the halfspaces $H_e^+ = \{x : \sum_{i=1}^d \frac{x_i}{b_i(e)} < 1\}$ and $H_e^- = \{x : \sum_{i=1}^d \frac{x_i}{b_i(e)} > 1\}$. We consider the hyperplane arrangement $\mathcal{H} = \{H_e : e \in U\}$ and its intersection with the positive orthant $K = \{x : x_i > 0, \forall 1 \leq i \leq d\}$. We assert that $(K, \mathcal{H})$ realizes $\mathcal{A} = (U, \mathcal{C})$. The proof is a consequence of the two following claims.

Claim 36. *For each $C \in \mathcal{C}$, there exists $y \in K$ such that*

(i) for each $e \in C$, $y \in H_e^+$, i.e., $\sum_{i=1}^d \frac{y_i}{b_i(e)} < 1$, and

(ii) for each $e \notin C$, $y \in H_e^-$, i.e., $\sum_{i=1}^d \frac{y_i}{b_i(e)} > 1$.

Proof. If $C = \varnothing$, let $y \in K$ such that $y_i = a_{|U|} + 1$ for all $1 \leq i \leq d$. Observe that for each $e \in U$ and each $1 \leq i \leq d$, $b_i(e) \leq a_{|U|} < y_i$. Consequently, for each $e \in U$, $\sum_{i=1}^d \frac{y_i}{b_i(e)} > 1$, and $y \in H_e^-$. Thus the claim holds for $C = \varnothing$.

Assume now that $C \in \mathcal{C} \setminus \{\varnothing\}$. For each $1 \leq \ell \leq d$, let $e_\ell = \min_{\leq_\ell} C$. Observe that $U \setminus C = \bigcup_{\ell=1}^d \{e \in U : e \leq_\ell e_\ell\}$. Let $c_\ell = b_\ell(e_\ell)$. If $e_\ell = \min_{\leq_\ell} U$, let $c_\ell^- = 0$ and if $e_\ell \neq \min_{\leq_\ell} U$, let e_ℓ^- be the predecessor of e_ℓ in $\leq_\ell$ and let $c_\ell^- = b_\ell(e_\ell^-)$. Consider the point $y \in K$ such that $y_\ell = 1 + c_\ell^-$ for each $1 \leq \ell \leq d$. For each $e \in C$, note that $e_\ell \leq_\ell e$ for all $1 \leq \ell \leq d$. Consequently, $c_\ell = b_\ell(e_\ell) \leq b_\ell(e)$ for all $1 \leq \ell \leq d$. Note that $c_\ell = d + 1 + dc_\ell^- = d(1 + c_\ell^-) + 1 > dy_\ell$. Consequently, $\sum_{\ell=1}^d \frac{y_\ell}{b_\ell(e)} \leq \sum_{\ell=1}^d \frac{y_\ell}{c_\ell} < \sum_{\ell=1}^d \frac{1}{d} = 1$, establishing (i). For each $e \notin C$, there exists ℓ such that $e \leq_\ell e_\ell^-$. Therefore $b_\ell(e) \leq b_\ell(e_\ell^-) = c_\ell^- < y_\ell$. Consequently, $\frac{y_\ell}{b_\ell(e)} > 1$ and since $\frac{y_i}{b_i(e)} > 0$ for all $1 \leq i \leq d$ with $i \neq \ell$, we have $\sum_{i=1}^d \frac{y_i}{b_i(e)} > 1$, establishing (ii). $\Diamond$

Claim 37. *Consider a point $x \in K$ such that $\sum_{i=1}^{d} \frac{x_i}{b_i(e)} \neq 1$ for all $e \in U$ and let $C = \{e \in U : \sum_{i=1}^{d} \frac{x_i}{b_i(e)} < 1\}$. Then $C \in \mathcal{C}$.*

Proof. If $C = \varnothing$, then $C \in \mathcal{C}$ because $\varnothing$ is a convex of $\mathcal{C}$. Assume now that $C \neq \varnothing$. For each $1 \leq \ell \leq d$, let $e_\ell = \min_{\leq_\ell} C$. To prove the claim, we establish that $C = \{e \in U : e_\ell \leq_\ell e, \forall 1 \leq \ell \leq d\}$. If $e \in C$, then for all $1 \leq \ell \leq d$, $e_\ell \leq_\ell e$ and thus $C \subseteq \{e \in U : e_\ell \leq_\ell e, \forall 1 \leq \ell \leq d\}$

Conversely, suppose that $e_\ell \leq_\ell e$ for all $1 \leq \ell \leq d$. If there exists ℓ such that $e = e_\ell$, then by definition of e_ℓ, we have $e \in C$. Assume now that for every $1 \leq \ell \leq d$, $e_\ell \leq_\ell e$. For each $1 \leq \ell \leq d$, let e_ℓ^+ be the successor of e_ℓ in $\leq_\ell$ and note that $e_\ell^+ \leq_\ell e$. Let $c_\ell = b_\ell(e_\ell)$ and let $c_\ell^+ = b_\ell(e_\ell^+)$. By the definition of e_ℓ, $e_\ell \in C$ and thus $\sum_{i=1}^{d} \frac{y_i}{b_i(e_\ell)} < 1$. Since $y \in K$, we have $\frac{y_i}{b_i(e_\ell)} > 0$ for each $1 \leq i \leq d$ with $i \neq \ell$ and consequently, $\frac{y_\ell}{c_\ell} = \frac{y_\ell}{b_\ell(e_\ell)} < 1$. Observe that $c_\ell < c_\ell^+ \leq b_\ell(e)$ for all $1 \leq \ell \leq d$. Moreover, $c_\ell^+ = dc_\ell + d + 1$ and thus $b_\ell(e) \geq c_\ell^+ > dc_\ell$. Consequently, $\sum_{\ell=1}^{d} \frac{y_\ell}{b_\ell(e)} < \sum_{\ell=1}^{d} \frac{y_\ell}{dc_\ell} = \frac{1}{d} \sum_{\ell=1}^{d} \frac{y_\ell}{c_\ell} < \frac{1}{d} d = 1$. Therefore, $e \in C$ and thus we have $C = \{e \in U : e_\ell \leq_\ell e, \forall 1 \leq \ell \leq d\}$, establishing that $C \in \mathcal{C}$. ◊

By Claim 36, for any convex set $C \in \mathcal{C}$, there exists a region corresponding to C in $(K, \mathcal{H})$. By Claim 37, any region of $(K, \mathcal{H})$ corresponds to a convex C of $\mathcal{A} = (U, \mathcal{C})$, concluding the proof of the theorem. ☐

5.3 Realization of ideals of convex geometries and VC-dimension

In this subsection, we consider the realizations of ideals $\mathcal{I} = (U, \mathcal{C}')$ of convex geometries by minimizing or bounding the dimension of the realizing space $\mathbb{R}^d$ by a dimension parameter of $\mathcal{I} = (U, \mathcal{C}')$. Since such $\mathcal{I} = (U, \mathcal{C}')$ are ample [4], the VC-dimension $\dim_{\mathrm{VC}}(\mathcal{I})$ of $\mathcal{I}$ coincides with the dimension of the largest cube of $\mathcal{C}'$. We are motivated by the following:

Corollary 38. *For every distributive lattice L, we have $\dim_{\mathrm{VC}}(L) = \dim_{\mathbb{E}}(L)$.*

Proof. The inequality $\dim_{\mathrm{VC}}(L) \leq \dim_{\mathbb{E}}(L)$ follows from Theorem 10. For $\dim_{\mathrm{VC}}(L) \geq \dim_{\mathbb{E}}(L)$ we use the fact the convex dimension and the VC-dimension of a distributive lattice coincide [13] together with Theorem 35. ☐

This leads us to believe:

Conjecture 39. The Euclidean dimension $\dim_{\mathbb{E}}(\mathcal{I})$ of an ideal $\mathcal{I} = (U, \mathcal{C}')$ of a convex geometry is always upper bounded by a function of its VC-dimension $\dim_{\mathrm{VC}}(\mathcal{I})$.

As a first approach we can get a bound for some situations:

Corollary 40. *If $\mathcal{A} = (U, \mathcal{C})$ is a bouquet of downset alignments with ℓ maxima, e.g., a median set systems with ℓ maxima, then $\mathcal{A}$ is realizable and $\dim_{\mathbb{E}}(\mathcal{A}) \leq \min(|U|, \ell \dim_{\mathrm{VC}}(\mathcal{A}))$.*

Proof. From Theorem 23, Corollary 25, and Theorem 26 we get realizability and the dimension bound of $|U|$. Now, using Theorem 23 and Corollary 38 we obtain the bounds of the form $\ell \dim_{\mathrm{VC}}(\mathcal{A}))$. ☐

In this subsection we confirm Conjecture 39 for trees and multisimplicial complexes (alias, ideals of the grid). Indeed, both our bounds are best-possible.

A *tree* is a connected acyclic graph. A rooted tree is a tree with a distinguished vertex r. A family of sets $\mathcal{A} = (U, \mathcal{C})$ is a *tree* if $\mathcal{A}$ is a median system whose 1-inclusion graph is a tree.

Proposition 41. *Every tree T can be realized in $\mathbb{R}^2$, i.e., $\dim_{\mathbb{E}}(T) \leq 2$ holds.*

Proof. Consider T rooted at r. We will use the basic observation that T can be constructed iteratively from the star of r (i.e., r with all its neighbors) by picking a leaf ℓ of the current tree and attaching to it all its children in T at once. We describe a representation of T following this construction sequence with the invariant that every current leaf ℓ is represented by an isosceles triangular cell whose base is in $\mathcal{H}$.

Let say $\deg(r) = k$. Then we start with K as a convex k-gon and we use $\mathcal{H}$ to cut off, i.e., truncate, each vertex of K such that an isosceles cell is created. Hence, we end up with a representation of the star of r satisfying the invariant, i.e. there is a triangle with base in $\mathcal{H}$. If ℓ has m children, then we add m halfspaces to K forming a convex curve between the intersection points of $\mathcal{H}$ and K, and staying inside the triangular region for ℓ. Now, truncate each corner with a new element of $\mathcal{H}$ such that isosceles regions are created. We have represented the children of ℓ in the desired manner. See Figure 2 for an illustration. ☐

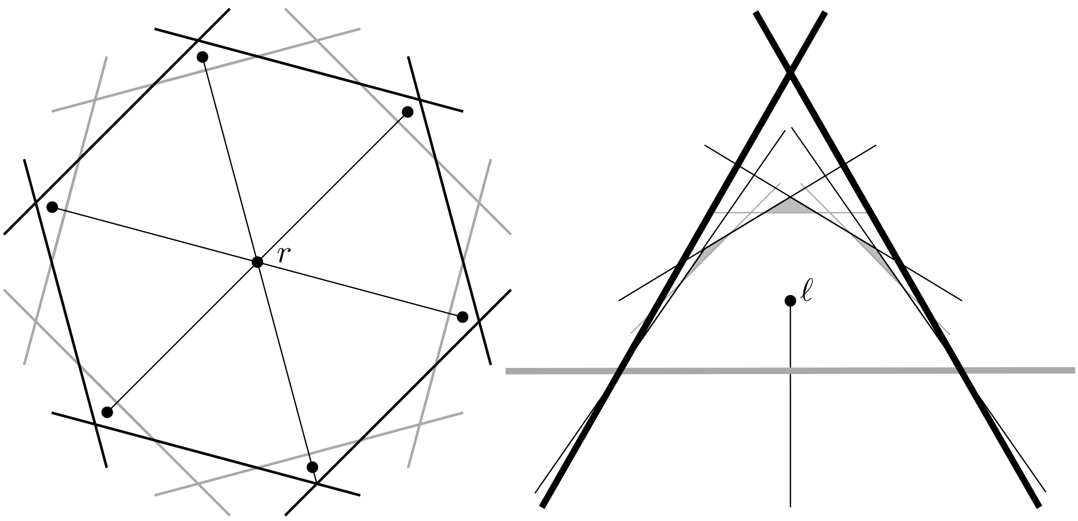

Figure 2: A two-dimensional Euclidean realization for trees.

Median set systems are realizable by Corollary 40. Since median systems are ample, their VC-dimension coincides with the dimension of a largest cube, and thus with their (topological) dimension. Corollary 38 and Proposition 41 motivate the following strengthening of Conjecture 39 in the case of median systems:

Conjecture 42. Every median system of dimension d admits a realization in $\mathbb{R}^{O(d)}$.

Recall that there exists a bijection between the ideals of the grid $(\mathbb{N}^n, \leq)$ and the multisimplicial complexes on a set X of size n.

Theorem 43. *Every ideal L of $\mathbb{N}^n$ with $\dim_{\mathrm{VC}}(L) = d$ has a representation in $\mathbb{R}^{2d}$, i.e., $\dim_{\mathbb{E}}(L) \leq 2\dim_{\mathrm{VC}}(L)$ holds.*

Proof. Let $\dim_{\mathrm{VC}}(L) = r$. By Dilworth's Theorem, L may be seen as a semilattice of ideals of a poset X consisting of n disjoint chains of some length k, such that each ideal intersects at most r chains. Denote by L_i the sublattice of L consisting of all the ideals of L intersecting each chain in at most i elements.

Let $P \subseteq \mathbb{R}^{2r}$ be the polar polytope of an r-neighborly polytope $P^* \subseteq \mathbb{R}^{2r}$ on n vertices containing the origin. The latter exists [17] and therefore P has the property that for any set of $\ell \leq r$ facets of P, their intersection defines a face of dimension $r - \ell$ of P. Now consider the dilates $P_i := iP$ for all $1 \leq i \leq k$ and for each i, let $\mathcal{H}_i$ be the arrangement of facet-defining hyperplanes of P_i. Set $\mathcal{H} := \bigcup_{i=1}^{k} \mathcal{H}_i$.

We will now proceed by induction on $i = 1, \ldots, k$ by introducing halfspaces of the convex set K and eventually changing the dilation for some P_j with $j > i$ in such a way that for any i we have $\mathcal{M}(\bigcup_{j=1}^{i} \mathcal{H}_j, K) = L_i$. We start with the basis case $i = 1$.

Case 1. *$i = 1$, i.e., L_1 is a simplicial complex.*

Proof. First, note that every element of L_1 is represented by a cell of $\mathcal{H}_1$. Namely, if $x \in L_1$ corresponds to taking one element from each of the chains $C_{i_1}, \ldots C_{i_\ell}$ with $\ell \leq r$, then take the face f of P corresponding to the intersection of facets $f_{i_1}, \ldots f_{i_\ell}$. The corresponding facet defining hyperplanes $H_{i_1}, \ldots H_{i_\ell}$ define a cone over f whose intersection with P is f. This cell represents x. Since the polytope P is simple (because it is the polar of the simplicial polytope P^*), in an ϵ-neighborhood of f we see the ideal of x in L_1, where the minimum is represented by the interior of P.

Now, suppose that some cell C of $\mathcal{H}_1$ represents an ideal of X that does not correspond to an element of L_1. Since C and P are open and convex we can choose a hyperplane H that separates C and P and intersects the closure $\overline{P}$ of P in $\overline{P} \cap \overline{C}$. Thus, if we add to K the halfspace defined by H that contains P, then we remove C from the arrangement and all cells of $\mathcal{H}_1$ that correspond to faces not included in $\overline{P} \cap \overline{C}$ are still present. The only further cells of $\mathcal{H}_1$ that are removed are those that correspond to ideals of X corresponding to supersets of the ideal corresponding do C. Since L_1 is an ideal, we did not want these regions. Consequently, taking as K the intersection of all open halfspaces containing P and defined by hyperplanes separating P from cells C representing ideals of X that does not correspond to elements of L_1, we will obtain that $L_1 = \mathcal{M}(\mathcal{H}_1, K)$. $\Diamond$

Case 2. $i > 1$.

Proof. By induction hypothesis, we can define the dilates $P_1, P_2, \ldots, P_{i-1}$ of P, their arrangements of supporting hyperplanes $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_{i-1}$, and the convex set K in such a way that $\mathcal{M}(\bigcup_{j=1}^{i-1} \mathcal{H}_j, K) = L_{i-1}$. First, suppose that there exists an element $x \in L_i$ that is represented by a cell C touching P_i and that lies outside of K. Since L is an ideal, the element $y \in L_{i-1}$ cutting the ideal of x to height at most $i-1$ on each chain is in L. In particular, this point p is represented by a cell touching P_{i-1} within K. Hence, if q is a point in the cell representing x, a line from p to q must cross the boundary of K. We can thus dilate P_i to a smaller polytope such that the cell representing x intersects the interior of K. In order to avoid now some of the regions touching P_i from the outside, we proceed as in the case of P_1 noting that all the hyperplanes that we add to K are disjoint from the interior of P_i hence they do not interfere with the so far obtained representation of L_{i-1}. $\Diamond$

This concludes the proof of the theorem. $\square$

The *full simplicial complex* $U_{r,n}$ of dimension r consists of all subsets of size at most r of a set of size n.

Theorem 44. *For the full simplicial complex $U_{r,n}$ of dimension r on a set $X = \{f_1, \ldots, f_n\}$, we have* $\dim_{\mathbb{E}}(U_{r,n}) \geq \min(2r, n-1)$.

Proof. Let $\mathcal{M}(\mathcal{H}, K)$ be a realization of $U_{r,n}$. We assume that K is full-dimensional, otherwise we intersect with the affine hull of K. Consider the cell P representing the empty set $\varnothing \in U_{r,n}$. Every 1-simplex of $U_{r,n}$ must be represented by a cell intersecting P in a facet. More generally, for any $1 \leq \ell \leq r$ every ℓ-simplex $\sigma = \{f_{i_1}, \ldots, f_{i_\ell}\}$ must be represented by a cell that is separated exactly by the facet-defining hyperplanes $H_{i_1}, \ldots, H_{i_\ell}$ from P. Hence, the intersection of $H_{i_1}, \ldots, H_{i_\ell}$ must be a face of dimension $r - \ell$ of P. Consider now the polyhedron P' by removing all the facet-defining hyperplanes of K from P. Hence, the polar polytope P^* of P' must be r-neighborly. It is known, that for all n, the smallest dimension in which an r-neighborly polytope P^* on n vertices exists is $2r$ or P^* is a simplex and hence has dimension $n-1$, see e.g. [17]. This lower bound carries over to P', hence also to P. Hence this is a lower bound for the dimension of K. $\square$

6 Closing remarks

Concepts of dimension. We have shown that the Euclidean dimension $\dim_{\mathbb{E}}(\mathcal{A})$ of a convex geometry $\mathcal{A} = (U, \mathcal{C})$ lies between its VC-dimension $\dim_{\mathrm{VC}}(\mathcal{A})$ and the minimum of $|U|$ and $\mathrm{cdim}(\mathcal{A})$. It is known that convex geometries with bounded convex dimension and unbounded $|U|$ exist. On the other hand also there are convex geometries with $\mathrm{cdim}(\mathcal{A})$ exponential in $|U|$, see [22]. Hence none of our bounds on $\dim_{\mathbb{E}}(\mathcal{A})$ dominates the other. We believe that $\dim_{\mathbb{E}}(\mathcal{A})$ can be bounded by a function of $\dim_{\mathrm{VC}}(\mathcal{A})$, see Conjecture 39. Further, it would be interesting to compare $\dim_{\mathbb{E}}(\mathcal{A})$ with other concepts of dimension for convex geometries, e.g., Dushnik-Miller dimension, Boolean dimension, local dimension, and fractional dimension. Their behavior on convex geometries has been studied recently in [22]. Another direction are representations by spheres and ellipsoids as studied in [1]

From ideals to bouquets. We believe that bouquets as a generalization of ideals deserve further investigation in their own right as a combinatorial structure. In the context of realizability we dare to state the following:

Conjecture 45. Every bouquet of convex geometries is realizable.

Example 46. The bouquet of convex geometries $(U, \mathcal{C}')$ of the right of Figure 1 is realizable. The difference between $(U, \mathcal{C}')$ and the convex geometry $(U, \mathcal{C})$ of the left of Figure 1 is that $\{1,2,3,4\} \in \mathcal{C} \setminus \mathcal{C}'$ and $\{1,4\} \in \mathcal{C}' \setminus \mathcal{C}$. Therefore, one cannot obtain a representation of $(U, \mathcal{C}')$ by adding linear constraints to the representation of $(U, \mathcal{C})$.

However, we claim that $(U, \mathcal{C}')$ is realized in $\mathbb{R}^4$ by the convex set K defined by the following two inequalities:

$$x_1 + 3x_3 < x_2 \qquad\qquad 3x_2 + x_4 < x_3$$

Observe that there is no point x in the realization such that $x_1 > 0$, $x_3 > 0$, and $x_2 < 0$ (respectively, $x_2 > 0$, $x_4 > 0$, and $x_3 < 0$). Consequently, if X is one of the sets $\{1,3\}$ or $\{1,3,4\}$ (respectively, $\{2,4\}$ or $\{1,2,4\}$), then K does not contain any point of the X-orthant. Moreover, by adding the two inequalities, we get $x_1 + 2x_2 + 2x_3 + x_4 < 0$, and thus for $X = \{1, 2, 3, 4\}$, there is no point $x \in K$ in the X-orthant. Therefore, the convex set K does not intersect any X-orthant such that $X \notin \mathcal{C}'$. We now consider a set $X \in \mathcal{C}'$ and we construct a point of the X-orthant that belongs to K. If X contains at most two elements, let $x_i = 1$ for each $i \in X$. We can then find negative values for the remaining coordinates so that both equations are satisfied. Consider now the set $X = \{1, 2, 3\}$ (respectively, $\{2, 3, 4\}$) and the point $x = (1, 5, 1, -15) \in \mathbb{R}^4$ (respectively, $(-15, 1, 5, 1) \in \mathbb{R}^4$). One can check that x corresponds to X and satisfies both inequalities. Consequently, the bouquet of convex geometries $(U, \mathcal{C})$ is realizable.

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