

# Geometry of convex geometries

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## Abstract

We prove that any convex geometry  $\mathcal{A} = (U, \mathcal{C})$  on  $n$  points and any ideal  $\mathcal{I} = (U', \mathcal{C}')$  of  $\mathcal{A}$  can be realized as the intersection pattern of an open convex polyhedral cone  $K \subseteq \mathbb{R}^n$  with the orthants of  $\mathbb{R}^n$ . Furthermore, we show that  $K$  can be chosen to have at most  $m$  facets, where  $m$  is the number of critical rooted circuits of  $\mathcal{A}$ . We also show that any convex geometry of convex dimension  $d$  is realizable in  $\mathbb{R}^d$  and that any multisimplicial complex (a basic example of an ideal of a convex geometry) of dimension  $d$  is realizable in  $\mathbb{R}^{2d}$  and that this is best possible. From our results it also follows that distributive lattices of dimension  $d$  are realizable in  $\mathbb{R}^d$  and that median systems are realizable. We leave open whether each median system of dimension  $d$  is realizable in  $\mathbb{R}^{O(d)}$ .

## 1 Introduction

Several fundamental combinatorial structures constitute abstract generalizations of geometric settings. Matroids generalize the linear independence in vector spaces, oriented matroids (OMs) capture the combinatorics of regions in a central hyperplane arrangement in  $\mathbb{R}^d$ , ample/lopsided sets (AMPs) encode the regions of the arrangement of coordinate hyperplanes intersected with a convex set, and convex geometries/antimatroids represent an abstraction of Euclidean convexity restricted to a finite set. Finally, complexes of oriented matroids (COMs) are a common generalization of oriented matroids and ample sets and capture the combinatorics of regions in an arbitrary hyperplane arrangement in  $\mathbb{R}^d$  restricted to a convex set  $K$ . Although this geometric model is a desirable property for a respective combinatorial structure, the realizability question is hard. For example, the problem of characterizing which oriented matroids come from hyperplane arrangements is intractable.

We investigate the realizability question for convex geometries and generalize them to ideals of convex geometries. Convex geometries (alias antimatroids) have been introduced and investigated by Edelman and Jamison [11] in the context of abstract convexity and by Korte and Lovasz [20, 21] in combinatorics. Kashiwabara, Nakamura, and Okamoto [16] proved that any convex geometry  $\mathcal{A} = (U, \mathcal{C})$  on  $n$  points can be realized in  $\mathbb{R}^n$  using a generalized convex shelling. Richter and Rogers [24] proved that  $\mathcal{A}$  can be realized in this way in  $\mathbb{R}^d$ , where  $d$  is the convex dimension of  $\mathcal{A}$ . In this paper, we consider a simpler (and dual) version of realizability via hyperplane arrangements and convex sets, as in the case of OMs and COMs. We prove that any convex geometry  $\mathcal{A}$  is realizable in this way in  $\mathbb{R}^n$ . Furthermore, we show that the convex set realizing  $\mathcal{A}$  can be chosen to be a polyhedral cone with at most  $m$  facets, where  $m$  is the number of critical rooted circuits of  $\mathcal{A}$ . We also establish that any ideal  $\mathcal{I} = (U, \mathcal{C}')$  of  $\mathcal{A}$  is realizable in  $\mathbb{R}^n$  by a convex polyhedron with  $m + k$  facets where  $k$  is the number of positive circuits. We also show that any convex geometry of convex dimension  $d$  is realizable in  $\mathbb{R}^d$ . As an application of our results on ideals, we show that any multisimplicial complex of dimension  $d$  is realizable in  $\mathbb{R}^{2d}$  and this is optimal. It follows that distributive lattices of dimension  $d$  are realizable in  $\mathbb{R}^d$  and that median systems are realizable. We leave open whether each median system of dimension  $d$  is realizable in  $\mathbb{R}^{O(d)}$  but show that any tree (median system of dimension 1) is realizable in  $\mathbb{R}^2$ .

## 2 Preliminaries

In this section, we define the main combinatorial structures investigated in this paper and their realizability.

## 2.1 Set families and systems of sign vectors

Let  $U$  be a set of size  $n$ . A *set family*  $\mathcal{S}$  is any collection of subsets of  $U$ . We denote by  $\mathcal{S}^*$  the complement  $2^U \setminus \mathcal{S}$  of the family  $\mathcal{S}$ . Any set family  $\mathcal{S} \subseteq 2^U$  can be viewed as a subset of vertices of the  $n$ -dimensional hypercube  $Q_n = Q(U)$ . Denote by  $G(\mathcal{S})$  the subgraph of  $Q_n$  induced by the vertices of  $Q_n$  corresponding to the sets of  $\mathcal{S}$ ;  $G(\mathcal{S})$  is called the *1-inclusion graph* of  $\mathcal{S}$ . A set family  $\mathcal{S}$  is called *isometric* if  $G(\mathcal{S})$  is an *isometric subgraph* of the hypercube  $Q_n$ , i.e., the distances in  $G(\mathcal{S})$  and in  $Q_n$  between any two vertices of  $G(\mathcal{S})$  are equal. An  $X$ -*cube*  $Q$  of  $Q_n$  is the 1-inclusion graph of the set family  $\{Y \cup X' : X' \subseteq X\}$ , where  $Y$  is a subset of  $U \setminus X$ , called the *support* of  $Q$ . If  $|X| = n'$ , then any  $X$ -cube is a  $n'$ -dimensional subcube of  $Q_n$  and  $Q_n$  contains  $2^{n-n'}$   $X$ -cubes.

Let  $\mathcal{L}$  be a *system of sign vectors* on  $U$ , i.e., maps from  $U$  to  $\{-1, 0, +1\}$ . The elements of  $\mathcal{L}$  are referred to as *covectors* and denoted by capital letters  $X, Y, Z$ . For  $X \in \mathcal{L}$ , the subset  $\underline{X} = \{e \in U : X_e \neq 0\}$  is the *support* of  $X$  and its complement  $X^0 = U \setminus \underline{X} = \{e \in U : X_e = 0\}$  is the *zero set* of  $X$ .

The systems of sign vectors generalize set families. Indeed, any set family  $(U, \mathcal{S})$  can be encoded as a system of sign vectors by setting for each set  $X \in \mathcal{S}$ ,  $X_e = -1$  if  $e \notin X$  and  $X_e = +1$  if  $e \in X$ . In this representation, each set  $X$  is encoded by a  $\{-1, +1\}$ -vector. The encoding with  $\{-1, 0, +1\}$ -vectors is useful when encoding cubes of  $Q_n$ . Indeed, each  $X$ -cube  $Q$  with support  $Y$  can be encoded by  $\{-1, 0, +1\}$ -vector  $X(Q)$ , where  $X(Q)_e = +1$  if  $e \in Y$ ,  $X(Q)_e = -1$  if  $e \in (U \setminus X) \setminus Y$ , and  $X(Q)_e = 0$  if  $e \in X$ .

For each subset  $X \subset U$ , the (open)  $X$ -*orthant* of  $\mathbb{R}^n$  is the set  $\mathcal{O}(X)$  of all points  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  such that  $x_e > 0$  if  $e \in X$  and  $x_e < 0$  if  $e \notin X$ . More generally, for a covector  $X \in \{-1, 0, +1\}^U$ , the  $X$ -*generalized orthant* is the set of all points  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  such that  $x_e > 0$  if  $X_e = +1$  and  $x_e < 0$  if  $X_e = -1$ . Notice that the  $X$ -generalized orthant is the union of all  $Y$ -orthants  $\mathcal{O}(Y)$  such that  $Y = \underline{X} \cup Y'$  with  $Y' \subseteq X^0$ .

## 2.2 OMs and COMs

We recall the basic theory of OMs and COMs from [5] and [3], respectively. Co-invented by Bland and Las Vergnas [6] and Folkman and Lawrence [14], and further investigated by Edmonds and Mandel [13] and many other authors, oriented matroids represent a unified combinatorial theory of orientations of ordinary matroids, which simultaneously captures the basic properties of sign vectors representing the regions in a hyperplane arrangement in  $\mathbb{R}^d$  and of sign vectors of the circuits in a directed graph. OMs provide a framework for the analysis of combinatorial properties of geometric configurations occurring in discrete geometry and in machine learning. Point and vector configurations, order types, hyperplane and pseudo-line arrangements, convex polytopes, directed graphs, and linear programming find a common generalization in this language. The Topological Representation Theorem of [14] connects the theory of OMs on a deep level to arrangements of pseudohyperplanes and distinguishes it from the theory of ordinary matroids.

Let  $\mathcal{L} \subseteq \{-1, 0, +1\}^U$  be a system of sign vectors. For the sake of this paper we assume that  $\mathcal{L}$  is *simple*, i.e., for all  $e \in U$  there exists  $X \in \mathcal{L}$  such that  $X_e \neq 0$ . For  $X, Y \in \mathcal{L}$ ,  $\text{Sep}(X, Y) = \{e \in U : X_e Y_e = -1\}$  is the *separator* of  $X$  and  $Y$ . The *composition* of  $X$  and  $Y$  is the sign vector  $X \circ Y$ , where for all  $e \in U$ ,  $(X \circ Y)_e = X_e$  if  $X_e \neq 0$  and  $(X \circ Y)_e = Y_e$  if  $X_e = 0$ .

**Definition 1.** An *oriented matroid* (OM) [5] is a system  $\mathcal{M} = (U, \mathcal{L})$  of sign vectors satisfying

(C) (Composition)  $X \circ Y \in \mathcal{L}$  for all  $X, Y \in \mathcal{L}$ .

(SE) (Strong elimination) for each pair  $X, Y \in \mathcal{L}$  and for each  $e \in \text{Sep}(X, Y)$ , there exists  $Z \in \mathcal{L}$  such that  $Z_e = 0$  and  $Z_f = (X \circ Y)_f$  for all  $f \in U \setminus \text{Sep}(X, Y)$ .

(Sym) (Symmetry)  $-\mathcal{L} = \{-X : X \in \mathcal{L}\} = \mathcal{L}$ , that is,  $\mathcal{L}$  is closed under sign reversal.

Let  $\leq$  be the product ordering on  $\{-1, 0, +1\}^U$  relative to  $0 \leq -1, +1$ . The poset  $(\mathcal{L}, \leq)$  of an OM  $\mathcal{M} = (U, \mathcal{L})$  with an artificial maximum  $\hat{1}$  forms the (graded) *big face lattice*  $\mathcal{F}_{\text{big}}(\mathcal{M})$ . The *topes*  $\mathcal{T}$  of  $\mathcal{L}$  are the co-atoms of  $\mathcal{F}_{\text{big}}(\mathcal{M})$ . By simplicity the topes are  $\{-1, +1\}$ -vectors and  $\mathcal{T}$  can be seen as a family of subsets of  $U$ . For each  $T \in \mathcal{T}$ , an element  $e \in U$  belongs to the corresponding set if and only if  $T_e = +1$ . The *tope graph*  $G(\mathcal{M})$  of an OM  $\mathcal{M} = (U, \mathcal{L})$  is the 1-inclusion graph of the set  $\mathcal{T}$  of topes of  $\mathcal{L}$ , i.e., the subgraph of the hypercube induced by the vertices corresponding to  $\mathcal{T}$ . Then  $G(\mathcal{M})$  is an isometric subgraph of  $Q_n$ .

Complexes of Oriented Matroids (COMs) were introduced by Bandelt, Chepoi, and Knauer [3] as a natural common generalization of ample sets and OMs. They are defined by replacing the global axiom (Sym) with a weaker local axiom:

**Definition 2.** A *complex of oriented matroids* (COM) [3] is a system of sign vectors  $\mathcal{M} = (U, \mathcal{L})$  satisfying (SE) and the following axiom:

(FS) (Face symmetry)  $X \circ -Y \in \mathcal{L}$  for all  $X, Y \in \mathcal{L}$ .

One can see that OMs are exactly the COMs containing the zero vector  $\mathbf{0}$ , see [3]. The twist between (Sym) and (FS) allows to keep using the same concepts, such as topes, tope graphs, the sign-order and the big face (semi)lattice in a completely analogous way. On the other hand, it leads to a combinatorial and geometric structure that is build from OMs as cells but is much richer than OMs. Let  $\mathcal{M} = (U, \mathcal{L})$  be a COM and  $X \in \mathcal{L}$  a covector. The *face* of  $X$  is  $\uparrow X := \{Y \in \mathcal{L} : X \leq Y\}$ , sometimes denoted  $F(X)$ , see [3, 5]. A *facet* is an inclusion maximal proper face. From the definition, any face  $\uparrow X$  consists of the sign vectors of all faces of the subcube of  $[-1, +1]^U$  with barycenter  $X$ . By [3, Lemma 4], each face  $\uparrow X$  of a COM  $\mathcal{M}$  is an OM. The *topes*  $\mathcal{T}$  and the *tope graph*  $G(\mathcal{M})$  of a COM are defined as for OMs. Again, the COM  $\mathcal{M} = (U, \mathcal{L})$  can be recovered from  $G(\mathcal{M})$  and  $G(\mathcal{M})$  is an isometric subgraph of  $\mathbb{Q}_n$ , see [3, 18].

## 2.3 Ample sets

Ample sets [2] (originally introduced as lopsided sets by Lawrence [22]) are combinatorial structures somewhat opposed to oriented matroids. They capture an important variety of combinatorial objects, e.g., diagrams of (upper locally) distributive lattices, median graphs or CAT(0) cube complexes, convex geometries and conditional antimatroids, see [2]. *Ample sets* are exactly the COMs, in which all faces are cubes. Ample sets can be defined and characterized in a multitude of combinatorial ways in the language of set families, i.e., in terms of topes; see [2, 7, 22]. One of them is via shattering and strong shattering.

Let  $\mathcal{S}$  be a family of subsets of an  $n$ -element set  $U$ . For a set  $Y \subset U$ , the *trace* of  $\mathcal{S}$  to  $Y$  is defined as  $\mathcal{S}|_Y = \{X \cap Y : X \in \mathcal{S}\}$ . A subset  $X$  of  $U$  is *shattered* by  $\mathcal{S}$  if for all  $Y \subseteq X$  there exists  $S \in \mathcal{S}$  such that  $S \cap X = Y$ , i.e.  $\mathcal{S}_X = 2^X$ . The *Vapnik-Chervonenkis dimension* (the *VC-dimension* for short)  $\text{VC-dim}(\mathcal{S})$  of  $\mathcal{S}$  is the cardinality of the largest subset of  $U$  shattered by  $\mathcal{S}$ . A subset  $X$  of  $U$  is *strongly shattered* by  $\mathcal{S}$  if the 1-inclusion graph  $G(\mathcal{S})$  of  $\mathcal{S}$  contains an  $X$ -cube. Denote by  $\overline{\mathcal{X}}(\mathcal{S})$  and  $\underline{\mathcal{X}}(\mathcal{S})$  the families consisting of all shattered and of all strongly shattered sets of  $\mathcal{S}$ , respectively. Clearly,  $\underline{\mathcal{X}}(\mathcal{S}) \subseteq \overline{\mathcal{X}}(\mathcal{S})$  and both  $\overline{\mathcal{X}}(\mathcal{S})$  and  $\underline{\mathcal{X}}(\mathcal{S})$  are closed under taking subsets, i.e.,  $\overline{\mathcal{X}}(\mathcal{S})$  and  $\underline{\mathcal{X}}(\mathcal{S})$  are *abstract simplicial complexes*. The VC-dimension  $\text{VC-dim}(\mathcal{S})$  of  $\mathcal{S}$  is thus the size of a largest set shattered by  $\mathcal{S}$ , i.e., the dimension of the simplicial complex  $\overline{\mathcal{X}}(\mathcal{S})$ . A family  $\mathcal{S}$  of subsets of  $U$  is *ample* whenever the simplicial complexes  $\overline{\mathcal{X}}(\mathcal{S})$  and  $\underline{\mathcal{X}}(\mathcal{S})$  coincide, i.e., each shattered set is strongly shattered. From this definition it follows that the VC-dimension of an ample set  $\mathcal{S}$  coincides with the dimension of a largest cube included in  $\mathcal{S}$ . The complement  $\mathcal{S}^* = 2^U \setminus \mathcal{S}$  of an ample set  $\mathcal{S}$  is also ample. Finally, ample sets are isometric.

## 2.4 Convex geometries

Convex geometries, introduced and investigated by Edelman and Jamison [11], are the abstract convexity spaces satisfying one of the most important properties of Euclidean convexity: each convex set is the convex hull of its extremal points. A *convex geometry* [11] is a pair  $\mathcal{A} = (U, \mathcal{C})$  of a finite *universe*  $U$  and a collection of *convex sets*  $\mathcal{C} \subseteq 2^U$  satisfying the following two conditions:

(C1)  $\emptyset \in \mathcal{C}$  and  $U \in \mathcal{C}$ ;

(C2)  $X \cap Y \in \mathcal{C}$  for all  $X, Y \in \mathcal{C}$ ; (intersection-closed)

(C3) if  $X \in \mathcal{C} \setminus \{U\}$ , then there exists  $e \in U \setminus X$  such that  $X \cup e \in \mathcal{C}$ . (extendable)

The *convex hull*  $\text{conv}(A)$  of a set  $A \subset U$  is the intersection of all convex sets of  $\mathcal{A}$  containing  $A$ . A point  $x$  of a convex set  $X$  is called an *extreme point* of  $X$  if  $X \setminus \{x\}$  is also convex. Denote by  $\text{ex}(X)$  the set of all extreme points of  $X$ . Under axioms (C1) and (C2), the axiom (C3) is equivalent to the following axiom:

(C4)  $X = \text{conv}(\text{ex}(X))$  for any convex set  $X$ .

Convex geometries can be also characterized by the *anti-exchange* axiom:

(C5) if  $X \in \mathcal{C}$  and  $p, q$  are two different points of  $U \setminus X$ , then  $q \in \text{conv}(X \cup p)$  implies that  $p \notin \text{conv}(X \cup q)$ .

For other characterizations of convex geometries, their properties, and examples, see the foundational paper by Edelman and Jamison [11]. Convex geometries are ample sets [2]. If  $\mathcal{A} = (U, \mathcal{C})$  is a convex geometry, then the set family  $\mathcal{A}^*$  on  $U$  consisting of the complements  $U \setminus C$  of sets  $C$  of  $\mathcal{C}$  is union closed and is called an *antimatroid*. For a nice exposition of the theory of convex geometries/antimatroids, see the book by Korte, Lovász, and Schrader [21]. Below we will present a characterization of convex geometries via rooted circuits and the notion of convex dimension.

A pair  $\mathcal{I} = (U', \mathcal{C}')$  is called an *ideal of a convex geometry* if there is a convex geometry  $\mathcal{A} = (U, \mathcal{C})$  such that  $U' \subseteq U$ ,  $\mathcal{C}' \subseteq \mathcal{C}$  and if  $X \in \mathcal{C}, Y \in \mathcal{C}'$  and  $X \subseteq Y$ , then  $X \in \mathcal{C}'$  (equivalently, for any  $X \in \mathcal{C}'$ , the intervals in  $\mathcal{C}'$  and  $\mathcal{C}$  between  $\emptyset$  and  $X$  coincide). One can show that ideals of convex geometries are ample sets because they are particular instances of conditional antimatroids, see [2] and Section 6.2. This will also be a consequence of our realizability results.

### 3 Realizability

In this section, we recall the notion of realizability of COMs and investigate some general properties of realizable COMs. We also compare the notion of realizability with the notion of generalized convex shelling, which can be seen as a dual realizability.

#### 3.1 Realizability of COMs

We continue with the definition of realizable COMs given in the paper [3], which generalizes realizability of oriented and affine oriented matroids [5] and of ample sets [22]. An *affine arrangement of hyperplanes*  $\mathcal{H}$  is a finite set of affine hyperplanes of  $\mathbb{R}^d$ . We will denote by  $\mathcal{H}_d$  the set of  $d$  coordinate hyperplanes of  $\mathbb{R}^d$ . Let  $K$  be an open convex set of  $\mathbb{R}^d$ . Each (oriented) hyperplane  $H = \{x \in \mathbb{R}^d : ax = b\}$  in an arrangement of hyperplanes  $\mathcal{H}$  splits  $\mathbb{R}^d$  into the positive part  $H^+ = \{x \in \mathbb{R}^d : ax > b\}$ , the negative part  $H^- = \{x \in \mathbb{R}^d : ax < b\}$ , and the zero part  $H$ . The arrangement  $\mathcal{H}$  partitions  $\mathbb{R}^d$  into open convex regions, called *cells*. All points belonging to the same cell have the same sign vector with respect to the hyperplanes of  $\mathcal{H}$ . Restrict the arrangement pattern to  $K$ , that is, remove all sign vectors which represent the open regions of the partition disjoint from  $K$ . The resulting set of sign vectors of the cells of  $K$  is denoted  $\mathcal{L}(\mathcal{H}, K)$  and constitutes a COM  $\mathcal{M}(\mathcal{H}, K) = (H, \mathcal{L}(\mathcal{H}, K))$ , see [3].

If  $\mathcal{H}$  is a central arrangement with  $K$  being any open convex set containing the origin, then  $\mathcal{M}(\mathcal{H}, K)$  coincides with the notion of *realizable oriented matroid* [5]. If the arrangement  $\mathcal{H}$  is affine and  $K$  is the entire space, then  $\mathcal{M}(\mathcal{H}, K)$  coincides with the *realizable affine oriented matroid*. Finally, the *realizable ample sets* arise by taking the central arrangement  $\mathcal{H}_d$  of coordinate hyperplanes restricted to an arbitrary open convex set  $K$  of  $\mathbb{R}^d$  (this model was first considered in [22]). A COM  $\mathcal{M}$  is called *realizable* if there exists an affine arrangement of hyperplanes  $\mathcal{H}$  and an open convex set  $K$  of  $\mathbb{R}^d$  such that  $\mathcal{M} = \mathcal{M}(\mathcal{H}, K)$ . Then the pair  $(\mathcal{H}, K)$  is called a *realization* of  $\mathcal{M}$ . As noticed in [3], for a realizable COM the open convex set  $K$  always can be selected to be an open polyhedron (alias open polyhedral cone). For a realizable COM  $\mathcal{M}$ , we denote by  $\dim_{\mathbb{E}}(\mathcal{M})$  the smallest  $d$  such that  $\mathcal{M}$  has a realization in  $\mathbb{R}^d$  and call it the *Euclidean dimension* of  $\mathcal{M}$ .

In realizable COMs,  $X \leq Y$  for two covectors  $X, Y$  if and only if the cell corresponding to  $Y$  is contained in the cell corresponding to  $X$ . Consequently, the topes of realizable COMs are the covectors of the inclusion maximal cells (which all have dimension  $d$ ), called *regions*. Therefore, the tope graph of a realizable COM with realization  $(\mathcal{H}, K)$  can be viewed as the adjacency graph of regions: the vertices of this graph are the regions of  $K$  defined by the hyperplane arrangement  $\mathcal{H}$  and two regions are adjacent in this graph if they are separated by a unique hyperplane of the arrangement (recall that a hyperplane  $H$  *separates* two disjoint open convex sets  $A$  and  $B$  if  $A$  and  $B$  belong to distinct halfspaces defined by  $H$ ).

To realize a COM  $\mathcal{M}$ , it is necessary to find an arrangement of hyperplanes  $\mathcal{H}$ , an open convex set  $K$  (which may be selected to be an open polyhedron), and a sign-preserving bijection between the topes of  $\mathcal{M}$  and the sign vectors of the regions (maximal cells) defined by  $\mathcal{H}$  and included in  $K$ . Now, we will show that  $\mathcal{H}$  can be selected to be the set of coordinate hyperplanes  $\mathcal{H}_d$  of  $\mathbb{R}^d$  and  $K$  to be an open polyhedron (but not necessarily of full dimension) of  $\mathbb{R}^d$ . By this we mean that  $K$  is an open set in the induced topology on the affine hull of  $K$ . This immediately follows from the following lemma:

**Lemma 3.** *Let  $\mathcal{M} = (U, \mathcal{L})$  be a realizable COM. Then  $\mathcal{M}$  has a realization  $(\mathcal{H}', K')$ , where  $K'$  is an open polyhedron, indeed a cone, and  $\mathcal{H}'$  is a central arrangement of hyperplanes.*

*Proof.* Let  $\mathcal{M} = (U, \mathcal{L})$  be a realizable COM with a realization  $\mathcal{M} = \mathcal{M}(\mathcal{H}, K)$ , where  $\mathcal{H}$  is an arrangement of affine hyperplanes and  $K$  is an open polyhedron of  $\mathbb{R}^d$ . First, we consider  $\mathbb{R}^d$  as the hyperplane  $x_{d+1} = 1$  of  $\mathbb{R}^{d+1}$ . Second, we transform each hyperplane  $H_e \in \mathcal{H}, e \in U$  to a central hyperplane  $H'_e$  of  $\mathbb{R}^{d+1}$  passing via  $H_e$  and the origin of coordinates. Denote the resulting arrangement by  $\mathcal{H}'$ . The supporting hyperplanes  $S_j$  of the closure of  $K$  define a second arrangement of hyperplanes  $\mathcal{S}$  of  $\mathbb{R}^d$ . Analogously to  $\mathcal{H}$ , the arrangement  $\mathcal{S}$  can be transformed into an arrangement  $\mathcal{S}'$  of central hyperplanes of  $\mathbb{R}^{d+1}$ . Then  $K$  belongs to one halfspace defined by each hyperplane  $S'_j \in \mathcal{S}'$ . The intersection of those positive open halfspaces is an open polyhedron  $K'$  containing  $K$ . Then  $\mathcal{L} = \mathcal{L}(\mathcal{H}', K')$  holds. Each cell  $C \subset K$  of the realization  $\mathcal{M}(\mathcal{H}, K)$  gives rise to an open polyhedral cone  $C' \subset K'$  (containing  $C$ ) of the realization  $\mathcal{M}(\mathcal{H}', K')$ . The cone  $C'$  is the intersection of positive or negative halfspaces defined by the hyperplanes of  $\mathcal{H}'$  and by the halfspaces of the hyperplanes of  $\mathcal{S}'$  containing  $K'$ .  $\square$

**Proposition 4.** *Let  $\mathcal{M} = (U, \mathcal{L})$  be a realizable COM. Then  $\mathcal{M}$  has a realization  $(\mathcal{H}_n, K'')$ , where  $K''$  is an open polyhedron, indeed a cone, and  $\mathcal{H}_n$  is the set of the coordinate hyperplanes.*

*Proof.* Let  $(\mathcal{H}', K')$  be a realization of  $\mathcal{M}$  in  $\mathbb{R}^d$  as in Lemma 3. Since the vectors normal to the central hyperplanes of  $\mathcal{H}'$  are independent, they form a basis of  $\mathbb{R}^d$ , thus there exists a linear transformation  $f$  of  $\mathbb{R}^d$  mapping these vectors to the coordinate vectors. Then  $f$  maps the hyperplanes of  $\mathcal{H}'$  to the coordinate hyperplanes  $\mathcal{H}_d$  and the open polyhedron  $K'$  to an open polyhedron  $K''$ . Furthermore, each open cell  $C'$  of  $K'$  bounded by a set  $\mathcal{H}'_0$  of hyperplanes of  $\mathcal{H}'$  is mapped to a nonempty open cell  $C''$  of  $K''$  bounded by the set  $\mathcal{H}''_0$  of hyperplanes of  $\mathcal{H}_d$ , which are images of the hyperplanes of  $\mathcal{H}'_0$ . Consequently,  $(\mathcal{H}_d, K'')$  realizes  $\mathcal{M}$ .  $\square$

### 3.2 General properties of realizability

We continue with some properties of realizable COMs and OMs. For this we continue with the notions of restriction, contraction, and minors for COMs. Let  $\mathcal{M} = (U, \mathcal{L})$  be a COM and  $A \subseteq U$ . Given a sign vector  $X \in \{\pm 1, 0\}^U$  by  $X \setminus A$  we refer to the *restriction* of  $X$  to  $U \setminus A$ , that is  $X \setminus A \in \{\pm 1, 0\}^{U \setminus A}$  with  $(X \setminus A)_e = X_e$  for all  $e \in U \setminus A$ . Note that in the realizable setting  $\mathcal{M}(\mathcal{H}, K)$  this operation corresponds to adding some hyperplanes from  $\mathcal{H}$  as halfspaces to  $K$ . The *deletion* of  $A$  is defined as  $(U \setminus A, \mathcal{L} \setminus A)$ , where  $\mathcal{L} \setminus A := \{X \setminus A : X \in \mathcal{L}\}$ . Note that in the realizable setting  $\mathcal{M}(\mathcal{H}, K)$  this operation corresponds to removing some hyperplanes from  $\mathcal{H}$ . The *contraction* of  $A$  is defined as  $(U \setminus A, \mathcal{L}/A)$ , where  $\mathcal{L}/A := \{X \setminus A : X \in \mathcal{L} \text{ and } \underline{X} \cap A = \emptyset\}$ . Note that in the realizable setting  $\mathcal{M}(\mathcal{H}, K)$  this operation corresponds to intersecting  $K$  with some hyperplanes from  $\mathcal{H}$ . If  $\mathcal{L}'$  arises by deletions and contractions from  $\mathcal{L}$ ,  $\mathcal{L}'$  is said to be *minor* of  $\mathcal{L}$ . With the arguments about realizability together with [3] we get.

**Lemma 5** ([3, Lemma 1]). *The class of realizable COMs is closed under taking minors and restrictions. Moreover, if  $\mathcal{M}'$  is obtained by such operations from  $\mathcal{M}$ , then  $\dim_{\mathbb{E}}(\mathcal{M}') \leq \dim_{\mathbb{E}}(\mathcal{M})$ .*

A COM  $\mathcal{M} = (U, \mathcal{L})$  is called *free* of dimension  $d$  if its tope graph is isomorphic to  $Q_d$ . A COM  $\mathcal{M}$  has VC-dimension  $d$  if the topes of  $\mathcal{M}$  define a set-family of VC-dimension  $d$ .

**Lemma 6** ([8, Lemma 1]). *A COM has VC-dimension  $\leq d$  if and only if it does not have a free COM of dimension  $d + 1$  as a minor.*

**Lemma 7.** *The Euclidean dimension of a free COM of dimension  $d$  is at least  $d$ .*

*Proof.* A hyperplane arrangement of  $d$  hyperplanes in  $\mathbb{R}^{d-1}$  has less than  $2^d$  maximal cells, see e.g., [5, Exercise 4.3.5]. Thus, the free COM (which has  $2^d$  topes) cannot be realized in  $\mathbb{R}^{d-1}$ .  $\square$

The previous lemmas together imply:

**Theorem 8.** *For every realizable COM  $\mathcal{M}$ , we have  $\dim_{\text{VC}}(\mathcal{M}) \leq \dim_{\mathbb{E}}(\mathcal{M}) \leq |U|$ .*

*Proof.* For the lower bound, if  $\mathcal{M}$  has  $\dim_{\text{VC}}(\mathcal{M}) = r$ , then by Lemma 6 we can remove hyperplanes from its realization and obtain a realization of  $Q_r$ , which is an OM of VC-dimension  $r$  by Lemma 7, hence by Lemma 5 it has  $r \leq \dim_{\mathbb{E}}(Q_r) \leq \dim_{\mathbb{E}}(\mathcal{M})$ . The upper bound is Proposition 4.  $\square$

As a side remark we provide a quick answer to a question communicated to us privately by Kunin, Lienkaemper, and Rosen, i.e., that a non-realizable OM cannot be realized as a COM either:

**Remark 9.** *If  $\mathcal{M}$  is an OM realizable as a COM, then  $\mathcal{M}$  is realizable by a central hyperplane arrangement.*

*Proof.* Let  $\mathcal{M} = (U, \mathcal{L})$  be an OM and suppose that it can be realized as a (realizable) COM  $\mathcal{M} = \mathcal{M}(\mathcal{H}, K)$  in  $\mathbb{R}^d$ . Suppose without loss of generality that  $K$  is full-dimensional, otherwise we project into the affine hull of  $K$ . Since  $\mathcal{M}$  is an OM, we have that  $\mathbf{0} \in \mathcal{L}$  (recall that  $\mathbf{0}$  is the all-zeros vector). Hence, there is a point  $x \in K$  in which all the hyperplanes of  $\mathcal{H}$  intersect. After translating  $x$  to the origin of  $\mathbb{R}^d$  we see that  $\mathcal{H}$  is a central hyperplane arrangement. Hence, its combinatorics is determined by any arbitrary small sphere  $S$  around  $x$ , see e.g. [5, Section 1.2.(c)]. Therefore, we can set  $K = \mathbb{R}^d$  and get a representation of  $\mathcal{M}$  by a central hyperplane arrangement.  $\square$

### 3.3 Generalized convex shellings

Convex geometries are considered as generalizations of Euclidean convexity in the following sense. Let  $\mathcal{A} = (U, \mathcal{C})$  be a convex geometry. We say that  $\mathcal{A}$  is *CG-realizable* if  $U$  can be viewed as a finite subset of points of  $\mathbb{R}^d$  such that  $X \subseteq U$  belongs to  $\mathcal{C}$  if and only if the Euclidean convex hull of  $X$  does not contain other points of  $U$ :  $\text{conv}_{\mathbb{R}^d}(X) \cap U = X$ . Not every convex geometry can be realized in this way. Kashiwabara et al. [16] gave a realization theorem for all convex geometries using the notion of generalized convex shellings. Let  $P$  and  $Q$  be two finite sets of  $\mathbb{R}^d$  such that the Euclidean convex hull of  $Q$  does not intersect  $P$ . Then the family of subsets  $\mathcal{C}(P, Q) = \{X \subseteq P : \text{conv}(X \cup Q) \cap P = X\}$  of  $P$  is called the *generalized convex shelling on  $P$  with respect to  $Q$* . It is easy to see that any generalized convex shelling is a convex geometry. Conversely, Kashiwabara et al. [16] proved that any convex geometry arises as a generalized convex shelling. If  $Q = \emptyset$ , then this leads to the notion of CG-realizability. If  $\mathcal{C} = \mathcal{C}(P, Q)$ , then we will say that  $\mathcal{C}(P, Q)$  is a *generalized convex shelling* of  $\mathcal{A} = (P, \mathcal{C})$ . Richter and Rogers [24] proved that any convex geometry of (combinatorial) dimension  $d$  has a generalized convex shelling in  $\mathbb{R}^d$ .

Extending the notation for generalized convex shellings, given two finite point sets  $P$  and  $Q$  of  $\mathbb{R}^d$ , let

$$\mathcal{M}(P, Q) = \{X \subseteq P : \exists \text{ halfspace } H^+ \text{ s.t. } Q \subset H^+ \text{ and } P \cap H^+ = X\}.$$

Trivially,  $P \in \mathcal{M}(P, Q)$ . We say that  $(P, Q)$  is a *point representation* of a COM  $\mathcal{M}$  if  $\mathcal{M} = \mathcal{M}(P, Q)$ , i.e., the set of topes of  $\mathcal{M}$  consists precisely of the  $\{-1, +1\}$ -vectors of all sets of  $\mathcal{M}(P, Q)$ . Now, we establish a correspondence between (hyperplane) realizations and point realizations of COMs and a link with generalized convex shellings. For this purpose we define a COM to be *acyclic* if it has the all plus-vector  $(+1, \dots, +1)$  as a tope.

**Proposition 10** (Point representations of COMs). *An acyclic COM  $\mathcal{M} = (U, \mathcal{L})$  is (hyperplane) realizable if and only if  $\mathcal{M}$  is point realizable.*

*Proof.* Let  $\mathcal{M} = (U, \mathcal{L})$  be an acyclic realizable COM. We have  $\mathcal{M} = \mathcal{M}(\mathcal{H}', K')$  where  $\mathcal{H}'$  is a central hyperplane arrangement  $\mathcal{H}'$  and  $K'$  an open polyhedral cone in  $\mathbb{R}^{d+1}$  as in Lemma 3. Now, the vectors  $v_e$ , normal to the hyperplanes  $H'_e, e \in U$  of the arrangement  $\mathcal{H}'$ , gives raise to a set  $\mathcal{V}$  of vectors of  $\mathbb{R}^{d+1}$ . Analogously, the vectors  $u_j$ , normal to the hyperplanes  $S'_j$  of  $\mathcal{S}'$  defining the polyhedron  $K'$  and such that their product with the vectors with ends inside  $K'$  is positive, gives raise to a set  $\mathcal{U}$  of vectors of  $\mathbb{R}^{d+1}$ . For a cell  $C$  of the initial realization, each point  $p$  of  $C$  defines a vector  $v_p$  belonging to the cone  $C'$ . Let  $H_{v_p}$  be the central hyperplane of  $\mathbb{R}^{d+1}$  having  $v_p$  as the normal vector. Then all vectors of  $\mathcal{U}$  belong to a halfspace defined by  $H_{v_p}$ . Analogously, each vector  $v_e \in \mathcal{V}$  belongs to the positive side of the hyperplane  $H_{v_p}$  if and only if the cell  $C$  is located on the positive side of the hyperplane  $H_e \in \mathcal{H}$  (and thus the cone  $C'$  is on the positive side of the central hyperplane  $H'_e \in \mathcal{H}'$ ). Therefore, the topes of  $\mathcal{M}$  can be identified (in the set-theoretical language) with the subsets  $X$  of  $U$  such that there exists a central hyperplane  $H'$  of  $\mathbb{R}^{d+1}$  such that all  $U$  belong to one halfspace defined by  $H'$  and a vector  $v_e, e \in U$  belongs to the same halfspace defined by  $H'$  if and only if  $e \in X$ .

Now, we can return back to  $\mathbb{R}^d$  by considering a hyperplane  $H_0$  of  $\mathbb{R}^{d+1}$  parallel to  $\mathbb{R}^d$ , which intersects all vectors of  $\mathcal{V} \cup \mathcal{U}$ . This hyperplane exists, since by acyclicity of  $\mathcal{M}$  all elements of  $\mathcal{V}$  have a positive last coordinate and the same holds for the elements of  $\mathcal{U}$  by the choice of normal vectors for the defining hyperplanes of  $K'$ . Let  $p_e \in H_0$  denote the point defined by the vector  $v_e \in \mathcal{V}$  and let  $q_j \in H_0$  denote the point defined by the vector  $u_j \in \mathcal{U}$ . Denote by  $P$  and  $Q$  the resulting point configurations of  $\mathbb{R}^d$ . The correspondence between points in the model  $(\mathcal{H}', K')$  and hyperplanes in the model  $(P, Q)$  establishes that for any point  $p$  in a cell  $C$  of  $(\mathcal{H}', K')$ , and thus for any tope  $X$  of  $\mathcal{M}(\mathcal{H}', K')$ , there is a hyperplane  $H_p$  such that  $H_p$  induces the sign-pattern of  $X$  in the model  $(P, Q)$ .

Conversely, the preceding transformation can be reversed. Namely, given sets  $P, Q$  in  $H_0 \subseteq \mathbb{R}^{d+1}$ , we can define the hyperplane arrangements  $\mathcal{H}$  and  $\mathcal{S}$  having  $P$  and  $Q$  as normal vectors, respectively. Letting  $K$  be the polyhedral cone defined by  $\mathcal{S}$ , we get a realization  $\mathcal{M} = \mathcal{M}(\mathcal{H}, K)$ .  $\square$

**Remark 11.** The essential difference between point representation of Proposition 10 provided by  $\mathcal{M}(P, Q)$  and generalized convex shelling of [16] provided by  $\mathcal{C}(P, Q)$  is that instead of a single halfspace  $H^+$  containing  $Q$  and intersecting  $P$  in  $X$ , the generalized convex shelling requires that  $\text{conv}(X \cup Q) \cap P = X$ . This is equivalent to the existence of a set of halfspaces  $H_1^+, \dots, H_k^+$ , all containing  $Q$  and such that  $X = P \cap (\bigcap_{i=1}^k H_i^+)$ . (As a set of halfspaces  $H_1^+, \dots, H_k^+$  one can take the halfspaces defined by the support hyperplanes of the facets of the polytope  $\text{conv}(X \cup Q)$  and containing this polytope). This is a natural requirement for intersection-closed set families, because any set family admitting a generalized convex shelling is closed by intersections: if  $H_1^+, \dots, H_k^+$  and  $S_1^+, \dots, S_m^+$  are two sets of halfspaces such that  $Q \subset (\bigcap_{i=1}^k H_i^+) \cap (\bigcap_{j=1}^m S_j^+)$  and  $P \cap (\bigcap_{i=1}^k H_i^+) = X$ ,  $P \cap (\bigcap_{i=1}^m S_j^+) = Y$ , then  $P \cap (\bigcap_{i=1}^k H_i^+) \cap (\bigcap_{i=1}^m S_j^+) = X \cap Y$ .

Therefore, the hyperplane realizations and point representations of COMs are much simpler than generalized convex shellings and correspond to classical OM-realizations. The main results of our paper (Theorems 25 and 34) must be considered from this point of view, while compared to the results of [16] and [24] in the case of convex geometries. On the other hand, the generalized convex shellings characterize convex geometries [16].

A *relaxed convex shelling* is defined as a generalized convex shelling without requiring that  $\text{conv}(Q) \cap P = \emptyset$ . Analogously to generalized convex shellings, relaxed convex shellings can be reformulated as follows: given two arbitrary finite sets  $P$  and  $Q$  of  $\mathbb{R}^d$ , a set  $X$  belongs to  $\mathcal{C}(P, Q) = \{X \subseteq P : \text{conv}(X \cup Q) \cap P = X\}$  if and only if there exists a set of halfspaces  $H_1^+, \dots, H_k^+$ , all containing  $Q$  and such that  $X = P \cap (\bigcap H_i^+)$ . A COM  $\mathcal{M}$  is called *intersection-closed* if the set of topes of  $\mathcal{M}$  defines an intersection-closed family of sets and  $\emptyset \in \mathcal{M}$ . Obviously, convex geometries are acyclic and their ideals are intersection closed COMs. Note however, that acyclicity corresponds to the universe being convex.

**Lemma 12.** *Any point representation  $(P, Q)$  of an acyclic intersection-closed COM  $\mathcal{M}$  yields a relaxed convex shelling representation of  $\mathcal{M}$ . If  $\mathcal{M}$  is a convex geometry, then a relaxed convex shelling is a generalized convex shelling.*

*Proof.* Since  $(P, Q)$  is a point representation of  $\mathcal{M}$ , we have  $\mathcal{M} = \mathcal{M}(P, Q)$ . From the definition of the point representations and relaxed convex shellings, we conclude that  $\mathcal{M}(P, Q) \subseteq \mathcal{C}(P, Q)$ . By Remark 11,  $\mathcal{C}(P, Q)$  is closed by taking intersections. To prove the converse inclusion  $\mathcal{C}(P, Q) \subseteq \mathcal{M}$ , pick any set  $X$  from  $\mathcal{C}(P, Q)$ . Then there exists a set  $H_1^+, \dots, H_k^+$  of halfspaces, all containing  $Q$  and such that  $X = P \cap (\bigcap H_i^+)$ . Let  $X_i = P \cap H_i^+$ ,  $i = 1, \dots, k$ . Then each  $X_i$  belongs to  $\mathcal{M}(P, Q)$ . Since  $\mathcal{M}(P, Q) = \mathcal{M}$ , each  $X_i$  belongs to  $\mathcal{M}$ . Since  $\mathcal{M}$  is intersection-closed and  $\bigcap_{i=1}^k X_i = X$ , we conclude that  $X \in \mathcal{M}$ , establishing that  $\mathcal{C}(P, Q) \subseteq \mathcal{M}$ . Since  $\emptyset \in \mathcal{M}$ , necessarily  $\text{conv}(Q) \cap P = \emptyset$ , establishing that the relaxed convex shelling is a generalized convex shelling.  $\square$

## 4 Convex geometries and their ideals

In this section, we present further results about convex geometries and their ideals that will be used in our proofs. We also present examples of ideals of convex geometries.

### 4.1 Rooted circuits

In the proof of realizability of convex geometries, we will use the characterization of convex geometries via rooted circuits and critical rooted circuits. Let  $\mathcal{A} = (U, \mathcal{C})$  be a convex geometry. A *rooted set* is a pair  $(C, r)$  consisting of a subset  $C$  of  $U$  and an element  $r$  of  $C$ . A convex geometry  $\mathcal{C}$  is *reconstructed* from a collection of rooted sets  $\mathcal{F}$  if  $\mathcal{C} = \{X \subseteq U : (C, r) \in \mathcal{F} \Rightarrow X \cap C \neq C \setminus \{r\}\}$ , i.e.,  $X \subseteq U$  belongs to  $\mathcal{C}$  if and only if no rooted set  $(C, r) \in \mathcal{F}$  meets  $X$  in  $C \setminus \{r\}$ . A rooted set  $(C, r)$  is a *rooted circuit* [20] of  $\mathcal{C}$  if  $\mathcal{C}|_C = 2^C \setminus \{C \setminus \{r\}\}$ . Denote by  $\mathcal{R}(\mathcal{A})$  the set of all rooted circuits of  $\mathcal{A}$ .  $\mathcal{R}(\mathcal{A})$  has the following properties established in [20] (for proofs, see Lemma 2 and Proposition 4 of [9]):

**Theorem 13** ([20, 21]). *Let  $\mathcal{A} = (U, \mathcal{C})$  be a convex geometry. Then:*

- (i) *If  $(C, r)$  is a rooted circuit of  $\mathcal{C}$ , then  $r \in \text{conv}(C \setminus \{r\})$ ;*
- (ii)  *$\mathcal{A}$  can be reconstructed from the family  $\mathcal{R}(\mathcal{A})$  of its rooted circuits.*

Dietrich [9] provided the following axiomatization of convex geometries via the rooted circuits:

**Theorem 14** ([9]). *Let  $\mathcal{R}$  be a set of rooted subsets of a finite set  $U$ . Then  $\mathcal{R}$  is the set of rooted circuits of a convex geometry if and only if  $\mathcal{R}$  satisfies the following two properties:*

- (1)  $(C_1, r_1), (C_2, r_2) \in \mathcal{R}$  and  $C_1 \subseteq C_2$  implies  $C_1 = C_2$  and  $r_1 = r_2$ ;
- (2)  $(C_1, r_1), (C_2, r_2) \in \mathcal{R}$  and  $r_1 \in C_2 \setminus \{r_2\}$  implies that there exists  $(C_3, r_2) \in \mathcal{R}$  such that  $C_3 \subseteq (C_1 \cup C_2) \setminus \{r_1\}$ .

Korte and Lovász [20] (see also the book [21]) established that antimatroids can be reconstructed from a unique minimal subset of rooted circuits, which they call critical rooted circuits. In terms of convex geometries, a rooted circuit  $(C, a)$  of a convex geometry  $\mathcal{A} = (U, \mathcal{C})$  is a *critical rooted circuit* if  $\text{conv}(C) \setminus \{a\}$  does not belong to  $\mathcal{C}$  but  $\text{conv}(C) \setminus \{a, b\}$  belongs to  $\mathcal{C}$  for any  $b \in C \setminus \{a\}$ . Denote by  $\mathcal{R}_0(\mathcal{A})$  the set of all critical rooted circuits of  $\mathcal{A}$ .

**Theorem 15** ([20, 21]). *Let  $\mathcal{A} = (U, \mathcal{C})$  be a convex geometry. Then  $\mathcal{A}$  can be reconstructed from the family  $\mathcal{R}_0(\mathcal{A})$  of its critical rooted circuits. Furthermore, if  $\mathcal{A}$  can be reconstructed from  $\mathcal{R}' \subseteq \mathcal{R}(\mathcal{A})$ , then  $\mathcal{R}_0(\mathcal{A}) \subseteq \mathcal{R}'$ .*

To each rooted set  $(C, r)$  of a convex geometry  $\mathcal{A} = (U, \mathcal{C})$  we can associate the cube  $Q(C, r)$ :  $Q(C, r)$  consists of all subsets of  $U$  of the form  $(C \setminus r) \cup X'$  with  $X' \subseteq U \setminus C$ . The cube  $Q(C, r)$  can be encoded by the sign vector  $X(C, r)$ , where  $X_e(C, r) = +1$  if  $e \in C \setminus \{r\}$ ,  $X_r(C, r) = -1$ , and  $X_e(C, r) = 0$  if  $e \in U \setminus C$ . If  $(C, r)$  is a rooted circuit, then the trace of  $\mathcal{C}$  on  $C$  coincides with  $2^C \setminus \{C \setminus \{r\}\}$ , thus the cube  $Q(C, r)$  belongs to the complement  $\mathcal{A}^*$  of  $\mathcal{A}$  and, furthermore,  $Q(C, r)$  is a maximal by inclusion cube of  $\mathcal{A}^*$ . Consequently,  $Q(C, r)$  is an  $(U \setminus C)$ -cube of  $\mathcal{A}^*$ . The converse also holds:

**Lemma 16.** *Let  $\mathcal{A} = (U, \mathcal{C})$  be a convex geometry. If  $Q$  is a maximal cube of  $\mathcal{A}^*$ , then there exists  $(C, r) \in \mathcal{R}(\mathcal{A})$  such that  $Q = Q(C, r)$ .*

*Proof.* Suppose that  $Q$  consists of the sets  $\{Y \cup X' : X' \subseteq X\}$  for  $Y \subseteq U \setminus X$ . Since  $Y \notin \mathcal{C}$ , by Theorem 13(ii), there exists a rooted circuit  $(C, r)$  such that  $Y \cap C = C \setminus \{r\}$ . But then  $Q$  is contained in the cube  $Q(C, r)$  of  $\mathcal{A}^*$  whose vertices are the sets  $\{(C \setminus \{r\}) \cup X' : X' \subseteq U \setminus C\}$ . Since  $Q$  is maximal,  $Q = Q(C, r)$ .  $\square$

## 4.2 Positive circuits

In case of ideals of convex geometries, along with rooted circuits we also have to define positive circuits. Let  $\mathcal{I} = (U', \mathcal{C}')$  be an ideal of a convex geometry  $\mathcal{A} = (U, \mathcal{C})$ . A *positive circuit* of  $\mathcal{I}$  with respect to  $\mathcal{A}$  is a subset  $P \subseteq U$  such that  $\mathcal{C}'_{|P} = 2^P \setminus \{P\}$ . In particular, for any  $e \in U \setminus U'$ ,  $\{e\}$  is a positive circuit of  $\mathcal{I}$  with respect to  $\mathcal{A}$ . Given an ideal  $\mathcal{I}$  of a convex geometry  $\mathcal{A}$ , we denote by  $\mathcal{P}(\mathcal{I})$  the set of positive circuits of  $\mathcal{I}$  with respect to  $\mathcal{A}$ . We now show that  $\mathcal{C}'$  is precisely the set of elements of  $\mathcal{C}$  not containing positive circuits.

**Lemma 17.** *For any ideal  $\mathcal{I} = (U', \mathcal{C}')$  of a convex geometry  $\mathcal{A} = (U, \mathcal{C})$ , we have*

$$\mathcal{C}' = \{X \in \mathcal{C} : \forall P \in \mathcal{P}(\mathcal{I}), P \not\subseteq X\}.$$

*Proof.* Let  $\mathcal{C}'' = \{X \in \mathcal{C} : \forall P \in \mathcal{P}(\mathcal{I}), P \not\subseteq X\}$ . Observe that  $\mathcal{C}''$  is trivially included in  $\mathcal{C}$ . For any  $X \in \mathcal{C}'$  and any  $P \subseteq U$ , if  $P \subseteq X$ , then  $P \notin \mathcal{P}(\mathcal{I})$ . Consequently,  $X \in \mathcal{C}''$  and thus  $\mathcal{C}' \subseteq \mathcal{C}''$ .

Suppose now that there exists  $Y \in \mathcal{C}'' \setminus \mathcal{C}'$  and assume that  $Y$  is such a set of minimal size. By minimality of  $Y$ , for any  $e \in Y$ ,  $Y \setminus \{e\} \in \mathcal{C}$  if and only if  $Y \setminus \{e\} \in \mathcal{C}'$ . Let  $P_0 = \{e \in Y : Y \setminus \{e\} \in \mathcal{C}'\}$ . Since  $\mathcal{C}'$  is intersection-closed, we conclude that for any  $\emptyset \subsetneq P' \subseteq P_0$ , we have  $Y \setminus P' \in \mathcal{C}'$ . Consequently,  $2^{P_0} \setminus \{P_0\} \subseteq \mathcal{C}'_{|P_0}$ . Since  $Y \cap P_0 = P_0$  and  $Y \in \mathcal{C}''$ , necessarily  $P_0 \notin \mathcal{P}(\mathcal{I})$ . Consequently, there exists  $X \in \mathcal{C}'$  such that  $P_0 \subseteq X$ . Let  $Z = X \cap Y$  and observe that  $P_0 \subseteq Z$ . Since  $\mathcal{C}$  is intersection-closed,  $Z \in \mathcal{C}$ . If  $Z = Y$ , then  $Y \subseteq X \in \mathcal{C}'$ , contradicting the fact that  $\mathcal{C}'$  is an ideal of  $\mathcal{C}$ . Thus, by the minimality of  $Y$ , we have  $Z \subsetneq Y$ . Since  $Z \in \mathcal{C}'$  and  $Y \notin \mathcal{C}'$ , there exists a set  $Z \subseteq A \subseteq Y$  and an element  $e \in Y \setminus A$  such that  $A \in \mathcal{C}'$  and  $B = A \cup \{e\} \in \mathcal{C} \setminus \mathcal{C}'$  ( $A$  and  $e$  exist since  $\mathcal{C}$  is ample and thus isometric). By minimality of  $Y$ , necessarily  $B = Y$ . Consequently  $e \in P_0$ , but this is impossible since  $P_0 \subseteq Z$ . This establishes that  $\mathcal{C}'' \subseteq \mathcal{C}'$ .  $\square$

## 4.3 Convex dimension

Edelman and Jamison [11] provided a nice characterization of convex geometries via order convexity. Given a universe  $U$  of size  $n$  and a total order  $e_1 \leq e_2 \leq \dots \leq e_n$ , call an *ending interval* of  $\leq$  any set of the form  $\{e \in U : e_i \leq e\}$  for



some  $e_i \in U$ . Given a set  $\Upsilon = \{\leq_1, \dots, \leq_d\}$  of  $d$  total orders on  $U$ , we say that a set family  $\mathcal{C} \subseteq 2^U$  is *generated* by the set  $\Upsilon$  if  $\emptyset \in \mathcal{C}$  and a nonempty set  $C$  belongs to  $\mathcal{C}$  if and only if  $C$  is the intersection of  $d$  ending intervals, one from each order  $\leq_i$  of  $\Upsilon$ , i.e., if there exists  $(e_i)_{1 \leq i \leq d} \in U^d$  such that  $C = \{e \in U : e_i \leq_i e, \forall 1 \leq i \leq d\}$ . Edelman and Jamison proved the following result:

**Theorem 18** ([11, Theorem 5.2]). *Any set family  $\mathcal{C} \subseteq 2^U$  generated by a set of total orders on  $U$  is a convex geometry and, conversely, any convex geometry  $\mathcal{A} = (U, \mathcal{C})$  can be generated by a set of total orders on  $U$ .*

The *convex dimension*  $\text{cdim}(\mathcal{A})$  of a convex geometry  $\mathcal{A} = (U, \mathcal{C})$  is the least number of total orders generating  $\mathcal{C}$ .

#### 4.4 (Ideals of) convex geometries as meet-(semi)lattices

A poset  $L = (X, \leq)$  is a *meet-semilattice* if for every  $x, y \in L$  there is unique largest element  $x \wedge y \in L$  such that  $x \wedge y \leq x, y$ ;  $x \wedge y$  is called the *meet* of  $x$  and  $y$ . A poset  $L = (X, \leq)$  is a *join-semilattice* if for every  $x, y \in L$  there is unique smallest element  $x \vee y \in L$  such that  $x \vee y \geq x, y$ ;  $x \vee y$  is called the *join* of  $x$  and  $y$ . A poset is a *lattice* if it is both a join- and a meet-semilattice. Note that both join and meet are associative and commutative operations, so in order to take the join or meet over all the elements of a set  $Y$  we sometimes just write  $\bigvee Y$  and  $\bigwedge Y$ , respectively. Note that if a poset has a global minimum  $\hat{0}$  or a global maximum  $\hat{1}$ , then we set  $\bigvee \emptyset = \hat{0}$  and  $\bigwedge \emptyset = \hat{1}$ , respectively. An element  $x \in L$  is called *join-irreducible* if  $x = \bigvee Y$  implies  $x \in Y$  for all  $Y \subseteq L$ . The set of join-irreducible elements is denoted by  $\mathcal{J}(L)$ . Similarly, an element  $x \in L$  is called *meet-irreducible* if  $x = \bigwedge Y$  implies  $x \in Y$  for all  $Y \subseteq L$ . The set of meet-irreducible elements is denoted  $\mathcal{M}(L)$ . Following Dilworth a *lower locally distributive lattice* (LLD) is a lattice  $L$  such that for all  $x \in L$  there is a unique inclusion-minimal  $J_x \subseteq \mathcal{J}(L)$  such that  $x = \bigvee J_x$  (in fact, Dilworth introduced the dual lattices, the *upper locally distributive lattices* (ULD)). This definition is equivalent to Edelman's notion of semi-distributive lattice, but since there are several notions of semidistributivity in the literature, we prefer the name LLD. We refer to [17, 23, 26] for different equivalent characterizations, different names, as well as many different instances of LLDs and ULDs. We continue with the characterization of lattices of convex geometries provided by Edelman [10]:

**Theorem 19** ([10, Theorem 3.3]). *A lattice is lower locally distributive if and only if it is isomorphic to the inclusion-order of convex sets of a convex geometry.*

The meet-irreducible elements of  $L = (\mathcal{C}, \subseteq)$  correspond to the copoints of the convex geometry  $\mathcal{A} = (U, \mathcal{C})$ ; a *copoint* attached at point  $p$  is a maximal by inclusion convex set of  $\mathcal{C}$  not containing  $p$ . Edelman and Jamison [11] characterized convex geometries as the convexity spaces for which each copoint has a unique attaching point. Furthermore, Edelman and Saks [12] characterized the convex dimension  $\text{cdim}(\mathcal{C})$  of a convex geometry  $\mathcal{A}$  in term of the poset of all meet-irreducibles in the following nice way:

**Theorem 20** ([12]). *For a convex geometry  $\mathcal{A} = (U, \mathcal{C})$ ,  $\text{cdim}(\mathcal{C})$  is equal to the size of the largest antichain of the poset  $(\mathcal{M}(L), \subseteq)$ , where  $L = (\mathcal{C}, \subseteq)$ .*

An *ideal*  $I$  of a poset is a subset that is downwards-closed, i.e. if  $x \leq y$  and  $y \in I$ , then  $x \in I$ . From Theorem 19 by definition we get a justification of the name *ideals* of convex geometries:

**Remark 21.** A meet-semilattice is an ideal of a lower locally distributive lattice if and only if it is isomorphic to the inclusion-order of convex sets of an ideal of a convex geometry.

#### 4.5 Examples of ideals of convex geometries

Edelman and Jamison [11] and Korte et al. [21] presented numerous examples of convex geometries and antimatroids, arising from geometry, language theory, and chip firing games. Bandelt et al. [2] presented simplicial complexes and median set systems as examples of conditional antimatroids. We will show below that multisimplicial complexes and median set systems are in fact ideals of ideal alignments, a particular class of convex geometries.

A *simplicial complex*  $L$  on a set  $U$  is a family of subsets of  $U$ , called *simplices* or *faces* of  $L$ , such that if  $\sigma \in L$  and  $\sigma' \subseteq \sigma$ , then  $\sigma' \in L$ . The *facets* of  $L$  are the maximal (by inclusion) faces of  $L$ . The *dimension*  $d$  of  $L$  is the size of its largest face. A multi-subset  $\sigma$  of  $U$  is a subset of  $U$  such that each element  $e \in U$  is given with its multiplicity  $n_\sigma(e)$  (the number of times,  $e$  occurs in  $\sigma$ ). A *multisimplicial complex*  $L$  on a set  $U$  is a family of multi-subsets of  $U$ ,

such that if  $\sigma \in L$  and  $\sigma'$  is a multi-subset of  $U$  such that  $n_{\sigma'}(e) \leq n_{\sigma}(e)$  for each  $e \in U$ , then  $\sigma' \in L$ . The *size* of a face  $\sigma$  of  $L$  is the size of  $\{e \in U : n_{\sigma}(e) > 0\}$ . The *dimension*  $d$  of  $L$  is the size of its largest face.

A *median set system* is a set family  $(U, \mathcal{C})$  satisfying (C1'), (C2), and

(C6) for any  $x \neq y$  in  $U$ , there exists some  $K \in \mathcal{C}$  with  $|\{x, y\} \cap K| = 1$ ,

(C7)  $K_i, M_i \in \mathcal{C}$  ( $i = 1, 2, 3$ ) with  $K_i \cup K_j \subseteq M_k$  for  $\{i, j, k\} = \{1, 2, 3\}$  implies  $K_1 \cup K_2 \cup K_3 \in \mathcal{C}$ .

The name is justified by the fact that by virtue of (C2) and (C7),  $\mathcal{C}$  is closed under the median operation  $m$  of  $2^U$  defined by

$$m(L_1, L_2, L_3) := (L_1 \cap L_2) \cup (L_1 \cap L_3) \cup (L_2 \cap L_3).$$

The resulting meet-semilattice is called a *median semilattice* [1, 4, 25]. Median semilattices are locally lower distributive. Indeed, by a result of Birkhoff and Kiss [4] they can be characterized by the property that all intervals  $[\hat{0}, x]$  are distributive lattices and three elements have an upper bound whenever each pair of them does.

Every abstract finite median algebra (for which the former set-theoretic ternary operation is axiomatized) or, equivalently, any median graph can be represented by a median set system via the Sholander embedding [25] into some power set  $2^U$ . An inherent feature of median algebras/graphs is that they may be oriented so that any element can serve as the empty set in the associated set representation: a median set system  $\mathcal{C}$  is mapped onto another one  $\mathcal{C} \Delta Z := \{A \Delta Z : K \in \mathcal{C}\}$ , by the automorphism of  $2^U$  taking the symmetric difference with a fixed set  $Z \in \mathcal{C}$ .

Let  $\leq$  be a partial order on  $U$ . The *ideal alignment*  $\mathcal{D}_P$  [11] of the poset  $P = (U, \leq)$  consists of all ideals of  $P$ . It was noticed in [11] that the ideal alignments are convex geometries and it was shown in [11, Theorem 3.2] that a convex geometry  $\mathcal{A} = (U, \mathcal{C})$  is an ideal alignment if and only if  $\mathcal{C}$  is union-closed. From the lattice point of view, ideal alignments simply correspond to distributive lattices.

We say that a pair  $\mathcal{A} = (U, \mathcal{C})$  is an *ideal semialignment* if  $\mathcal{C}$  contains  $\emptyset$ , is intersection-closed, *locally extendable* in the sense that if  $X, Y \in \mathcal{C}$  such that  $X \subseteq Y$ , then there is  $e \in Y \setminus X$  such that  $X \cup e \in \mathcal{C}$ , and *locally union-closed*, in the sense that if  $X, Y \in \mathcal{C}$  and there exists  $Z \in \mathcal{C}$  such that  $X, Y \subseteq Z$ , then  $X \cup Y \in \mathcal{C}$ . The next result shows that ideal semialignments are ideals of ideal alignments, hence in particular ideals of convex geometries:

**Proposition 22.** *Every ideal semialignment  $\mathcal{A} = (U, \mathcal{C})$  with  $k$  maximal convex sets is an ideal of an ideal alignment  $\mathcal{A}^+$  such that  $\text{VC-dim}(\mathcal{A}^+) \leq k \cdot \text{VC-dim}(\mathcal{C})$ .*

*Proof.* We will prove that  $\mathcal{A} = (U, \mathcal{C})$  is an ideal of an ideal alignment by induction on the number of inclusion-maximal convex sets in  $\mathcal{C}$ . If there is only one such set, then  $\mathcal{A}$  is a convex geometry and we are done. Otherwise, take a maximal set  $\mathcal{M} = \{X_1, \dots, X_k\}$  of at least two maxima such that  $\bigcap_{i=1}^k X_i = M$  is maximal. Now define  $\mathcal{C}^+$  by adding all sets of the form  $Y_1 \cup \dots \cup Y_k$  with  $M \subseteq Y_i \subseteq X_i$  for  $1 \leq i \leq k$  to  $\mathcal{C}$ . We set  $\mathcal{A}^+ = (U, \mathcal{C}^+)$ .

First notice that  $\mathcal{C}^+$  has a new maximal element  $\bigcup_{i=1}^k X_i$ , while none of the sets in  $\mathcal{M}$  is maximal anymore. Hence,  $\mathcal{C}^+$  has less inclusion-maximal convex sets than  $\mathcal{C}$ . Let us show that  $\mathcal{C}$  is an ideal of  $\mathcal{C}^+$ . For this purpose let  $X$  be an inclusion-maximal convex set from  $\mathcal{C}$  and  $R = Y_1 \cup \dots \cup Y_k \subseteq X$  with  $R \in \mathcal{C}^+$ . Then for all  $1 \leq i \leq k$  we have  $Y_i \subseteq X_i \cap X$ . But by the maximality in the choice of  $\mathcal{M}$  this implies  $X_i \cap X = M$  and hence  $X \in \mathcal{M}$ , say  $X = X_k$ . But then  $Y_i = M$  for  $1 \leq i \leq k-1$  and hence  $R = Y_k \in \mathcal{C}$ .

Now, we show that  $\mathcal{C}^+$  is intersection-closed. Let  $A, B \in \mathcal{C}^+$ . If  $A, B \in \mathcal{C}$ , then there is nothing to show, so suppose first  $A \in \mathcal{C}$  and  $B \in \mathcal{C}^+ \setminus \mathcal{C}$ . Hence we can represent  $B = Y_1 \cup \dots \cup Y_k$  and we consider  $A \cap (Y_1 \cup \dots \cup Y_k) = (A \cap Y_1) \cup \dots \cup (A \cap Y_k)$ . Since all of these sets are subsets of  $A$ , since  $\mathcal{C}$  is locally union-closed, their union is in  $\mathcal{C}$ . If  $A, B \in \mathcal{C}^+ \setminus \mathcal{C}$ , then  $A = Y_1 \cup \dots \cup Y_k$  and  $B = Y'_1 \cup \dots \cup Y'_k$  and  $(Y_1 \cup \dots \cup Y_k) \cap (Y'_1 \cup \dots \cup Y'_k) = Y_1 \cap (Y'_1 \cup \dots \cup Y'_k) \cup \dots \cup Y_k \cap (Y'_1 \cup \dots \cup Y'_k)$  but for each  $i$  here we have  $M \subseteq Y_i \cap (Y'_1 \cup \dots \cup Y'_k) \subseteq X_i$ , hence their union is in  $\mathcal{C}^+$  by definition of  $\mathcal{C}^+$ .

In order to show that  $\mathcal{C}^+$  is locally union-closed let  $A, B \in \mathcal{C}^+$  and let  $A, B \subseteq Z$ . Since we can assume that  $A \notin \mathcal{C}$ , without loss of generality we can set  $Z = \bigcup_{i=1}^k X_i$  and  $A = Y_1 \cup \dots \cup Y_k$  as before. Now,  $B = (B \cap X_1) \cup \dots \cup (B \cap X_k)$ , where each term is in  $\mathcal{C}$ . Then,  $A \cup B = (Y_1 \cup (B \cap X_1)) \cup \dots \cup Y_k \cup (B \cap X_k)$ , where each term is in  $\mathcal{C}$  by local union-closedness. Hence by definition of  $\mathcal{C}^+$ ,  $A \cup B \in \mathcal{C}^+$ .

Finally, let us show that  $\mathcal{C}^+$  is locally extendable. For this purpose let  $A, B \in \mathcal{C}^+$  distinct sets such that  $B$  is a maximal convex set and  $A \subseteq B$ . If  $B \in \mathcal{C}$ , then since  $\mathcal{C}$  is an ideal, also  $A \in \mathcal{C}$  and there is nothing to show. So suppose  $A \in \mathcal{C}^+$  and  $B \in \mathcal{C}^+ \setminus \mathcal{C}$ . We know that  $B = \bigcup_{i=1}^k X_i$  if  $A \in \mathcal{C}$ , then if  $A \cap X_i \subsetneq X_i$  for some  $i$ , then there is an element  $e \in X_i \setminus A \subseteq B \setminus A$  such that  $(A \cap X_i) \cup e \in \mathcal{C}$  since  $\mathcal{C}$  is locally extendable. Now,  $\mathcal{C}^+$  is locally union-closed also  $A \cup (A \cap X_i) \cup e = A \cup e \in \mathcal{C}^+$ . If  $A \in \mathcal{C}^+ \setminus \mathcal{C}$ , then  $A = Y_1 \cup \dots \cup Y_k$  as usual and  $Y_i \subsetneq X_i$  for some  $i$ . Since  $\mathcal{C}$  is locally extendable, there is  $e \in X_i \setminus Y_i \subseteq B \setminus A$  such that  $Y_i \cup e \in \mathcal{C}$ . But then by definition of  $\mathcal{C}^+$ , we have that  $A \cup e \in \mathcal{C}^+$ . The bound on the VC-dimension follows by construction.  $\square$

The grid  $\mathbb{N}^n$  can be viewed as the covering graph of the poset  $(\mathbb{N}^n, \leq)$ , where for  $x, y \in \mathbb{N}^n$  with  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$ , we set  $x \leq y$  if and only  $x_i \leq y_i$  for  $i = 1, \dots, n$ . Since ideals of the grid  $(\mathbb{N}^n, \leq)$  are ample, the VC-dimension  $\dim_{\text{VC}}(L)$  of any such ideal  $L$  is the largest subcube of  $L$ . Notice that  $\dim_{\text{VC}}(L)$  can be smaller than the dimension  $n$  of the grid. It is obvious that there exists a bijection between the ideals of the grid  $(\mathbb{N}^n, \leq)$  and the multisimplicial complexes on a set  $X$  of size  $n$ . The simplicial complexes are in bijection with the ideals of the  $n$ -cube  $\{0, 1\}^n \subset \mathbb{N}^n$ . The VC-dimension of any (multi)simplicial complex coincides with its dimension. Hence, we can apply Proposition 22 to obtain the following result:

**Corollary 23.** *Every multisimplicial complex of dimension  $d$  and  $k$  facets is an ideal of an ideal alignment of VC-dimension at most  $kd$ .*

We get an analogous bound for median set systems:

**Corollary 24.** *Every median set system of VC-dimension  $d$  and having  $k$  maximal elements is an ideal of an ideal alignment of VC-dimension at most  $kd$ .*

## 5 Realization of convex geometries and of their ideals

In this section we prove the main results of the paper. First, we prove that any convex geometry  $\mathcal{A} = (U, \mathcal{C})$  admits a realization  $(\mathcal{H}_n, K(\mathcal{R}_0))$  in  $\mathbb{R}^n$ , where  $n = |U|$ ,  $\mathcal{H}_n$  are the coordinate hyperplanes of  $\mathbb{R}^n$ , and  $K(\mathcal{R}_0)$  has as many facets as the number of critical rooted circuits of  $\mathcal{A}$ . Then we prove that any ideal  $\mathcal{I}$  of  $\mathcal{A}$  is also realizable in  $\mathbb{R}^n$  and the polyhedron  $K_0$  realizing  $\mathcal{I}$  has as many facets as the number of critical rooted circuits of  $\mathcal{A}$  and positive circuits of  $\mathcal{I}$ . Furthermore, we prove that each convex geometry is realizable in dimension equal to its convex dimension. Finally, we prove that trees and multisimplicial complexes have realizations in dimension bounded linearly in their VC-dimension.

### 5.1 Realization of ideals of convex geometries

Let  $U = \{1, \dots, n\}$  and let  $\mathcal{A} = (U, \mathcal{C})$  be a convex geometry on  $U$  having  $\mathcal{R} := \mathcal{R}(\mathcal{A})$  as the set of rooted circuits and  $\mathcal{R}_0 := \mathcal{R}_0(\mathcal{A})$  as the set of critical rooted circuits. For each rooted circuit  $(C, r)$  of  $\mathcal{A}$  consider the hyperplane  $H(C, r)$  defined by the equation  $|U|x_r = \sum_{e \in C \setminus \{r\}} x_e$ . Denote by  $K(C, r)$  the open halfspace of  $\mathbb{R}^n$  determined by  $H(C, r)$  as

$$K(C, r) = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n : |U|x_r > \sum_{e \in C \setminus \{r\}} x_e \right\}.$$

Let  $K(\mathcal{R}) = \bigcap_{(C, r) \in \mathcal{R}} K(C, r)$  and  $K(\mathcal{R}_0) = \bigcap_{(C, r) \in \mathcal{R}_0} K(C, r)$  be the open polyhedra that are respectively the intersection of all  $K(C, r)$  taken over all rooted circuits  $(C, r)$  of  $\mathcal{A}$  and the intersection of all  $K(C, r)$  taken over all critical rooted circuits  $(C, r)$  of  $\mathcal{A}$ . Obviously,  $K(\mathcal{R}) \subseteq K(\mathcal{R}_0)$ .

Consider now an ideal  $\mathcal{I} = (U', \mathcal{C}')$  of the convex geometry  $\mathcal{A} = (U, \mathcal{C})$ . Let  $\mathcal{P} := \mathcal{P}(\mathcal{I})$  be the set of positive circuits of  $\mathcal{I}$  with respect to  $\mathcal{A}$ . For each  $P \in \mathcal{P}$ , the hyperplane  $H(P)$  is defined by the equation  $\sum_{e \in P} x_e = 0$ . Denote by  $K(P)$  the open halfspace of  $\mathbb{R}^n$  determined by  $H(P)$  as

$$K(P) = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n : \sum_{e \in P} x_e < 0 \right\}.$$

Finally, consider the open polyhedra  $K(\mathcal{P}) = \bigcap_{P \in \mathcal{P}} K(P)$ ,  $K = K(\mathcal{R}) \cap K(\mathcal{P})$ , and  $K_0 = K(\mathcal{R}_0) \cap K(\mathcal{P})$ . Clearly,  $K \subseteq K_0 \subseteq K(\mathcal{P})$ .

**Theorem 25.** *For a convex geometry  $\mathcal{A} = (U, \mathcal{C})$ , the pairs  $(\mathcal{H}_n, K(\mathcal{R}))$  and  $(\mathcal{H}_n, K(\mathcal{R}_0))$  are realizations of  $\mathcal{A}$ . More generally, for any ideal  $\mathcal{I} = (U', \mathcal{C}')$  of  $\mathcal{A}$ , the pairs  $(\mathcal{H}_n, K)$  and  $(\mathcal{H}_n, K_0)$  are realizations of  $\mathcal{I}$ .*

The proof of the first assertion is a direct consequence of the second assertion. The proof that  $(\mathcal{H}_n, K(\mathcal{R}))$  and  $(\mathcal{H}_n, K)$  are realizations of  $\mathcal{A}$  and  $\mathcal{I}$  is inspired by (but is simpler than) the proof of Kashiwabara et al. [16] that convex geometries can be represented via generalized convex shellings. The proof of the second assertion follows from four lemmas, which we prove first. We first show that for any  $X \in \mathcal{C}'$ , the convex set  $K$  intersects the  $X$ -orthant  $\mathcal{O}(X)$  of  $\mathbb{R}^n$ .

**Lemma 26.** For any  $X \in \mathcal{C}'$ , the  $X$ -orthant  $\mathcal{O}(X)$  of  $\mathbb{R}^n$  intersects the convex polyhedron  $K$ .

*Proof.* We prove that the  $X$ -orthant  $\mathcal{O}(X)$  intersects the convex set  $K$ , i.e., that there exists a point  $x^* = (x_1^*, \dots, x_n^*) \in \mathcal{O}(X) \cap K$ . Let  $|X| = k$  and suppose without loss of generality that  $X = \{e_1, \dots, e_k\}$ . First, we set  $x_{e_1}^* = x_{e_2}^* = \dots = x_{e_k}^* = 1$ . In the following, we set a negative value for each  $x_e^*, e \notin X$ . Observe that for any rooted circuit  $(C, r)$  such that  $C \subseteq X$ , we have  $|U|x_r^* = |U| > (|C| - 1) = \sum_{e \in C \setminus \{r\}} x_e^*$  and thus  $x^* \in K(C, r)$ , independently of the values of the remaining  $n - k$  coordinates of  $x^*$ . Notice also that, since  $X$  belongs to  $\mathcal{C}'$ ,  $X$  does not contain any positive circuit of  $\mathcal{C}'$ . Since  $X \in \mathcal{C}$ , by axiom (C3) of convex geometries, the elements of  $U \setminus X$  can be ordered  $e_{k+1}, \dots, e_n$ , such that all the sets  $X_i = X \cup \{x_{k+1}, \dots, x_{k+i}\} = X_{i-1} \cup \{x_{k+i}\}, i = 1, \dots, n - k$  are convex sets of  $\mathcal{C}$ . We define the remaining  $n - k$  coordinates of  $x^*$  (which have to be negative) following the order  $e_{k+1}, \dots, e_n$ . Suppose that after  $i - 1$  steps the coordinates  $x_{e_{k+1}}^*, \dots, x_{e_{k+i-1}}^*$  have been defined. At stage  $i$  we need to define the value of  $x_{e_{k+i}}^*$  in order to satisfy

- all constraints of the form  $|U|x_r^* > \sum_{e \in C \setminus \{r\}} x_e^*$ , where  $(C, r) \in \mathcal{R}(\mathcal{A})$ ,  $e_{k+i} \in C$ , and  $C \subset X_{k+i}$ ;
- all constraints of the form  $\sum_{e \in P} x_e^* < 0$ , where  $P \in \mathcal{P}$ ,  $e_{k+i} \in P$ , and  $P \subset X_{k+i}$ .

For any rooted circuit  $(C, r)$  such that  $e_{k+i} \in C \subseteq X_{k+i}$ , the values of  $x_e^*$  for all elements  $e \in C \setminus \{e_{k+i}\}$  have been already defined. Since  $X_{k+i-1} \in \mathcal{C}$  and since  $(C, r)$  is a rooted circuit of  $\mathcal{A}$ , we know that  $r \neq e_{k+i}$ . Consequently, if we set  $x_{e_{k+i}}^* < |U|x_r^* - \sum_{e \in C \setminus \{r, e_{k+i}\}} x_e^*$ , the point  $x^*$  is in the halfspace  $K(C, r)$ .

Similarly, for any positive circuit  $P$  of  $\mathcal{I}$  such that  $e_{k+i} \in P \subseteq X_{k+i}$ , the values of  $x_e^*$  for all elements  $e \in P \setminus \{e_{k+i}\}$  have been already defined. Setting  $x_{e_{k+i}}^* < -\sum_{e \in P \setminus \{e_{k+i}\}} x_e^*$  ensures that the point  $x^*$  is in the halfspace  $K(P)$ . By choosing a negative value for  $x_{e_{k+i}}^*$  that is small enough, we can satisfy all the constraints over the rooted circuits  $(C, r)$  and the positive circuits  $P$  contained in the set  $X_{k+i}$ .

Consequently, we have constructed a point  $x^*$  belonging to the  $X$ -orthant  $\mathcal{O}(X)$  and to the convex polyhedron  $K$ .  $\square$

We now show that if a set  $Y$  is not in  $\mathcal{C}$ , then we can find a rooted circuit  $(C, r)$  whose hyperplane  $H(C, r)$  separates the  $Y$ -orthant  $\mathcal{O}(Y)$  from the polyhedron  $K(\mathcal{R})$  (and thus from  $K$  and  $K_0$ ).

**Lemma 27.** For any  $Y \in \mathcal{C}^*$ , there exists a rooted circuit  $(C, r)$  of  $\mathcal{A}$  such that  $\mathcal{O}(Y) \cap K(C, r) = \emptyset$ . Consequently, the hyperplane  $H(C, r)$  separates  $\mathcal{O}(Y)$  from  $K(\mathcal{R})$ .

*Proof.* Since  $Y \notin \mathcal{C}$ , by Theorem 13 there exists a rooted circuit  $(C, r)$  such that  $C \cap Y = C \setminus \{r\}$ . Pick any point  $y = (y_1, \dots, y_n)$  from the  $Y$ -orthant  $\mathcal{O}(Y)$  of  $\mathbb{R}^n$  (i.e.,  $y_e > 0$  for  $e \in Y$  and  $y_e < 0$  for  $e \notin Y$ ). Since  $C \cap Y = C \setminus \{r\}$ , we have  $y_r < 0$  and  $y_e > 0$  for any  $e \in C \setminus \{r\}$ . Consequently,  $y$  does not satisfy the inequality  $|U|y_r > \sum_{e \in C \setminus \{r\}} y_e$ , and thus  $y \notin K$ . As a result, each orthant  $\mathcal{O}(Y)$  with  $Y \in \mathcal{C}^*$  is separated from  $K(\mathcal{R})$  by a hyperplane of the form  $H(C, r)$  with  $(C, r) \in \mathcal{R}$ .  $\square$

Similarly, if  $Y \in \mathcal{C} \setminus \mathcal{C}'$ , then we can find a positive circuit  $P \in \mathcal{P}$  such that the hyperplane  $H(P)$  separates the  $Y$ -orthant  $\mathcal{O}(Y)$  from the polyhedron  $K(\mathcal{P})$  (and thus from  $K$  and  $K_0$ ).

**Lemma 28.** For any  $Y \in \mathcal{C} \setminus \mathcal{C}'$ , there exists a positive circuit  $P \in \mathcal{P}$  such that  $\mathcal{O}(Y) \cap K(P) = \emptyset$ . Consequently, the hyperplane  $H(P)$  separates  $\mathcal{O}(Y)$  from  $K(\mathcal{P})$ .

*Proof.* By Lemma 17, there exists a positive circuit  $P \in \mathcal{P}$  such that  $P \subseteq Y$ . Consider a point  $y = (y_1, \dots, y_n)$  from the  $Y$ -orthant  $\mathcal{O}(Y)$  of  $\mathbb{R}^n$ . Since  $P \subseteq Y$ , we have  $y_e > 0$  for any  $e \in P$ . Consequently,  $y$  does not satisfy the inequality  $\sum_{e \in P} y_e < 0$ , and thus  $y \notin K(\mathcal{P})$ . As a result, each orthant  $\mathcal{O}(Y)$  with  $Y \in \mathcal{C} \setminus \mathcal{C}'$  is separated from  $K(\mathcal{P})$  by a hyperplane of the form  $H(P)$  with  $P \in \mathcal{P}$ .  $\square$

Finally, we show that if  $Y \in \mathcal{C}^*$ , then the orthant  $\mathcal{O}(Y)$  does not intersect the polyhedron  $K(\mathcal{R}_0)$  defined by the critical rooted circuits of  $\mathcal{A}$ . This follows from the following result.

**Lemma 29.** For any  $Y \in \mathcal{C}^*$ , there exists a rooted set  $(A, r)$  and real numbers  $(a_g)_{g \in A \setminus \{e\}}$  such that

- (1)  $Y \cap A = A \setminus \{r\}$ ,
- (2)  $0 < a_g \leq \frac{1}{|Y|}$  for each  $g \in A \setminus \{r\}$ ,

(3) for any  $x \in K(\mathcal{R}_0)$ ,  $x_r > \sum_{g \in A \setminus \{r\}} a_g x_g$ .

*Proof.* In order to prove the lemma, we employ an *elimination procedure* similar to Gaussian elimination. Note that the rooted sets  $(A, r)$  we consider are not necessarily circuits. We show the lemma by reverse induction on  $|Y|$ .

The following claim ensures that the property holds for any  $Y \in \mathcal{C}^*$  such that  $Y \cup \{r\} \in \mathcal{C}$  for some  $r \in U \setminus Y$ .

**Claim 30.** *For any  $Y \in \mathcal{C}^*$ , if there exists  $r \in U \setminus Y$  such that  $Y \cup \{r\} \in \mathcal{C}$ , then there exists a critical rooted circuit  $(C, r)$  such that  $C \subseteq Y$ , and thus for each  $x \in K(\mathcal{R}_0)$ , we have  $x_r > \sum_{g \in C \setminus \{r\}} \frac{1}{|U|} x_g$ .*

*Proof.* Consider a minimal subset  $Z \subseteq Y$  such that  $Z \in \mathcal{C}^*$  and  $Z \cup \{r\} \in \mathcal{C}$ . Let  $C = \text{ex}(Z \cup \{r\})$  and note that  $Z \cup \{r\} = \text{conv}(C)$  by (C4). By the definition of  $C$ , for any  $z \in C$ , we have  $Z \cup \{r\} \setminus \{z\} \in \mathcal{C}$ , and by minimality of  $Z$ , we have  $Z \setminus \{z\} \in \mathcal{C}$ . Consequently,  $(C, r)$  is a critical rooted circuit of  $\mathcal{C}$  and thus by the definition of  $K(\mathcal{R}_0)$ , for each  $x \in K(\mathcal{R}_0)$ , we have  $x_r > \sum_{g \in C \setminus \{r\}} \frac{1}{|U|} x_g$ .  $\square$

Suppose now that  $Y \in \mathcal{C}^*$  and that for any  $e \in U \setminus Y$ , we also have  $Y \cup \{e\} \in \mathcal{C}^*$ . By induction hypothesis, for any such set  $Y \cup \{e\}$ , there exists a rooted set  $(A_e, r_e)$  and coefficients  $(a_g)_{g \in A_e \setminus \{r_e\}}$  such that

- $(Y \cup \{e\}) \cap A_e = A_e \setminus \{r_e\}$ ,
- $0 < a_g \leq \frac{1}{|Y|+1}$  for each  $g \in A_e \setminus \{r_e\}$ ,
- $x_{r_e} > \sum_{g \in A_e \setminus \{r_e\}} a_g x_g$  for any  $x \in K(\mathcal{R}_0)$ .

If there exists  $e \in U \setminus Y$  such that  $e \notin A_e$ , then  $Y \cap A_e = A_e \setminus \{r_e\}$  and we are done. Consequently, we now assume that for any  $e \in U \setminus Y$ , we have  $e \in A_e$ , i.e., for any  $e \in U \setminus Y$ , we have  $Y \cap A_e = Y \setminus \{e, r_e\}$ . Consider the digraph  $D_Y$  having  $U \setminus Y$  as the vertex-set and where there is an arc  $(e, f)$  precisely when  $f = r_e$ . Note that in  $D_Y$  every vertex has an out-neighbor and consequently,  $D_Y$  contains a cycle. Consider a shortest cycle  $(e_0, e_1, \dots, e_k, e_{k+1} = e_0)$  in  $D_Y$ . By induction hypothesis, for each  $0 \leq i \leq k$ , there exists a rooted set  $(A_{e_i}, e_{i+1})$  and coefficients  $(a_{i,g})_{g \in A_{e_i} \setminus \{e_{i+1}\}}$  such that

- $(Y \cup \{e_i\}) \cap A_{e_i} = A_{e_i} \setminus \{e_{i+1}\}$ ,
- $0 < a_{i,g} \leq \frac{1}{|Y|+1}$  for each  $g \in A_{e_i} \setminus \{e_{i+1}\}$ ,
- for any  $x \in K(\mathcal{R}_0)$ ,  $x_{e_{i+1}} > \sum_{g \in A_{e_i} \setminus \{e_{i+1}\}} a_{i,g} x_g = a_{i,e_i} x_{e_i} + \sum_{g \in A_{e_i} \setminus \{e_i, e_{i+1}\}} a_{i,g} x_g$ .

For each  $0 \leq i \leq k$ , let  $b_i = a_{i,e_i}$  and observe that  $0 < b_i \leq \frac{1}{|Y|+1} < 1$ . Consequently,  $0 < \prod_{i=0}^k b_i < 1$ . For each  $0 \leq i \leq k$  and each  $g \in Y \setminus A_{e_i}$ , let  $a_{i,g} = 0$ . Observe that for any  $0 \leq i \leq k$ , for every  $x \in K(\mathcal{R}_0)$ , we thus obtain  $x_{e_{i+1}} > b_i x_{e_i} + \sum_{g \in Y} a_{i,g} x_g$ . We have the following inequality:

**Claim 31.** *For every  $0 \leq i \leq k$  and every point  $x \in K(\mathcal{R}_0)$ , we have:*

$$x_{e_{i+1}} > \left( \prod_{j=0}^i b_j \right) x_{e_0} + \sum_{g \in Y} \left( \sum_{j=0}^i \left( \prod_{\ell=j+1}^i b_\ell \right) a_{j,g} \right) x_g.$$

*Proof.* We prove the claim by induction on  $i$ . The claim trivially holds for  $i = 0$ . Assume that the claim holds for  $i < k$  and note that for any point  $x \in K(\mathcal{R}_0)$ , we have:

$$\begin{aligned} x_{e_{i+2}} &> b_{i+1} x_{e_{i+1}} + \sum_{g \in Y} a_{i+1,g} x_g \\ &> b_{i+1} \left( \left( \prod_{j=0}^i b_j \right) x_{e_0} + \sum_{g \in Y} \left( \sum_{j=0}^i \left( \prod_{\ell=j+1}^i b_\ell \right) a_{j,g} \right) x_g \right) + \sum_{g \in Y} a_{i+1,g} x_g \\ &= \left( \prod_{j=0}^{i+1} b_j \right) x_{e_0} + \sum_{g \in Y} \left( \sum_{j=0}^i \left( \prod_{\ell=j+1}^{i+1} b_\ell \right) a_{j,g} \right) x_g + \sum_{g \in Y} a_{i+1,g} x_g \\ &= \left( \prod_{j=0}^{i+1} b_j \right) x_{e_0} + \sum_{g \in Y} \left( \sum_{j=0}^{i+1} \left( \prod_{\ell=j+1}^{i+1} b_\ell \right) a_{j,g} \right) x_g. \end{aligned} \quad \square$$

Consequently, for every point  $x \in K(\mathcal{R}_0)$ , we have

$$x_{e_0} = x_{e_{k+1}} > \left( \prod_{j=0}^k b_j \right) x_{e_0} + \sum_{g \in Y} \left( \sum_{j=0}^k \left( \prod_{\ell=j+1}^k b_\ell \right) a_{j,g} \right) x_g.$$

This implies the inequality

$$\left( 1 - \prod_{j=0}^k b_j \right) x_{e_0} > \sum_{g \in Y} \left( \sum_{j=0}^k \left( \prod_{\ell=j+1}^k b_\ell \right) a_{j,g} \right) x_g,$$

yielding

$$x_{e_0} > \frac{1}{1 - \prod_{j=0}^k b_j} \sum_{g \in Y} \left( \sum_{j=0}^k \left( \prod_{\ell=j+1}^k b_\ell \right) a_{j,g} \right) x_g.$$

For any  $g \in Y$ , let  $a'_g = \frac{1}{1 - \prod_{j=0}^k b_j} \sum_{j=0}^k \left( \prod_{\ell=j+1}^k b_\ell \right) a_{j,g}$ . Let  $C' \subseteq Y$  be the support of  $(a'_g)_{g \in Y}$ , i.e., the set  $\{g \in Y : a'_g \neq 0\}$ . Observe that the rooted set  $(C' \cup \{e_0\}, e_0)$  and the coefficients  $(a'_g)_{g \in C'}$  satisfy Conditions (1) and (3) of the lemma. Note that for each  $0 \leq i \leq k$  and each  $g \in Y$ , we have  $0 < b_i \leq \frac{1}{|Y|+1}$  and  $0 < a_{i,g} \leq \frac{1}{|Y|+1}$ . It is obvious that  $a'_g > 0$  for any  $g \in C'$ . In order to establish Condition (2), it is then enough to show that  $a'_g \leq \frac{1}{|Y|}$  for any  $g \in Y$ . Let  $\Delta = \frac{1}{|Y|+1}$  and note that  $\frac{1}{1 - \prod_{j=0}^k b_j} \leq \frac{1}{1 - \Delta^{k+1}}$ . Consequently, for every  $g \in Y$ , we have

$$\begin{aligned} a'_g &= \frac{1}{1 - \prod_{j=0}^k b_j} \sum_{j=0}^k \left( \prod_{\ell=j+1}^k b_\ell \right) a_{j,g} \leq \frac{1}{1 - \Delta^{k+1}} \sum_{j=0}^k \left( \prod_{\ell=j+1}^k \Delta \right) \Delta = \frac{\Delta}{1 - \Delta^{k+1}} \sum_{j=0}^k \Delta^{k-j} \\ &= \frac{\Delta}{1 - \Delta^{k+1}} \sum_{j=0}^k \Delta^j = \frac{\Delta}{1 - \Delta^{k+1}} \cdot \frac{1 - \Delta^{k+1}}{1 - \Delta} = \frac{\Delta}{1 - \Delta} = \frac{\frac{1}{|Y|+1}}{1 - \frac{1}{|Y|+1}} = \frac{1}{|Y|+1} \cdot \frac{|Y|+1}{|Y|+1-1} \\ &= \frac{1}{|Y|}, \end{aligned} \quad \square$$

which concludes the proof of the lemma.

*Proof of Theorem 25.* The proof that  $(\mathcal{H}_n, K)$  is a realization of  $\mathcal{I}$  is obtained by combining Lemmas 26, 27, and 28 and the equality  $K = K(\mathcal{R}) \cap K(\mathcal{P})$ . The proof that  $(\mathcal{H}_n, K_0)$  is a realization of  $\mathcal{I}$  is obtained by combining Lemmas 26, 29, and 28 and the equality  $K_0 = K(\mathcal{R}_0) \cap K(\mathcal{P})$ . Finally, the proof of the first assertion of the theorem that the pairs  $(\mathcal{H}_n, K(\mathcal{R}))$  and  $(\mathcal{H}_n, K(\mathcal{R}_0))$  are realizations of  $\mathcal{A}$  is a direct consequence of the second assertion because a convex geometry  $\mathcal{A}$  is an ideal of itself not containing positive circuits (and thus  $K = K(\mathcal{R})$  and  $K_0 = K(\mathcal{R}_0)$ ).  $\square$

**Example 32.** Consider the convex geometry  $\mathcal{A} = (U, \mathcal{C})$  of Figure 1. Here,  $U = \{1, 2, 3, 4\}$  and  $\mathcal{C} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$ . The rooted circuits of  $\mathcal{A}$  are  $(\{1, 2, 3\}, 2)$ ,  $(\{2, 3, 4\}, 3)$ ,  $(\{1, 2, 4\}, 2)$ , and  $(\{1, 3, 4\}, 3)$ . Then in  $\mathbb{R}^4$ ,  $\mathcal{A}$  is realized by the convex set defined by the following inequalities:

$$x_1 + x_3 < 3x_2 \qquad x_2 + x_4 < 3x_3 \qquad x_1 + x_4 < 3x_2 \qquad x_1 + x_4 < 3x_3$$

On the other hand, among the rooted circuits of  $\mathcal{A}$ , only  $(\{1, 2, 3\}, 2)$  and  $(\{2, 3, 4\}, 3)$  are critical. The two first inequalities above correspond to these two critical rooted circuits. These two inequalities are sufficient to have a realization of  $\mathcal{A}$  in  $\mathbb{R}^4$ . Indeed, from the two first inequalities, we obtain  $3x_1 + x_4 < 8x_2$  and  $x_1 + 3x_4 < 8x_3$ . This implies that there is no point in the realization corresponding to the sets  $\{1, 4\}$ ,  $\{1, 2, 4\}$ , and  $\{1, 3, 4\}$ , that are precisely the subsets of  $U$  that are forbidden by the circuits  $(\{1, 2, 4\}, 2)$ , and  $(\{1, 3, 4\}, 3)$ .

From Proposition 10, we obtain the following corollary.

**Corollary 33.** *Any convex geometry  $\mathcal{A} = (U, \mathcal{C})$  admits a point representation  $\mathcal{M}(U, Q_0)$  where  $|Q_0|$  is the number of critical rooted circuits of  $\mathcal{A}$ .*

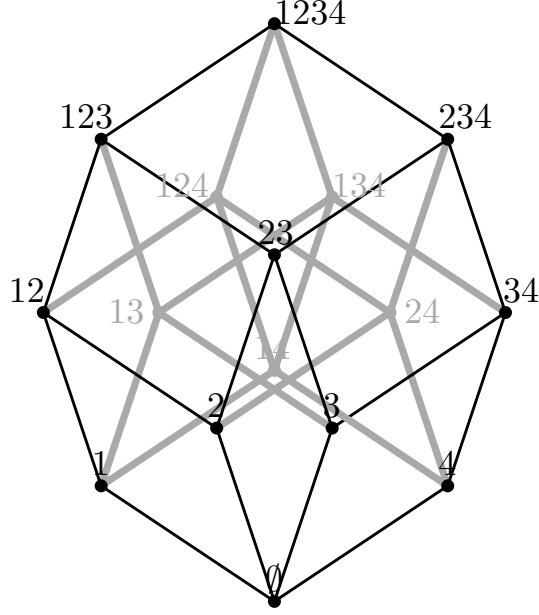


Figure 1: A convex geometry represented in black as subgraph of the 4-dimensional hypercube (in gray)

## 5.2 Realization of convex geometries and convex dimension

In the spirit of Kashiwabara et al. [16], Richter and Rogers [24] established that every convex geometry of convex dimension  $d := \text{cdim}(\mathcal{C})$  can be represented as a generalized convex shelling in  $\mathbb{R}^d$ . The proof of the following result can be seen as a dualization of the proof of [24].

**Theorem 34.** *A convex geometry  $\mathcal{A} = (U, \mathcal{C})$  of convex dimension  $d := \text{cdim}(\mathcal{C})$  is realizable in  $\mathbb{R}^d$ .*

*Proof of Theorem 34.* Let  $\mathcal{A} = (U, \mathcal{C})$  be defined by a set of  $d$  total orders  $(\leq_i)_{1 \leq i \leq d}$ . For each  $e \in \mathcal{C}$ , let  $j_i(e)$  be the index of  $e$  in  $\leq_i$ , (i.e., there are precisely  $j_i(e)$  elements  $e' \in U$  such that  $e' \leq_i e$ ). Consider the sequence of integers  $(a_j)_{0 \leq j \leq |U|}$  such that  $a_j = \frac{d+1}{d-1}(d^j - 1)$  for each  $0 \leq j \leq |U|$ . Observe that  $a_{j+1} = da_j + d + 1$  for any  $0 \leq j < |U|$ . For each  $e \in U$ , let  $b_i(e) = a_{j_i(e)}$  for each  $1 \leq i \leq d$  and consider the hyperplane  $H_e = \{x : \sum_{i=1}^d \frac{x_i}{b_i(e)} = 1\}$  and the halfspaces  $H_e^+ = \{x : \sum_{i=1}^d \frac{x_i}{b_i(e)} < 1\}$  and  $H_e^- = \{x : \sum_{i=1}^d \frac{x_i}{b_i(e)} > 1\}$ . We consider the hyperplane arrangement  $\mathcal{H} = \{H_e : e \in U\}$  and its intersection with the positive orthant  $K = \{x : x_i > 0, \forall 1 \leq i \leq d\}$ . We assert that  $(K, \mathcal{H})$  realizes  $\mathcal{A} = (U, \mathcal{C})$ . The proof is a consequence of the two following claims.

**Claim 35.** *For each  $C \in \mathcal{C}$ , there exists  $y \in K$  such that*

(i) *for each  $e \in C$ ,  $y \in H_e^+$ , i.e.,  $\sum_{i=1}^d \frac{y_i}{b_i(e)} < 1$ , and*

(ii) *for each  $e \notin C$ ,  $y \in H_e^-$ , i.e.,  $\sum_{i=1}^d \frac{y_i}{b_i(e)} > 1$ .*

*Proof.* If  $C = \emptyset$ , let  $y \in K$  such that  $y_i = a_{|U|} + 1$  for all  $1 \leq i \leq d$ . Observe that for each  $e \in U$  and each  $1 \leq i \leq d$ ,  $b_i(e) \leq a_{|U|} < y_i$ . Consequently, for each  $e \in U$ ,  $\sum_{i=1}^d \frac{y_i}{b_i(e)} > 1$ , and  $y \in H_e^-$ . Thus the claim holds for  $C = \emptyset$ .

Assume now that  $C \in \mathcal{C} \setminus \{\emptyset\}$ . For each  $1 \leq \ell \leq d$ , let  $e_\ell = \min_{\leq_\ell} C$ . Observe that  $U \setminus C = \bigcup_{\ell=1}^d \{e \in U : e \leq_\ell e_\ell\}$ . Let  $c_\ell = b_\ell(e_\ell)$ . If  $e_\ell = \min_{\leq_\ell} U$ , let  $c_\ell^- = 0$  and if  $e_\ell \neq \min_{\leq_\ell} U$ , let  $e_\ell^-$  be the predecessor of  $e_\ell$  in  $\leq_\ell$  and let  $c_\ell^- = b_\ell(e_\ell^-)$ . Consider the point  $y \in K$  such that  $y_\ell = 1 + c_\ell^-$  for each  $1 \leq \ell \leq d$ . For each  $e \in C$ , note that  $e_\ell \leq_\ell e$  for all  $1 \leq \ell \leq d$ . Consequently,  $c_\ell = b_\ell(e_\ell) \leq b_\ell(e)$  for all  $1 \leq \ell \leq d$ . Note that  $c_\ell = d + 1 + dc_\ell^- = d(1 + c_\ell^-) + 1 > dy_\ell$ . Consequently,  $\sum_{\ell=1}^d \frac{y_\ell}{b_\ell(e)} \leq \sum_{\ell=1}^d \frac{y_\ell}{c_\ell} < \sum_{\ell=1}^d \frac{1}{d} = 1$ , establishing (i). For each  $e \notin C$ , there exists  $\ell$  such that  $e \leq_\ell e_\ell^-$ . Therefore  $b_\ell(e) \leq b_\ell(e_\ell^-) = c_\ell^- < y_\ell$ . Consequently,  $\frac{y_\ell}{b_\ell(e)} > 1$  and since  $\frac{y_i}{b_i(e)} > 0$  for all  $1 \leq i \leq d$  with  $i \neq \ell$ , we have  $\sum_{i=1}^d \frac{y_i}{b_i(e)} > 1$ , establishing (ii).  $\square$

**Claim 36.** Consider a point  $x \in K$  such that  $\sum_{i=1}^d \frac{x_i}{b_i(e)} \neq 1$  for all  $e \in U$  and let  $C = \{e \in U : \sum_{i=1}^d \frac{x_i}{b_i(e)} < 1\}$ . Then  $C \in \mathcal{C}$ .

*Proof.* If  $C = \emptyset$ , then  $C \in \mathcal{C}$  because  $\emptyset$  is a convex of  $\mathcal{C}$ . Assume now that  $C \neq \emptyset$ . For each  $1 \leq \ell \leq d$ , let  $e_\ell = \min_{\leq_\ell} C$ . To prove the claim, we establish that  $C = \{e \in U : e_\ell \leq_\ell e, \forall 1 \leq \ell \leq d\}$ . If  $e \in C$ , then for all  $1 \leq \ell \leq d$ ,  $e_\ell \leq_\ell e$  and thus  $C \subseteq \{e \in U : e_\ell \leq_\ell e, \forall 1 \leq \ell \leq d\}$

Conversely, suppose that  $e_\ell \leq_\ell e$  for all  $1 \leq \ell \leq d$ . If there exists  $\ell$  such that  $e = e_\ell$ , then by definition of  $e_\ell$ , we have  $e \in C$ . Assume now that for every  $1 \leq \ell \leq d$ ,  $e_\ell \leq_\ell e$ . For each  $1 \leq \ell \leq d$ , let  $e_\ell^+$  be the successor of  $e_\ell$  in  $\leq_\ell$  and note that  $e_\ell^+ \leq_\ell e$ . Let  $c_\ell = b_\ell(e_\ell)$  and let  $c_\ell^+ = b_\ell(e_\ell^+)$ . By the definition of  $e_\ell$ ,  $e_\ell \in C$  and thus  $\sum_{i=1}^d \frac{y_i}{b_i(e_\ell)} < 1$ . Since  $y \in K$ , we have  $\frac{y_i}{b_i(e_\ell)} > 0$  for each  $1 \leq i \leq d$  with  $i \neq \ell$  and consequently,  $\frac{y_\ell}{c_\ell} = \frac{y_\ell}{b_\ell(e_\ell)} < 1$ . Observe that  $c_\ell < c_\ell^+ \leq b_\ell(e)$  for all  $1 \leq \ell \leq d$ . Moreover,  $c_\ell^+ = dc_\ell + d + 1$  and thus  $b_\ell(e) \geq c_\ell^+ > dc_\ell$ . Consequently,  $\sum_{\ell=1}^d \frac{y_\ell}{b_\ell(e)} < \sum_{\ell=1}^d \frac{y_\ell}{dc_\ell} = \frac{1}{d} \sum_{\ell=1}^d \frac{y_\ell}{c_\ell} < \frac{1}{d}d = 1$ . Therefore,  $e \in C$  and thus we have  $C = \{e \in U : e_\ell \leq_\ell e, \forall 1 \leq \ell \leq d\}$ , establishing that  $C \in \mathcal{C}$ .  $\square$

By Claim 35, for any convex set  $C \in \mathcal{C}$ , there exists a region corresponding to  $C$  in  $(K, \mathcal{H})$ . By Claim 36, any region of  $(K, \mathcal{H})$  corresponds to a convex  $C$  of  $\mathcal{A} = (U, \mathcal{C})$ , concluding the proof of the theorem.  $\square$

### 5.3 Realization of ideals of convex geometries and VC-dimension

In this subsection, we consider the realizations of ideals  $\mathcal{I} = (U, \mathcal{C}')$  of convex geometries by minimizing or bounding the dimension of the realizing space  $\mathbb{R}^d$  by a dimension parameter of  $\mathcal{I} = (U, \mathcal{C}')$ . Since such  $\mathcal{I} = (U, \mathcal{C}')$  are ample [2], the VC-dimension  $\dim_{\text{VC}}(\mathcal{I})$  of  $\mathcal{I}$  coincides with the dimension of the largest cube of  $\mathcal{C}'$ . We are motivated by the following:

**Corollary 37.** For every distributive lattice  $L$ , we have  $\dim_{\text{VC}}(L) = \dim_{\mathbb{E}}(L)$ .

*Proof.* The inequality  $\dim_{\text{VC}}(L) \leq \dim_{\mathbb{E}}(L)$  follows from Theorem 8. For  $\dim_{\text{VC}}(L) \geq \dim_{\mathbb{E}}(L)$  we use the fact the convex dimension and the VC-dimension of a distributive lattice coincide [11] together with Theorem 34.  $\square$

This leads us to believe:

**Conjecture 38.** The Euclidean dimension  $\dim_{\mathbb{E}}(\mathcal{I})$  of an ideal  $\mathcal{I} = (U, \mathcal{C}')$  of a convex geometry is always upper bounded by a function of its VC-dimension  $\dim_{\text{VC}}(\mathcal{I})$ .

As a first approach we can get a bound for some situations:

**Corollary 39.** If  $\mathcal{A} = (U, \mathcal{C})$  is an ideal semialignment with  $k$  facets or a median set systems with  $k$  maxima, then  $\mathcal{A}$  is realizable and  $\dim_{\mathbb{E}}(\mathcal{A}) \leq \min(|U|, k \dim_{\text{VC}}(\mathcal{A}))$ .

*Proof.* From Corollaries 23 and 24 and Theorem 25 we get realizability and the dimension bound of  $|U|$ . Now, using Corollary 37 we obtain the bounds of the form  $k \dim_{\text{VC}}(\mathcal{A})$ .  $\square$

In this subsection we confirm Conjecture 38 for trees and multisimplicial complexes (alias, ideals of the grid). Indeed, both our bounds are best-possible.

Trees are the simplest examples of ideals of convex geometries that are not convex geometries. They are also the median set systems of VC-dimension 1.

**Proposition 40.** Every tree  $T$  can be realized in  $\mathbb{R}^2$ , i.e.,  $\dim_{\mathbb{E}}(T) \leq 2$  holds.

*Proof.* Consider  $T$  rooted at  $r$ . We will use the basic observation that  $T$  can be constructed iteratively from the star of  $r$  (i.e.,  $r$  with all its neighbors) by picking a leaf  $\ell$  of the current tree and attaching to it all its children in  $T$  at once. We describe a representation of  $T$  following this construction sequence with the invariant that every current leaf  $\ell$  is represented by an isosceles triangular cell whose base is in  $\mathcal{H}$ .

Let say  $\deg(r) = k$ . Then we start with  $K$  as a convex  $k$ -gon and we use  $\mathcal{H}$  to cut off, i.e., truncate, each vertex of  $K$  such that an isosceles cell is created. Hence, we end up with a representation of the star of  $r$  satisfying the invariant, i.e. there is a triangle with base in  $\mathcal{H}$ . If  $\ell$  has  $m$  children, then we add  $m$  halfspaces to  $K$  forming a convex curve between the intersection points of  $\mathcal{H}$  and  $K$ , and staying inside the triangular region for  $\ell$ . Now, truncate each corner with a new element of  $\mathcal{H}$  such that isosceles regions are created. We have represented the children of  $\ell$  in the desired manner. See Figure 2 for an illustration.  $\square$



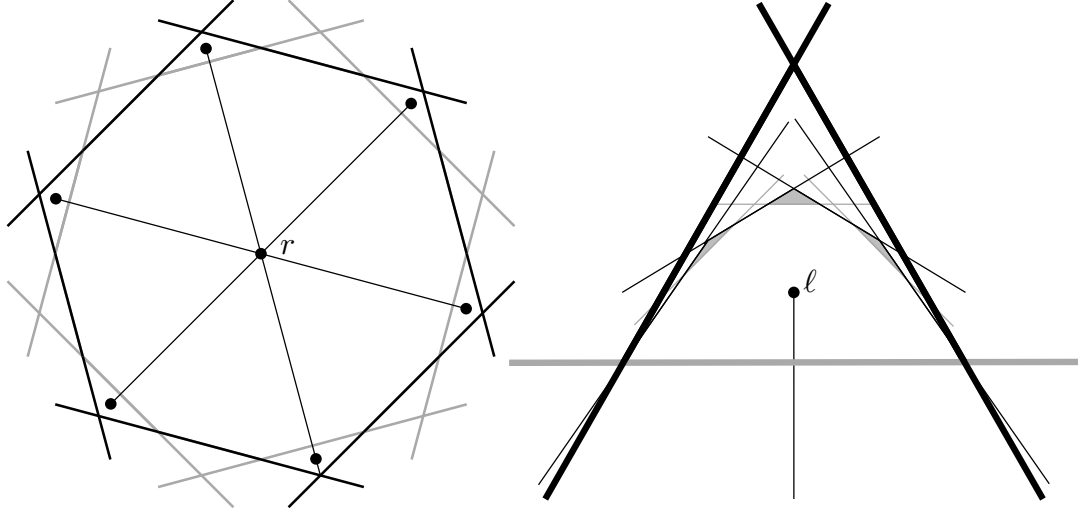


Figure 2: A two-dimensional Euclidean realization for trees.

Median set systems are realizable by Corollary 39. Since median systems are ample, their VC-dimension coincides with the dimension of a largest cube, and thus with their (topological) dimension. Corollary 37 and Proposition 40 motivate the following strengthening of Conjecture 38 in the case of median systems:

**Conjecture 41.** Every median system of dimension  $d$  admits a realization in  $\mathbb{R}^{O(d)}$ .

Recall that there exists a bijection between the ideals of the grid  $(\mathbb{N}^n, \leq)$  and the multisimplicial complexes on a set  $X$  of size  $n$ .

**Theorem 42.** Every ideal  $L$  of  $\mathbb{N}^n$  with  $\dim_{\text{VC}}(L) = d$  has a representation in  $\mathbb{R}^{2d}$ , i.e.,  $\dim_{\mathbb{E}}(L) \leq 2 \dim_{\text{VC}}(L)$  holds.

*Proof.* Let  $\dim_{\text{VC}}(L) = r$ . By Dilworth's Theorem,  $L$  may be seen as a semilattice of ideals of a poset  $X$  consisting of  $n$  disjoint chains of some length  $k$ , such that each ideal intersects at most  $r$  chains. Denote by  $L_i$  the sublattice of  $L$  consisting of all the ideals of  $L$  intersecting each chain in at most  $i$  elements.

Let  $P \subseteq \mathbb{R}^{2r}$  be the polar polytope of an  $r$ -neighborly polytope  $P^* \subseteq \mathbb{R}^{2r}$  on  $n$  vertices containing the origin. The latter exists [15] and therefore  $P$  has the property that for any set of  $\ell \leq r$  facets of  $P$ , their intersection defines a face of dimension  $r - \ell$  of  $P$ . Now consider the dilates  $P_i := iP$  for all  $1 \leq i \leq k$  and for each  $i$ , let  $\mathcal{H}_i$  be the arrangement of facet-defining hyperplanes of  $P_i$ . Set  $\mathcal{H} := \bigcup_{i=1}^k \mathcal{H}_i$ .

We will now proceed by induction on  $i = 1, \dots, k$  by introducing halfspaces of the convex set  $K$  and eventually changing the dilation for some  $P_j$  with  $j > i$  in such a way that for any  $i$  we have  $\mathcal{M}(\bigcup_{j=1}^i \mathcal{H}_j, K) = L_i$ . We start with the basis case  $i = 1$ .

**Case 43.**  $i = 1$ , i.e.,  $L_1$  is a simplicial complex.

*Proof.* First, note that every element of  $L_1$  is represented by a cell of  $\mathcal{H}_1$ . Namely, if  $x \in L_1$  corresponds to taking one element from each of the chains  $C_{i_1}, \dots, C_{i_\ell}$  with  $\ell \leq r$ , then take the face  $f$  of  $P$  corresponding to the intersection of facets  $f_{i_1}, \dots, f_{i_\ell}$ . The corresponding facet defining hyperplanes  $H_{i_1}, \dots, H_{i_\ell}$  define a cone over  $f$  whose intersection with  $P$  is  $f$ . This cell represents  $x$ . Since the polytope  $P$  is simple (because it is the polar of the simplicial polytope  $P^*$ ), in an  $\epsilon$ -neighborhood of  $f$  we see the ideal of  $x$  in  $L_1$ , where the minimum is represented by the interior of  $P$ .

Now, suppose that some cell  $C$  of  $\mathcal{H}_1$  represents an ideal of  $X$  that does not correspond to an element of  $L_1$ . Since  $C$  and  $P$  are open and convex we can choose a hyperplane  $H$  that separates  $C$  and  $P$  and intersects the closure  $\bar{P}$  of  $P$  in  $\bar{P} \cap \bar{C}$ . Thus, if we add to  $K$  the halfspace defined by  $H$  that contains  $P$ , then we remove  $C$  from the arrangement and all cells of  $\mathcal{H}_1$  that correspond to faces not included in  $\bar{P} \cap \bar{C}$  are still present. The only further cells of  $\mathcal{H}_1$  that are removed are those that correspond to ideals of  $X$  corresponding to supersets of the ideal corresponding to  $C$ . Since  $L_1$  is an ideal, we did not want these regions. Consequently, taking as  $K$  the intersection of all open halfspaces containing  $P$  and defined by hyperplanes separating  $P$  from cells  $C$  representing ideals of  $X$  that do not correspond to elements of  $L_1$ , we will obtain that  $L_1 = \mathcal{M}(\mathcal{H}_1, K)$ .  $\square$

**Case 44.**  $i > 1$ .

*Proof.* By induction hypothesis, we can define the dilates  $P_1, P_2, \dots, P_{i-1}$  of  $P$ , their arrangements of supporting hyperplanes  $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_{i-1}$ , and the convex set  $K$  in such a way that  $\mathcal{M}(\bigcup_{j=1}^{i-1} \mathcal{H}_j, K) = L_{i-1}$ . First, suppose that there exists an element  $x \in L_i$  that is represented by a cell  $C$  touching  $P_i$  and that lies outside of  $K$ . Since  $L$  is an ideal, the element  $y \in L_{i-1}$  cutting the ideal of  $x$  to height at most  $i-1$  on each chain is in  $L$ . In particular, this point  $p$  is represented by a cell touching  $P_{i-1}$  within  $K$ . Hence, if  $q$  is a point in the cell representing  $x$ , a line from  $p$  to  $q$  must cross the boundary of  $K$ . We can thus dilate  $P_i$  to a smaller polytope such that the cell representing  $x$  intersects the interior of  $K$ . In order to avoid now some of the regions touching  $P_i$  from the outside, we proceed as in the case of  $P_1$  noting that all the hyperplanes that we add to  $K$  are disjoint from the interior of  $P_i$  hence they do not interfere with the so far obtained representation of  $L_{i-1}$ .  $\square$

This concludes the proof of the theorem.  $\square$

The full simplicial complex  $U_{r,n}$  of dimension  $r$  consists of all subsets of size at most  $r$  of a set of size  $n$ .

**Theorem 45.** For the full simplicial complex  $U_{r,n}$  of dimension  $r$  on a set  $X = \{f_1, \dots, f_n\}$ , we have  $\dim_{\mathbb{E}}(U_{r,n}) \geq \min(2r, n-1)$ .

*Proof.* Let  $\mathcal{M}(\mathcal{H}, K)$  be a realization of  $U_{r,n}$ . We assume that  $K$  is full-dimensional, otherwise we intersect with the affine hull of  $K$ . Consider the cell  $P$  representing the empty set  $\emptyset \in U_{r,n}$ . Every 1-simplex of  $U_{r,n}$  must be represented by a cell intersecting  $P$  in a facet. More generally, for any  $1 \leq \ell \leq r$  every  $\ell$ -simplex  $\sigma = \{f_{i_1}, \dots, f_{i_\ell}\}$  must be represented by a cell that is separated exactly by the facet-defining hyperplanes  $H_{i_1}, \dots, H_{i_\ell}$  from  $P$ . Hence, the intersection of  $H_{i_1}, \dots, H_{i_\ell}$  must be a face of dimension  $r - \ell$  of  $P$ . Consider now the polyhedron  $P'$  by removing all the facet-defining hyperplanes of  $K$  from  $P$ . Hence, the polar polytope  $P^*$  of  $P'$  must be  $r$ -neighborly. It is known, that for all  $n$ , the smallest dimension in which an  $r$ -neighborly polytope  $P^*$  on  $n$  vertices exists is  $2r$  or  $P^*$  is a simplex and hence has dimension  $n-1$ , see e.g. [15]. This lower bound carries over to  $P'$ , hence also to  $P$ . Hence this is a lower bound for the dimension of  $K$ .  $\square$

## 6 Closing remarks

### 6.1 Concepts of dimension

We have shown that the Euclidean dimension  $\dim_{\mathbb{E}}(\mathcal{A})$  of a convex geometry  $\mathcal{A} = (U, \mathcal{C})$  lies between its VC-dimension  $\dim_{\text{VC}}(\mathcal{A})$  and the minimum of  $|U|$  and  $\text{cdim}(\mathcal{A})$ . It is known that convex geometries with bounded convex dimension and unbounded  $|U|$  exist. On the other hand also there are convex geometries with  $\text{cdim}(\mathcal{A})$  exponential in  $|U|$ , see [19]. Hence none of our bounds on  $\dim_{\mathbb{E}}(\mathcal{A})$  dominates the other. It would be interesting to compare  $\dim_{\mathbb{E}}(\mathcal{A})$  with other concepts of dimension for convex geometries, e.g., Dushnik-Miller dimension, Boolean dimension, local dimension, and fractional dimension. Their behavior on convex geometries has been studied recently in [19].

### 6.2 From ideals to bouquets

To finish we want to extend our vision from ideals of convex geometries. A pair  $\mathcal{A} = (U, \mathcal{C})$  of a finite universe  $U$  and a collection of convex sets  $\mathcal{C} \subseteq 2^U$  is called *bouquet of convex geometries* if it satisfies the following two conditions:

(C1')  $\emptyset \in \mathcal{C}$ ;

(C2)  $X \cap Y \in \mathcal{C}$  for all  $X, Y \in \mathcal{C}$ ;

(C3') for all  $X, Y \in \mathcal{C}$  with  $Y \subset X$ , there is  $e \in X \setminus Y$  such that  $Y \cup e \in \mathcal{C}$ ; (locally extendable)

These structures were called *conditional antimatroids* in [2]. It is easy to see that ideals of convex geometries are bouquets of convex geometries. Note that in a sense ideal semialignments were defined as bouquets of convex geometries with the additional property of being locally union-closed. We then showed that they are indeed ideals of convex geometries in Proposition 22. However, there are bouquets of convex geometries that are not ideals.

**Proposition 46.** *There exist bouquets of convex geometries that are not ideals of convex geometries.*

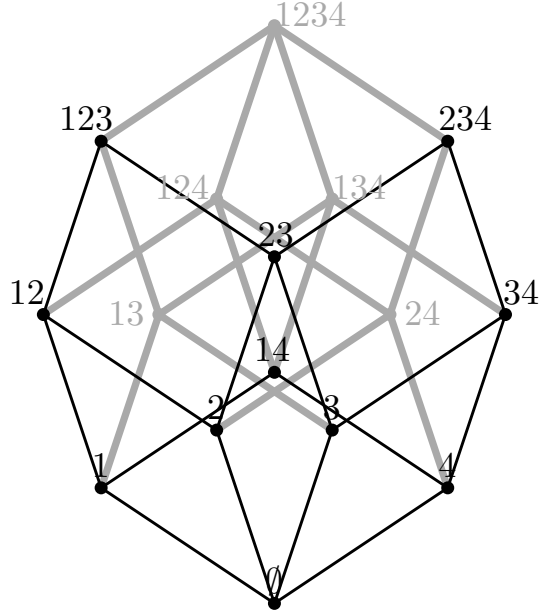


Figure 3: A bouquet of convex geometries that is not an ideal of a convex geometry. The bouquets of convex geometries is represented in black as isometric subgraph of the 4-dimensional hypercube (in gray)

*Proof.* Consider the bouquet of convex geometries in Figure 3, right, i.e.,  $E' = \{1, 2, 3, 4\}$  and

$$\mathcal{C}' = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3\}, \{2, 3, 4\}\}.$$

Suppose that  $\mathcal{A}' = (E', \mathcal{C}')$  is an ideal of a convex geometry  $\mathcal{A} = (E, \mathcal{C})$ . One can remove elements from  $E'$  until  $E = E'$ . Hence,  $E \in \mathcal{C}'$  by (C1). By (C3) applied to  $\{1, 4\}$ , one of the sets  $\{1, 2, 4\}$  or  $\{1, 3, 4\}$  must be in  $\mathcal{C}$ . Now applying (C2) with respect to  $\{1, 2, 4\}$  and  $\{2, 3, 4\}$  or  $\{1, 3, 4\}$  and  $\{1, 2, 3\}$ , we get that  $\{2, 4\}$  or  $\{1, 3\}$  are in  $\mathcal{C}$ . Hence  $\mathcal{A}' = (E', \mathcal{C}')$  is not an ideal of  $\mathcal{A}$ .  $\square$

We believe that bouquets as a generalization of ideals deserve further investigation in their own right as a combinatorial structure. In the context of realizability we dare to state the following:

**Conjecture 47.** Every bouquet of convex geometries is realizable.

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