Wreath products of semigroup varieties

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Let X be a countable alphabet. A semigroup identity is

$$u \approx v, \tag{1}$$

where u and v are any words in the alphabet X. A semigroup identity (1) is true in a semigroup S if any map $\varphi : X \to S$ can be continue up to homomorphism $\overline{\varphi} : X^* \to S$ and identity (1) becomes a true equality

$$\overline{\varphi}(u) = \overline{\varphi}(v) \tag{2}$$

This continuation of the map φ realized by the equality $\overline{\varphi}(w_1 w_2) = \overline{\varphi}(w_1)\overline{\varphi}(w_2)$. Note that any periodic group variety is a semigroup variety.

Definition 1 *A semigroup variety is called finitely based if all its identities are followed from a finite number of identities.*

Definition 2 *A semigroup variety is called hereditarily finitely based if every its semigroup is finitely based.*

Definition 3 *A semigroup variety is called a Cross variety if it is finitely based, is generated by a finite semigroup and has a finite lattice of subvarieties.*

Theorem 1 *(Sh. Oates, M. Powell,1964, [10]). The group variety* $\mathbf{V}$ *generated by a finite group is Cross.*

For semigroup varieties the situation is different. In 1969 Perkins demonstrated that the Brand monoid

$$B_2^1 = \langle a, b, 1 | a^2 = b^2 = 0, aba = a, bab = b \rangle$$

of order six is non-finitely based (see [11]).

Theorem 2 *(Jackson M., 2000, [4]). Let* $varB_2^1$ *be five-element Brandt semigroup with identity element adjoined. Then the semigroup variety* $varB_2^1$ *contains continuum subvarieties.*

Let $P_2^1 = \langle a, b, 1 | a^2 = ab = a, b^2a = b^2 \rangle$.

Theorem 3 *(Lee Edmond W.H. [5][theorem 1.2]). Every semigroup S of order five or less that is distinct from $varP_2^1$ is hereditarily finitely based. Therefore, the variety $varS$ for such semigroup contains finite or countable many subvarieties.*

The atoms of the lattice of all semigroup varieties are well known. There are the class $\mathbf{N_2}$ of all semigroups with zero multiplication, the class $\mathbf{L_1}$ of all left zero semigroups, the class $\mathbf{R_1}$ of all right zero semigroups, the class $\mathbf{Sl}$ of all semilattices, the class $\mathbf{A_p}$ of all commutative groups with exponent p, where p is a prime number.

Theorem 4 *(Tishchenko A.V., [16] [theorem 1.2]). If* $\mathbf{U}$ *and* $\mathbf{V}$ *are atoms of the lattice of semigroup varieties, then the wreath product* $\mathbf{UwV}$ *is a Cross variety, except in the following cases:*

1) $\mathbf{U} = \mathbf{V} = \mathbf{A_p}$*, then the variety* $\mathbf{A_p w A_p} = \mathbf{A_p^2}$ *is finitely based but is not generated by a finite semigroup and has an infinite lattice of subvarieties;*

2) $\mathbf{U} = \mathbf{V} = \mathbf{Sl}$ *and* $\mathbf{U} = \mathbf{Sl}, \mathbf{V} = \mathbf{R_1}$*, then each of the varieties* $\mathbf{SlwSl} = \mathbf{Sl^2}$ *and* $\mathbf{SlwR_1}$ *is finitely based, is generated by a finite semigroup and has an infinite lattice of subvarieties;*

3) $\mathbf{U} = \mathbf{Sl}, \mathbf{V} = \mathbf{A_p}$*, then the variety* $\mathbf{SlwA_p}$ *is essentially infinitely based, is generated by a finite semigroup and has an infinite lattice of subvarieties.*

Now we can give some additional information on this question and to formulate some open problems.

Proposition 1 *1) The variety* $\mathbf{A_p w A_p}$ *has the countable many subvarieties. 2) The variety* $\mathbf{SlwR_1}$ *has the countable many subvarieties. 3) The variety* $\mathbf{SlwA_p}$ *has the continuum many subvarieties any simple* p*.*

Problem 1. How many subvarieties has the variety $\mathbf{SlwSl}$?

Problem 2 (see [5]). How many subvarieties has the variety $varP_2^1$?
Now we can note the following fact.

Proposition 2 *The variety* $varP_2^1 \subseteq \mathbf{SlwSl}$*.*

Problem 3. How many subvarieties has the variety $\mathbf{SlwN_2}$?

In [16] it was proved that the lattice $L(\mathbf{SlwN_2})$ is finite. In [17] it was proved that the lattice $L(\mathbf{SlwN_2})$ has at the least 33 elements.

Theorem 5 *([18]). The lattice L of subvarieties of $\mathbf{W}$ contains at the least 39 elements.*

In 1939-40 Malcev A. has given necessary and sufficient conditions for a semigroup can be embedded into a group. But these conditions were not finite axiomatizable. Then he has supposed more simple conditions for semigroups can be embedded into a nilpotent group.

Theorem 6 *([8]). A semigroup can be embedded into a nilpotent group of step n as group of quotients if and only if this semigroup satisfy the identity*

$$U_n \approx V_n \tag{5}$$

Here $U_0 = x, V_0 = y$,

$$U_{n+1} = U_n z_{n+1} V_n, V_{n+1} = V_n z_{n+1} U_n, \tag{6}$$

Theorem 7 *([3]Baumslag, 1959). Standard wreath product of two groups of power more than one is a nilpotent group if and only if the active group of the wreath product is finite and the passive group is a nilpotent p-group of the bounded exponent.*

We can suppose the following generalization of this Baumslag theorem.

Theorem 8 *([19]). An extended wreath product* $\mathbf{Sw_1R}$ *of two semigroups of power more than one from a variety of finite step is nilpotent if and only if each of them is nilpotent in the sense due to Malcev and one of the following conditions hold:*

1) S *is a nilpotent in usual sense;*

2) R *is a finite nilpotent group of odd order and* S *a semilattice of nilpotent semigroups;*

3) R *is a finite* p*-group for some odd simple number* p *and* S *a semilattice of ideal nilpotent extensions of nilpotent* p*-groups of bounded exponent;*

4) R *is a finite* 2*-group and* S *is an ideal nilpotent extension of nilpotent* 2*-group of bounded exponent.*

In the next paper there are given of Malcev nilpotent wreath products of semigroups where a passive semigroup is not a semigroup of a finite step. The considered examples allow to set a problem of generalizing the result proved early on Malcev nilpotency of wreath product of finite step semigroups to semiarchimedian semigroups. These examples complement the criterion received early by the author for the wreath product of semigroups of finite step to be Malcev nilpotent semigroup.

Problem 4. To generalize the last theorem on Malcev nilpotency of wreath product of finite step semigroups to semiarchimedian semigroups.

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