COLORING MINIMAL CAYLEY GRAPHS

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ABSTRACT. We show that any minimal Cayley graph of a (finitely generated) generalized dihedral or nilpotent group has chromatic number at most 3, while 4 colors are sometimes necessary for soluble groups. On the other hand we construct graphs of unbounded chromatic number that admit a proper edge coloring such that each cycle has some color at least twice. The latter disproves a conjecture of Babai'78 that would have implied that all minimal Cayley graphs have bounded chromatic number – a problem that remains open.

1. INTRODUCTION

Given a group Γ and a *connection set* $C \subseteq \Gamma$ the (undirected, right) *Cayley graph* $\mathrm{Cay}(\Gamma, C)$ has vertex set Γ and $a, b \in \Gamma$ form an edge if $a^{-1}b \in C$. A Cayley graph is *minimal* (a.k.a irreducible) if C is an inclusion-minimal generating set of Γ and in this paper we only consider the case where C is finite. Minimal Cayley graphs appear naturally: a famous and open problem often attributed to Lovász is (equivalent to) whether minimal Cayley graphs are Hamiltonian, see [27, Section 4]. Other areas in which minimal Cayley graphs naturally occur are the genus of a group, see the book [32] or concerning sensitivity [14, Questions 7.7 and 8.2]. Different aspects of these graphs (sometimes of particular groups) and their distinguishing features compared to general Cayley graphs have been considered in [12, 15, 19, 21, 24, 30]. The present work is motivated by a question that has been brought up by Babai [3, 5]:

Question 1.1. Does there exist a constant c such that every minimal Cayley graph has chromatic number at most c?

Note that assuming minimality here is essential. Indeed, every graph is an induced subgraph of some Cayley graph [4, 6, 16], this is far from being the case for minimal Cayley graphs, which are known to be sparse [3, 31]. However, there also exist (non-minimal) sparse Cayley graphs (of arbitrary girth) that are expanders and hence have unbounded chromatic number [22]. The chromatic number of random Cayley graphs has been studied by Alon [2] and Green [17] as well as some concrete Cayley graphs have been considered recently [9, 10]. However, none of these are minimal. Concerning, the chromatic number of minimal Cayley graphs in [5, Section 3.4] Babai mentions the following:

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Conjecture 1.2. For every $\varepsilon > 0$ there exists a minimal Cayley graph $G = (V, E)$ such that $\alpha(G) \leq \varepsilon|V|$ where $\alpha(G)$ denotes the size of the largest independent set of G.

Hence, this in particular would imply a negative answer to Question 1.1. However, previously in [3] Babai proposed an opposing conjecture, for which we introduce some notions first. A graph G is called *no lonely color* if G admits an edge coloring satisfying:

(1) each vertex is incident with at most two edges of any color,
(2) each circuit contains no color exactly once.

In a Cayley graph the edges can be naturally colored by the elements of C, which makes it easy to see that any subgraph G of a minimal Cayley graph is *no lonely color*. A Cayley graph $\mathrm{Cay}(\Gamma, C)$ is *semiminimal* if C is a generating set of Γ that can be linearly ordered such that none of its elements is generated by its predecessors. With the same ideas one can show that any subgraph G of a semiminimal Cayley graph is *one popular color*[1], this is, G admits an edge coloring that satisfies (1) and

(2') each circuit contains at least one color more than once.

Babai observes that $K_4 - e$ and $K_{3,5}$ are not no lonely color and $K_{5,17}$ is not one popular color, hence they cannot be subgraphs of minimal and semiminimal Cayley graphs, respectively. Later using these ideas Spencer [31] shows that for every $g \geq 3$ there exists a finite graph G of girth g that is not a subgraph of any (semi)minimal Cayley graph. In [3, Conjecture 3.5] Babai puts forward the following:

Conjecture 1.3. There is a constant c such that any one popular color graph (no lonely color graph) has chromatic number at most c.

Note that Conjecture 1.3 in particular implies that (semi)minimal Cayley graphs have bounded chromatic number and hence would disprove the more recent Conjecture 1.2 and give a positive answer to Question 1.1.

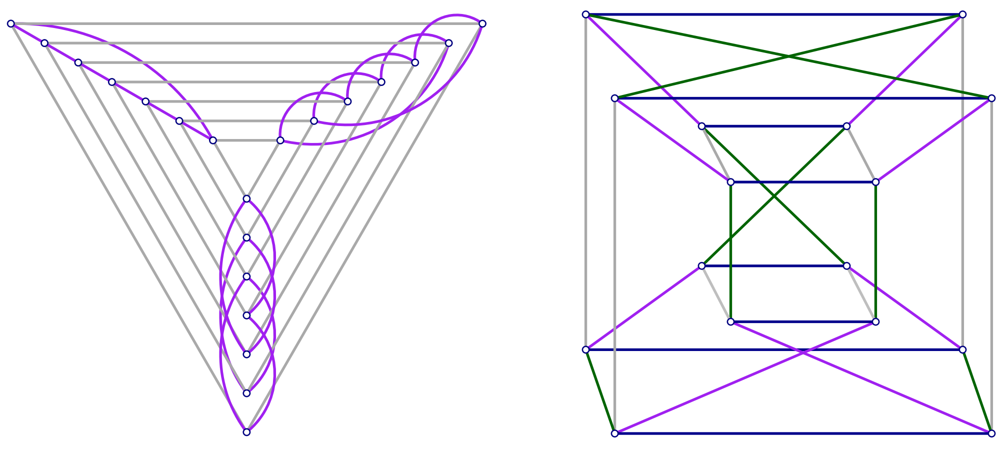

FIGURE 1. Left: $\mathrm{Cay}(\mathbb{Z}_3 \rtimes \mathbb{Z}_7, \{(0,1),(1,0)\})$ is a minimal Cayley graph of chromatic number 4. Right: G, such that $G \boxtimes K_2 = \mathrm{Cay}(Q_{32}, C)$ is a semiminimal Cayley graph of chromatic number 7, being $Q_{32} = \langle a, b \mid a^{16} = b^4 = 1, a^8 = b^2, aba = b\rangle$ the generalized quaternion group or order 32 and $C = (b^2, a^4, a^5b, a^3b, a^6b)$. The graph G is induced by vertices $\{b^i a^j \mid 0 \leq i \leq 1, 0 \leq j \leq 7\}$, edges corresponding to right multiplication by a^4, a^5b, a^4b and $a^{14}b$ are depicted in blue, gray, green and violet, respectively.

This is the starting point for the present work: First, we observe that every group admits a minimal Cayley graph of chromatic number at most 3 and characterize the bipartite case (Theorem

[1] Babai called these *no pied circuit*.

2.2). Then we show that minimal Cayley graphs of finitely generated nilpotent groups (Theorem 2.5) as well as generalized dihedral groups (Theorem 2.6) have chromatic number at most 3. On the other hand, we show a minimal Cayley graph of a soluble group of chromatic number 4 and a semiminimal Cayley graph of a nilpotent group of chromatic number 7 (see Figure 1). Finally, we disprove the stronger statement of Conjecture 1.3 by constructing a family of one popular color graphs of unbounded chromatic number (Theorem 3.2). We further remark, that the clique number of a semiminimal Cayley graph is at most 4 (Proposition 3.1).

2. Positive results

Let us begin with the question whether every group admits a minimal Cayley graph of bounded chromatic number. This was already answered by Babai [3,5]. Indeed, he proved that every group has a 3-colorable minimal Cayley graph contingent upon the truth of some facts about simple groups that were conjectured at that time (see [3, Conjecture 4.3]). Years later the classification of finite simple groups was settled and those conjectures were verified. For completeness we include a proof following Babai's lines. At the basis of this we have the following [3, Lemma 4.2], which will also use later on. For a group Γ denote its *minimum chromatic number* y $\chi_{\min}(\Gamma)$ the minimum $\chi(\mathrm{Cay}(\Gamma, C))$ over all generating sets C of Γ.

Lemma 2.1. *Let $(\Gamma, \cdot)$ be a group, N a normal subgroup of Γ and $C \subseteq \Gamma/N$. Then,*

$$\chi(\mathrm{Cay}(\Gamma, C \cdot N)) \leq \chi(\mathrm{Cay}(\Gamma/N, C));$$

where $C \cdot N := \{c \cdot n \mid c \cdot N \in C,\, n \in N\}$. In particular, $\chi_{\min}(\Gamma) \leq \chi_{\min}(\Gamma/N)$.

Combining this also with ideas present in [12] we get:

Theorem 2.2. *For a group Γ we have:*

$$\chi_{\min}(\Gamma) = \begin{cases} 1 & \text{if } \Gamma \text{ is trivial,} \\ 2 & \text{if } \Gamma \text{ has a subgroup of index } 2, \\ 3 & \text{otherwise.} \end{cases}$$

Proof. The statement that $\chi_{\min}(\Gamma) = 1$ if and only if Γ is trivial, is trivial. Now, by Lemma 2.1 $\chi_{\min}(\Gamma) \leq \chi_{\min}(\Gamma/N)$ for any normal subgroup N of Γ. If Γ has a subgroup of index 2, then it is normal an the quotient is $\mathbb{Z}_2$ and has a bipartite Cayley graph. In general, this argument reduces the question to simple groups. Since each such group Γ' is either cyclic or has a generating set C' of size 2 containing an involution [20], $\mathrm{Cay}(\Gamma', C')$ has maximum degree 3 and is different from K_4. Hence $\chi(\mathrm{Cay}(\Gamma', C')) \leq 3$. It remains to show that if $\chi_{\min}(\Gamma) = 2$, then Γ has a subgroup of index 2. In this case, Γ may be partitioned into two independent sets A, B, which by vertex transitivity of Γ are of equal size. If say $e \in A$, then A consists of all elements of Γ that can be expressed as an word of even length in C. Hence, A is a subgroup of index 2. □

We now turn to the *maximum chromatic number* of a group Γ, i.e., $\chi_{\max}(\Gamma)$ the maximum $\chi(\mathrm{Cay}(\Gamma, C))$ over all minimal generating sets C of Γ. We will show that this is at most 3 for Dedekind groups, generalized dihedral groups, and nilpotent groups.

If $H < \Gamma$ is a subgroup and $C \subseteq \Gamma$, the *Schreier (coset) graph* $\mathrm{Cay}(\Gamma/H, C)$ has as vertices the left cosets of H and there is an edge between two cosets if they can be represented as gH, $g'H$ and $g^{-1}g' \in C$.

Lemma 2.3. *Let C be a minimal generating set of Γ, then*

$$\chi(\mathrm{Cay}(\Gamma, C)) \leq \max\{\chi(\mathrm{Cay}(\Gamma/\langle C - \{c\}\rangle, c)) \mid c \in C\}.$$

Proof. Since $C = \{c_1, \ldots, c_k\}$ is minimal, for each $c \in C$ the graph $\mathrm{Cay}(\Gamma, C - \{c\})$ is disconnected and its connected components correspond to the vertices of $\mathrm{Cay}(\Gamma/\langle C - \{c\}\rangle, c)$. Moreover, if two vertices x, y of $\mathrm{Cay}(\Gamma, C)$ are connected with an edge corresponding to c, then x, y are contained in different components of $\mathrm{Cay}(\Gamma, C - \{c\})$ corresponding to adjacent vertices x_c, y_c of $\mathrm{Cay}(\Gamma/\langle C - \{c\}\rangle, c)$. Hence, if we map every vertex of $x \in \mathrm{Cay}(\Gamma, C)$ to the tuple of vertices $(x_{c_1}, \ldots, x_{c_k})$ we obtain a graph homomorphism into the Cartesian product $\mathrm{Cay}(\Gamma/\langle C - \{c_1\}\rangle, c_1)\Box\cdots\Box\mathrm{Cay}(\Gamma/\langle C - \{c_k\}\rangle, c_k)$. Hence, the chromatic number of the latter is an upper bound for $\chi(\mathrm{Cay}(\Gamma, C))$. Finally, by a well-known result of Sabidussi [29], the chromatic number of a Cartesian product is the maximum chromatic number of its factors. $\square$

Remark 2.4. Note that Lemma 2.3 alone does not give a useful upper bound for general minimal Cayley graphs. Consider the so-called star graph (see [1],) which is the bipartite graph $\mathrm{Cay}(S_n, C)$ of the symmetric group of degree n with respect to the set of transpositions involving 1, i.e., $C = \{(12), (13), \ldots, (1n)\}$). However, $\mathrm{Cay}(S_n/\langle C - \{c\}\rangle, c)) = K_n$ for all n and $c \in C$.

We can however use the above lemma to bound the maximum chromatic in some cases. A group is called *Dedekind* if all its subgroups are normal. Clearly, this includes all abelian groups, and by results of Dedekind [11] and Baer [7], actually not much more.

Theorem 2.5. *Every minimal Cayley graph of a Dedekind group is* 3*-colorable.*

Proof. Consider $\mathrm{Cay}(\Gamma, C)$ a minimal Cayley graph of a Dedekind group Γ. Since Γ is Dedekind, for all $c \in C$ we have that $\langle C - \{c\}\rangle$ is a normal subgroup of Γ. Hence, $\Gamma/\langle C - \{c\}\rangle$ is a cyclic group generated by the coset of c, i.e., the Schreier coset graph (which is a Cayley graph) is a cycle and $\chi(\mathrm{Cay}(\Gamma/\langle C - \{c\}\rangle, c)) \leq 3$. The statement follows from Lemma 2.3. $\square$

For any abelian group Γ, the generalized dihedral group of Γ, written $\mathrm{Dih}(\Gamma)$, is the semidirect product of Γ and $\mathbb{Z}_2$, with $\mathbb{Z}_2$ acting on Γ by inverting elements, i.e., $\mathrm{Dih}(\Gamma) = \Gamma \rtimes_\phi \mathbb{Z}_2$ with $\phi(0)$ the identity, and $\phi(1)$ the inversion. Thus we get:

$(g_1, 0) * (g_2, t_2) = (g_1 + g_2, t_2)$, and
$(g_1, 1) * (g_2, t_2) = (g_1 - g_2, 1 + t_2)$,

for all $g_1, g_2 \in \Gamma$, and $t_2 \in \mathbb{Z}_2$.

Theorem 2.6. *Every minimal Cayley graph of a generalized dihedral group is* 3*-colorable.*

Proof. Let $G = \mathrm{Cay}(\mathrm{Dih}(\Gamma), C)$ be the Cayley graph of a generalized dihedral group $\mathrm{Dih}(\Gamma) = \Gamma \rtimes \mathbb{Z}_2$ minimally generated by C. We denote $C_i = C \cap (\Gamma \times \{i\})$ for $i = 0, 1$, and consider $H := (\Gamma \times \{0\}) \cap \langle C_1\rangle$. By the minimality of C we have that:

(a) the vertices corresponding to elements of H form an independent set in G,
(b) The cosets of C_0 minimally generate the abelian group $(\Gamma \times \{0\})/H$.

By (b) and Theorem 2.5, it follows that $\mathrm{Cay}((\Gamma \times \{0\})/H, C_0)$ is 3-colorable. We consider $\tilde{f} : (\Gamma \times \{0\})/H \longrightarrow \{0, 1, 2\}$ a proper coloring. By (a), we have that $\tilde{f} : \Gamma \times \{0\} \longrightarrow \{0, 1, 2\}$ defined as $f(g, 0) := \tilde{f}((g, 0) * H)$ is also a proper coloring of $\mathrm{Cay}(\Gamma \times \{0\}, C_0)$. Now we choose $(y, 1) \in C_1$ and consider

$$\begin{array}{rccl} h: & \mathrm{Dih}(\Gamma) & \longrightarrow & \{0,1,2\} \\ & (g,0) & \mapsto & f(g,0) \\ & (g,1) & \mapsto & f(g-y,0) + 1 \bmod 3. \end{array}$$

We claim that h is a proper 3-coloring of G. Indeed, consider $(g_1, t_1), (g_2, t_2)$ two adjacent vertices of G and let us prove that $f(g_1, t_1) \neq f(g_2, t_2)$. We separate the proof in three cases:

(1) If $t_1 = t_2 = 0$, then $(g_1, 0)$ and $(g_2, 0)$ are adjacent in $\mathrm{Cay}(\Gamma \times \{0\}, C_0)$ and, hence, $h(g_1, 0) = f(g_1, 0) \neq f(g_2, 0) = h(g_2, 0)$.
(2) If $t_1 = t_2 = 1$, then $(g_1, 1)$ and $(g_2, 1)$ are adjacent if and only if there exist $(x, 0) \in C_0$ such that $(g_1, 1) = (g_2, 1) * (x, 0) = (g_2 - x, 1)$. Then,
$$(g_1 - y, 0) = (g_1, 1) * (y, 1) = (g_2 - x, 1) * (y, 1) = (g_2 - y - x, 0)$$
and, hence, $(g_1 - y, 0)$ and $(g_2 - y, 0)$ are adjacent in $\mathrm{Cay}(\Gamma \times \{0\}, C_0)$. Thus, we conclude that $h(g_1, 1) = f(g_1 - y, 0) + 1 \neq f(g_2 - y, 0) + 1 = h(g_2, 1)$.

(3) If $t_1 = 0, t_2 = 1$, then $(g_1, 0)$ and $(g_2, 1)$ are adjacent if and only if there exist $(z, 1) \in C_1$ such that $(g_2, 1) = (g_1, 0) * (z, 1) = (g_1 + z, 1)$. Then,

$$h(g_2, 1) = f(g_1 + z - y, 0) + 1 \bmod 3 = f(g_1, 0) + 1 \bmod 3 \neq f(g_1, 0) = h(g_1, 0),$$

where the equality $f(g_1 + z - y, 0) = f(g_1, 0)$ follows from the fact that $(z - y, 0) = (z, 1) * (y, 1) \in H$.

□

For the next result, we denote by $\Phi(\Gamma)$ the *Frattini subgroup* of Γ, that is, the intersection of all maximal proper subgroups of Γ, or $\Phi(\Gamma) = \{e\}$ if it has no maximal subgroups.

Lemma 2.7. *Let Γ be a group with Frattini subgroup $\Phi(\Gamma)$, then:*

$$\chi_{\max}(\Gamma) \leq \chi_{\max}(\Gamma/\Phi(\Gamma)).$$

Proof. For Γ a group and C any minimal generating set. The following remarkable properties of the Frattini subgroup are well known (see, e.g., [28, Section 5.2]):

(1) $\Phi(\Gamma)$ is a characteristic subgroup of Γ and, hence, $\Phi(\Gamma) \trianglelefteq \Gamma$,
(2) $\Phi(\Gamma) \cap C = \emptyset$, and
(3) $C/\Phi(\Gamma) = \{c \cdot \Phi(\Gamma) \mid c \in C\}$ is a minimal generating set of $\Gamma/\Phi(\Gamma)$.

By (1), (2) and Lemma 2.1, one has that $\chi(\mathrm{Cay}(\Gamma, C)) \leq \chi(\mathrm{Cay}(\Gamma/\phi(\Gamma), C/\Phi(\Gamma)))$. By (3) $\chi(\mathrm{Cay}(\Gamma/\phi(\Gamma), C/\Phi(\Gamma))) \leq \chi_{\max}(\Gamma/\phi(\Gamma))$, and the result follows. □

For a group $(\Gamma, \cdot)$, we denote by Γ' its commutator subgroup, i.e.,

$$\Gamma' = \{x \cdot y \cdot x^{-1} \cdot y^{-1} \mid x, y \in \Gamma\}.$$

Theorem 2.8. *Let Γ be a finitely generated group such that its commutator Γ' is contained in its Frattini subgroup $\Phi(\Gamma)$. Then, every minimal Cayley graph of Γ is 3-colorable. This includes nilpotent groups.*

Proof. Let $G = \mathrm{Cay}(\Gamma, C)$ be the Cayley graph of Γ with respect to a minimal set of generators C. Since $\Gamma' \subseteq \Phi(\Gamma)$, then it follows that $\Gamma/\Phi(\Gamma)$ is commutative and minimally generated by $C/\Phi(\Gamma)$. Then, by Theorem 2.5, $\mathrm{Cay}(\Gamma/\Phi(\Gamma), C/\Phi(\Gamma))$ is 3-colorable. Since $C \subset (C/\Phi(\Gamma)) \cdot \Phi(\Gamma)$, by Lemma 2.1 we conclude that $\mathrm{Cay}(\Gamma, C)$ is 3-colorable. □

A group Γ satisfies that $\Gamma' < \Phi(\Gamma)$ if and only if all its maximal subgroups have prime index (see [25, Theorem A] for other characterizations of these groups). In particular, every nilpotent group satisfies that $\Gamma' \leq \Phi(\Gamma)$. A result of Wielandt [28, 5.2.16] proves that for a finite group Γ, one has that $\Gamma' \leq \Phi(\Gamma)$ if and only if Γ is nilpotent. However, there are non-nilpotent infinite groups whose commutator is contained in its corresponding Frattini subgroup. A famous such group is the Grigorchuk group, which is a finitely generated 2-group in which all maximal subgroups have index 2. In fact $\Gamma' = \Phi(\Gamma)$, and has index 8 in Γ [18]. Other examples are considered in [13].

Let us end this section with a general upper bound for the chromatic number of (semi)minimal Cayley graphs. For this purpose given a positive integer n, we denote by $W_b(n)$ the *binary Lambert W function*, i.e., $n = W_b(n)2^{W_b(n)}$.

Proposition 2.9. *Let Γ be a group of order n and C be a generating set. We have*

$$\chi(\Gamma, C) \leq \begin{cases} 2\log_2 n & \text{if } C \text{ is semiminimal,} \\ 2W_b(n) & \text{if } C \text{ is minimal.} \end{cases}$$

Moreover, $W_b(n) < \log_2 n - \log_2 \log_2(\frac{n}{\log_2 n})$.

Proof. Let $C = (c_1, \ldots, c_k)$ a semiminimal generating set of Γ. One has that $\Gamma_{i-1} := \langle c_1, \ldots, c_{i-1} \rangle$ is a proper subgroup of Γ_i for all $1 \leq i \leq k$ and, by Lagrange's Theorem, we have $2^i \leq |\Gamma_i|$. Hence $k = |C| \leq \log_2(|\Gamma|) = \log_2 n$. Thus, the maximum degree of $\mathrm{Cay}(G, C)$ is at most

$2\log_2 n$. By Brook's Theorem, this is an upper bound for $\chi(\mathrm{Cay}(G, C))$ except if $\mathrm{Cay}(G, C)$ is an odd cycle or a clique. However, in the first case $\chi(\mathrm{Cay}(G, C)) = 3 \leq 2\log_2(2\ell + 1)$ for all $\ell \geq 1$. If otherwise $\mathrm{Cay}(G, C) = K_n$ is a clique, by Proposition 3.1 we know that $n = 4$ and $\chi(\mathrm{Cay}(G, C)) = 4 \leq 2\log_2(4)$.

Now suppose C minimal. Again by Lagrange's Theorem, for any $c \in C$ the subgroup $\langle C - \{c\}\rangle$ has order at least 2^{k-1}. Hence, the Schreier graph of this subgroup has at most $\frac{n}{2^{k-1}}$ vertices, and $\chi(\mathrm{Cay}(\Gamma/\langle C - \{c\}\rangle, c)) \leq \frac{n}{2^{k-1}}$. Thus, with the first part and Lemma 2.3 we have that $\chi(\mathrm{Cay}(\Gamma, C)) \leq \min(2k, \frac{n}{2^{k-1}})$, which is maximised exactly if $k = W_b(n)$. By definition we have that $W_b(n) = \log_2(\frac{n}{W_b(n)}) = \log_2(\frac{n}{\log_2(\frac{n}{W_b(n)})})$. Moreover we clearly have clearly $W_b(n) < \log_2(n)$, then we finally get that $W_b(n) < \log_2(\frac{n}{\log_2(\frac{n}{\log_2(n)})}) = \log_2 n - \log_2 \log_2(\frac{n}{\log_2 n})$. □

Note that the bounds in the previous proposition depend on the maximal size of a minimal generating set of a group Γ. This parameter has been studied, see e.g. [8, 23].

3. Lower bounds

Already in [3] it is shown that minimal Cayley graphs have clique number at most 3 and it also follows from the results there, that semiminimal Cayley graphs have bounded clique number. We first make this precise.

Proposition 3.1. *For any one popular color graph G we have $\omega(G) \leq 5$ and this is tight. For a semiminimal Cayley graph $\mathrm{Cay}(\Gamma, C)$ we have $\omega(\mathrm{Cay}(\Gamma, C)) \leq 4$ and this is tight.*

Proof. Suppose a one popular color coloring of K_6 and consider an edge ab of color 1 and the four triangles abc, abd, abe, abf. At most two among these four triangles have popular color 1. If two of these triangles, say abc, abd have one popular color among different colors $2, 3$, then the triangle acd has no popular color. Hence two triangles say abc, abd have popular color 2 then the triangles abe, abf must have popular color 1. So assume that the edge ae is colored in 1. But then the edges ec and ed both must be of color 1 to make triangles aec and aed have a popular color. But then the degree of e in color 1 is 3.

To show tightness just edge-color K_5 with two colors each inducing a cycle of length 5. Since we are only using two colors, it is straight-forward to check that all cycles have a popular color. Let us now, see that this is the only way to one popular color color the K_5. Let ab an edge of color 1 and consider the three triangles abc, abd, abe. If two of these triangles, say abc, abd have one popular color because of different colors $2, 3$, then the triangle acd has no popular color. If two triangles say abc, abd have one popular color because of color 2 then the triangle abe must have popular color 1, say the edge be is of color 1 and both the edges ce, de must be of color 1 in order to make triangle bce, bde have a popular color. But then the degree of e in color 1 is 3. Suppose now that only the triangle abc has popular color 2, then both abd, abe have popular color 1 and without loss of generality we have edges ad, be of color 1. Now bd cannot be of color 1 (because the degree of b in color 1 would be 3), and cannot be of color 3, because this would force the edge cd to be of color 3, but then the triangle acd would have no popular color. Hence bd is of color 2, and by an analogous argument also ae is of color 2. Now, cd is forced to be of color 1, de of color 2, and ce of color 1. The resulting two-coloring is a decomposition into two cycles of length 5.

Suppose now that a semiminimal Cayley graph $\mathrm{Cay}(\Gamma, C)$ contains a K_5, then as argued above its one popular color coloring consists of two cycles of length 5. Hence the corresponding elements $c, c' \in C$ generate the same cyclic subgroup of order 5 of Γ. Hence C is not semiminimal. Thus, $\omega(\mathrm{Cay}(\Gamma, C)) \leq 4$. To see that this is tight simply consider $\mathrm{Cay}(\mathbb{Z}_4, \{2, 1\}) = K_4$. □

We finish disproving [3, Conjecture 3.5], i.e., that one popular color graphs have bounded chromatic number. The construction we provide is based on one of the fundamental constructions for triangle-free graphs of arbitrary chromatic number due to Tutte (alias Blanche Descartes) and independently Zykov, see Nešetřil's survey [26] for a more detailed discussion.

Theorem 3.2. *For any k there exists a one popular color graph G_k with $\chi(G_k) \geq k$.*

Proof. We proceed by induction on $k \geq 1$. Chose G_1 to be the graph with a single vertex and $G_2 = C_4$ with edges colored alternatingly with two colors. If $k \geq 3$ then denote by n the order of G_{k-1}. Define X as a set of size $(k-1)(n-1)+1$. Now, for every subset of Y size n of X take a copy G' of G_{k-1} (where all copies can be considered to be edge-colored with the same set) and add a perfect matching between Y and the copy G'. Each of these new matchings will be edge colored with its own private color.

To see that G_k is a one popular color graph, note first that every color class is a matching, hence the coloring satisfies property (1). Now, observe that by induction hypothesis all cycles within a single copy G' have one color at least twice. Since X is an independent set any other cycle must enter and leave some copy G', but then it uses two edges of the same matching, hence repeats at least one color. This proves property (2').

The fact that $\chi(G_k) \geq k$ is well-known, see e.g. [26].

□

While we have disproved the strong variant of Conjecture 1.3 its weak variant remains open. Let us propose a strengthening of it.

Conjecture 3.3. There is a function f, such that if the edges of a graph G can be colored such the subgraph induced by any color has maximum degree d, and no color appears exactly once on a cycle of G, then $\chi(G) \leq f(d)$.

Note that this conjecture for $d = 2$ is the weak variant of Conjecture 1.3. However, it is open even for the case $d = 1$.

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