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Ehrhart theory of lattice path matroid polytopes

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Abstract. We study the h^*-vector of lattice path matroid polytopes. On the one hand, we give a combinatorial interpretation for its components in the case of rank 2. This allows to find recursive formulas for the volume of these polytopes. On the other hand, we characterize Gorenstein lattice path matroid polytopes, which yields a new class of matroids satisfying the unimodality conjecture of de Loera, Haws, and Köppe.

Keywords: Lattice path matroids, matroid polytope, h^*-vector, Gorenstein polytope.

1 Introduction

Matroids are combinatorial objects at the core of many branches in mathematics such as graph theory, polyhedral geometry, optimization, algebraic geometry, etc. Originally, they arise as a combinatorial axiomatization of the concept of independence from linear algebra. However, their versatility allows one to explore problems of different natures evolving around them. In particular, a matroid M can be studied geometrically via its *matroid polytope* P_M and the associated h^* polynomial $h^*(z) = h_0^* + h_1^* z + \ldots + h_d^* z^d$.

In this manuscript we focus on the family of *lattice path matroids* (LPMs), introduced in [3]. Given a lattice path matroid M of rank 2, one of our main results provides a combinatorial interpretation of the tuple $(h_0^*, \ldots, h_d^*)$, known as the h^*-vector of M. This result is given in Theorem 3.5, where each h_i^* is interpreted as counting a certain set of permutations. One application of this result is that the normalized volume of P_M is equal to $h_0^* + \cdots + h_d^*$. Hence, as a byproduct of our work we obtain recursive formulas for the volume of LPMs of rank 2.

On the other hand, it was conjectured by de Loera et al. [6, Conjecture 2] that the h^*-vector of any matroid polytope is unimodal. This conjecture remains wide-open and has only been verified for small matroids [6, Theorem 3], sparse paving matroids of rank 2 [7, Theorem 1.3], and certain LPMs related to order polytopes [12]. Our second main result stated in Theorem 5.1, characterizes Gorenstein LPMs, i.e., those LPMs whose h^*-vector is unimodal and symmetric. This complements known results for graphic matroids (see [9, 13]).

Both our main results make use of the fact that if M is an LPM, then P_M is *alcoved* and therefore P_M is naturally endowed with a regular and unimodal triangulation Δ, known as the alcoved triangulation of P_M. In particular, when M is the uniform matroid

$U_{2,n}$ (which is an LPM) we are able to interpret the collection of simplices in Δ as the set $P_{1,n-1}$ of permutations of $[n-1] := \{1, \ldots, n-1\}$ with one descent. For an arbitrary LPM of rank 2, the collection of simplices in Δ is a subset of $P_{1,n-1}$. This lies at the core of our combinatorial interpretation of the h^*-vector of LPMs of rank 2.

This manuscript is organized as follows. Section 2 provides the necessary background needed to state and comprehend our results. In particular, we present necessary results on alcoved polytopes. In Section 3 we provide a combinatorial characterization of the h^*-vector of any LPM of rank 2. Section 4 makes use of this combinatorial characterization and provides a recursive formula for the h_i^*, which in turn, is used to build an array of numbers whose entries correspond to the volume of each LPM of rank 2. This array of numbers is constructed via a recursive formula satisfied by the volumes of these LPMs, which we are able to obtain along the way. Finally, in Section 5, we characterize Gorenstein LPMs.

2 Preliminaries

2.1 Lattice path matroids

Matroids arose as a mean to generalize the notion of linear independence in a vector space. Below we give one of many equivalent definitions of matroids stemming from this idea. The interested reader is invited to consult the book [16] for an account of the classic theory and [1] for a recent survey on the geometric aspects.

Definition 2.1. A *matroid* is a pair $M = (E, \mathcal{B})$ where E is a finite set and $\mathcal{B} \neq \emptyset$ is a set of subsets of E that satisfies:

- Let $B_1, B_2 \in \mathcal{B}$. For every $b_1 \in B_1 \setminus B_2$ there is $b_2 \in B_2 \setminus B_1$ such that $B_1 \setminus \{b_1\} \cup \{b_2\} \in \mathcal{B}$.

We refer to the elements of $\mathcal{B}$ as the bases of the matroid M.

Definition 2.2. Given a matroid $M = (E, \mathcal{B})$, its *matroid base polytope* P_M is defined as the convex hull

$$P_M := \operatorname{conv} \{\vec{e}_B \mid B \in \mathcal{B}\}$$

where $\vec{e}_B = \sum_{i \in B} \vec{e}_i$ for $B \subseteq [n]$ and $\{\vec{e}_i\}_{i=1}^n$ is the canonical basis for $\mathbb{R}^n$.

Let $B, B' \in \binom{[n]}{k}$. We say that $B \leq B'$ if $b_i \leq b'_i$, for all $i \in [k]$, where $B = b_1 \leq \ldots \leq b_r$ and $B' = b'_1 \leq \ldots \leq b'_r$.

For $0 \leq k \leq n$ fixed, let $U \leq L$ where $U, L \in \binom{[n]}{k}$. We think of U as the lattice path from $(0,0)$ to $(n-k, k)$ whose north steps are the elements in U (similarly for L). The collection $\mathcal{B} = \left\{B \in \binom{[n]}{k} \ : \ U \leq B \leq L\right\}$ is the collection of bases of a matroid on

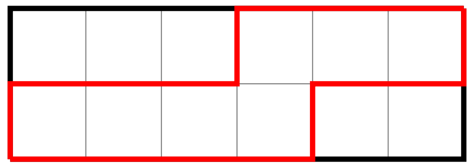

Figure 1: The diagram of the snake $S(4,2,3)$ inside $U_{2,8}$.

the ground set $[n]$ (see [3]). This matroid is the *lattice path matroid* $M = M[U, L]$. The *diagram* of $M[U, L]$ is the region in the rectangle from $(0, 0)$ to $(n-k, k)$ that lies between L and U. We say that M is connected if U and L only intersect in $(0, 0)$ and $(n-k, k)$. As a first example, the *uniform matroid* $U_{k,n}$ of rank k over $[n]$ is a lattice path matroid (or LPM, for short) given by $U_{k,n} = M[U, L]$ where $U = [k]$ and $L = \{n-k+1, \ldots, n-1, n\}$. An LPM is a *snake* if its diagram does not contain a 2×2 square. Thus the diagram of a snake S, read from southwest to northeast consists of horizontal and vertical ribbons concatenated. Therefore we think of S as the sequence $(t_1, \ldots, t_r)$ where t_i is the size of the i-th ribbon in S (see Figure 1).

Since the main results of this manuscript evolve around combinatorial properties of LPM polytopes, we now establish some well-known facts that will motivate our results.

Given a d-dimensional polytope $P \subset \mathbb{R}^d$ and $t \geq 0$, let $L_P(t) := \#\big(tP \cap \mathbb{Z}^d\big)$, where tP is the dilation of P by t. When P is a lattice polytope, i.e. its vertices are integer vectors, it is known that $L_P(t)$ is a polynomial in t known as the *Ehrhart polynomial* of P. Moreover we have that

$$L_P(t) = h_0^* \binom{t+d}{d} + h_1^* \binom{t+d-1}{d} + \ldots + h_{d-1}^* \binom{t+1}{d} + h_d^* \binom{t}{d} \tag{2.1}$$

for some real numbers $h_0^*, \ldots, h_d^*$. Also, the generating function of the polynomials $\{L_P(t)\}_{t \geq 0}$, known as the Ehrhart series of P and denoted by $\mathrm{Ehr}_P(z)$, satisfies

$$\mathrm{Ehr}_P(z) := \sum_{t \geq 0} L_P(t) z^t = \frac{h_0^* + h_1^* z + \ldots + h_d^* z^d}{(1-z)^{d+1}}.$$

The polynomial $h^*(z) := h_0^* + h_1^* z + \ldots + h_d^* z^d$ is known as the *h^*-polynomial of P*, and the tuple $(h_0^*, h_1^*, \ldots, h_d^*)$ will be referred to as the *h^*-vector of P*, $h^*(P)$. We invite the curious reader to check [2] for diving into lattice polytopes.

Theorem 2.3. *[17, Theorem 2.1] Let P be a lattice d-polytope in $\mathbb{R}^d$. Then $h^*(P) \in \mathbb{Z}^d_{\geq 0}$.*

Given P as in the Theorem 2.3 a natural question to ask for is to provide a combinatorial interpretation of the coefficients of $h^*(z)$. This question inspired the results that we provide in Section 4. A motivation to answer this question is given by the fact that $h^*(1) = h_0^* + h_1^* + \ldots + h_d^*$ recovers the (normalized) volume of P. Throughout this document, when we refer to volume, we mean normalized volume.

The family of LPMs enjoys several useful properties. LPMs are positroids, and as such LPM polytopes are alcoved, see [14, Theorem 2.1]). This family of polytopes will be introduced further in Section 2.2 and many of the properties of their h^*-vector are used throughout the manuscript. Moreover, LPMs have a canonical matroid decomposition in terms of snakes [5]. Furthermore, matroid polytopes of snakes are order-polytopes [12, Theorem 4.7]. Thus, the canonical alcoved triangulation of snakes can be interpreted as linear extensions of the underlying poset [18, Section 5]. These results are crucial in Section 3. Also, the fact of LPM polytopes being alcoved together with the minor-closedness of the family are useful for the result of Section 5.

2.2 Alcoved polytopes

For $0 \le i < j \le n$ and $l \in \mathbb{Z}$ consider the infinite arrangement of hyperplanes in $\mathbb{R}^n$ given by

$$H_{ij}^{l} = \left\{ \vec{x} \in \mathbb{R}^n \mid x_i - x_j = l \right\}$$

where we set $x_0 = 0$. We define an *alcoved polytope* as a polytope in $\mathbb{R}^n$ such that its defining hyperplanes are all of the form H_{ij}^{l}. In other words, P is an alcoved polytope if it can be described as

$$P = \left\{ \vec{x} \in \mathbb{R}^n \mid a_{ij} \le x_i - x_j \le b_{ij} \right\}$$

for a given set of integers a_{ij} and b_{ij}. The *alcoved triangulation* Δ of P is obtained by subdividing P with hyperplanes of the form H_{ij}^{l}. .

Given any triangulation T of a polytope P, the *dual graph* of T, denoted by G_T, is the graph whose vertices are the maximal cells of T and two vertices are joined by an edge if their corresponding cells in T share a facet. Given a vertex α_0 of G_T let $D_T(\alpha_0)$ be the orientation of the graph G_T *away from* α_0. That is, an edge $\{\alpha, \beta\}$ is directed from α to β if and only if $d(\alpha_0, \beta) = d(\alpha_0, \alpha) + 1$ where $d(\alpha, \beta)$ is the minimal length among the paths in G_T from α to β.

The alcoved triangulation Δ of P is known to be unimodular and regular. Using these properties along with [17, Corollary 2.5], we obtain the following result which will be used in Section 3.

Theorem 2.4. *Let P be an alcoved polytope with alcoved triangulation Δ and fix a vertex α_0 in G_Δ. Let $D_\Delta(\alpha_0)$ be the orientation of G_Δ away from α_0. Then*

$$h_k^*(P) = \left\{ \alpha \in G_\Delta \;\middle|\; \alpha \text{ has } k \text{ incoming arcs in } D_\Delta(\alpha_0) \right\}.$$

The fact that LPM polytopes are alcoved will be used in Section 5 in order to deduce from the Gorenstein property that the h^*-vector is unimodal. Theorem 2.4 will be used to give a combinatorial meaning to the h^*-vector of any $M[U, L]$ of rank 2. We will make use of the alcoved triangulation Δ of the matroid polytope, and an orientation of its dual graph G_Δ away from a specific vertex.

3 The h^*-vector of lattice path matroids of rank 2

The main result of this section, given in Theorem 3.5, provides a combinatorial way to compute the h^*-vector of any LPM of rank 2. This result will be achieved in three main steps:

1. We provide a new combinatorial interpretation for $h^*(U_{2,n})$. The coefficients of $h^*(U_{k,n})$ appeared previously in [10, Corollary 2.9], although this interpretation is not obtained combinatorially. On the other hand, in [11] a combinatorial formula for the coefficients of $h^*(U_{k,n})$ is derived from [10, Corollary 2.9].

2. We extend the interpretation of $h^*(U_{2,n})$ to any Schubert matroid of rank 2.

3. We prove that *opposite* Schubert matroids are such that its h^*-vector can be obtained from a Schubert matroid. Combining the above we obtain an analogous combinatorial interpretation for the h^*-vector of arbitrary LPMs of rank 2.

3.1 Uniform matroids of rank 2

Letting $M = U_{2,n}$, it is known that the volume of P_M is the cardinality of the set $P_{1,n-1}$ of permutations of $[n-1]$ with one descent [15], that is, the Eulerian number $A_{1,n-1}$. Thus our strategy to interpret combinatorially the coefficients of $h^*(M)$ will be to give a partition $(A_0, A_1, \dots, A_d)$ of the set $P_{1,n-1}$ in such a way that $|A_i| = h_i^*$, for $i = 0, \dots, d$.

First of all, as noticed previously, the uniform matroid $U_{2,n} = M[12, \{n-1, n\}]$. Then each snake $S = (t_1, \dots, t_r)$ inside the diagram of $U_{2,n}$ is such that $r = 2$ or $r = 3$ and S is completely determined by t_r. Thus we denote such snakes as $S_{t_r,n}$. Now, given $M = M[U, L]$ of rank 2 and its alcoved triangulation Δ it holds that the set of simplices in Δ are in bijective correspondence with *labelled snakes*. A labelled snake $S_{n-k,n}$ is a filling of its diagram from southwest to northeast with a set of integers $\{a_i\}_{i=1}^{n-1}$ satisfying $a_1 > a_2 > \dots > a_k$, $a_k < a_{k+1}$ and $a_{k+1} > \dots > a_{n-1}$. When $M = U_{2,n}$ such filled snakes give rise to the following bijection:

$$\{\text{Labelled snakes } S_{n-k,n} \text{ inside } U_{2,n}\} \longrightarrow \left\{\begin{matrix}\text{Permutations of } [n-1] \text{ with} \\ \text{1 descent in one-line notation}\end{matrix}\right\}$$

$$\begin{matrix} & & & a_{k+1} & \dots & a_{n-1} \\ a_1 & a_2 & \dots & a_k & & \end{matrix} \longmapsto a_{n-1}a_{n-2}\dots a_{k+1}|a_k a_{k-1}\dots a_1$$

where the vertical line in the permutation marks its unique descent.

This bijection along with a statistic on the set $P_{1,n-1}$ will provide us with the combinatorial description of $h^*(U_{2,n})$. Given $\pi \in P_{1,n-1}$, the number of *initial gaps* of π is the number of non-consecutive pairs of values before its descent. Similarly, the number of

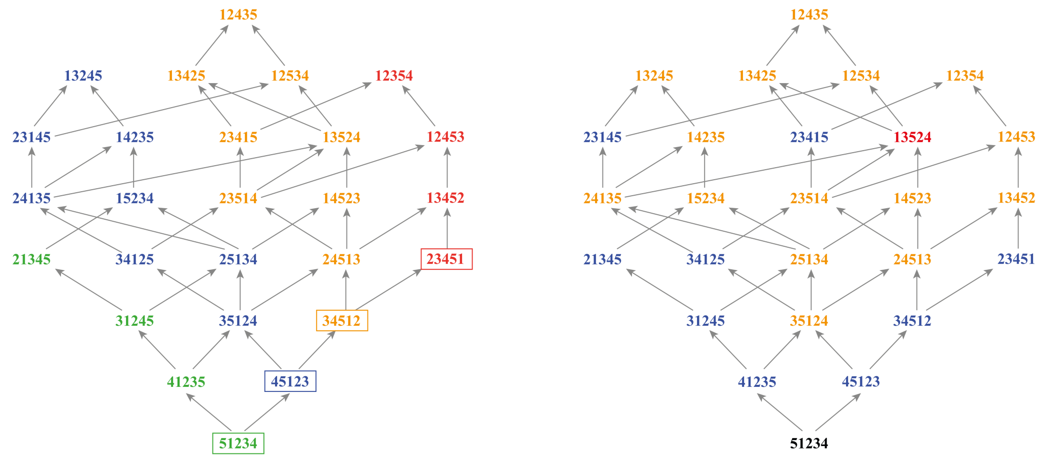

Figure 2: Dual graph G_Δ of $U_{2,6}$ oriented away from $\pi_0 = 5\,1\,2\,3\,4$. The graph on the left distinguishes the four different snakes, by color. The graph on the right gives that $h_0^* = 1$, $h_1^* = 9$, $h_2^* = 15$ and $h_3^* = 1$ accounting for the number of black, blue, yellow and red vertices, respectively.

final gaps is the number of gaps after the descent. For example, the permutation $2\,3\,4|1\,5$ has no initial gaps and one final gap, whereas $1\,3\,5|2\,4$ has 2 initial gaps and one final gap.

Theorem 3.1. *Let $P = \Delta_{2,n}$ be the matroid polytope of $U_{2,n}$. Then $h^*(P)$ is given by:*

- *$h_0^* + h_1^*$ counts the set of permutations on $n-1$ elements without initial gaps.*
- *h_k^* counts the set of permutations on $n-1$ elements with k-1 initial gaps, for $k \geq 2$.*

The idea of the proof makes use of the dual graph G_Δ of the alcoved triangulation of P. We orient G_Δ away from the simplex whose labelled snake corresponds to the permutation $\pi_0 := n-1|1\,2\,\ldots\,n-2 \in P_{1,n-1}$. In particular h_0^* accounts for π_0. Then we proceed to use Theorem 2.4. We make use of the fact that the collection of all simplices in Δ indexed by a given snake S (independently of its labellings) form a subpolytope of P, called the snake polytope P_S (see [12, Theorem 4.7]). Figure 2 exemplifies, on one hand, the mentioned orientation of G_Δ along with the four different snake polytopes, and the (simplices) permutations that conform it. On the other hand, it shows how to compute $h^*(P)$.

By a simple combinatorial counting, Theorem 3.1 leads us to the following corollary.

Corollary 3.2. *Let $P = \Delta_{2,n}$ be the matroid polytope of $U_{2,n}$. Then, $h^*(P)$ is given by*

$$h_k^* = \begin{cases} 1 & \textit{if } k = 0 \\ \binom{n}{2} - n & \textit{if } k = 1 \\ \binom{n}{2k} & \textit{if } k \geq 2. \end{cases}$$

We now proceed to make use of the ideas just developed, and use them to analyze more general h^*-vectors.

3.2 Lattice path matroids of rank 2

We are ready to establish our description of $h^*(P)$ where P is an LPM of rank 2. Throughout we make use of the following notation.

Definition 3.3. Let $n \geq 4$, $0 \leq k \leq n-3$ and $1 \leq \ell \leq n-2$. We denote by $M_n[k, \ell]$ the LPM of rank 2 obtained by removing a horizontal stripe of size $n - 2 - \ell$ from the north-west corner and a horizontal stripe of size $k - 1$ from the south-east corner of the diagram of $U_{2,n}$. In particular, Schubert matroids (of rank 2) are obtained when $k = 1$.

Theorem 3.4. *(h^*-vector of Schubert matroids of rank 2) Let $1 \leq \ell \leq n-2$ and $M = M_n[1, \ell]$. Then, the h^*-vector of P_M can be described as follows:*

- *$h_0^* + h_1^*$ counts the permutations of $n-1$ elements with one descent in position $1, 2, \ldots, \ell$ and without initial gaps*
- *h_k^* counts the permutations of $n-1$ elements with one descent in position $1, 2, \ldots, \ell$ and $k-1$ initial gaps, for $k \geq 2$.*

This result follows after identifying the polytope P_M as the subpolytope of $P_{U_{2,n}}$ obtained by removing the snake polytopes $\{S_{t,n} : t = \ell + 1, \ldots, n-2\}$. For example, using Figure 2 we can conclude that $h^*(M_6[1, 2]) = (1, 6, 6)$ obtained by removing from G_Δ the orange and red vertices. Theorem 3.4 then tells us how to obtain the h^*-vector of Schubert matroids $M_n[1, \ell]$. Moreover, the h^*-vector of a rank 2 *opposite Schubert matroid*, that is a matroid of the form $M_n[k, n-2]$, is the h^*-vector of $M_n[1, n-k-1]$. This observation is crucial as, unlike for matroids $M_n[1, \ell]$ whose h^*-vector was obtained by deleting "top" snakes from G_Δ, deletion of "bottom" snakes from G_Δ does not allow us to recover the h^*-vector of $M_n[k, n-2]$. Our fix to this requires two different orientations of the dual graph of a certain lattice path matroid polytope. These results combined give rise to our main result:

Theorem 3.5. *(h^*-vector of LPMs of rank 2) Let $M = M_n[k, \ell]$. Then, the h^*-vector of P_M can be computed as follows:*

- *$h_0^* + h_1^*$ equals the number of permutations of $n-1$ elements with one descent in position $1, 2, \ldots, \ell$ without initial gaps **minus** the number of permutations of $n-1$ elements with one descent in position $1, 2, \ldots, k-1$ without final gaps.*

- *h_m^* equals the number of permutations of $n-1$ elements with one descent in position $1, 2, \ldots, \ell$ and $m-1$ initial gaps **minus** the number of permutations of $n-1$ elements with one descent in position $1, 2, \ldots, k-1$ and $m-1$ final gaps, for $m \geq 2$.*

Remark 3.6. In Theorem 3.5 the description of h_m^* is given by counting the difference of two combinatorial sets unlike the preceding results in this section. We also want to point out that Theorems 3.2 and 3.4 are a consequence of this, however we could not have achieved Theorem 3.5 avoiding these two.

Corollary 3.7. *The h^*-vector of the matroid $M = M_n[k, \ell]$ satisfies*

$$h_m^*\big(M_n[k,\ell]\big) = h_m^*\big(M_n[1,\ell]\big) + h_m^*\big(M_n[1, n-k-1]\big) - h_m^*\big(U_{2,n}\big)$$

for $m = 0, 1, \ldots, d$.

4 Volumes of Lattice Path Matroid Polytopes

We give recursive formulas for the h^*-vector of Schubert matroids of rank 2, by interpreting the permutations in the set $P_{1,n-1}$ as Young diagrams. These recursive formulas will be used to provide enumerative results concerning the volume of LPMs of rank 2.

4.1 Recursive formulas for $h^*(M_n[k,l])$

Given $\pi : a_1\, a_2\, \ldots\, a_k \,|\, b_1\, b_2\, \ldots\, b_{n-1-k} \in P_{1,n-1}$ with descent in position k we associate to it the Young diagram T_π (in English notation) inside the rectangle from $(0,0)$ to $k \times (n-1-k)$ whose lower boundary is the lattice path $\{b_1, b_2, \ldots, b_{n-1-k}\}$.

Under this correspondence we have that the width of T_π is the position of the descent of π. Moreover, the number of initial gaps of π corresponds to the number of outer corners minus 1. Also, the dual graph G_Δ can be described easily in terms of these tableaux, although we leave this reinterpretation to the reader.

With tableaux as combinatorial gadget, we derive the following recursions.

Theorem 4.1. *The h^*-vector of the Schubert matroid $M = M_n[1,\ell]$ is given by $h_k^*(M) = \sum_{t=1}^{\ell} f_k(t,n)$ where the $f_k(t,n)$are defined recursively by:*

- $f_1(t,n) = (n-1-t) - \frac{1}{\ell}$,

- $f_2(t,n) = \frac{1}{2}(n-t)(n-t-1)(t-1)$,

- $f_k(t,n) = \sum_{r=1}^{t-1}\sum_{m=0}^{n-t-2} f_{k-1}(t-r, n-m-r-1)$ *for* $k \geq 3$*,*

subject to the restriction that $f_k(t,n) = 0$ *if* $k-1 \geq t$ *or if* $k-1 \geq n-t-1$*.*

Now we will provide relations satisfied by the volume of Schubert matroids of rank 2. We were able to obtain these relations experimentally by making use of the formulas in Theorem 4.1.

4.2 Volumes of Schubert matroids of rank 2

Throughout this section we let $\sigma_k(n,\ell) := \mathrm{Vol}(M_n[k,\ell])$. In particular, $\sigma_k(n,\ell) = 0$ for $\ell < k$ and $n < \ell + 2$.

Theorem 4.2. *The sequence* $\sigma_k(n,\ell)$ *satisfies*

1. $\sigma_k(n,k) = \binom{n-1}{k} - 1$.
2. $\sigma_k(n,\ell-1) + \sigma_k(n,\ell) + \left(\binom{n-1}{k-1} + \ell - k\right) = \sigma_k(n+1,\ell)$.
3. *For fixed* k *and* ℓ*,* $\sigma_k(n,\ell)$ *is a polynomial in* n *of degree* ℓ*.*

The first part of the theorem can be proved by direct counting of the number of labellings of the snake $S_{k,n} = M_n[k,k]$. For the second part, note that $\sigma_k(n+1,\ell)$ counts the number of labelled snakes inside the diagram of $M_{n+1}[l,\ell]$. Each of those snakes can be mapped either to a labelled snake inside $M_n[k,\ell-1]$ or $M_n[k,\ell]$, except for a set of labelled snakes of size $\left(\binom{n-1}{k-1} + \ell - k\right)$ which do not admit such mapping. Thus, the proof of the first two parts of the theorem is fully combinatorial. For the third part, as well as for Corollary 4.4 follows from [19, Proposition 1.9.2].

Notice that when $k = 1$, we have arbitrary Schubert matroids of rank 2. In this case, setting $\sigma(n,\ell) := \sigma_1(n,\ell)$ we obtain the following.

Corollary 4.3. *The sequence* $\sigma(n,\ell)$ *satisfies*

$$\sigma(n,\ell-1) + \sigma(n,\ell) + \ell = \sigma(n+1,\ell), \tag{4.1}$$

where $\sigma(n,1) = n-2$ *for* $n \geq 3$*. Moreover,* $A_{1,n-1} = \sigma(n,\ell) + \sigma(n,n-\ell-2)$ *for any* $1 \leq \ell \leq n-3$*.*

Although Equation 4.1 is stated for positive integers n and ℓ, we can extend $\sigma(n,\ell)$ to a function $f_\ell : \mathbb{Z} \to \mathbb{Z}$ defined by $f_\ell(n) = \sigma(n,\ell)$ using the recursive relation. Below we computed $f_\ell(n)$ for $0 \leq n \leq 10$ and $1 \leq \ell \leq 8$. In particular, the first columns corresponds to the values of $\sigma(n,1)$ and the blue diagonal is the sequence $A_{1,n-1}$.

−2	−2	−4	−4	−6	−8	0	−28
−1	−2	−3	−4	−5	−8	−1	−20
0	−1	−2	−3	−4	−7	−2	−13
1	1	0	−1	−2	−5	−2	−7
2	4	4	3	2	1	0	−1
3	8	11	11	10	9	8	7
4	13	22	26	26	25	24	23
5	19	38	52	57	57	56	55
6	26	60	94	114	120	120	119
7	34	89	158	213	240	247	247
8	43	126	251	376	459	494	502

Corollary 4.4. *For a fixed ℓ, $\sigma(n,\ell)$ is a polynomial in n of degree ℓ. Moreover,*

$$\sigma(n,\ell) = \sum_{k=0}^{\ell} c_k \binom{n}{k}$$

where $c_0 = \sigma(0,\ell)$, $c_1 = \sigma(0,\ell-1)+\ell$, and $c_k = \sigma(1,\ell-k+1) - \sigma(0,\ell-k+1)$ for $k \geq 2$.

5 Gorenstein LPMs and unimodality

We finalize this manuscript by providing a characterization of LPMs whose h^*-vector is unimodal and symmetric. In order to do this, we introduce the notion of Gorenstein matroid.

Let $P \subset \mathbb{R}^d$ be a lattice polytope containing 0 in its interior. Then P is called *reflexive* if its dual (polar) polytope is also a lattice polytope. In general, one says that a lattice polytope P is reflexive if, a translation of P is reflexive. If P is a lattice polytope we say that P is *δ-Gorenstein* if δP is reflexive, for some $\delta \in \mathbb{Z}_{>0}$. We say that P is a *Gorenstein polytope* if it is δ-Gorenstein for some δ. We say that a matroid M is *Gorenstein matroid* if its matroid polytope is Gorenstein.

As mentioned before, if M is an LPM then its matroid polytope has an alcoved triangulation which is is regular and unimodular. Thus, using [4, Theorem 1] one concludes that if M is Gorenstein then $h^*(M)$ is symmetric and unimodal. The main result of this final section, characterizes connected LPMs of any rank that are Gorenstein, and thus, posses a unimodal h^*-vector. Using the minor-closedness property of the family of LPMs and the hyperplane description of LPM polytopes given in [12, Theorem 3.3], we obtain the following. See Figure 3 for an illustration.

Theorem 5.1. *Let $M = M[U,L]$ be a connected LPM. M is δ-Gorenstein if and only if one of the following conditions holds:*

1. *the diagonal of the diagram does not touch U and L except in the initial and final points of intersection, and every concave corner of the paths touches a diagonal square in the diagram and $\delta = 2$,*

2. *either M or M^* is isomorphic to the snake $S(\delta - 1, \ldots, \delta - 1)$ and $\delta > 2$.*

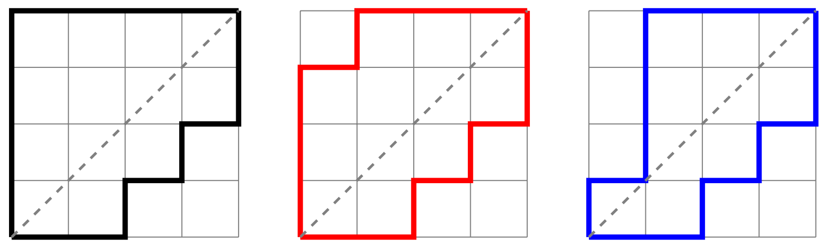

Figure 3: On the left a LPM satisfying condition 1. of Theorem 5.1. The other two LPMs are not Gorenstein.

As uniform matroids are LPMs, we recover a central result of [8, Theorem 2.4].

Corollary 5.2. *A uniform matroid is Gorenstein if and only if it is $U_{n,2n}$, $U_{1,n}$ or $U_{n-1,n}$.*

6 Towards positroids

As mentioned before, matroid polytopes that are also alcoved give rise to the family of positroid polytopes [14, Theorem 2.1]. Hence it is natural to extend our results to the family of positroids. Thus we propose the following problems:

- find a combinatorial interpretation for the h^*-vector of positroids of rank 2,
- find a recursive formula for the volume of positroids of rank 2,
- characterize positroids that are Gorenstein.

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