Local-to-Global Aspects in Metric Graph Theory and Distributed Computing

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“Local-to-Global ?”

Distributed Computing

- Local observations/actions
- Detection of global properties/Performing a global computation

“Local-to-Global ?”

Metric Graph Theory

- Graphs defined by metric properties similar to existing properties of classical metric geometries
- When can we check these properties locally ?
- Similar results exist in geometry: Cartan-Hadamard theorem
- These classes of graphs appear in other fields: concurrency theory, learning theory, phylogeny, geometric group theory

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A common tool

The notion of coverings is fundamental in both cases

Graph Exploration

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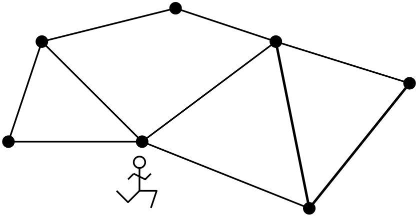

- An agent is moving along the edges of a graph
- Goal: visit all the nodes and stop

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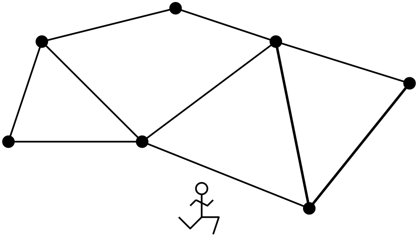

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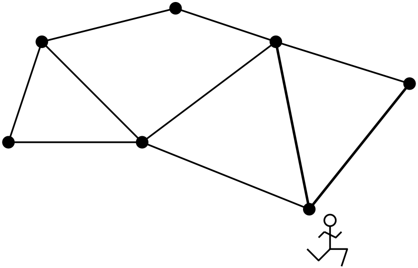

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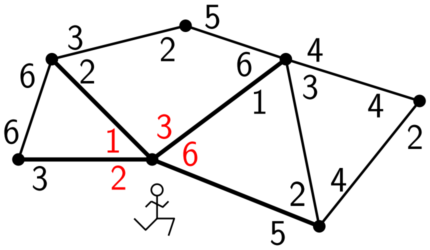
How to navigate in the graph?



- Anonymous graph
- Port-numbering
- The agent knows its incoming port number
- It has an infinite memory

Exploration without information

Exploration of a graph G

Visit every node of G and stop

Question

What graphs can we explore without information?

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What graphs can we explore without information?

An algorithm $\mathcal{A}$ is an exploration algorithm for a family $\mathcal{F}$

- for every graph G, if $\mathcal{A}$ stops, then the agent has visited all the nodes of G
- for every graph $G \in \mathcal{F}$, $\mathcal{A}$ visits all nodes of G and stops

Known Results [Folklore]

If nodes can be marked :

- every graph is explorable by DFS in $O(m)$ moves

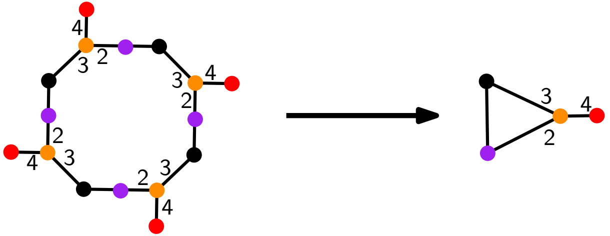
If nodes cannot be marked :

- Trees can be explored by DFS in $O(n)$ moves
- Non tree graphs: it is impossible to detect when all nodes have been visited

Graph Coverings

Definition

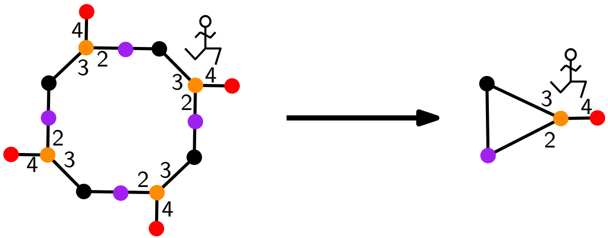
A graph covering is a locally bijective homomorphism $\varphi : G \to H$



Lifting Lemma

Lifting Lemma (from Angluin)

If G is a graph cover of H, then an agent cannot decide if it starts on $v \in V(G)$ or on $\varphi(v) \in V(H)$



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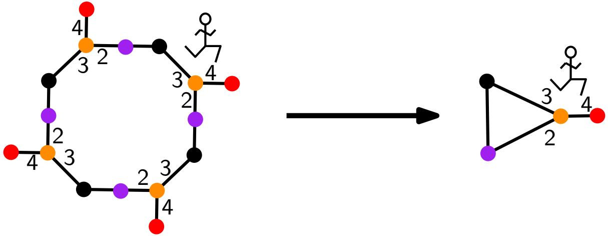


Corollary

If an exploration algorithm $\mathcal{A}$ stops in r steps in H, $r \geq |V(G)|$

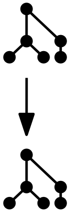
Explorable graphs without global information

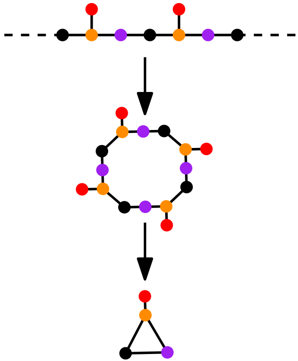
G is explorable

$\iff$ G has a unique graph cover (itself)

$\iff$ G has no infinite graph cover

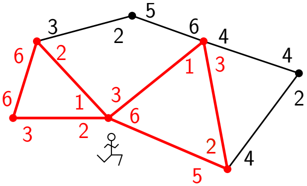
$\iff$ G is a tree





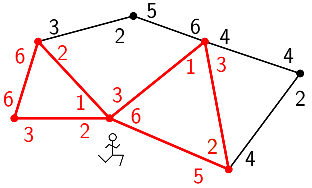
Mobile Agent with Binoculars

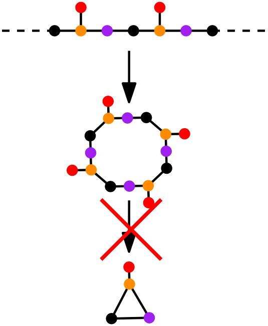
- the agent sees the subgraph induced by its neighbors



Mobile Agent with Binoculars

- the agent sees the subgraph induced by its neighbors
- One can detect triangles
- Graph covering is no longer the good notion

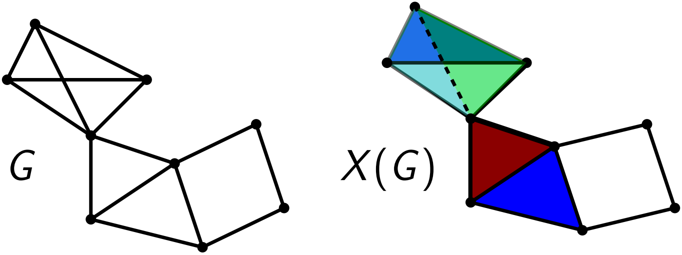




Clique complexes

Definition

The clique complex $X(G)$ of G is a simplicial complex s.t. the simplices of $X(G)$ are the cliques of G



G is the 1-skeleton of $X(G)$

Coverings of Simplicial Complexes

Definition

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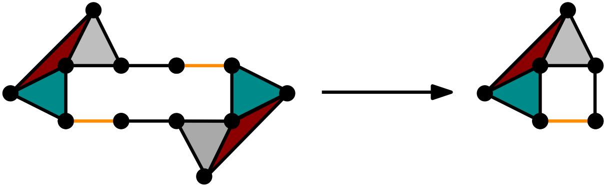

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If $X(G)$ is a cover of $X(H)$, then an agent with binoculars cannot decide if it starts on $v \in V(G)$ or on $\varphi(v) \in V(H)$

Universal covers

Theorem (from Topology)

- *Any complex X has a universal cover $\widetilde{X}$ such that if Y is a cover of X then $\widetilde{X}$ is a cover of Y*
- $\widetilde{X} = X \iff X$ *is simply connected*

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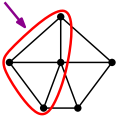
- $X(G)$ is simply connected if all cycles of G are contractible
- a cycle is contractible if it can be contracted to a point by a sequence of elementary deformations:
  - Pushing across a triangle
  - Deleting a pending edge



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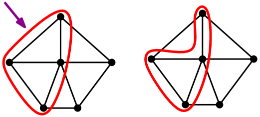

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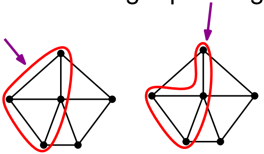

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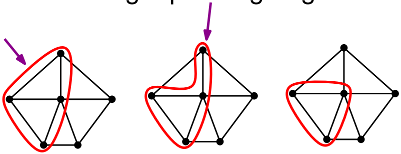

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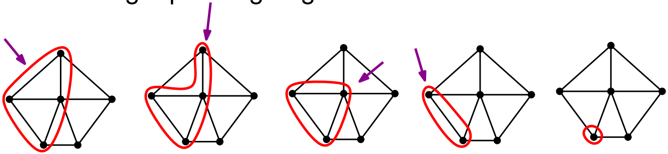

Exploration with binoculars: Characterization

Theorem (C., Godard, Naudin '15)

G is explorable with binoculars $\iff$ $\widetilde{X}(G)$ is finite
In particular, G is explorable if $X(G)$ is simply connected

- a large family of graphs: chordal graphs, (weakly) bridged graphs, Helly graphs, cop-win graphs, triangulations of the (projective) plane, ...
- a Universal Exploration Algorithm
- No efficient universal exploration algorithm: the exploration time cannot be bounded by a computable function

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What subclasses can be explored efficiently ?

Theorem (C., Godard, Naudin '17)

Weetman graphs can be explored in linear time

- chordal graphs, (weakly) bridged graphs, Helly graphs

Local-to-Global Characterizations of Classes of Graphs

Helly Property

Definition

A family $\mathcal{F}$ of subsets of a ground set X has the Helly Property if for any $\mathcal{F}' \subseteq \mathcal{F}$,

$$\forall S, S' \in \mathcal{F}', S \cap S' \neq \emptyset \iff \bigcap_{S \in \mathcal{F}'} S \neq \emptyset$$

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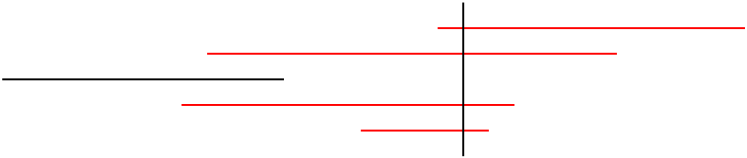

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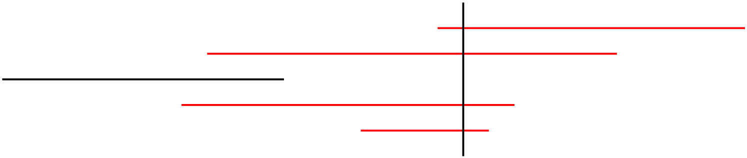
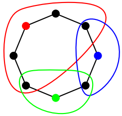


- Axis-parallel boxes in $\mathbb{R}^d$

Helly, 1-Helly and clique-Helly Graphs

Definitions

- A graph G is (ball-)Helly if its family of balls $\{B_r(v) \mid v \in V(G), r \in \mathbb{N}\}$ has the Helly property



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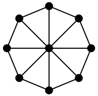

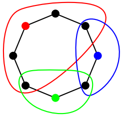

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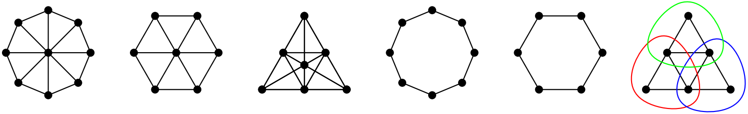

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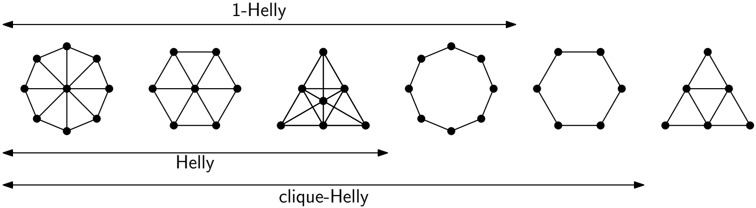

Helly, 1-Helly and clique-Helly graphs

Remarks

- Helly $\implies$ 1-Helly
- 1-Helly $\implies$ clique-Helly
- being 1-Helly or clique-Helly is a local property
- being Helly is a global property

- trees are Helly graphs
- cycles C_n are not Helly when $n \geq 4$ but they are clique-Helly and even 1-Helly when $n \geq 7$.

Local-to-Global Characterization

We cannot characterize Helly graphs using only local properties.

- locally a cycle and a long path look the same

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Theorem (C., Chepoi, Hirai, Osajda '17)

G is Helly $\iff$ G is clique-Helly and $X(G)$ is simply connected

We characterize a global metric condition by local conditions and a global topological condition

This answers a question of
[Prisner '92; Larrión, Pizaña, Villarroel-Flores '10, Chepoi]

Characterization of Helly Graphs

Theorem

For a graph G, the followings are equivalent

(1) *G is Helly*

(2) *G is 1-Helly and weakly modular* [Bandelt Pesch'89]

(3) *G is clique-Helly and cop-win* [Bandelt-Prisner'91]

(4) *G is clique-Helly and $X(G)$ is simply connected*

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The difficult part of the proof is (4) $\implies$ (2)

Proposition

If G is a clique-Helly graph, the 1-skeleton $\widetilde{G}$ of $\widetilde{X}(G)$ is weakly modular and 1-Helly

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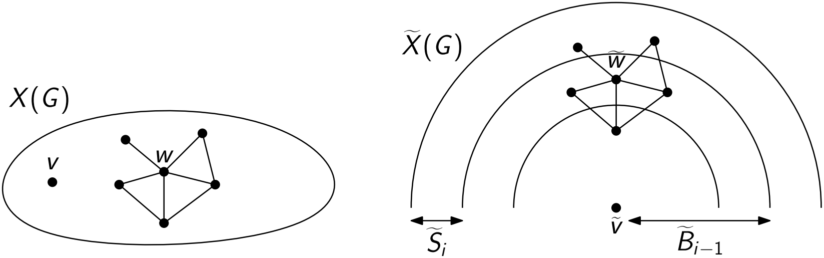
If G is a clique-Helly graph, the 1-skeleton $\widetilde{G}$ of $\widetilde{X}(G)$ is weakly modular and 1-Helly

We build inductively the universal cover $\widetilde{X}(G)$ of $X(G)$ from a basepoint $\tilde{v}$



We prove some properties w.r.t $\tilde{v}$ and use the unicity of $\widetilde{X}(G)$

Other Local-to-Global Characaterizations

Previous characterizations proved via disk diagrams:

- □ Median graphs [Chepoi '00]
- △ (Weakly) Bridged graphs [Chepoi '00; Chepoi, Osajda '15]

Characterizations proved via universal covers:

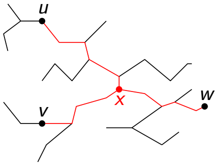
- △□ Basis graphs of matroids [C., Chepoi, Osajda '15]
  (conjectured by [Maurer '73])
- △□ (Weakly) modular graphs [C., Chepoi, Hirai, Osajda '17]
- △ Helly graphs [C., Chepoi, Hirai, Osajda '17]
- △ Prime pre-median graphs [C., Chepoi, Hirai, Osajda '17]
- △□ Dual-Polar graphs [C., Chepoi, Hirai, Osajda '17]
- △□ Bucolic graphs [Brešar, C., Chepoi, Gologranc, Osajda '13]

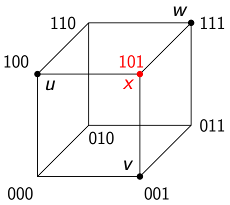
Median Graphs and Event Structures

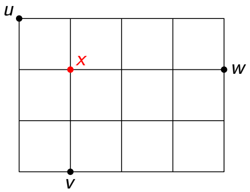
Median graphs

Definition

A graph $G = (V, E)$ is median if for all $u, v, w \in V$, there exists a unique $x \in V$ lying on a (u, v)-shortest path, a (u, w)-shortest path, and a (v, w)-shortest path



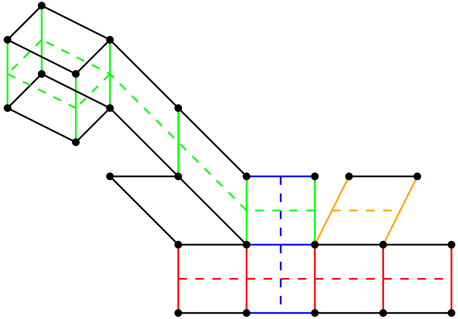




Hyperplanes [Sageev]

In a median graph G, the Djoković-Winkler relation Θ is defined as follows:

- $e_1 \Theta_1 e_2$ if e_1 and e_2 are two opposite edges of a square
- $\Theta = \Theta_1^*$
- a hyperplane of G is an equivalence class of Θ

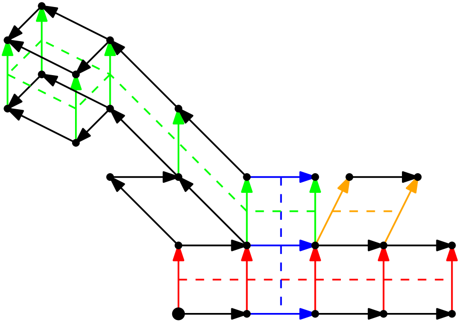


Median Graphs and Event Structures

Event structures are a model of concurrent computation

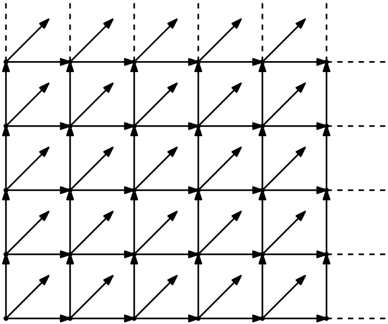
Theorem (Barthélémy and Constantin '93)

- *the domain of an event structure is a pointed median graph*
- *Any pointed median graph is the domain of an event structure*



Domains of Regular Event Structures

An event structure $\mathcal{E}$ is regular if in its domain $D(\mathcal{E})$, the degree is bounded and there is a finite number of equivalence classes of futures



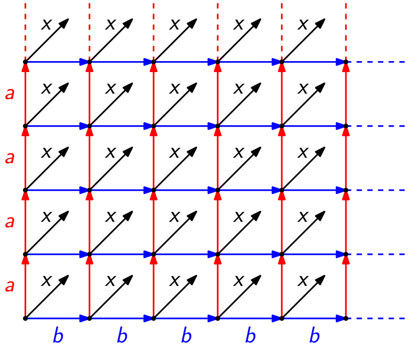
Idea: when considering the executions of a finite state system (like finite state automata or 1-safe Petri nets), there should be some regularity

Regular Nice Labelings

A nice labeling λ is a coloring of the edges of $D(\mathcal{E})$

- two edges with the same origin have distinct colors
- two opposite edges of a square have the same color

A nice labeling is regular if in $D(\mathcal{E})$, there is a finite number of equivalence classes of labeled futures



Thiagarajan’s regularity conjecture

Thiagarajan’s regularity conjecture ’96 (reworded)

Any regular event structure admits a regular nice labeling

Thiagarajan's regularity conjecture

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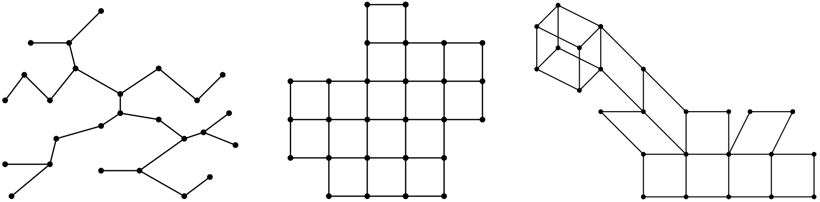
Any regular event structure admits a regular nice labeling

Our results (C., Chepoi '17 & '19)

- The conjecture is false
- A characterization of event structures admitting a regular nice labeling
- We disprove another conjecture of Thiagarajan about the decidability of the MSO theory of regular labeled event structures

CAT(0) cube complexes

A cube complex is a cell complex where each cell is a cube and when two cubes intersect, they intersect on a common face.

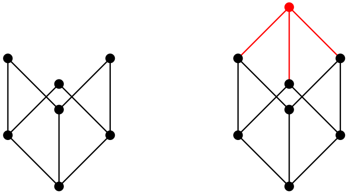


CAT(0) cube complexes

A cube complex is a cell complex where each cell is a cube and when two cubes intersect, they intersect on a common face.

A cube complex X is CAT(0) if

- X is nonpositively curved (NPC) [Gromov]
- X is simply connected



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Theorem (Chepoi '00)

Median graphs are exactly the 1-skeletons of CAT(0) cube complexes

Constructing Event Structures from NPC complexes

- Starting from a finite NPC cube complex X, its universal cover $\widetilde{X}$ is a CAT(0) cube complex
- We have a finite number of equivalence classes of vertices in $\widetilde{X}$ up to isomorphism

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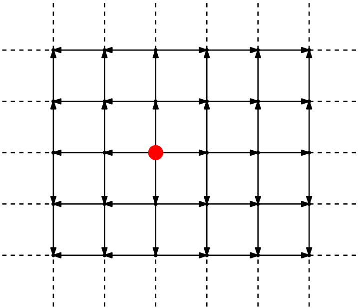
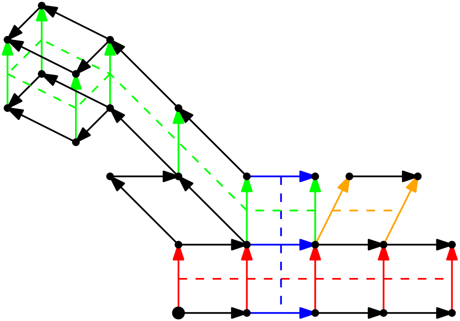
- Problem: we need to have some orientation on the edges to get the domain of an event structure



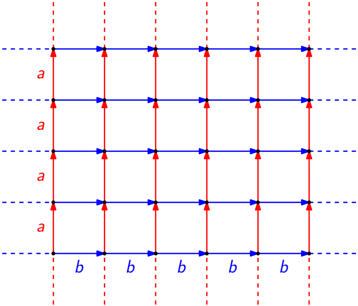
Directed NPC complexes

A directed NPC complex is a complex such that each edge is directed in such a way that two opposite edges of a square have the same direction



Constructing Regular Event Structures

- Starting from a finite directed NPC complex X, we construct its universal cover $\widetilde{X}$
- We have a finite number of classes of futures
- But vertices can have an infinite past ...



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- Starting from a finite directed NPC complex X, we construct its universal cover $\widetilde{X}$
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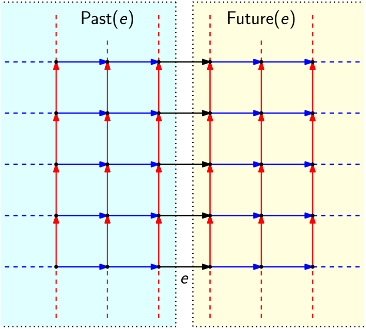
- We cut along hyperplanes



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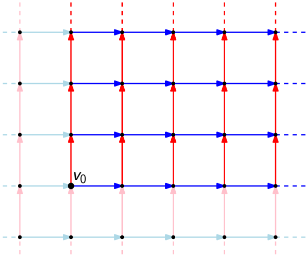
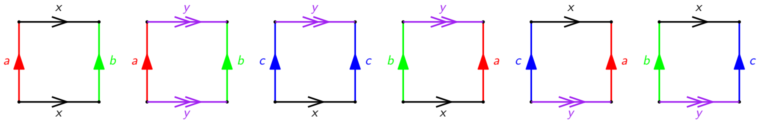


Constructing Regular Event Structures

- Starting from a finite directed NPC complex X, we construct its universal cover $\widetilde{X}$
- We have a finite number of classes of futures
- We cut along hyperplanes
- We have constructed a pointed CAT(0) cube complex $\widetilde{X}_{v_0}$, i.e., the domain of an event structure
- The number of classes of futures is bounded by $|V(X)|$

Wise's directed NPC complex X

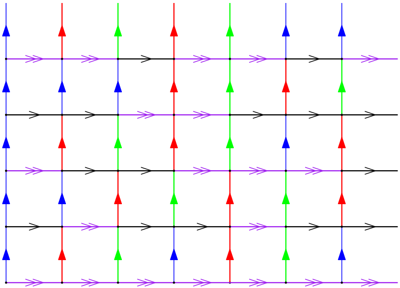
A colored directed NPC complex with 1 vertex, 2 "horizontal" edges (x and y), 3 "vertical" edges (a, b, and c), 6 squares



- it is a directed NPC square complex
- Colors have nothing to do with a nice labeling
- We encode the colors by a trick to get a (colorless) directed NPC complex W
- We construct the domain $\widetilde{W}_v$ of a regular event structure

An aperiodic tiling in the universal cover $\widetilde{X}$ of X

In the universal cover $\widetilde{X}$ of X, the quarter of plane defined by y^ω and c^ω is aperiodic



Proposition (Wise '96)

All horizontal words starting on the side of the quarter of plane are distinct

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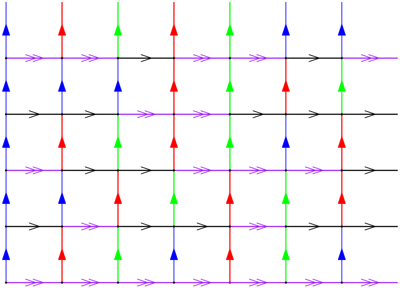


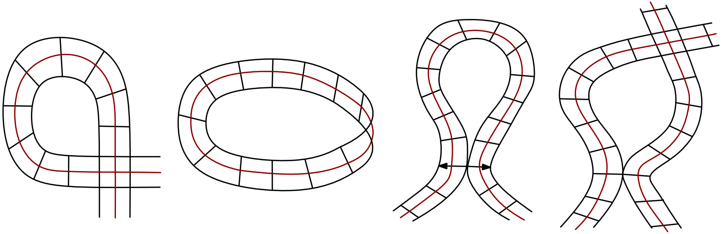
Theorem (C., Chepoi '17)

$\widetilde{W_v}$ does not admit a regular nice labeling

On the positive side: special cube complexes

A NPC complex is special if its hyperplanes behave nicely

[Haglund, Wise ’08]



(a) no self-intersection

(b) no 1-sided hyperplane

(c) no direct self-osculation

(d) no interosculation

On the positive side: special cube complexes

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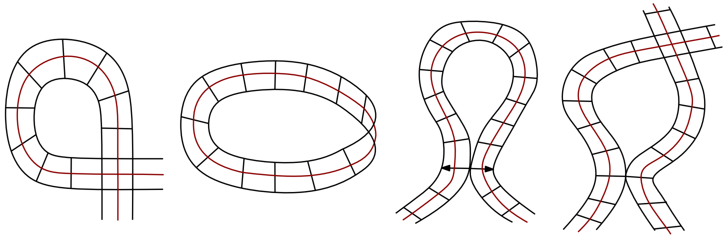


Theorem (C., Chepoi '19)

- *If X is a finite special cube complex, then $\widetilde{X}_v$ has a regular nice labeling*
- *If a domain $D(\mathcal{E})$ has a regular nice labeling, then $D(\mathcal{E}) \simeq \widetilde{X}_v$ for some finite special cube complex X*

Cop and Robber Game and Hyperbolicity

Cop & Robber Game with Speeds

A game between one cop **C** moving at speed **s'** and one robber **R** moving at speed **s**

Initialization:

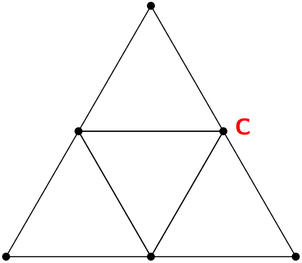
- **C** chooses a vertex
- **R** chooses a vertex

Step-by-step:

- **C** traverses at most **s'** edges
- **R** traverses at most **s** edges

Winning Condition:

- **C** wins if it is on the same vertex as **R**
- **R** wins if it can avoid **C** forever



- **C** has speed $s' = 1$
- **R** has speed $s = 2$

Cop & Robber Game with Speeds

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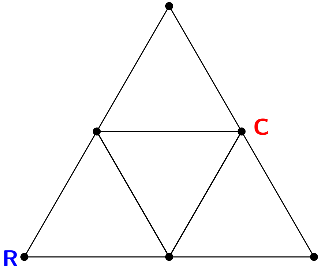

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Step-by-step:

- **C** traverses at most **s′** edges
- **R** traverses at most **s** edges

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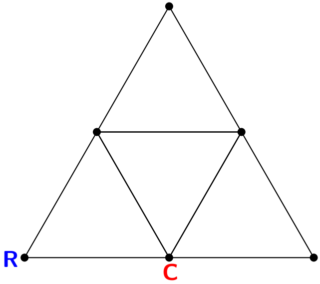

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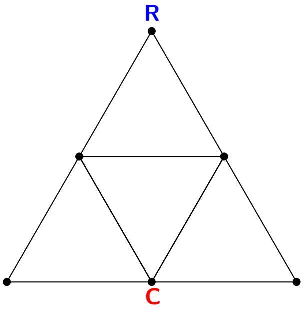

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$(\mathbf{s}, \mathbf{s'})$-Cop-win Graphs

A graph G is $(\mathbf{s},\mathbf{s'})$-cop-win if **C** (moving at speed $\mathbf{s'}$) can win whatever **R** (moving at speed $\mathbf{s}$) does

- If $\mathbf{s} = \mathbf{s'} = 1$, this is the classical Cop and Robber game [Nowakowski and Winkler '83; Quilliot '83]
  - cop-win graphs are exactly the dismantlable graphs
  - chordal, (weakly) bridged, Helly graphs are cop-win
- If $\mathbf{s} = \mathbf{s'}$, this is the classical game played in $G^{\mathbf{s}}$
- If $\mathbf{s} < \mathbf{s'}$, every graph is $(\mathbf{s},\mathbf{s'})$-cop-win

Question

What are the $(\mathbf{s}, \mathbf{s'})$-cop-win graphs with $\mathbf{s} > \mathbf{s'}$?

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Question

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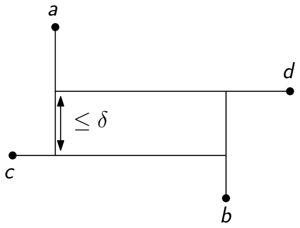
- A characterization of $(\mathbf{s},\mathbf{s'})$-cop-win graphs [C., Chepoi, Nisse, Vaxès '11]
  (in the same spirit as the characterization of cop-win graphs when $\mathbf{s} = \mathbf{s'} = 1$)

δ-hyperbolic graphs [Gromov]

A graph (or a metric space) is δ-hyperbolic if for every four points a, b, c, d,

$$d(a,b) + d(c,d) \leq \max\{d(a,c) + d(b,d), d(a,d) + d(b,c)\} + 2\delta$$

The hyperbolicity $\delta^*(G)$ of a graph G is the minimal value of δ such that G is δ-hyperbolic



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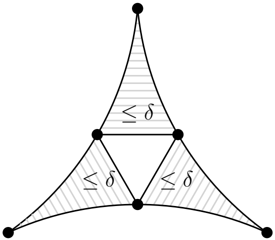
Remark

Many definitions of δ-hyperbolicity; equivalent up to a multiplicative factor

$\delta^*(G)$ measures how G is metrically close from a tree



Hyperbolic graphs are $(\mathbf{s}, \mathbf{s}')$-cop-win graphs

Proposition (from Chepoi, Estellon '07)

Any δ-hyperbolic graph is $(2s, s+2\delta)$-cop-win

Theorem (C., Chepoi, Papasoglu, Pecatte '14)

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We use the following theorem

Theorem (Gromov)

Hyperbolic graphs are the graphs satisfying a linear isoperimetric inequality

If we consider the small cycles of G as 2-dimensional cells, each cycle of G can be contracted to a point with a linear number of elementary deformations

Approximating $\delta^*(G)$

Theorem (C., Chepoi, Papasoglu, Pecatte '14)

One can compute a $O(1)$-approximation of $\delta^*(G)$ in $O(n^2)$

- a "local" algorithm once a BFS has been computed
- the approximation factor is large (1569)
- existing algorithms had a better approximation factor, but a worse complexity
  - a $(2+\epsilon)$-approx. in $O(\frac{1}{\epsilon}n^{2.38})$ [Duan '14]

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Theorem (C., Chepoi, Dragan, Ducoffe, Mohammed, Vaxès '18)

One can compute an 8-approximation of $\delta^*(G)$ in $O(n^2)$

- a simple algorithm also based on a BFS, but not "local"

Open Questions

Distributed Computing

Graph Exploration with binoculars

- What happens if we enlarge the vision of the agent?
  - we believe the results would be qualitatively the same
- Find large subclasses that can be explored more efficiently (with a linear or polynomial number of moves)
  - Weakly modular graphs, basis graphs of matroids
  - δ-hyperbolic graphs

Distributed Computing

What properties can be computed locally with a BFS

- With a BFS at hand, one can distinguish 1569δ-hyperbolic graphs from non δ-hyperbolic graphs by looking at a $O(\delta)$-ball around each node
- Can we approximate $\delta^*(G)$ in such a way ?
- What other global properties can we verify once a BFS has been computed?
  - recognition of Helly graphs and bridged graphs
  - What about other classes of graphs?

Metric classes of graphs

- Can we find other local-to-global characterizations
  - For a class containing weakly modular graphs and basis graphs of matroids?
  - For graphs with convex balls? (△⬠)

Metric classes of graphs

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  - For a class containing weakly modular graphs and basis graphs of matroids?
  - For graphs with convex balls? (△⬠)
- For several classes, we can associate cell complexes of higher dimension and establish some nice properties
  - Can we associate a canonical cell complex of higher dimension to a weakly modular graph?
  - When are the cell complexes contractible?
  - When are the groups acting on such complexes (bi)automatic?

Regular Event Structures

- Nice connections between event structures and NPC complexes
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Thank you! Questions?