The λ-calculus: from simple types to non-idempotent intersection types

Days 4–5: Non-idempotent intersection types for the λ-calculus

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Outline

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The λ-calculus between simple types and the untyped one

The simply typed λ-calculus:

1. has very nice operational properties (e.g. normalization, confluence);
2. has a clear logical meaning (Curry-Howard correspondence);
3. is not very expressive (recursion cannot be represented, Turing-completeness fails).

The untyped λ-calculus:

1. has some very nice properties (e.g. confluence, Turing-completeness);
2. misses some nice properties (e.g. normalization);
3. has no logical meaning;
4. contains diverging terms without any meaning (e.g. $\delta\delta$).

Questions.

1. Is there a more liberal type system which only takes the pros of the two worlds?
2. Can it characterize all and only the "meaningful" terms of the untyped λ-calculus?

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The syntax for non-idempotent intersection types

We fix a countably infinite set of atoms, denoted by $X, Y, Z, \ldots$.

$$
\begin{aligned}
\text{Linear types:} \quad & A, B ::= X \mid M \multimap A \\
\text{Multi types:} \quad & M, N ::= [A_1, \ldots, A_n] \qquad (\text{with } n \in \mathbb{N}) \\
\text{(Non-idempotent intersection) types:} \quad & S, T ::= A \mid M
\end{aligned}
$$

where $[A_1, \ldots, A_n]$ with $n \in \mathbb{N}$ is a finite multiset ($[\,]$ is the empty multiset for $n = 0$).

Idea. $[A_1, \ldots, A_n]$ stands for a conjunction $A_1 \land \cdots \land A_n$ where $\land$ is:

- commutative $A \land B \equiv B \land A$ (multisets do not take order into account);
- associative $A \land (B \land C) \equiv (A \land B) \land C$ (multisets are associative);
- non-idempotent $A \land A \not\equiv A$ (multisets take multiplicites into account).

Def. A judgment is a sequent of the form $\Gamma \vdash t : T$ where

1. t is a term, T is a type, Γ is a type context, that is,
2. Γ is a map from variables to multi types such that the set $\{x \mid \Gamma(x) \neq [\,]\}$ is finite.

Notation. $\uplus$ is the multiset union (e.g. $[A, B] \uplus [A] = [A, A, B] \neq [A, B]$) whose unit is $[\,]$.
Extended to type contexts pointwise: $(\Gamma \uplus \Delta)(x) = \Gamma(x) \uplus \Delta(x)$.

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The typing rules for non-idempotent intersection type system NI

Notation. A context Γ is denoted by $x_1 : M_1, \ldots, x_n : M_n$ if:
variables $x_1, \ldots, x_n$ are pariwise distinct and $\Gamma(x) = \begin{cases} M_i & \text{if } x = x_i \text{ for some } 1 \leq i \leq n, \\ [\,] & \text{otherwise.} \end{cases}$

Typing rules for NI:

$$\frac{}{x : [A] \vdash x : A}\,\text{var}$$

$$\frac{\Gamma, x : M \vdash t : A}{\Gamma \vdash \lambda x.t : M \multimap A}\,\lambda \qquad \frac{\Gamma \vdash s : M \multimap A \qquad \Delta \vdash t : M}{\Gamma \uplus \Delta \vdash st : A}\,@ \qquad \frac{(\Gamma_i \vdash t : A_i)_{1 \leq i \leq n} \qquad n \in \mathbb{N}}{\biguplus_{i=1}^{n} \Gamma_i \vdash t : [A_1, \ldots, A_n]}\,!$$

Idea. A term typed $t : [A, A, B]$ means that, during evaluation, t can be used:

- once as a data of type B, and
- twice as a data of type A.

Notation. $\mathcal{D} \triangleright_{\mathsf{NI}} \Gamma \vdash t : T$ means that $\mathcal{D}$ is a derivation in NI with conclusion $\Gamma \vdash t : T$.
$\Gamma \vdash_{\mathsf{NI}} t : T$ means that there is a derivation $\mathcal{D} \triangleright_{\mathsf{NI}} \Gamma \vdash t : T$.

Rmk. $\vdash_{\mathsf{NI}} t : [\,]$ for every term t (take ! with no premises).

Def. The size $|\mathcal{D}|$ of a derivation $\mathcal{D}$ is the number of its rules, not counting the rules !.
$|\mathcal{D}|_{\text{var}}$ (resp. $|\mathcal{D}|_\lambda$; $|\mathcal{D}|_@$) is the number of rules var (resp. λ; @) in $\mathcal{D}$.

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Some examples of derivations in NI

Ex. Find all the derivations with conclusion $\vdash \lambda x.x : C$, for any linear type C.

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$$\mathcal{D}^I_A = \dfrac{\dfrac{}{x : [A] \vdash x : A}\,\text{var}}{\vdash \lambda x.x : [A] \multimap A}\,\lambda \qquad \text{for any linear type } A.$$

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Ex. Find all the derivations with conclusion $\vdash \lambda x.xx : C$, for any linear type C.

$$\mathcal{D}^{\delta,n}_{A_0,\dots,A_n} = \cfrac{\cfrac{\cfrac{}{x : [[A_1,\dots,A_n] \multimap A] \vdash x : [A_1,\dots,A_n] \multimap A_0}\,\text{var} \quad \cfrac{\left(\cfrac{}{x : [A_i] \vdash x : A_i}\,\text{var}\right)_{1 \leq i \leq n}}{x : [A_1,\dots,A_n] \vdash x : [A_1,\dots,A_n]}\,!}{x : [[A_1,\dots,A_n] \multimap A_0, A_1,\dots,A_n] \vdash xx : A_0}\,@}{\vdash \lambda x.xx : [[A_1,\dots,A_n] \multimap A_0, A_1,\dots,A_n] \multimap A_0}\,\lambda$$

for any $n \in \mathbb{N}$ and any linear types $A_0,\dots,A_n$ (in particular, for $n = 0$, $\vdash \lambda x.x : [[\,] \multimap A_0] \multimap A_0$).

More examples of derivations in NI

Ex. Find all the derivations with conclusion $\vdash (\lambda x.x)\lambda y.y : C$, for any linear type C.

Ex. Find a derivation with conclusion $\vdash (\lambda x.xx)\lambda y.y : C$, for some linear type C.

Rmk. In the derivation $\mathcal{D}_A^{II}$ (resp. $\mathcal{D}_A^{\delta,I}$) the rule ! has 1 premise (resp. 2 premises) because 1 copy (resp. 2 copies) of $\lambda y.y$ is (resp. are) needed in the evaluation $(\lambda x.x)\lambda y.y \to_{h\beta} \lambda y.y$ (resp. $(\lambda x.xx)\lambda y.y \to_{h\beta} (\lambda y.y)\lambda y.y$).

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for any linear type A (actually, all derivations for $(\lambda x.xx)\lambda y.y$ have the form above).

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Oh no! More examples of derivations in NI!

Ex. Find a derivation with conclusion $\vdash (\lambda x.xx)((\lambda y.y)\lambda z.z) : C$, for some linear type C.

Rmk. In the derivation $\mathcal{D}_A^{\delta,II}$, the rule ! has 2 premises because 2 copies of $(\lambda y.y)\lambda z.z$ are needed in the evaluation $(\lambda x.xx)((\lambda y.y)\lambda z.z) \to_{h\beta} ((\lambda y.y)\lambda z.z)((\lambda y.y)\lambda z.z)$.

In turn, in each of the derivations $\mathcal{D}^{II}_{[[A]\multimap A]\multimap[A]\multimap A}$ and $\mathcal{D}^{II}_{[A]\multimap A}$ the rule ! has 2 premises, hence the derivation $\mathcal{D}_A^{\delta,II}$ has 4 subderivations with conclusion $\lambda x.x$, because 4 copies of $\lambda x.x$ are needed in the evaluation $(\lambda x.xx)((\lambda y.y)\lambda z.z) \to_{h\beta} ((\lambda y.y)\lambda z.z)((\lambda y.y)\lambda z.z)$.

Ex. Find a derivation with conclusion $\vdash (\lambda x.xx)\lambda y.yy : C$, for some linear type C.

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Ex. Find a derivation with conclusion $\vdash (\lambda x.xx)((\lambda y.y)\lambda z.z) : C$, for some linear type C.

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Good luck!

Enough with the examples of derivations, old man!

Ex. Find all the derivations with conclusion $\lambda x.x((\lambda y.yy)\lambda z.zz) : C$ for any linear type C.

Ex. Find a derivation for $A = \lambda a.\lambda f.f(aaf)$ and one for $\Theta = AA$.

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This is a good exercise, old man!

Outline

What can we do with non-idempotent intersection types?

Goal. We want to characterize the all and only the $h\beta$-normalizing terms via NI.

Motivation. There are many theoretical reasons to say "meaningful" = $h\beta$-normalizing.

To achieve this qualitative characterization, we need to prove two properties.

1. Correctness: if a term is typable in NI then it is $h\beta$-normalizing.
2. Completeness: if a term is $h\beta$-normalizing then it is typable in NI.

Bonus. We can extract some quantitative information from NI about:

1. the length of evaluation (the number of $h\beta$-steps to reach the $h\beta$-normal form);
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Ingredients to prove correctness

Def. The head size $|t|_{h\beta}$ of a term t is defined by induction on t as follows:

$$|x|_{h\beta} = 0 \qquad |\lambda x.t|_{h\beta} = 1 + |t|_{h\beta} \qquad |st|_{h\beta} = 1 + |s|_{h\beta}$$

Lemma (Typing $h\beta$-normal forms)

Let t be $h\beta$-normal. If $\mathcal{D} \triangleright_{\mathsf{NI}} \Gamma \vdash t : A$ then $|t|_{h\beta} \leq |\mathcal{D}|$.

Proof. Every $h\beta$-normal term is of the form $t = \lambda x_n. \dots \lambda x_1.yt_1 \dots t_m$ for some $m, n \in \mathbb{N}$. The lemma is proved by induction on $(m, n) \in \mathbb{N}^2$ with the lexicographical order. □

Notation. For a finite multiset M over a set X, its cardinality is $|M| = \sum_{x \in X} M(x) \in \mathbb{N}$.

Lemma (Substitution)

If $\mathcal{D} \triangleright_{\mathsf{NI}} \Gamma, x : M \vdash t : A$ and $\mathcal{D}' \triangleright_{\mathsf{NI}} \Delta \vdash s : M$, then there is $\mathcal{D}'' \triangleright_{\mathsf{NI}} \Gamma \uplus \Delta \vdash t\{s/x\} : A$ with $|\mathcal{D}''| = |\mathcal{D}| + |\mathcal{D}'| - |M|$.

Proof. By structural induction on $\mathcal{D}$. The base case is when the last rule of $\mathcal{D}$ is var. The other cases follow easily from the inductive hypothesis. □

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A graphical view to the substitution lemma

Like natural deduction, derivations in NI can be depicted by a tree-like structure where:

- edges are labeled by typed terms, nodes are the typing rules,
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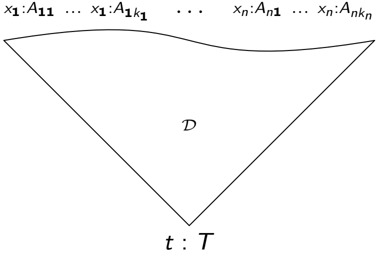

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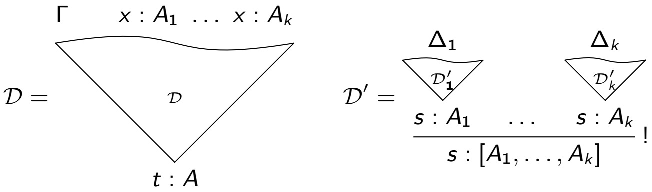

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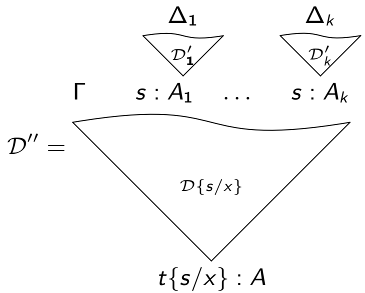

Correctness of NI: typability implies $h\beta$-normalization

Proposition (Quantitative subject reduction)

If $\mathcal{D} \triangleright_{\mathsf{NI}} \Gamma \vdash t : A$ and $t \to_{h\beta} t'$, then there is $\mathcal{D}' \triangleright_{\mathsf{NI}} \Gamma \vdash t' : A$ with $|\mathcal{D}| > |\mathcal{D}'|$.

Proof. By induction on the definition $t \to_{h\beta} t'$. The only non-trivial case is when $t = (\lambda x.u)s \to_{h\beta} u\{s/x\} = t'$: then, $\mathcal{D}$ must have the form below, with $\Gamma = \Gamma' \uplus \Gamma''$.

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Rmk. The quantitative aspect of subject reduction (i.e. $|\mathcal{D}| > |\mathcal{D}'|$) is false:

- if $t \to_{\beta} t'$ instead of $t \to_{h\beta} t'$, e.g. $\lambda x.x(\delta\delta) \to_{\beta} \lambda x.x(\delta\delta)$ with $\delta = \lambda z.zz$, see p. 10;
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If $\mathcal{D} \triangleright_{\mathsf{NI}} \Gamma \vdash t : A$ then there is s $h\beta$-normal such that $t \overbrace{\to_{h\beta} \cdots \to_{h\beta}}^{k\ h\beta\text{-steps}} s$ and $|\mathcal{D}| \geq k + |s|_{h\beta}$.

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Proof. By induction on $|\mathcal{D}|$. If t is $h\beta$-normal, then the claim follows from the lemma about typing $h\beta$-normal forms, taking $s = t$ and $k = 0$.

Correctness of NI: typability implies $h\beta$-normalization

Proposition (Quantitative subject reduction)

If $\mathcal{D} \rhd_{\mathsf{NI}} \Gamma \vdash t : A$ and $t \to_{h\beta} t'$, then there is $\mathcal{D}' \rhd_{\mathsf{NI}} \Gamma \vdash t' : A$ with $|\mathcal{D}| > |\mathcal{D}'|$.

Proof. By induction on the definition $t \to_{h\beta} t'$. The only non-trivial case is when $t = (\lambda x.u)s \to_{h\beta} u\{s/x\} = t'$: then, $\mathcal{D}$ must have the form below, with $\Gamma = \Gamma' \uplus \Gamma''$.

$$\mathcal{D} = \dfrac{\dfrac{\dfrac{\vdots\, \mathcal{D}_u}{\Gamma', x : M \vdash u : A}}{\Gamma' \vdash \lambda x.u : M \multimap A}\,\lambda \qquad \dfrac{\vdots\, \mathcal{D}_s}{\Gamma'' \vdash s : M}}{\Gamma' \uplus \Gamma'' \vdash (\lambda x.u)s : A}\,@$$

By substitution lemma, there is $\mathcal{D}' \rhd_{\mathsf{NI}} \Gamma \vdash u\{s/x\} : A$ with $|\mathcal{D}'| = |\mathcal{D}_u| + |\mathcal{D}_s| - |M| < |\mathcal{D}_u| + |\mathcal{D}_s| + 2 = |\mathcal{D}|$. □

Rmk. The quantitative aspect of subject reduction (i.e. $|\mathcal{D}| > |\mathcal{D}'|$) is false:

- if $t \to_{\beta} t'$ instead of $t \to_{h\beta} t'$, e.g. $\lambda x.x(\delta\delta) \to_{\beta} \lambda x.x(\delta\delta)$ with $\delta = \lambda z.zz$, see p. 10;
- if $\mathcal{D}$ and $\mathcal{D}'$ are derivations in the simply typed λ-calculus, instead of NI.

Theorem (Correctness of NI)

If $\mathcal{D} \rhd_{\mathsf{NI}} \Gamma \vdash t : A$ then there is s $h\beta$-normal such that $t \overbrace{\to_{h\beta} \cdots \to_{h\beta}}^{k\ h\beta\text{-steps}} s$ and $|\mathcal{D}| \geq k + |s|_{h\beta}$.

Proof. By induction on $|\mathcal{D}|$. If t is $h\beta$-normal, then the claim follows from the lemma about typing $h\beta$-normal forms, taking $s = t$ and $k = 0$.

Otherwise, $t \to_{h\beta} t'$ and by quantitative subject reduction there is $\mathcal{D}' \rhd_{\mathsf{NI}} \Gamma \vdash t' : A$ with $|\mathcal{D}| > |\mathcal{D}'|$. By induction hypothesis, $t' \to_{h\beta}^* s$ in k $h\beta$-steps for some $h\beta$-normal s with $|\mathcal{D}'| \geq k + |s|_{h\beta}$. Hence, $t \to_{h\beta}^* s$ in $k{+}1$ $h\beta$-steps and $|\mathcal{D}| \geq |\mathcal{D}'| + 1 \geq k + 1 + |s|_{h\beta}$. □

Ingredients to prove completeness

Rmk. Completeness is the converse of correctness, so their needed ingredients are "dual".

Lemma (Typability of $h\beta$-normal forms)

If t is $h\beta$-normal, then there is $\mathcal{D} \rhd_{\mathsf{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| = |t|_{h\beta} + 1 = |\mathcal{D}| + |\mathcal{D}|_{\mathsf{var}}$.

Proof. Every $h\beta$-normal term is of the form $t = \lambda x_n. \dots \lambda x_1.yt_1 \dots t_m$ for some $m, n \in \mathbb{N}$. For $n = 0$, we prove (by induction on $m \in \mathbb{N}$) the stronger property that, for every $k \in \mathbb{N}$ and formula A, there is $\mathcal{D} \rhd_{\mathsf{NI}} y : [A_k] \vdash yt_1 \dots t_m : A_k$ with $|\mathcal{D}| = m + 1 = m + |\mathcal{D}|_{\mathsf{var}}$ and

$$A_k = \overbrace{[\,] \multimap \cdots \multimap [\,]}^{k \text{ times } [\,]} \multimap A \qquad \text{(note that } |yt_1 \dots t_m|_{h\beta} = m\text{)}.$$

The statement of the lemma is then proved by induction on $n \in \mathbb{N}$. □

Lemma (Anti-substitution)

If $\mathcal{D} \rhd_{\mathsf{NI}} \Gamma \vdash t\{s/u\} : A$, then there are contexts Γ' and Γ'', a multi type M and derivations $\mathcal{D}' \rhd_{\mathsf{NI}} \Gamma', x : M \vdash t : A$ and $\mathcal{D}'' \rhd_{\mathsf{NI}} \Gamma'' \vdash s : M$ such that $\Gamma = \Gamma' \uplus \Gamma''$ and $|\mathcal{D}| = |\mathcal{D}'| + |\mathcal{D}''| - |M|$.

Proof. By structural induction on t. The base case is when t is a variable (either x or other than x). The other cases follow easily from the inductive hypothesis. □

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Proof. By structural induction on t. The base case is when t is a variable (either x or other than x). The other cases follow easily from the inductive hypothesis. □

Completeness of NI: $h\beta$-normalization implies typability

Proposition (Quantitative subject expansion)

If $\mathcal{D}' \rhd_{\mathsf{NI}} \Gamma \vdash t' : A$ and $t \to_{h\beta} t'$, then there is $\mathcal{D} \rhd_{\mathsf{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| > |\mathcal{D}'|$.

Proof. By induction on the definition $t \to_{h\beta} t'$. The only non-trivial case is when $t = (\lambda x.u)s \to_{h\beta} u\{s/x\} = t'$: by the anti-substitution lemma, since $\mathcal{D}' \rhd_{\mathsf{NI}} \Gamma \vdash t' : A$,

$$\mathcal{D} = \dfrac{\dfrac{\dfrac{\vdots\, \mathcal{D}_u}{\Gamma' x : M \vdash u : A}}{\Gamma' \vdash \lambda x.u : M \multimap A}\lambda \qquad \dfrac{\vdots\, \mathcal{D}_s}{\Gamma'' \vdash s : M}}{\Gamma' \uplus \Gamma'' \vdash (\lambda x.u)s : A}@$$

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Rmk. We have seen (in day 1) that subject expansion fails with simple types.

Notation. Given $k \in \mathbb{N}$, we write $t \to_{h\beta}^{k} s$ if $t \overbrace{\to_{h\beta} \cdots \to_{h\beta}}^{k\ h\beta\text{-steps}} s$ (thus $t \to_{h\beta}^{0} s$ means $t = s$).

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If $t \to_{h\beta}^{k} s$ with s $h\beta$-normal, then there is $\mathcal{D} \rhd_{\mathsf{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| \geq k + |s|_{h\beta}$.

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Completeness of NI: $h\beta$-normalization implies typability

Proposition (Quantitative subject expansion)

If $\mathcal{D}' \rhd_{\mathsf{NI}} \Gamma \vdash t' : A$ and $t \to_{h\beta} t'$, then there is $\mathcal{D} \rhd_{\mathsf{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| > |\mathcal{D}'|$.

Proof. By induction on the definition $t \to_{h\beta} t'$. The only non-trivial case is when $t = (\lambda x.u)s \to_{h\beta} u\{s/x\} = t'$: by the anti-substitution lemma, since $\mathcal{D}' \rhd_{\mathsf{NI}} \Gamma \vdash t' : A$,

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Completeness of NI: $h\beta$-normalization implies typability

Proposition (Quantitative subject expansion)

If $\mathcal{D}' \rhd_{\mathsf{NI}} \Gamma \vdash t' : A$ and $t \to_{h\beta} t'$, then there is $\mathcal{D} \rhd_{\mathsf{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| > |\mathcal{D}'|$.

Proof. By induction on the definition $t \to_{h\beta} t'$. The only non-trivial case is when $t = (\lambda x.u)s \to_{h\beta} u\{s/x\} = t'$: by the anti-substitution lemma, since $\mathcal{D}' \rhd_{\mathsf{NI}} \Gamma \vdash t' : A$,

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If $t \to_{h\beta}^{k} s$ with s $h\beta$-normal, then there is $\mathcal{D} \rhd_{\mathsf{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| \geq k + |s|_{h\beta}$.

Proof. By induction on $k \in \mathbb{N}$. If $k = 0$, then $t = s$ and typability of $h\beta$-normal concludes.

Completeness of NI: $h\beta$-normalization implies typability

Proposition (Quantitative subject expansion)

If $\mathcal{D}' \rhd_{\mathsf{NI}} \Gamma \vdash t' : A$ and $t \to_{h\beta} t'$, then there is $\mathcal{D} \rhd_{\mathsf{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| > |\mathcal{D}'|$.

Proof. By induction on the definition $t \to_{h\beta} t'$. The only non-trivial case is when $t = (\lambda x.u)s \to_{h\beta} u\{s/x\} = t'$: by the anti-substitution lemma, since $\mathcal{D}' \rhd_{\mathsf{NI}} \Gamma \vdash t' : A$,

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Proof. By induction on $k \in \mathbb{N}$. If $k = 0$, then $t = s$ and typability of $h\beta$-normal concludes. Otherwise $k > 0$ and $t \to_{h\beta} t' \to_{h\beta}^{k-1} s$. By induction hypothesis, there is $\mathcal{D}' \rhd_{\mathsf{NI}} \Gamma \vdash t' : A$ with $|\mathcal{D}'| \geq k - 1 + |s|_{h\beta}$. By quantitative subject expansion, there is $\mathcal{D} \rhd_{\mathsf{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| > |\mathcal{D}'|$, therefore $|\mathcal{D}| \geq |\mathcal{D}'| + 1 \geq k + |s|_{h\beta}$. □

Summing up: characterization of head normalization

Putting together correctness and completeness of NI, we obtain:

Corollary (Characterization of head normalization)

A term t is $h\beta$-normalizing if and only if there is $\mathcal{D} \triangleright_{\mathsf{NI}} \Gamma \vdash t : A$. Moreover, $|\mathcal{D}| \geq k + |s|_{h\beta}$ if $t \to_{h\beta}^{k} s$ with s $h\beta$-normal.

Rmk. The quantitative information about

- the length k of evaluation (head reduction) from t to its $h\beta$-normal form s, and
- the head size $|s|_{h\beta}$ of the $h\beta$-normal term s

are in the size $|\mathcal{D}|$ of $\mathcal{D}$ without performing head reduction $\to_{h\beta}$ or knowing s.

Rmk. $|\mathcal{D}|$ is an upper bound to k plus $|s|_{h\beta}$ together. NI can be refined so that one can:

1. disentangle the information about k and $|s|_{h\beta}$ by means of two different sizes of $\mathcal{D}$,
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Outline

Bibliography

- For an (almost gentle) introduction to non-idempotent intersection types:
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  - Beniamino Accattoli, Stéphan Graham-Lengrand, Delia Kesner. *Tight typings and split bounds, fully developed*. Journal of Functional Programming, vol. 30, 14 pages, 2020. https://doi.org/10.1017/S095679682000012X