# The $\lambda$-calculus: from simple types to non-idempotent intersection types

## Day 1: The simply typed $\lambda$-calculus and the Curry-Howard correspondence


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# Outline

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# Objectives of the course

This is an introductory course to the $\lambda$-calculus and to its links to proof-theory.

- I present the simply typed $\lambda$-calculus as a shorthand for natural deduction in minimal logic (Curry-Howard correspondence).
- I forget the logical content of the simply typed $\lambda$-calculus and we get the untyped $\lambda$-calculus, a Turing-complete model of computation.
- I introduce a more liberal typing system, non-idempotent intersection types, to characterize termination of evaluation in the untyped $\lambda$-calculus.
- I extract some quantitative information from non-idempotent intersection type system, such the length of the evaluation or the size of the result.
- I show that non-idempotent intersection types provide a denotational semantics for the untyped $\lambda$-calculus and we study the type inhabitation problem.

I do not assume any knowledge on natural deduction, $\lambda$-calculus and type systems. But a familiarity with propositional logic and proofs by induction is welcome!

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# The day-by-day plan of the course

1. Day 1: The simply typed $\lambda$-calculus and the Curry-Howard correspondence.
2. Day 2: The untyped $\lambda$-calculus.
3. Day 3: Non-idempotent intersection types for the $\lambda$-calculus.
4. Day 4: Quantitative information from non-idempotent intersection types.
5. Day 5: Denotational semantics and the type inhabitation problem.

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- If you already know natural deduction and its normalization, skip day 1.
- If you already know the $\lambda$-calculus, skip day 2.
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# Other courses of interest at ESSLLI (more or less related to this one)

1. Beniamino Accattoli: *Time and space for the $\lambda$-calculus*.
   A natural (and more advanced) continuation of my course.
2. Matteo Acclavio & Paolo Pistone: *An introduction to proof equivalence*.
   A deep look at a fundamental question in proof-theory
3. Anupam Das: *Proof theory of arithmetic*.
   An advanced course about consistency of arithmetic via proof-theoretic means.

# Outline

# Minimal logic: the implicational fragment of propositional intuitionistic logic

Language of minimal logic: implicational fragment of propositional intuitionistic logic.

Def. Given a countably infinite set of propositional variables, denoted by $X, Y, Z, \dots$, formulas are defined by the BNF grammar below:

$$A, B, C ::= X \mid (A \Rightarrow B)$$

This is a shorthand for an inductive definition of the set of formulas. That is:

- Every propositional variable is a formula.
- If $A$ and $B$ are formulas, then $(A \Rightarrow B)$ is a formula (called implication).
- Nothing else is a formula.

Notation.

- The outermost parentheses are often omitted: $A \Rightarrow B \coloneqq (A \Rightarrow B)$.
- $\Rightarrow$ is right-associative: $A \Rightarrow B \Rightarrow C \coloneqq (A \Rightarrow (B \Rightarrow C))$.

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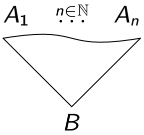
# Natural deduction for minimal logic, informally

Natural deduction (ND) is a formalism to represent proofs in minimal logic (and others).

A proof in ND is a finite, vaguely tree-like structure (this is more a graphical illusion):

- edges are labeled by formulas, nodes are inference rules $\dfrac{A_1 \quad \overset{n\in\mathbb{N}}{\ldots} \quad A_n}{B}$ ;
- leaves are hypotheses (they are finitely many, possibly none) or dead leaves;
- the root is the (unique) conclusion.

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- Syntax-directed: By the tree-like structure, the last rule depends on the conclusion.

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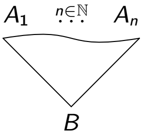


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# Natural deduction for minimal logic, slightly more formally

**Notation.** $\begin{array}{c}\vdots\,\mathcal{D}\\ B\end{array}$ means that $\mathcal{D}$ is a derivation with conclusion $B$ and some hypotheses.

**Def.** A derivation $\mathcal{D}$ in ND is

- either $A$ (for any formula $A$), which is both the conclusion and the hypothesis of $\mathcal{D}$;
- or it is obtained from derivations $\mathcal{D}'$, $\mathcal{D}_1$, $\mathcal{D}_2$ by applying one of the inference rules

$$\dfrac{\begin{array}{c}\vdots\,\mathcal{D}_1\\ A \Rightarrow B\end{array} \quad \begin{array}{c}\vdots\,\mathcal{D}_2\\ A\end{array}}{B}\Rightarrow_e \qquad\qquad \dfrac{\begin{array}{c}[A]^*\\ \vdots\,\mathcal{D}'\\ B\end{array}}{A \Rightarrow B}\Rightarrow_i^*$$

$\Rightarrow$ elimination $\qquad\qquad$ $\Rightarrow$ introduction

where the hypotheses of $\mathcal{D}$ are

- in $\Rightarrow_e$, the union of the ones of $\mathcal{D}_1$ and $\mathcal{D}_2$,
- in $\Rightarrow_i$, the ones of $\mathcal{D}'$ minus an arbitrary number (possibly 0) of occurrences of $A$.

**Rmk.** ND marks when an hypothesis is discharged (becoming a dead leaf) by a $\Rightarrow_i$.

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# Examples of derivations in ND

Notation. $A_1, \ldots, A_n \vdash B$ means that there is a derivation in ND with conclusion $B$ and hypothesis *among* $A_1, \ldots, A_n$.

1. Prove that $A \vdash A$, and $\vdash A \Rightarrow A$, and $B \vdash A \Rightarrow A$, and $\vdash A \Rightarrow B \Rightarrow A$.

2. Give two distinct derivations of $A \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.

3. Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

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$$A \qquad\qquad \dfrac{[A]^*}{A \Rightarrow A}\Rightarrow_i^* \qquad\qquad \dfrac{[A]^*}{A \Rightarrow A}\Rightarrow_i^* \qquad\qquad \dfrac{\dfrac{[A]^*}{B \Rightarrow A}\Rightarrow_i}{A \Rightarrow B \Rightarrow A}\Rightarrow_i^*$$

2. Give two distinct derivations of $A \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.

$$\dfrac{A}{A \Rightarrow A}\Rightarrow_i \qquad \dfrac{[A]^*}{A \Rightarrow A}\Rightarrow_i^* \qquad\qquad \dfrac{\dfrac{[A]^*}{A \Rightarrow A}\Rightarrow_i}{A \Rightarrow A \Rightarrow A}\Rightarrow_i^* \qquad \dfrac{\dfrac{[A]^*}{A \Rightarrow A}\Rightarrow_i^*}{A \Rightarrow A \Rightarrow A}\Rightarrow_i$$

3. Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{[B \Rightarrow C]^* \qquad \dfrac{[A]^\circ \qquad [A \Rightarrow B]^\dagger}{B}\Rightarrow_e}{C}\Rightarrow_e}{A \Rightarrow C}\Rightarrow_i^\circ}{(B \Rightarrow C) \Rightarrow A \Rightarrow C}\Rightarrow_i^*}{(A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C}\Rightarrow_i^\dagger$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A]^\circ \qquad [A \Rightarrow (B \Rightarrow C)]^\dagger}{B \Rightarrow C}\Rightarrow_e \qquad \dfrac{[A \Rightarrow B]^* \qquad [A]^\circ}{B}\Rightarrow_e}{C}\Rightarrow_e}{A \Rightarrow C}\Rightarrow_i^\circ}{(A \Rightarrow B) \Rightarrow (A \Rightarrow C)}\Rightarrow_i^*}{(A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)}\Rightarrow_i^\dagger$$

# Soundness and completeness of ND with respect to minimal logic

Def. A (finite) multiset over a set $X$ is a (finite) set of occurrences of elements of $X$.
Idea. A multiset takes into account the number of copies (not the order) of its elements.

Notation. Given a finite multiset $\Gamma = A_1, \ldots, A_n$ of formulas, with $n \in \mathbb{N}$ ($\Gamma = \emptyset$ if n = 0)

- $\mathcal{D} \triangleright_{\mathrm{ND}} \Gamma \vdash A$ means that $\mathcal{D}$ is a derivation with conclusion $A$ and hypotheses among the formulas in $\Gamma$;
- $\Gamma \vdash A$ means that there is a derivation $\mathcal{D} \triangleright_{\mathrm{ND}} \Gamma \vdash A$.

**Theorem (Soundness and completeness)**

$A_1, \ldots, A_n \vdash B$ if and only if $A_1 \Rightarrow \cdots \Rightarrow A_n \Rightarrow B$ is valid in minimal logic.

Proof. Omitted. □

Moral. The syntactic approach (ND) is equivalent to the semantic one (Kripke models).

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Moral. The syntactic approach (ND) is equivalent to the semantic one (Kripke models).

# An alternative presentation of ND via sequents

Def. A sequent is a pair $\Gamma \vdash A$ of a finite multiset $\Gamma$ of formulas and a formula $A$.

Def. A derivation in $\mathrm{ND_{seq}}$ is a tree built up from the inference rules below.

$$\frac{}{A \vdash A}\,\mathrm{ax} \qquad \frac{\Gamma \vdash A \Rightarrow B \qquad \Gamma \vdash A}{\Gamma \vdash B}\Rightarrow_e \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\Rightarrow_i$$

Notation. $\mathcal{D} \rhd_{\mathrm{ND_{seq}}} \Gamma \vdash A$ means that $\mathcal{D}$ is a derivation in $\mathrm{ND_{seq}}$ with conclusion $\Gamma \vdash A$.

**Proposition**

$\Gamma \vdash A$ in ND if and only if $\Gamma \vdash A$ is derivable in $\mathrm{ND_{seq}}$.

Proof. Every $\mathcal{D} \rhd_{\mathrm{ND}} \Gamma \vdash A$ can be translated into a $\mathcal{D}' \rhd_{\mathrm{ND_{seq}}} \Gamma \vdash A$ and vice versa.

$$A \rightsquigarrow \frac{}{\Gamma, A \vdash A}\,\mathrm{ax} \qquad \frac{\begin{matrix}\Gamma \\ \vdots\,\mathcal{D}_1 \\ A \Rightarrow B\end{matrix} \quad \begin{matrix}\Gamma \\ \vdots\,\mathcal{D}_2 \\ A\end{matrix}}{B}\Rightarrow_e \rightsquigarrow \frac{\begin{matrix}\mathcal{D}'_1 \\ \Gamma \vdash A \Rightarrow B\end{matrix} \quad \begin{matrix}\mathcal{D}'_2 \\ \Gamma \vdash A\end{matrix}}{\Gamma \vdash B}\Rightarrow_e \qquad \frac{\begin{matrix}[A]^* \\ \vdots\,\mathcal{D}_1 \\ B\end{matrix}}{A \Rightarrow B}\Rightarrow_i^* \rightsquigarrow \frac{\begin{matrix}\mathcal{D}'_1 \\ \Gamma, A \vdash B\end{matrix}}{\Gamma \vdash A \Rightarrow B}\Rightarrow_i$$

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$$A \leadsto \frac{}{\Gamma, A \vdash A}\,\mathsf{ax}$$

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3. Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

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# Outline

# The inversion principle in natural deduction

Def. Let $\mathcal{D}$ a derivation in ND.

- A cut-formula is a formula in $\mathcal{D}$ that is conclusion of a $\Rightarrow_i$ and left premise of a $\Rightarrow_e$.
- A redex is a pair $\Rightarrow_i/\Rightarrow_e$ containing a cut-formula.

Inversion principle. A redex proving $B$ by means of $\Rightarrow_e$, having proved its premises $A \Rightarrow B$ and $A$, the former by means of $\Rightarrow_i$ with a proof of $B$ from $A$, amounts to concatenate a proof of $A$ with a proof of $B$ from $A$ (substitution of hypotheses $A$ for a derivation of $A$).

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Mathematically, "amounts to" can be seen as a rewrite relation $\rightarrow_{\text{cut}}$ (cut-elimination).

Rmk. All the discharged hypotheses are replaced by (copies of) the derivation of $A$.

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# Examples of cut-elimination steps in ND

The cut-formula is in blue.

$$\dfrac{\dfrac{[X \Rightarrow X]^*}{(X \Rightarrow X) \Rightarrow X \Rightarrow X}\Rightarrow_i^* \qquad \dfrac{[X]^\dagger}{X \Rightarrow X}\Rightarrow_i^\dagger}{X \Rightarrow X}\Rightarrow_e \quad \to_{\text{cut}} \quad \dfrac{[X]^\dagger}{X \Rightarrow X}\Rightarrow_i^\dagger$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A \Rightarrow (B \Rightarrow A)]^\dagger \quad [A]^\circ}{B \Rightarrow A}\Rightarrow_e \quad \dfrac{[A \Rightarrow B]^* \quad [A]^\circ}{B}\Rightarrow_e}{A}\Rightarrow_e}{A \Rightarrow A}\Rightarrow_i^\circ}{(A \Rightarrow B) \Rightarrow (A \Rightarrow A)}\Rightarrow_i^*}{(A \Rightarrow (B \Rightarrow A)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow A)}\Rightarrow_i^\dagger \quad \dfrac{\dfrac{[A]^\dagger}{B \Rightarrow A}\Rightarrow_i}{A \Rightarrow (B \Rightarrow A)}\Rightarrow_i^\dagger}{(A \Rightarrow B) \Rightarrow (A \Rightarrow A)}\Rightarrow_e \quad \to_{\text{cut}} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A]^\dagger}{B \Rightarrow A}\Rightarrow_i}{A \Rightarrow (B \Rightarrow A)}\Rightarrow_i^\dagger \quad [A]^\circ}{B \Rightarrow A}\Rightarrow_e \quad \dfrac{[A \Rightarrow B]^* \quad [A]^\circ}{B}\Rightarrow_e}{A}\Rightarrow_e}{A \Rightarrow A}\Rightarrow_i^\circ}{(A \Rightarrow B) \Rightarrow (A \Rightarrow A)}\Rightarrow_i^*$$

# Examples of cut-elimination steps in ND

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# Other examples of cut-elimination steps in ND

The cut-formula is in blue.

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X) \quad [X \Rightarrow X]^{\circ}}{B \Rightarrow X \Rightarrow X}\Rightarrow_e
      \quad
      \cfrac{(X \Rightarrow X) \Rightarrow B \quad [X \Rightarrow X]^{\circ}}{B}\Rightarrow_e
    }{X \Rightarrow X}\Rightarrow_e
  }{\color{blue}{(X \Rightarrow X) \Rightarrow X \Rightarrow X}}\Rightarrow_i^{\circ}
  \quad
  \cfrac{[X]^{*}}{X \Rightarrow X}\Rightarrow_i^{*}
}{X \Rightarrow X}\Rightarrow_e
$$

$$\downarrow_{\text{cut}}$$

$$
\cfrac{
  \cfrac{(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X) \quad \cfrac{[X]^{*}}{X \Rightarrow X}\Rightarrow_i^{*}}{B \Rightarrow X \Rightarrow X}\Rightarrow_e
  \quad
  \cfrac{(X \Rightarrow X) \Rightarrow B \quad \cfrac{[X]^{*}}{X \Rightarrow X}\Rightarrow_i^{*}}{B}\Rightarrow_e
}{X \Rightarrow X}\Rightarrow_e
$$

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$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X) \quad [X \Rightarrow X]^{\circ}}{B \Rightarrow X \Rightarrow X}{\Rightarrow_e}
      \qquad
      \dfrac{(X \Rightarrow X) \Rightarrow B \quad [X \Rightarrow X]^{\circ}}{B}{\Rightarrow_e}
    }{X \Rightarrow X}{\Rightarrow_e}
  }{\color{blue}(X \Rightarrow X) \Rightarrow X \Rightarrow X}{\Rightarrow_i^{\circ}}
  \qquad
  \dfrac{[X]^{*}}{X \Rightarrow X}{\Rightarrow_i^{*}}
}{X \Rightarrow X}{\Rightarrow_e}
$$

$$\downarrow_{\text{cut}}$$

$$
\dfrac{
  \dfrac{(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X) \quad \color{red}\dfrac{[X]^{*}}{X \Rightarrow X}{\Rightarrow_i^{*}}}{B \Rightarrow X \Rightarrow X}{\Rightarrow_e}
  \qquad
  \dfrac{(X \Rightarrow X) \Rightarrow B \quad \color{red}\dfrac{[X]^{*}}{X \Rightarrow X}{\Rightarrow_i^{*}}}{B}{\Rightarrow_e}
}{X \Rightarrow X}{\Rightarrow_e}
$$

# Cut-elimination (aka normalization) theorem in natural deduction

Def. The size of a formula $A$ is the number of occurrences of $\Rightarrow$ in $A$.
The weight of a redex is the size of its cut-formula.
The weight $w(\mathcal{D})$ of a derivation $\mathcal{D}$ is the finite multiset of the weights of its redexes.

Rmk. A multiset over a set $S$ can be seen as a function $m \colon X \to \mathbb{N}$.

Idea. $m(x) \in \mathbb{N}$ is the multiplicity of $x$, the number of copies of $x$ in the multiset $m$.

Def. Let $(S, \prec)$ be an ordered set and $m, n$ be multisets over $S$: $m \prec_{mul} n$ if $m \neq n$ and for all $x \in S$ such that $n(x) < m(x)$ there is $y \in S$ such that $x \prec y$ and $m(y) < n(y)$.

Ex. $[1, 2, 2, 3, 3, 3] \prec_{mul} [2, 3, 3, 3, 3]$.

Prop. If $(S, \prec)$ is well-ordered, then $\prec_{mul}$ is well-ordered.

**Theorem (Cut-elimination [Gentzen 1936, Prawitz 1965])**

If $\mathcal{D} \rhd_{\mathsf{ND}} \Gamma \vdash A$, then there is $\mathcal{D}' \rhd_{\mathsf{ND}} \Gamma \vdash A$ without redexes such that $\mathcal{D} \to^*_{\mathsf{cut}} \mathcal{D}'$.

Proof. If $\mathcal{D}$ is without redexes, we are done. Otherwise, take a redex $r$ in $\mathcal{D}$ such that there are no redexes above the $\Rightarrow_e$ in $r$ (such a $r$ exists because $\mathcal{D}$ is finite!). Apply $\to_{\mathsf{cut}}$ to $r$ to get $\mathcal{D}_1 \rhd_{\mathsf{ND}} \Gamma \vdash A$ where redexes are not duplicated (as $r$ is an uppermost redex), new redexes can be created but have a lower weight (smaller cut-formula). Therefore, $w(\mathcal{D}) \succ_{mul} w(\mathcal{D}_1)$. By induction hypothesis on the weight of derivations, we conclude. □

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# Some consequences of cut-elimination

Prop. If $\mathcal{D} \triangleright_{\text{ND}} \Gamma \vdash A$ is without redexes, then in $\mathcal{D}$ there are only subformulas of $\Gamma$ or $A$.

**Corollary (Subformula property)**

If $\Gamma \vdash A$ in ND then there is $\mathcal{D} \triangleright_{\text{ND}} \Gamma \vdash A$ only containing subformulas of $\Gamma$ and $A$.

Proof. By cut-elimination, there is $\mathcal{D}$ with no redexes. By Prop. above, we conclude. □

Moral. If you are searching for a $\mathcal{D} \triangleright_{\text{ND}} \Gamma \vdash A$, just look at the subformulas of $\Gamma$ and $A$.

**Corollary (Consistency of ND)**

Some formulas are not provable in ND.

Proof. $\nvdash X$ in ND, otherwise there would be $\mathcal{D} \triangleright_{\text{ND}} \vdash X$ with the subformula property by Cor. above, but the last rule of $\mathcal{D}$ could neither be $\Rightarrow_i$ (because $X$ is not an implication) nor $\Rightarrow_e$ (by the subformula property) nor an hypothesis (since $\mathcal{D}$ has no hypotheses). □

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# Outline

# A computational interpretation of ND

Idea. A derivation $\mathcal{D} \triangleright_{\mathsf{ND}} A_1, \ldots, A_n \vdash B$ can be seen as a function $t(x_1, \ldots, x_n)$ that associates with derivations $\mathcal{D}_i \triangleright_{\mathsf{ND}} \vdash A_i$ a derivation $t(\mathcal{D}_1/x_1, \ldots, \mathcal{D}_n/x_n) \triangleright_{\mathsf{ND}} \vdash B$.

- A derivation consisting of a single hypothesis $A$ is represented by a variable $x$. Different formulas are associated with different variables. For several occurrences of $A$ as hypotheses, we chose the same $x$ or another variable. $\rightsquigarrow$ We work with parcel of hypotheses (of the same formula).
- If $\mathcal{D}$ ends in $\Rightarrow_i$ let $s(x, y_1, \ldots, y_n)$ be the function associated with the $\Rightarrow_i$-premise. Let $x$ be the variable associated with the parcel of hypotheses $A$ discharged by $\Rightarrow_i$. The function $t(y_1, \ldots, y_n)$ associated with $\mathcal{D}$ maps $\mathcal{D}' \triangleright_{\mathsf{ND}} \vdash A$ to $s(\mathcal{D}'/x, y_1, \ldots, y_n)$.

$$(\text{abstraction}) \quad t(y_1, \ldots, y_n) \coloneqq \lambda x.s(x, y_1, \ldots, y_n) \quad (\text{i.e. } x \mapsto s(x, y_1, \ldots, y_n))$$

- If $\mathcal{D}$ ends in $\Rightarrow_e$, let $s_1(x_1, \ldots, x_n)$ and $s_2(x_1, \ldots, x_n)$ be the functions associated with the two premises of $\Rightarrow_e$. The function $t(x_1, \ldots, x_n)$ associated with $\mathcal{D}$ is the application (noted as juxtaposition) of $s_1(x_1, \ldots, x_n)$ to $s_2(x_1, \ldots, x_n)$.

$$(\text{application}) \quad t(x_1, \ldots, x_n) \coloneqq s_1(x_1, \ldots, x_n)s_2(x_1, \ldots, x_n)$$

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Idea. A derivation $\mathcal{D} \triangleright_{\mathrm{ND}} A_1, \dots, A_n \vdash B$ can be seen as a function $t(x_1, \dots, x_n)$ that associates with derivations $\mathcal{D}_i \triangleright_{\mathrm{ND}} \vdash A_i$ a derivation $t(\mathcal{D}_1/x_1, \dots, \mathcal{D}_n/x_n) \triangleright_{\mathrm{ND}} \vdash B$.

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- If $\mathcal{D}$ ends in $\Rightarrow_i$ let $s(x, y_1, \dots, y_n)$ be the function associated with the $\Rightarrow_i$-premise. Let $x$ be the variable associated with the parcel of hypotheses $A$ discharged by $\Rightarrow_i$. The function $t(y_1, \dots, y_n)$ associated with $\mathcal{D}$ maps $\mathcal{D}' \triangleright_{\mathrm{ND}} \vdash A$ to $s(\mathcal{D}'/x, y_1, \dots, y_n)$.

$$(\text{abstraction}) \quad t(y_1, \dots, y_n) \coloneqq \lambda x.s(x, y_1, \dots, y_n) \quad (\text{i.e. } x \mapsto s(x, y_1, \dots, y_n))$$

- If $\mathcal{D}$ ends in $\Rightarrow_e$, let $s_1(x_1, \dots, x_n)$ and $s_2(x_1, \dots, x_n)$ be the functions associated with the two premises of $\Rightarrow_e$. The function $t(x_1, \dots, x_n)$ associated with $\mathcal{D}$ is the application (noted as juxtaposition) of $s_1(x_1, \dots, x_n)$ to $s_2(x_1, \dots, x_n)$.

$$(\text{application}) \quad t(x_1, \dots, x_n) \coloneqq s_1(x_1, \dots, x_n)s_2(x_1, \dots, x_n)$$

# A computational interpretation of ND

Idea. A derivation $\mathcal{D} \triangleright_{\mathrm{ND}} A_1, \dots, A_n \vdash B$ can be seen as a function $t(x_1, \dots, x_n)$ that associates with derivations $\mathcal{D}_i \triangleright_{\mathrm{ND}} \vdash A_i$ a derivation $t(\mathcal{D}_1/x_1, \dots, \mathcal{D}_n/x_n) \triangleright_{\mathrm{ND}} \vdash B$.

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# Cut-elimination as a computational step

$$\dfrac{\dfrac{\begin{array}{c}[A]^x \\ \vdots\ t(x,\vec{y}) \\ B\end{array}}{\lambda x.t(x,\vec{y}) : A \Rightarrow B}\Rightarrow_i^x \qquad \begin{array}{c}\vdots \\ s(\vec{y}) : A\end{array}}{(\lambda x.t(x,\vec{y}))s(\vec{y}) : B}\Rightarrow_e \qquad \to_{\text{cut}} \qquad \begin{array}{c}\vdots \\ s(\vec{y}) : A \\ \vdots \\ t(s(\vec{y})/x,\vec{y}) : B\end{array}$$

- We can decorate each formula in a derivation with a term.
  $\rightsquigarrow$ For every derivation $\mathcal{D}$, its term $(\mathcal{D})_\lambda$ is the decoration of its conclusion.
- This decoration commute with cut-elimination via the step:

$$(\lambda x.t)s \to_\beta t\{s/x\}$$

where $t\{s/x\}$ stands for the substitution of $s$ for the free occurrences of $x$ in $t$.

$$\begin{array}{ccc}\mathcal{D} & \xrightarrow[\text{cut}]{} & \mathcal{D}' \\ \text{decoration} \rightsquigarrow & & \rightsquigarrow \text{decoration} \\ \mathcal{D}_\lambda & \xrightarrow[\beta]{} & (\mathcal{D}_\lambda)' = (\mathcal{D}')_\lambda\end{array}$$

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$$

# Examples of decorations of derivations in ND

$$x : A \qquad\qquad \dfrac{[x : A]^x}{\lambda x.x : A \Rightarrow A}\Rightarrow_i^x$$

$$\dfrac{\dfrac{[x : A]^x}{\lambda y.x : A \Rightarrow A}\Rightarrow_i}{\lambda x.\lambda y.x : A \Rightarrow A \Rightarrow A}\Rightarrow_i^x \qquad \dfrac{\dfrac{[y : A]^y}{\lambda y.y : A \Rightarrow A}\Rightarrow_i^y}{\lambda x.\lambda y.y : A \Rightarrow A \Rightarrow A}\Rightarrow_i^x$$

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# Example of decorations of derivations in ND with cut-elimination

$$\dfrac{\dfrac{x : [X \Rightarrow X]^x}{\lambda x.x : (X \Rightarrow X) \Rightarrow X \Rightarrow X}\Rightarrow_i^x \qquad \dfrac{y : [X]^y}{\lambda y.y : X \Rightarrow X}\Rightarrow_i^y}{(\lambda x.x)\lambda y.y : X \Rightarrow X}\Rightarrow_e \quad \to_{\text{cut}} \quad \dfrac{y : [X]^x}{\lambda y.y : X \Rightarrow X}\Rightarrow_i^x$$

Rmk. $(\lambda x.x)\lambda y.y \to_\beta x\{\lambda y.y/x\} = \lambda y.y \rightsquigarrow$ cut-elimination commutes with decoration.

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# Inverse decoration: from terms to derivations

Question. Given the term $\lambda f.\lambda x.fx$, what is the derivation associated with it?

Problem. Without knowing the formulas associated with variables, there is no answer.

Question. Given the term $\lambda f^{X \Rightarrow X}.\lambda x^{X}.fx$, what is the derivation associated with it?
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# The simply typed $\lambda$-calculus

**Types:** $A, B ::= X \mid A \Rightarrow B$

**Terms:** $s, t ::= x^A \mid \lambda x^A.t : A \Rightarrow B \mid st : A$ (provided that $t : B$ and $s : B \Rightarrow A$)

Rmk. Types are uniquely determined once they are fixed for variables.

Equivalently, terms are the ones that can be constructed via the typing rules below.

$$\frac{}{\Gamma, x : A \vdash x : A}\,\text{var} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \Rightarrow B}\,\lambda \qquad \frac{\Gamma \vdash s : B \Rightarrow A \quad \Gamma \vdash t : B}{\Gamma \vdash st : A}\,@$$

The free variables of a term $t$ are the variables that are not bound to a $\lambda$. Formally,

$$\mathrm{fv}(x) = \{x\} \qquad \mathrm{fv}(st) = \mathrm{fv}(s) \cup \mathrm{fv}(t) \qquad \mathrm{fv}(\lambda x.t) = \mathrm{fv}(t) \setminus \{x\}$$

$\beta$-reduction ($t\{s/x\}$ is the capture-avoiding substitution of $s$ for the free occurrences of $x$ in $t$):

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Types: $A, B ::= X \mid A \Rightarrow B$

Terms: $s, t ::= x^A \mid \lambda x^A.t : A \Rightarrow B \mid st : A$ (provided that $t : B$ and $s : B \Rightarrow A$)

Rmk. Types are uniquely determined once they are fixed for variables.

Equivalently, terms are the ones that can be constructed via the typing rules below.

$$\frac{}{\Gamma, x : A \vdash x : A}\,\mathsf{var} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \Rightarrow B}\,\lambda \qquad \frac{\Gamma \vdash s : B \Rightarrow A \quad \Gamma \vdash t : B}{\Gamma \vdash st : A}\,@$$

The free variables of a term $t$ are the variables that are not bound to a $\lambda$. Formally,

$$\mathsf{fv}(x) = \{x\} \qquad \mathsf{fv}(st) = \mathsf{fv}(s) \cup \mathsf{fv}(t) \qquad \mathsf{fv}(\lambda x.t) = \mathsf{fv}(t) \setminus \{x\}$$

$\beta$-reduction ($t\{s/x\}$ is the capture-avoiding substitution of $s$ for the free occurrences of $x$ in $t$):

$$(\lambda x.t)s \to_\beta t\{s/x\}$$

# Some properties of the simply typed $\lambda$-calculus

**Lemma (Substitution)**

If $\Gamma \vdash t : A$ and $\Delta \vdash s : B$ are derivable, then so is $\Gamma, \Delta \vdash t\{s/x^B\} : A$.

*Proof.* By structural induction on $t$ (exercise!). □

**Theorem (Subject reduction)**

If $\Gamma \vdash t : A$ is derivable and $t \to_\beta s$, then $\Gamma \vdash s : A$ is derivable.

*Proof.* By structural induction on $t$, using the substitution lemma in the key-case. □

Rmk. The converse (subject expansion) does not hold: let $A = Y \Rightarrow (X \Rightarrow X)$, and $s = (\lambda x.\lambda y.\lambda z.(xz)(yz))\lambda x.\lambda y.x$ and $t = \lambda x.\lambda y.y$, then $s \to_\beta^* t$ and $\vdash t : A$, but $\nvdash s : A$.

**Theorem (Normalization)**

If $\Gamma \vdash t : A$ is derivable, then $t \to_\beta^* s$ and for some derivable $\Gamma \vdash s : A$ without redexes.

*Proof.* Exactly the proof of cut-elimination for ND. □

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# The Curry-Howard correspondence

| minimal logic | simply typed $\lambda$-calculus | computer science |
|---|---|---|
| formula | type | specification |
| derivation | term | program |
| cut-elimination step | $\beta$-reduction | computation step |
| derivation without redexes | normal form | result |
| cut-elimination theorem | normalization | termination |

Concerning the correspondence between derivations and terms:

| derivation in minimal logic | term in simply typed $\lambda$-calculus |
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| $\Rightarrow_i$ | abstraction $\lambda$ |
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# Outline

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