The λ-calculus: from simple types to non-idempotent intersection types

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Solutions to selected exercises — ECI 2024

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Exercises from Day 1 (https://pageperso.lis-lab.fr/~giulio.guerrieri/ECI2024/day1.pdf)

Exercise 1

Prove the following facts, using ND and ND_{seq}.

1. $\vdash X \Rightarrow ((X \Rightarrow Y) \Rightarrow Y)$.
2. $(X \Rightarrow Y) \Rightarrow (X \Rightarrow Z) \vdash Y \Rightarrow X \Rightarrow Z$.
3. $(X \Rightarrow Y) \Rightarrow X \vdash Y \Rightarrow X$.
4. $X \Rightarrow (Y \Rightarrow Z) \vdash Y \Rightarrow X \Rightarrow Z$.
5. $X \Rightarrow Y \Rightarrow Z, X \Rightarrow Y \vdash X \Rightarrow Z$.
6. $(X \Rightarrow X) \Rightarrow Y \vdash (Y \Rightarrow Z) \Rightarrow Z$.

Solution to Exercise 1

1. In ND and ND_{seq}, respectively:

$$\dfrac{\dfrac{\dfrac{[X \Rightarrow Y]^{\circ} \qquad [X]^{*}}{Y}\Rightarrow_e}{(X \Rightarrow Y) \Rightarrow Y}\Rightarrow_i^{\circ}}{X \Rightarrow ((X \Rightarrow Y) \Rightarrow Y)}\Rightarrow_i^{*} \qquad\qquad \dfrac{\dfrac{\dfrac{\dfrac{}{X, X \Rightarrow Y \vdash X \Rightarrow Y}\mathsf{ax} \qquad \dfrac{}{X, X \Rightarrow Y \vdash X}\mathsf{ax}}{X, X \Rightarrow Y \vdash Y}\Rightarrow_e}{X \vdash (X \Rightarrow Y) \Rightarrow Y}\Rightarrow_i}{\vdash X \Rightarrow ((X \Rightarrow Y) \Rightarrow Y)}\Rightarrow_i$$

2. In ND:

$$\dfrac{\dfrac{(X \Rightarrow Y) \Rightarrow (X \Rightarrow Z) \qquad \dfrac{[Y]^{*}}{X \Rightarrow Y}\Rightarrow_i}{X \Rightarrow Z}\Rightarrow_e}{Y \Rightarrow X \Rightarrow Z}\Rightarrow_i^{*}$$

In ND_{seq}:

$$\dfrac{\dfrac{\dfrac{}{(X \Rightarrow Y) \Rightarrow (X \Rightarrow Z), Y \vdash (X \Rightarrow Y) \Rightarrow (X \Rightarrow Z)}\mathsf{ax} \qquad \dfrac{\dfrac{}{(X \Rightarrow Y) \Rightarrow (X \Rightarrow Z), X, Y \vdash Y}\mathsf{ax}}{(X \Rightarrow Y) \Rightarrow (X \Rightarrow Z), Y \vdash X \Rightarrow Y}\Rightarrow_i}{(X \Rightarrow Y) \Rightarrow (X \Rightarrow Z), Y \vdash X \Rightarrow Z}\Rightarrow_e}{(X \Rightarrow Y) \Rightarrow (X \Rightarrow Z) \vdash Y \Rightarrow X \Rightarrow Z}\Rightarrow_i^{*}$$

3. In ND and ND_{seq}, respectively:

$$\dfrac{\dfrac{(X \Rightarrow Y) \Rightarrow X \qquad \dfrac{[Y]^*}{X \Rightarrow Y}\,{\Rightarrow_i}}{X}\,{\Rightarrow_e}}{Y \Rightarrow X}\,{\Rightarrow_i^*}
\qquad
\dfrac{\dfrac{\dfrac{}{(X \Rightarrow Y) \Rightarrow X, Y \vdash (X \Rightarrow Y) \Rightarrow X}\,\mathsf{ax} \qquad \dfrac{\dfrac{}{(X \Rightarrow Y) \Rightarrow X, X, Y \vdash Y}\,\mathsf{ax}}{(X \Rightarrow Y) \Rightarrow X, Y \vdash X \Rightarrow Y}\,{\Rightarrow_i}}{(X \Rightarrow Y) \Rightarrow X, Y \vdash X}\,{\Rightarrow_e}}{(X \Rightarrow Y) \Rightarrow X \vdash Y \Rightarrow X}\,{\Rightarrow_i}$$

4. In ND:

$$\dfrac{\dfrac{\dfrac{\dfrac{X \Rightarrow (Y \Rightarrow Z) \qquad [X]^\circ}{Y \Rightarrow Z}\,{\Rightarrow_e} \qquad [Y]^*}{Z}\,{\Rightarrow_e}}{X \Rightarrow Z}\,{\Rightarrow_i^\circ}}{Y \Rightarrow X \Rightarrow Z}\,{\Rightarrow_i^*}$$

In $\mathsf{ND}_{\mathrm{seq}}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{X \Rightarrow (Y \Rightarrow Z), X, Y \vdash X \Rightarrow (Y \Rightarrow Z)}\,\mathsf{ax} \qquad \dfrac{}{X \Rightarrow (Y \Rightarrow Z), X, Y \vdash X}\,\mathsf{ax}}{X \Rightarrow (Y \Rightarrow Z), X, Y \vdash Y \Rightarrow Z}\,{\Rightarrow_e} \qquad \dfrac{}{X \Rightarrow (Y \Rightarrow Z), X, Y \vdash Y}\,\mathsf{ax}}{X \Rightarrow (Y \Rightarrow Z), X, Y \vdash Z}\,{\Rightarrow_e}}{X \Rightarrow (Y \Rightarrow Z), Y \vdash X \Rightarrow Z}\,{\Rightarrow_i}}{X \Rightarrow (Y \Rightarrow Z) \vdash Y \Rightarrow X \Rightarrow Z}\,{\Rightarrow_i}$$

5. In ND:

$$\dfrac{\dfrac{\dfrac{X \Rightarrow Y \Rightarrow Z \qquad [X]^*}{Y \Rightarrow Z}\,{\Rightarrow_e} \qquad \dfrac{X \Rightarrow Y \qquad [X]^*}{Y}\,{\Rightarrow_e}}{Z}\,{\Rightarrow_e}}{X \Rightarrow Z}\,{\Rightarrow_i^*}$$

In $\mathsf{ND}_{\mathrm{seq}}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{X \Rightarrow Y \Rightarrow Z, X \Rightarrow Y, X \vdash X \Rightarrow Y \Rightarrow Z}\,\mathsf{ax} \quad \dfrac{}{X \Rightarrow Y \Rightarrow Z, X \Rightarrow Y, X \vdash X}\,\mathsf{ax}}{X \Rightarrow Y \Rightarrow Z, X \Rightarrow Y, X \vdash Y \Rightarrow Z}\,{\Rightarrow_e} \quad \dfrac{\dfrac{}{X \Rightarrow Y \Rightarrow Z, X \Rightarrow Y, X \vdash X \Rightarrow Y}\,\mathsf{ax} \quad \dfrac{}{X \Rightarrow Y \Rightarrow Z, X \Rightarrow Y, X \vdash X}\,\mathsf{ax}}{X \Rightarrow Y \Rightarrow Z, X \Rightarrow Y, X \vdash Y}\,{\Rightarrow_e}}{X \Rightarrow Y \Rightarrow Z, X \Rightarrow Y, X \vdash Z}\,{\Rightarrow_e}}{X \Rightarrow Y \Rightarrow Z, X \Rightarrow Y \vdash X \Rightarrow Z}\,{\Rightarrow_i}$$

6. In ND:

$$\dfrac{\dfrac{[Y \Rightarrow Z]^* \qquad \dfrac{(X \Rightarrow X) \Rightarrow Y \qquad \dfrac{[X]^\circ}{X \Rightarrow X}\,{\Rightarrow_i^\circ}}{Y}\,{\Rightarrow_e}}{Z}\,{\Rightarrow_e}}{(Y \Rightarrow Z) \Rightarrow Z}\,{\Rightarrow_i^*}$$

In $\mathsf{ND}_{\mathrm{seq}}$:

$$\dfrac{\dfrac{\dfrac{}{Y \Rightarrow Z, (X \Rightarrow X) \Rightarrow Y \vdash Y \Rightarrow Z}\,\mathsf{ax} \qquad \dfrac{\dfrac{}{Y \Rightarrow Z, (X \Rightarrow X) \Rightarrow Y \vdash (X \Rightarrow X) \Rightarrow Y}\,\mathsf{ax} \qquad \dfrac{\dfrac{}{Y \Rightarrow Z, (X \Rightarrow X) \Rightarrow Y, X \vdash X}\,\mathsf{ax}}{Y \Rightarrow Z, (X \Rightarrow X) \Rightarrow Y \vdash X \Rightarrow X}\,{\Rightarrow_i}}{Y \Rightarrow Z, (X \Rightarrow X) \Rightarrow Y \vdash Y}\,{\Rightarrow_e}}{Y \Rightarrow Z, (X \Rightarrow X) \Rightarrow Y \vdash Z}\,{\Rightarrow_e}}{(X \Rightarrow X) \Rightarrow Y \vdash (Y \Rightarrow Z) \Rightarrow Z}\,{\Rightarrow_i}$$

Exercise 2

Show that $\nvdash (X \Rightarrow Y) \Rightarrow X$, i.e. $(X \Rightarrow Y) \Rightarrow X$ is not derivable with no hypotheses.

Solution to Exercise 2

Suppose by absurd that $(X \Rightarrow Y) \Rightarrow X$ is derivable in ND with no hypothesis. The last rule of the derivation cannot be either an hypothesis (because there are no hypotheses) or $\Rightarrow_e$ (otherwise it would be it would contradict the subformula property), hence it could only be $\Rightarrow_i$ discharging the hypothesis $X \Rightarrow Y$, that is,

$$\dfrac{\begin{array}{c}[X \Rightarrow Y]^* \\ \vdots \\ X\end{array}}{(X \Rightarrow Y) \Rightarrow X}\Rightarrow_i^*$$

The rule whose conclusion is X cannot be either $\Rightarrow_i$ (otherwise its conclusion should be an arrow) or an hypothesis (because there is no hypothesis X), hence it could only be $\Rightarrow_e$ with premises $A \Rightarrow X$ and A for some formula A, that is,

$$\dfrac{\dfrac{\begin{array}{c}[X \Rightarrow Y]^* \\ \vdots \\ A \Rightarrow X\end{array} \qquad \begin{array}{c}[X \Rightarrow Y]^* \\ \vdots \\ A\end{array}}{X}\Rightarrow_e}{(X \Rightarrow Y) \Rightarrow X}\Rightarrow_i^*$$

For the subformula property applied to the derivation whose conclusion is X, A could only be a subformula of X or $X \Rightarrow Y$, that is,

- either $A = X$, but then $A \Rightarrow X = X \Rightarrow X$ is a formula of that derivation that is not a subformula of X or $X \Rightarrow Y$, which contradicts the subformula property;
- or $A = Y$, but then $A \Rightarrow X = Y \Rightarrow X$ is a formula of that derivation that is not a subformula of X or $X \Rightarrow Y$, which contradicts the subformula property;
- or $A = X \Rightarrow Y$, but then $A \Rightarrow X = (X \Rightarrow Y) \Rightarrow X$ is a formula of that derivation that is not a subformula of X or $X \Rightarrow Y$, which contradicts the subformula property.

Therefore, there is no derivation of $(X \Rightarrow Y) \Rightarrow X$ with no hypotheses.

Exercise 3

Perform all passible cut-elimination steps from the derivation on p. 24 of Day 1 slides, until you get a derivation without redexes. Is it always the same?

Solution to Exercise 3

The derivation on p. 24 of Day 1 slides is $\mathcal{D}$ below, where there are two redexes, marked as blue and red.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)]^\dagger \quad [X \Rightarrow X]^\circ}{B \Rightarrow (X \Rightarrow X)}\Rightarrow_e \quad \dfrac{[(X \Rightarrow X) \Rightarrow B]^* \quad [X \Rightarrow X]^\circ}{B}\Rightarrow_e}{X \Rightarrow X}\Rightarrow_e}{\color{red}(X \Rightarrow X) \Rightarrow (X \Rightarrow X)}{\color{red}\Rightarrow_i}^\circ \quad \dfrac{[X]^\bullet}{X \Rightarrow X}\Rightarrow_i^\bullet}{X \Rightarrow X}\color{red}\Rightarrow_e}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}\Rightarrow_i^*}{\color{blue}((X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)) \Rightarrow (((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X))}{\color{blue}\Rightarrow_i}^\dagger \quad \dfrac{\dfrac{[X \Rightarrow X]^\dagger}{B \Rightarrow X \Rightarrow X}\Rightarrow_i}{(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)}\Rightarrow_i^\dagger}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}\color{blue}\Rightarrow_e$$

If the red redex in $\mathcal{D}$ is fired, then $\mathcal{D}$ reduces to the derivation $\mathcal{D}_1$ below.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)]^\dagger \quad \dfrac{[X]^\bullet}{X \Rightarrow X}\Rightarrow_i^\bullet}{B \Rightarrow (X \Rightarrow X)}\Rightarrow_e \quad \dfrac{[(X \Rightarrow X) \Rightarrow B]^* \quad \dfrac{[X]^\bullet}{X \Rightarrow X}\Rightarrow_i^\bullet}{B}\Rightarrow_e}{X \Rightarrow X}\Rightarrow_e}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}\Rightarrow_i^*}{\color{blue}((X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)) \Rightarrow (((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X))}{\color{blue}\Rightarrow_i}^\dagger \quad \dfrac{\dfrac{[X \Rightarrow X]^\dagger}{B \Rightarrow X \Rightarrow X}\Rightarrow_i}{(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)}\Rightarrow_i^\dagger}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}\color{blue}\Rightarrow_e$$

If the blue redex in $\mathcal{D}_1$ is fired, then $\mathcal{D}_1$ reduces to the derivation $\mathcal{D}_1'$ below, with a new green redex.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[X \Rightarrow X]^{\dagger}}{B \Rightarrow X \Rightarrow X}{\scriptstyle\Rightarrow_i}}{(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)}{\scriptstyle\Rightarrow_i^{\dagger}} \quad \dfrac{[X]^{\bullet}}{X \Rightarrow X}{\scriptstyle\Rightarrow_i^{\bullet}}}{B \Rightarrow (X \Rightarrow X)}{\scriptstyle\Rightarrow_e} \quad \dfrac{[(X \Rightarrow X) \Rightarrow B]^{*} \quad \dfrac{[X]^{\bullet}}{X \Rightarrow X}{\scriptstyle\Rightarrow_i^{\bullet}}}{B}{\scriptstyle\Rightarrow_e}}{X \Rightarrow X}{\scriptstyle\Rightarrow_e}}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}{\scriptstyle\Rightarrow_i^{*}}$$

If the green redex in $\mathcal{D}_1'$ is fired, then $\mathcal{D}_1'$ reduces to derivation $\mathcal{D}_1''$ below, with a new gray redex.

$$\dfrac{\dfrac{\dfrac{\dfrac{[X]^{\bullet}}{X \Rightarrow X}{\scriptstyle\Rightarrow_i^{\bullet}}}{B \Rightarrow X \Rightarrow X}{\scriptstyle\Rightarrow_i} \quad \dfrac{[(X \Rightarrow X) \Rightarrow B]^{*} \quad \dfrac{[X]^{\bullet}}{X \Rightarrow X}{\scriptstyle\Rightarrow_i^{\bullet}}}{B}{\scriptstyle\Rightarrow_e}}{X \Rightarrow X}{\scriptstyle\Rightarrow_e}}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}{\scriptstyle\Rightarrow_i^{*}}$$

If the gray redex in $\mathcal{D}_1''$ is fired, then $\mathcal{D}_1''$ reduces to derivation $\mathcal{D}_0$ below, which is without redexes.

$$\dfrac{\dfrac{[X]^{\bullet}}{X \Rightarrow X}{\scriptstyle\Rightarrow_i^{\bullet}}}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}{\scriptstyle\Rightarrow_i}$$

If the blue redex in $\mathcal{D}$ is fired, then $\mathcal{D}$ reduces to the derivation $\mathcal{D}_2$ below, with a new green redex.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[X \Rightarrow X]^{\dagger}}{B \Rightarrow X \Rightarrow X}{\scriptstyle\Rightarrow_i}}{(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)}{\scriptstyle\Rightarrow_i^{\dagger}} \quad [X \Rightarrow X]^{\circ}}{B \Rightarrow (X \Rightarrow X)}{\scriptstyle\Rightarrow_e} \quad \dfrac{[(X \Rightarrow X) \Rightarrow B]^{*} \quad [X \Rightarrow X]^{\circ}}{B}{\scriptstyle\Rightarrow_e}}{X \Rightarrow X}{\scriptstyle\Rightarrow_e}}{(X \Rightarrow X) \Rightarrow (X \Rightarrow X)}{\scriptstyle\Rightarrow_i^{\circ}} \quad \dfrac{[X]^{\bullet}}{X \Rightarrow X}{\scriptstyle\Rightarrow_i^{\bullet}}}{X \Rightarrow X}{\scriptstyle\Rightarrow_e}}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}{\scriptstyle\Rightarrow_i^{*}}$$

If the red redex in $\mathcal{D}_2$ is fired, then $\mathcal{D}_2$ reduces to the derivation $\mathcal{D}_{21}$ below.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[X \Rightarrow X]^{\dagger}}{B \Rightarrow X \Rightarrow X}{\scriptstyle\Rightarrow_i}}{(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)}{\scriptstyle\Rightarrow_i^{\dagger}} \quad \dfrac{[X]^{\bullet}}{X \Rightarrow X}{\scriptstyle\Rightarrow_i^{\bullet}}}{B \Rightarrow (X \Rightarrow X)}{\scriptstyle\Rightarrow_e} \quad \dfrac{[(X \Rightarrow X) \Rightarrow B]^{*} \quad \dfrac{[X]^{\bullet}}{X \Rightarrow X}{\scriptstyle\Rightarrow_i^{\bullet}}}{B}{\scriptstyle\Rightarrow_e}}{X \Rightarrow X}{\scriptstyle\Rightarrow_e}}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}{\scriptstyle\Rightarrow_i^{*}}$$

If the green redex in $\mathcal{D}_{21}$ is fired, then $\mathcal{D}_{21}$ reduces to the derivation $\mathcal{D}_1''$ already shown above.
If the green redex in $\mathcal{D}_2$ is fired, then $\mathcal{D}_2$ reduces to the derivation $\mathcal{D}_{22}$ below, with a new gray redex.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[X \Rightarrow X]^{\circ}}{B \Rightarrow X \Rightarrow X}{\scriptstyle\Rightarrow_i} \quad \dfrac{[(X \Rightarrow X) \Rightarrow B]^{*} \quad [X \Rightarrow X]^{\circ}}{B}{\scriptstyle\Rightarrow_e}}{X \Rightarrow X}{\scriptstyle\Rightarrow_e}}{(X \Rightarrow X) \Rightarrow (X \Rightarrow X)}{\scriptstyle\Rightarrow_i^{\circ}} \quad \dfrac{[X]^{\bullet}}{X \Rightarrow X}{\scriptstyle\Rightarrow_i^{\bullet}}}{X \Rightarrow X}{\scriptstyle\Rightarrow_e}}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}{\scriptstyle\Rightarrow_i^{*}}$$

If the red redex in $\mathcal{D}_{22}$ is fired, then $\mathcal{D}_{22}$ reduces to the derivation $\mathcal{D}_{221}$ below.

$$\dfrac{\dfrac{\dfrac{\dfrac{[X]^{\bullet}}{X \Rightarrow X}{\scriptstyle\Rightarrow_i^{\bullet}}}{B \Rightarrow X \Rightarrow X}{\scriptstyle\Rightarrow_i} \quad \dfrac{[(X \Rightarrow X) \Rightarrow B]^{*} \quad \dfrac{[X]^{\bullet}}{X \Rightarrow X}{\scriptstyle\Rightarrow_i^{\bullet}}}{B}{\scriptstyle\Rightarrow_e}}{X \Rightarrow X}{\scriptstyle\Rightarrow_e}}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}{\scriptstyle\Rightarrow_i^{*}}$$

If the gray redex in $\mathcal{D}_{221}$ is fired, then $\mathcal{D}_{221}$ reduces to the derivation $\mathcal{D}_0$ below, which is without redexes.

$$\dfrac{\dfrac{[X]^\bullet}{X \Rightarrow X}\,{\scriptstyle \Rightarrow_i^\bullet}}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}\,{\scriptstyle \Rightarrow_i}$$

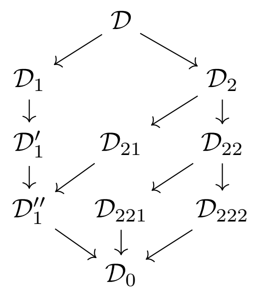
If the gray redex in $\mathcal{D}_{22}$ is fired, then $\mathcal{D}_{22}$ reduces to the derivation $\mathcal{D}_{222}$ below.

$$\dfrac{\dfrac{\dfrac{[X \Rightarrow X]^\circ}{(X \Rightarrow X) \Rightarrow (X \Rightarrow X)}\,{\scriptstyle \Rightarrow_i^\circ} \qquad \dfrac{[X]^\bullet}{X \Rightarrow X}\,{\scriptstyle \Rightarrow_i^\bullet}}{X \Rightarrow X}\,{\scriptstyle \Rightarrow_e}}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}\,{\scriptstyle \Rightarrow_i^*}$$

If the red redex in $\mathcal{D}_{222}$ is fired, then $\mathcal{D}_{222}$ reduces to the derivation $\mathcal{D}_0$ below, which is without redexes.

$$\dfrac{\dfrac{[X]^\bullet}{X \Rightarrow X}\,{\scriptstyle \Rightarrow_i^\bullet}}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}\,{\scriptstyle \Rightarrow_i}$$

All possible cut-elimination steps from $\mathcal{D}$ are the following:



In any case, every reduction sequence eventually reaches the same derivation $\mathcal{D}_0$ with no redexes.

Exercise 4

Order the following multisets over $\mathbb{N}$ according to the (strict) multiset order $\prec_{\text{mul}}$.

$$[1,1] \quad [0,2] \quad [1] \quad [0,0,2] \quad [\,] \quad [0,3] \quad [0,2,2]$$

Solution to Exercise 4

$$[\,] \prec_{\text{mul}} [1] \prec_{\text{mul}} [1,1] \prec_{\text{mul}} [0,2] \prec_{\text{mul}} [0,0,2] \prec_{\text{mul}} [0,2,2] \prec_{\text{mul}} [0,3].$$

Exercise 5

Prove in a rigorous way the proposition on p. 15 of Day 1 slides.

Solution to Exercise 5

Proposition. *Let Γ be a finite multiset of formulas and A be a formula: $\Gamma \vdash A$ in* ND *if and only if the sequent $\Gamma \vdash A$ is derivable in* ND_{seq}.

Proof. **$\Rightarrow$:** By induction on the number of rules of the smallest derivation $\mathcal{D}$ in ND proving that $\Gamma \vdash A$. Cases:

- $\mathcal{D}$ is just an hypothesis, that is, $\mathcal{D} = A$ and so $\Gamma = \Gamma', A$ for any finite multiset Γ'. Then, the derivation $\mathcal{D}_{\text{seq}}$ below derives the sequent $\Gamma \vdash A$ in ND_{seq}.

$$\mathcal{D}_{\text{seq}} \quad = \quad \dfrac{}{\Gamma', A \vdash A}\,{\scriptstyle \mathsf{ax}}$$

- The last rule in $\mathcal{D}$ is $\Rightarrow_i$, that is, $A = B \Rightarrow C$ and

$$\mathcal{D} \quad = \quad \dfrac{\begin{matrix}[B]^* \\ \vdots\;\mathcal{D}' \\ C\end{matrix}}{B \Rightarrow C}\,{\scriptstyle \Rightarrow_i^*}$$

where $\mathcal{D}'$ is the smallest derivation in ND that proves that $\Gamma, B \vdash C$, by minimality of $\mathcal{D}$. By induction hypothesis applied to $\mathcal{D}'$, there is a derivation $\mathcal{D}'_{\text{seq}}$ in ND_{seq} of the sequent $\Gamma, B \vdash C$. Then, the derivation $\mathcal{D}_{\text{seq}}$ below derives the sequent $\Gamma \vdash A$ in ND_{seq}.

$$\mathcal{D}_{\text{seq}} \quad = \quad \dfrac{\begin{matrix} \vdots\ \mathcal{D}'_{\text{seq}} \\ \Gamma, B \vdash C \end{matrix}}{\Gamma \vdash B \Rightarrow C} \Rightarrow_i$$

- The last rule in $\mathcal{D}$ is $\Rightarrow_e$, that is, for some formula B

$$\mathcal{D} \quad = \quad \dfrac{\begin{matrix} \vdots\ \mathcal{D}' \\ B \Rightarrow A \end{matrix} \qquad \begin{matrix} \vdots\ \mathcal{D}'' \\ B \end{matrix}}{A} \Rightarrow_e$$

where $\mathcal{D}'$ and $\mathcal{D}''$ are the smallest derivation in ND that prove that $\Gamma \vdash B \Rightarrow A$ and $\Gamma \vdash B$, respectively, by minimality of $\mathcal{D}$. By induction hypothesis applied to $\mathcal{D}'$ and $\mathcal{D}''$, respectively, there are derivations $\mathcal{D}'_{\text{seq}}$ and $\mathcal{D}''_{\text{seq}}$ in ND_{seq} of the sequents $\Gamma \vdash B \Rightarrow A$ and $\Gamma \vdash B$. Then, the derivation $\mathcal{D}_{\text{seq}}$ below derives the sequent $\Gamma \vdash A$ in ND_{seq}.

$$\mathcal{D}_{\text{seq}} \quad = \quad \dfrac{\begin{matrix} \vdots\ \mathcal{D}'_{\text{seq}} \\ \Gamma \vdash B \Rightarrow A \end{matrix} \qquad \begin{matrix} \vdots\ \mathcal{D}''_{\text{seq}} \\ \Gamma \vdash B \end{matrix}}{\Gamma \vdash A} \Rightarrow_e$$

$\Leftarrow$: By induction on the number of rules of the smallest derivation $\mathcal{D}$ in ND_{seq} proving the sequent $\Gamma \vdash A$. Cases:

- The last rule of $\mathcal{D}$ is ax, that is,

$$\mathcal{D} \quad = \quad \dfrac{}{\Gamma', A \vdash A} \mathsf{ax}$$

where $\Gamma = \Gamma', A$ for some finite multiset Γ'. Then, the derivation $\mathcal{D}_0 = A$ proves that $\Gamma \vdash A$ in ND.

- The last rule in $\mathcal{D}$ is $\Rightarrow_i$, that is, $A = B \Rightarrow C$ and

$$\mathcal{D} \quad = \quad \dfrac{\begin{matrix} \vdots\ \mathcal{D}' \\ \Gamma, B \vdash C \end{matrix}}{\Gamma \vdash B \Rightarrow C} \Rightarrow_i$$

where $\mathcal{D}'$ is the smallest derivation in ND_{seq} of the sequent $\Gamma, B \vdash C$, by minimality of $\mathcal{D}$. By induction hypothesis applied to $\mathcal{D}'$, there is a derivation $\mathcal{D}'_0$ in ND that proves that $\Gamma, B \vdash C$. Then, the derivation $\mathcal{D}_0$ below proves that $\Gamma \vdash A$ in ND.

$$\mathcal{D} \quad = \quad \dfrac{\begin{matrix} [B]^* \\ \vdots\ \mathcal{D}' \\ C \end{matrix}}{B \Rightarrow C} \Rightarrow_i^*$$

- The last rule in $\mathcal{D}$ is $\Rightarrow_e$, that is, for some formula B

$$\mathcal{D}_{\text{seq}} \quad = \quad \dfrac{\begin{matrix} \vdots\ \mathcal{D}' \\ \Gamma \vdash B \Rightarrow A \end{matrix} \qquad \begin{matrix} \vdots\ \mathcal{D}'' \\ \Gamma \vdash B \end{matrix}}{\Gamma \vdash A} \Rightarrow_e$$

where $\mathcal{D}'$ and $\mathcal{D}''$ are the smallest derivation in ND_{seq} that prove the sequents $\Gamma \vdash B \Rightarrow A$ and $\Gamma \vdash B$, respectively, by minimality of $\mathcal{D}$. By induction hypothesis applied to $\mathcal{D}'$ and $\mathcal{D}''$, respectively, there are derivations $\mathcal{D}'_0$ and $\mathcal{D}''_0$ in ND that prove $\Gamma \vdash B \Rightarrow A$ and $\Gamma \vdash B$. Then, the derivation $\mathcal{D}_0$ below prove that $\Gamma \vdash A$ in ND.

$$\mathcal{D} \quad = \quad \dfrac{\begin{matrix} \vdots\ \mathcal{D}' \\ B \Rightarrow A \end{matrix} \qquad \begin{matrix} \vdots\ \mathcal{D}'' \\ B \end{matrix}}{A} \Rightarrow_e$$

$\square$

Exercise 6

For any formula B, prove that if $\Gamma \vdash A$ is derivable in ND_{seq}, then so is $\Gamma, B \vdash A$.

Solution to Exercise 6

By induction on the number of rules of the smallest derivation $\mathcal{D}$ in ND_{seq} proving the sequent $\Gamma \vdash A$. Cases:

- The last rule of $\mathcal{D}$ is ax, that is,

$$\mathcal{D} \quad = \quad \frac{}{\Gamma', A \vdash A}\,\mathsf{ax}$$

  where $\Gamma = \Gamma', A$ for some finite multiset Γ'. Then, the derivation below proves the sequent $\Gamma, B \vdash A$ in ND_{seq}.

$$\frac{}{\Gamma', B, A \vdash A}\,\mathsf{ax}$$

- The last rule in $\mathcal{D}$ is $\Rightarrow_i$, that is, $A = D \Rightarrow C$ and

$$\mathcal{D} \quad = \quad \frac{\begin{matrix}\vdots\,\mathcal{D}' \\ \Gamma, D \vdash C\end{matrix}}{\Gamma \vdash D \Rightarrow C}\,\Rightarrow_i$$

  where $\mathcal{D}'$ is the smallest derivation in ND_{seq} of the sequent $\Gamma, D \vdash C$, by minimality of $\mathcal{D}$. By induction hypothesis applied to $\mathcal{D}'$, there is a derivation $\mathcal{D}_0$ in ND_{seq} that proves the sequent $\Gamma, B, D \vdash C$. Then, the derivation below proves the sequent $\Gamma, B \vdash A$ in ND_{seq}.

$$\frac{\begin{matrix}\vdots\,\mathcal{D}_0 \\ \Gamma, B, D \vdash C\end{matrix}}{\Gamma, B \vdash D \Rightarrow C}\,\Rightarrow_i$$

- The last rule in $\mathcal{D}$ is $\Rightarrow_e$, that is, for some formula C

$$\mathcal{D}_{\text{seq}} \quad = \quad \frac{\begin{matrix}\vdots\,\mathcal{D}' \\ \Gamma \vdash C \Rightarrow A\end{matrix} \qquad \begin{matrix}\vdots\,\mathcal{D}'' \\ \Gamma \vdash C\end{matrix}}{\Gamma \vdash A}\,\Rightarrow_e$$

  where $\mathcal{D}'$ and $\mathcal{D}''$ are the smallest derivations in ND_{seq} that prove the sequents $\Gamma \vdash C \Rightarrow A$ and $\Gamma \vdash C$, respectively, by minimality of $\mathcal{D}$. By induction hypothesis applied to $\mathcal{D}'$ and $\mathcal{D}''$, respectively, there are derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ in ND_{seq} that prove the sequents $\Gamma, B \vdash C \Rightarrow A$ and $\Gamma \vdash C$. Then, the derivation below prove the sequent $\Gamma, B \vdash A$ in ND_{seq}.

$$\frac{\begin{matrix}\vdots\,\mathcal{D}_1 \\ \Gamma, B \vdash C \Rightarrow A\end{matrix} \qquad \begin{matrix}\vdots\,\mathcal{D}_2 \\ \Gamma, B \vdash C\end{matrix}}{\Gamma, B \vdash A}\,\Rightarrow_e$$

Exercise 7

For any formula B, prove that if $\Gamma, B, B \vdash A$ is derivable in ND_{seq} then so is $\Gamma, B \vdash A$.

Solution to Exercise 7

By induction on the number of rules of the smallest derivation $\mathcal{D}$ in ND_{seq} proving the sequent $\Gamma, B, B \vdash A$. Cases:

- The last rule of $\mathcal{D}$ is ax, that is,

$$\mathcal{D} \quad = \quad \frac{}{\Gamma', B, B, A \vdash A}\,\mathsf{ax}$$

  where $\Gamma = \Gamma', A$ for some finite multiset Γ'. Then, the derivation below proves the sequent $\Gamma, B \vdash A$ in ND_{seq}.

$$\frac{}{\Gamma', B, A \vdash A}\,\mathsf{ax}$$

- The last rule in $\mathcal{D}$ is $\Rightarrow_i$, that is, $A = D \Rightarrow C$ and

$$\mathcal{D} \quad = \quad \dfrac{\begin{array}{c}\vdots\, \mathcal{D}' \\ \Gamma, B, D \vdash C\end{array}}{\Gamma, B, B \vdash D \Rightarrow C} \Rightarrow_i$$

where $\mathcal{D}'$ is the smallest derivation in ND_{seq} of the sequent $\Gamma, B, B, D \vdash C$, by minimality of $\mathcal{D}$. By induction hypothesis applied to $\mathcal{D}'$, there is a derivation $\mathcal{D}_0$ in ND_{seq} that proves the sequent $\Gamma, B, D \vdash C$. Then, the derivation below proves the sequent $\Gamma, B \vdash A$ in ND_{seq}.

$$\dfrac{\begin{array}{c}\vdots\, \mathcal{D}_0 \\ \Gamma, B, D \vdash C\end{array}}{\Gamma, B \vdash D \Rightarrow C} \Rightarrow_i$$

- The last rule in $\mathcal{D}$ is $\Rightarrow_e$, that is, for some formula C

$$\mathcal{D}_{\text{seq}} \quad = \quad \dfrac{\begin{array}{c}\vdots\, \mathcal{D}' \\ \Gamma, B, B \vdash C \Rightarrow A\end{array} \qquad \begin{array}{c}\vdots\, \mathcal{D}'' \\ \Gamma, B, B \vdash C\end{array}}{\Gamma, B, B \vdash A} \Rightarrow_e$$

where $\mathcal{D}'$ and $\mathcal{D}''$ are the smallest derivations in ND_{seq} that prove the sequents $\Gamma, B, B \vdash C \Rightarrow A$ and $\Gamma, B, B \vdash C$, respectively, by minimality of $\mathcal{D}$. By induction hypothesis applied to $\mathcal{D}'$ and $\mathcal{D}''$, respectively, there are derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ in ND_{seq} that prove the sequents $\Gamma, B \vdash C \Rightarrow A$ and $\Gamma \vdash C$. Then, the derivation below proves the sequent $\Gamma, B \vdash A$ in ND_{seq}.

$$\dfrac{\begin{array}{c}\vdots\, \mathcal{D}_1 \\ \Gamma, B \vdash C \Rightarrow A\end{array} \qquad \begin{array}{c}\vdots\, \mathcal{D}_2 \\ \Gamma, B \vdash C\end{array}}{\Gamma, B \vdash A} \Rightarrow_e$$

Exercises from Day 2 (https://pageperso.lis-lab.fr/~giulio.guerrieri/ECI2024/day2.pdf)

Exercise 1

Find the simply typed λ-terms (in Curry-style and Church-style) associated with the derivations in ND found for the facts below (see Exercise 1 from Day 1).

1. $\vdash X \Rightarrow ((X \Rightarrow Y) \Rightarrow Y)$.
2. $(X \Rightarrow Y) \Rightarrow (X \Rightarrow Z) \vdash Y \Rightarrow X \Rightarrow Z$.
3. $(X \Rightarrow Y) \Rightarrow X \vdash Y \Rightarrow X$.
4. $X \Rightarrow (Y \Rightarrow Z) \vdash Y \Rightarrow X \Rightarrow Z$.
5. $X \Rightarrow Y \Rightarrow Z, X \Rightarrow Y \vdash X \Rightarrow Z$.
6. $(X \Rightarrow X) \Rightarrow Y \vdash (Y \Rightarrow Z) \Rightarrow Z$.

Solution to Exercise 1

1. In Curry-style and Church-style for λ-terms, and ND for derivations:

$$\dfrac{\dfrac{\dfrac{[y : X \Rightarrow Y]^{\circ} \qquad [x : X]^{*}}{yx : Y} \Rightarrow_e}{\lambda y.yx : (X \Rightarrow Y) \Rightarrow Y} \Rightarrow_i^{\circ}}{\lambda x.\lambda y.yx : X \Rightarrow ((X \Rightarrow Y) \Rightarrow Y)} \Rightarrow_i^{*} \qquad \dfrac{\dfrac{\dfrac{[y : X \Rightarrow Y]^{\circ} \qquad [x : X]^{*}}{yx : Y} \Rightarrow_e}{\lambda y^{X \Rightarrow Y}.yx : (X \Rightarrow Y) \Rightarrow Y} \Rightarrow_i^{\circ}}{\lambda x^{X}.\lambda y^{X \Rightarrow Y}.yx : X \Rightarrow ((X \Rightarrow Y) \Rightarrow Y)} \Rightarrow_i^{*}$$

2. In Curry-style and Church-style for λ-terms, and ND for derivations:

$$\dfrac{\dfrac{z:(X\Rightarrow Y)\Rightarrow(X\Rightarrow Z) \qquad \dfrac{[y:Y]^*}{\lambda x.y : X\Rightarrow Y}\Rightarrow_i}{z(\lambda x.y):X\Rightarrow Z}\Rightarrow_e}{\lambda y.z(\lambda x.y):Y\Rightarrow X\Rightarrow Z}\Rightarrow_i^* \qquad \dfrac{\dfrac{z:(X\Rightarrow Y)\Rightarrow(X\Rightarrow Z) \qquad \dfrac{[y:Y]^*}{\lambda x^X.y : X\Rightarrow Y}\Rightarrow_i}{z(\lambda x^X.y):X\Rightarrow Z}\Rightarrow_e}{\lambda y^Y.z(\lambda x^X.y):Y\Rightarrow X\Rightarrow Z}\Rightarrow_i^*$$

3. In Curry-style and Church-style for λ-terms, and ND for derivations:

$$\dfrac{\dfrac{z:(X\Rightarrow Y)\Rightarrow X \qquad \dfrac{[y:Y]^*}{\lambda x.y : X\Rightarrow Y}\Rightarrow_i}{z(\lambda x.y):X}\Rightarrow_e}{\lambda y.z(\lambda x.y):Y\Rightarrow X}\Rightarrow_i^* \qquad \dfrac{\dfrac{z:(X\Rightarrow Y)\Rightarrow X \qquad \dfrac{[y:Y]^*}{\lambda x^X.y : X\Rightarrow Y}\Rightarrow_i}{z(\lambda x^X.y):X}\Rightarrow_e}{\lambda y^Y.z(\lambda x^X.y):Y\Rightarrow X}\Rightarrow_i^*$$

4. In Curry-style and Church-style for λ-terms, and ND for derivations:

$$\dfrac{\dfrac{\dfrac{\dfrac{z:X\Rightarrow(Y\Rightarrow Z) \qquad [x:X]^\circ}{zx:Y\Rightarrow Z}\Rightarrow_e \qquad [y:Y]^*}{zxy:Z}\Rightarrow_e}{\lambda x.zxy:X\Rightarrow Z}\Rightarrow_i^\circ}{\lambda y.\lambda x.zxy:Y\Rightarrow X\Rightarrow Z}\Rightarrow_i^* \qquad \dfrac{\dfrac{\dfrac{\dfrac{z:X\Rightarrow(Y\Rightarrow Z) \qquad [x:X]^\circ}{zx:Y\Rightarrow Z}\Rightarrow_e \qquad [y:Y]^*}{zxy:Z}\Rightarrow_e}{\lambda x^X.zxy:X\Rightarrow Z}\Rightarrow_i^\circ}{\lambda y^Y.\lambda x^X.zxy:Y\Rightarrow X\Rightarrow Z}\Rightarrow_i^*$$

5. In Curry-style and Church-style for λ-terms, and ND for derivations:

$$\dfrac{\dfrac{\dfrac{z:X\Rightarrow Y\Rightarrow Z \quad [x:X]^*}{zx:Y\Rightarrow Z}\Rightarrow_e \qquad \dfrac{y:X\Rightarrow Y \quad [x:X]^*}{yx:Y}\Rightarrow_e}{zx(yx):Z}\Rightarrow_e}{\lambda x.zx(yx):X\Rightarrow Z}\Rightarrow_i^* \qquad \dfrac{\dfrac{\dfrac{z:X\Rightarrow Y\Rightarrow Z \quad [x:X]^*}{zx:Y\Rightarrow Z}\Rightarrow_e \qquad \dfrac{y:X\Rightarrow Y \quad [x:X]^*}{yx:Y}\Rightarrow_e}{zx(yx):Z}\Rightarrow_e}{\lambda x^X.zx(yx):X\Rightarrow Z}\Rightarrow_i^*$$

6. In Curry-style and Church-style for λ-terms, and ND for derivations:

$$\dfrac{\dfrac{[z:Y\Rightarrow Z]^* \qquad \dfrac{y:(X\Rightarrow X)\Rightarrow Y \qquad \dfrac{[x:X]^\circ}{\lambda x.x:X\Rightarrow X}\Rightarrow_i^\circ}{y\,\lambda x.x:Y}\Rightarrow_e}{z(y\,\lambda x.x):Z}\Rightarrow_e}{\lambda z.z(y\,\lambda x.x):(Y\Rightarrow Z)\Rightarrow Z}\Rightarrow_i^* \qquad \dfrac{\dfrac{[z:Y\Rightarrow Z]^* \qquad \dfrac{y:(X\Rightarrow X)\Rightarrow Y \qquad \dfrac{[x:X]^\circ}{\lambda x^X.x:X\Rightarrow X}\Rightarrow_i^\circ}{y\,\lambda x^X.x:Y}\Rightarrow_e}{z(y\,\lambda x^X.x):Z}\Rightarrow_e}{\lambda z^{Y\Rightarrow Z}.z(y\,\lambda x^X.x):(Y\Rightarrow Z)\Rightarrow Z}\Rightarrow_i^*$$

Exercise 2

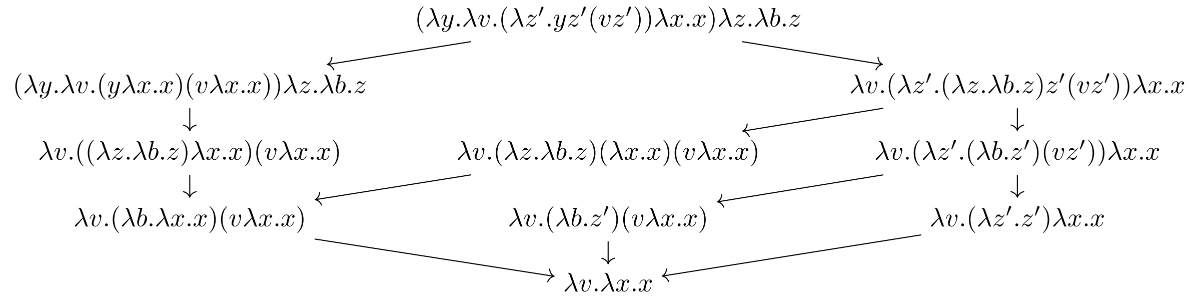
Perform all possible β-reduction steps from the λ-term decorating the derivation $\mathcal{D}$ in ND on p. 24 of Day 1, until you get a β-normal form. Is it always the same? Compare it with the normal derivation obtained by cut-elimination steps from $\mathcal{D}$.

Solution to Exercise 2

The derivation on p. 24 of Day 1 slides is $\mathcal{D}$ below, decorated with λ-terms is Curry-style.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[y:(X\Rightarrow X)\Rightarrow(B\Rightarrow X\Rightarrow X)]^\dagger \quad [z':X\Rightarrow X]^\circ}{yz':B\Rightarrow(X\Rightarrow X)}\Rightarrow_e \qquad \dfrac{[v:(X\Rightarrow X)\Rightarrow B]^* \quad [z':X\Rightarrow X]^\circ}{vz':B}\Rightarrow_e}{yz'(vz'):X\Rightarrow X}\Rightarrow_e}{\lambda z'.yz'(vz'):(X\Rightarrow X)\Rightarrow(X\Rightarrow X)}\Rightarrow_i^\circ \qquad \dfrac{[x:X]^\bullet}{\lambda x.x:X\Rightarrow X}\Rightarrow_i^\bullet}{(\lambda z'.yz'(vz'))\lambda x.x:X\Rightarrow X}\Rightarrow_e}{\lambda v.(\lambda z'.yz'(vz'))\lambda x.x:((X\Rightarrow X)\Rightarrow B)\Rightarrow(X\Rightarrow X)}\Rightarrow_i^*}{\lambda y.\lambda v.(\lambda z'.yz'(vz'))\lambda x.x:((X\Rightarrow X)\Rightarrow(B\Rightarrow X\Rightarrow X))\Rightarrow((X\Rightarrow X)\Rightarrow B)\Rightarrow(X\Rightarrow X))}\Rightarrow_i^\dagger \qquad \dfrac{\dfrac{[z:X\Rightarrow X]^\dagger}{\lambda b.z:B\Rightarrow X\Rightarrow X}\Rightarrow_i}{\lambda z.\lambda b.z:(X\Rightarrow X)\Rightarrow(B\Rightarrow X\Rightarrow X)}\Rightarrow_i^\dagger}{(\lambda y.\lambda v.(\lambda z'.yz'(vz'))\lambda x.x)\lambda z.\lambda b.z:((X\Rightarrow X)\Rightarrow B)\Rightarrow(X\Rightarrow X)}\Rightarrow_e$$

Thus, the λ-term decorating $\mathcal{D}$ is $t = (\lambda y.\lambda v.(\lambda z'.yz'(vz'))\lambda x.x)\lambda z.\lambda b.z$. All possible β-reduction steps from t are the following:



In any case, every β-reduction sequence eventually reaches the same β-normal term $\lambda v.\lambda x.x$. Note that $\lambda v.\lambda x.x$ is the decoration of the derivation $\mathcal{D}_0$ below, which is the derivation without redexes to which $\mathcal{D}$ eventually reduces via cut-elimination steps (see Exercise 3 from day 1).

$$\dfrac{\dfrac{[x : X]^{\bullet}}{\lambda x.x : X \Rightarrow X}\Rightarrow_i^{\bullet}}{\lambda v.\lambda x.x : ((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}\Rightarrow_i$$

Exercise 3

Prove rigorously the following facts ($f^n x = \overbrace{f(\dots(f\ x)\dots)}^{n \text{ times } f}$ for any $n \in \mathbb{N}$):

1. $\lambda x.xx$ is untypable in Curry-style, $\lambda x^A.xx$ is untypable in Church-style for any type A;
2. in Church-style, $\lambda f^Y.\lambda x^X.f^n x$ is not typable for any $n > 0$ but $\lambda f^Y.\lambda x^X.x$ is typable;
3. $\lambda f.\lambda x.f^n x$ is typable in Curry-style, for all $n \in \mathbb{N}$.

Solution to Exercise 3

1. *Curry-style:* Suppose by absurd that $\lambda x.xx$ is typable in the simply typed λ-calculus in Curry-style. Then there would be a derivation $\mathcal{D}$ of $\lambda x.xx$. Its last rule is necessarily λ (because the term in the derivation is an abstraction), and its second to last rule is necessarily @ (because the body of the abstraction in the derivation is an application), and its leaves are necessarily var rules (because the proper subterms of the application are variables), hence $\mathcal{D}$ has the form below, for some types A, B, C.

$$\dfrac{\dfrac{\dfrac{}{x : A \vdash x : C \Rightarrow B}\mathsf{var} \qquad \dfrac{}{x : A \vdash x : C}\mathsf{var}}{x : A \vdash xx : B}@}{\vdash \lambda x.xx : A \Rightarrow B}\lambda$$

   To make $\mathcal{D}$ a valid derivation, the two instances of the rule var must be correct, thus $A = C \Rightarrow B$ and $A = C$ must hold, which implies that $C = C \Rightarrow B$, but this is impossible for any type B, C.

   Church-style: Suppose by absurd that $\lambda x^A.xx$ is typable in the simply typed λ-calculus in Church-style. Then there would be a derivation $\mathcal{D}$ of $\lambda x^A.xx$. Its last rule is necessarily λ abstracting a variable of type A (because the term in the derivation is an abstraction of type A), and its second to last rule is necessarily @ (because the body of the abstraction is an application), and its leaves are necessarily var rules (because the proper subterms of the application are variables), hence $\mathcal{D}$ has the form below, for some types B, C.

$$\dfrac{\dfrac{\dfrac{}{x : A \vdash x : C \Rightarrow B}\mathsf{var} \qquad \dfrac{}{x : A \vdash x : C}\mathsf{var}}{x : A \vdash xx : B}@}{\vdash \lambda x^A.xx : A \Rightarrow B}\lambda$$

   To make $\mathcal{D}$ a valid derivation, the two instances of the rule var must be correct, thus $A = C \Rightarrow B$ and $A = C$ must hold, which implies that $C = C \Rightarrow B$, but this is impossible for any type B, C.

2. The term $\lambda f^Y.\lambda x^X.x$ is typable in Church-style, as shown by the derivation below.

$$\dfrac{\dfrac{\dfrac{}{f : Y, x : X \vdash x : X}\mathsf{var}}{f : Y \vdash \lambda x^X.x : X \Rightarrow X}\lambda}{\vdash \lambda f^Y.\lambda x^X.x : Y \Rightarrow X \Rightarrow X}\lambda$$

We prove by contradiction that $\lambda f^Y.\lambda x^X.f^n x$ is not typable in Church-style for any $n \in \mathbb{N}^+$. Since $n \in \mathbb{N}^+ = \mathbb{N} \setminus \{0\}$, then $f^n x = f(f^{n-1}x)$ where $n-1 \in \mathbb{N}$. Suppose by absurd that $\lambda f^Y.\lambda x^X.f^n x$ is typable in the simply typed λ-calculus in Church-style. Then there would be a derivation $\mathcal{D}$ of $\lambda f^Y.\lambda x^X.f^n x$. Its two last rules are necessarily λ (because the term in the derivation is a double abstraction), and its third to last rule is necessarily @ (because the body of the double abstraction is the application $f(f^{n-1}x)$), and the left premise of the @ rule is necessarily a var rule (because the left subterm of the application is a variable), hence $\mathcal{D}$ has the form below, for some types A, B.

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{f : Y,\, x : X \vdash f : B \Rightarrow A}\,\text{var} \qquad \dfrac{\vdots}{f : Y,\, x : X \vdash f^{n-1}x : B}}{f : Y,\, x : X \vdash f^n x : A}\,@}{f : Y \vdash \lambda x^X.x : X \Rightarrow A}\,\lambda}{\vdash \lambda f^Y.\lambda x^X.x : Y \Rightarrow X \Rightarrow A}\,\lambda
$$

To make $\mathcal{D}$ a valid derivation, the left instance of the rule var must be correct, thus $Y = B \Rightarrow A$ must hold for some types A, B, but this is impossible because Y is a ground type.

3. We first prove the following.

Fact. *For all $n \in \mathbb{N}$, there is a derivation of $f : X \Rightarrow X,\, x : X \vdash f^n x : X$ (in Curry-style and Church-style).*

Proof. By induction on $n \in \mathbb{N}$. Cases:

(a) $n = 0$: then, $f^0 x = x$ and hence the derivation below concludes.

$$
\dfrac{}{f : X \Rightarrow X,\, x : X \vdash x : X}\,\text{var}
$$

(b) $n > 0$: then $f^n x = f(f^{n-1}x)$ and by induction hypothesis there is a derivation $\mathcal{D}$ of $f : X \Rightarrow X,\, x : X \vdash f^{n-1}x : X$. The derivation below concludes.

$$
\dfrac{\dfrac{}{f : X \Rightarrow X,\, x : X \vdash f : X \Rightarrow X}\,\text{var} \qquad \dfrac{\vdots\,\mathcal{D}}{f : X \Rightarrow X,\, x : X \vdash f^{n-1}x : X}}{f : X \Rightarrow X,\, x : X \vdash f^n x : X}\,@
$$

□

We can now show that, for all $n \in \mathbb{N}$, the term $\lambda f.\lambda x.f^n x$ is typable in Curry-style. Indeed, by the fact above, there is a derivation $\mathcal{D}$ of $f : X \Rightarrow X,\, x : X \vdash f^n x : X$ for all $n \in \mathbb{N}$. The derivation below concludes:

$$
\dfrac{\dfrac{\dfrac{\vdots\,\mathcal{D}}{f : X \Rightarrow X,\, x : X \vdash f^n x : X}}{f : X \Rightarrow X \vdash \lambda x.f^n x : X \Rightarrow X}\,\lambda}{\vdash \lambda f.\lambda x.f^n x : (X \Rightarrow X) \Rightarrow X \Rightarrow X}\,\lambda
$$

Exercise 13

In a ARS $(A, \to)$, prove that $t \in A$ is SN if and only if for every $t' \in A$, if $t \to t'$ then t' is SN.

Solution to Exercise 13

$$
\begin{array}{cc}
 & t \text{ is not strongly normalizing} \\
\iff & \text{there is an infinite sequence } (t_i)_{i\in\mathbb{N}} \text{ such that } t_0 = t \text{ and } t_i \to t_{i+1} \text{ for all } i \in \mathbb{N} \\
\iff & \text{there is } t' \text{ such that } t \to t' \text{ and an infinite sequence } (t'_i)_{i\in\mathbb{N}} \text{ such that } t_0 = t' \text{ and } t'_i \to t'_{i+1} \text{ for all } i \in \mathbb{N} \\
\iff & \text{there is } t' \text{ such that } t \to t' \text{ and } t' \text{ is not strongly normalizing.}
\end{array}
$$

Exercises from Day 3 (https://pageperso.lis-lab.fr/~giulio.guerrieri/ECI2024/day3.pdf)

Exercise 1

Write the tree representation of following terms (as on p. 7 of Day 3), specifying $m, n \in \mathbb{N}$ and the subtrees corresponding to $h, t_1, \dots, t_m$: x, I, $\lambda x.Ixx$, $\lambda x.I(xx)$, $\lambda x.xxx(xx)$, II (where $I = \lambda z.z$).

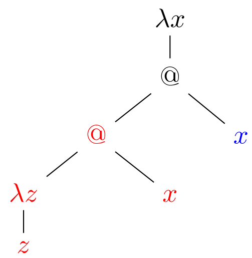
Solution to Exercise 1

The subtree corresponding to the head h (head variable or head redex) is marked in red, the ones corresponding to t_1, t_2 and t_3 (if any) are marked in blue, gray and green, respectively.

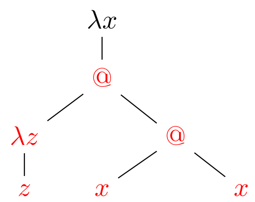
1. x: then $m = 0 = n$ and

2. $I = \lambda z.z$: then $n = 1$, $m = 0$ and

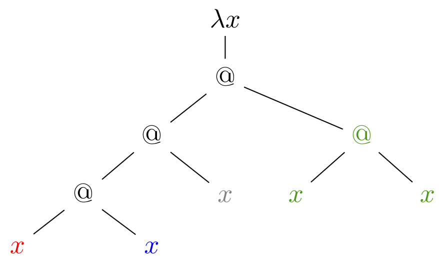
3. $\lambda x.Ixx = \lambda x.(\lambda z.z)xx$: then $n = 1$, $m = 1$ and



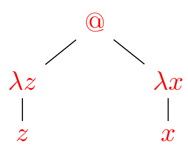
4. $\lambda x.I(xx) = \lambda x.(\lambda z.z)(xx)$: then $n = 1$, $m = 0$ and



5. $\lambda x.xxx(xx)$: then $n = 1$, $m = 3$ and



6. $II = (\lambda z.z)\lambda x.x$: then $n = 0$, $m = 0$ and



Exercise 3

Consider the η-reduction $\to_\eta$ defined below, which can be fired everywhere in a term. Prove that $\to_\eta$ is strongly normalizing.

$$\lambda x.tx \to_\eta t \qquad \text{if } x \notin \mathsf{fv}(t)$$

Solution to Exercise 3

Fact. *Let $\to$ be a reduction on a set A: $t \in A$ is strongly normalizing (for $\to$) if and only if every t' such that $t \to t'$ is strongly normalizing (for $\to$).*

Proof. Let $t \in A$.

$$\begin{array}{rc} & t \text{ is not strongly normalizing} \\ \iff & \text{there is an infinite sequence } (t_i)_{i\in\mathbb{N}} \text{ such that } t_0 = t \text{ and } t_i \to t_{i+1} \text{ for all } i \in \mathbb{N} \\ \iff & \text{there is } t' \text{ such that } t \to t' \text{ and an infinite sequence } (t'_i)_{i\in\mathbb{N}} \text{ such that } t_0 = t' \text{ and } t'_i \to t'_{i+1} \text{ for all } i \in \mathbb{N} \\ \iff & \text{there is } t' \text{ such that } t \to t' \text{ and } t' \text{ is not strongly normalizing.} \end{array}$$

□

Formally, *η-reduction* is defined on the terms of the untyped λ-calculus by the rules below.

$$\frac{x \notin \mathsf{fv}(t)}{\lambda x.tx \to_\eta t} \qquad \frac{t \to_\eta t'}{\lambda x.t \to_\eta \lambda x.t'} \qquad \frac{t \to_\eta t'}{ts \to_\eta t's} \qquad \frac{t \to_\eta t'}{st \to_\eta st'}$$

Let the *size* $|t| \in \mathbb{N}$ of a term t be defined by structural induction on t as follows:

$$|x| = 1 \qquad |\lambda x.t| = 1 + |t| \qquad |st| = 1 + |s| + |t|$$

Lemma. *If $t \to_\eta t'$ then $|t| > |t'|$.*

Proof. By induction on the definition of $t \to_\eta t'$. Cases:

- If $\lambda x.tx \to_\eta t$ with $x \notin \mathsf{fv}(t)$, then $|\lambda x.tx| = 3 + |t| > |t|$.
- If $\lambda x.t \to_\eta \lambda x.t'$ with $t \to_\eta t'$, then $|t| > |t'|$ by induction hypothesis, hence $|\lambda x.t| = 1 + |t| > 1 + |t'| = |\lambda x.t'|$.
- If $ts \to_\eta t's$ with $t \to_\eta t'$, then $|t| > |t'|$ by induction hypothesis, hence $|ts| = 1+|t|+|s| > 1+|t'|+|s| = |t's|$.
- If $st \to_\eta st'$ with $t \to_\eta t'$, then $|t| > |t'|$ by induction hypothesis, so $|st| = 1+|s|+|t| > 1+|s|+|t'| = |st'|$. □

Corollary. *$\to_\eta$ is strongly normalizing.*

Proof. Let t be a term. We prove that t is strongly η-normalizing by induction on $|t| \in \mathbb{N}$. Cases:

- If t is η-normal, we are done.
- If $t \to_\eta t'$, then $|t| > |t'|$ by the lemma above, and hence t' is strongly η-normalizing by induction hypothesis; we conclude that t is strongly η-normalizing thanks to the fact above. □

Exercise 4

Find a term r such that $rt \to_\beta^* t(tr)$ for every t (*Hint:* use the fixpoint combinator Θ).

Solution to Exercise 4

Saying that r is an term such that $rt \to_\beta^* t(tr)$ for every term t amounts to say that $rx \to_\beta^* x(xr)$ for any variable $x \notin \mathsf{fv}(r)$, which follows from $r \to_\beta^* \lambda x.x(xr)$, which in turn follows from $r \to_\beta^* (\lambda y.\lambda x.x(xy))r$. Note that r is a fixed point of $\lambda y.\lambda x.x(xy)$. Let $r = \Theta\, \lambda y.\lambda x.x(xy)$, where Θ is the fixpoint combinator, that is, $\Theta t \to_\beta^* t(\Theta t)$ for every term t. Now, $r = \Theta\, \lambda y.\lambda x.x(xy) \to_\beta^* (\lambda y.\lambda x.x(xy))(\Theta\, \lambda y.\lambda x.x(xy)) = (\lambda y.\lambda x.x(xy))r \to_\beta \lambda x.x(xr)$. Therefore, $rt \to_\beta^* (\lambda x.x(xr))t \to_\beta t(tr)$ for every term t.

Exercise 5

Prove that $\underline{succ}\,\underline{n} \to_\beta^* \underline{n+1}$ for all $n \in \mathbb{N}$, and $\underline{add}\,\underline{m}\,\underline{n} \to_\beta^* \underline{m+n}$ for all $m, n \in \mathbb{N}$.

Solution to Exercise 5

$$\begin{aligned} \underline{succ}\,\underline{n} = (\lambda m.\lambda f.\lambda x.f(mfx))\lambda g.\lambda y.g^n y &\to_\beta \lambda f.\lambda x.f((\lambda g.\lambda y.g^n y)fx) \\ &\to_\beta \lambda f.\lambda x.f((\lambda y.f^n y)x) \to_\beta \lambda f.\lambda x.f(f^n x) = \lambda f.\lambda x.f^{n+1}x = \underline{n+1} \end{aligned}$$

$$\begin{aligned} \underline{add}\,\underline{m}\,\underline{n} &= (\lambda m.\lambda n.\lambda f.\lambda x.mf(nfx))(\lambda g.\lambda y.g^m y)(\lambda h.\lambda z.h^n z) \\ &\to_\beta (\lambda n.\lambda f.\lambda x.(\lambda g.\lambda y.g^m y)f(nfx))(\lambda h.\lambda z.h^n z) \\ &\to_\beta (\lambda n.\lambda f.\lambda x.(\lambda y.f^m y)(nfx))(\lambda h.\lambda z.h^n z) \to_\beta (\lambda n.\lambda f.\lambda x.f^m(nfx))(\lambda h.\lambda z.h^n z) \\ &\to_\beta \lambda f.\lambda x.f^m((\lambda h.\lambda z.h^n z)fx) \to_\beta \lambda f.\lambda x.f^m((\lambda z.f^n z)x) \\ &\to_\beta \lambda f.\lambda x.f^m(f^n x) = \lambda f.\lambda x.f^{m+n}x = \underline{m+n} \end{aligned}$$

Exercise 6

Find terms t, t', s, s' such that $t =_\alpha t'$, $s =_\alpha s'$ and $t[s/x] \neq_\alpha t'[s'/x]$ (where $=_\alpha$ is α-equivalence and $t[s/x]$ is *naïve* substitution, see p. 10 on Day 2 slides).

Solution to Exercise 6

Let $t = \lambda y.x$ and $t' = \lambda z.x$ where x, y, z are pairwise distinct variables, let $s = z = s'$. Thus,

$$t[s/x] = (\lambda y.x)[z/x] = \lambda y.z \neq_\alpha \lambda z.z = (\lambda z.x)[z/x] = t'[s'/x].$$

Exercises from Day 4 (https://pageperso.lis-lab.fr/~giulio.guerrieri/ECI2024/day4.pdf)

Exercise 3

Prove that all derivations in NI for $(\lambda x.xx)\lambda y.y$ have the form $\mathcal{D}_A^{\delta,I}$ shown on p. 8 of Day 4, for any linear type A.

Solution to Exercise 3

Every derivation in NI for $(\lambda x.xx)\lambda y.y$ has the form below for some $m, n \in \mathbb{N}$ and some linear types $A_0, \dots, A_n, B_1, \dots, B_m$, where $\mathcal{D}^{\delta,n}_{A_0,\dots,A_n}$ and $\mathcal{D}^I_{B_i}$ are the derivations in NI defined on p. 7 of Day 4 slides:

$$\dfrac{\dfrac{\vdots\, \mathcal{D}^{\delta,n}_{A_0,\dots,A_n}}{\vdash \lambda x.xx : [[A_1,\dots,A_n] \multimap A_0, A_1, \dots, A_n] \multimap A_0} \qquad \dfrac{\left(\dfrac{\vdots\, \mathcal{D}^I_{B_i}}{\vdash \lambda y.y : [B_i] \multimap B_i}\right)_{1 \le i \le m}}{\vdash \lambda y.y : [[B_1] \multimap B_1, \dots, [B_m] \multimap B_m]}\,!}{\vdash (\lambda x.xx)\lambda z.z : A_0}\,@$$

To make the last rule @ valid, $[[A_1,\dots,A_n] \multimap A_0, A_1, \dots, A_n] = [[B_1] \multimap B_1, \dots, [B_m] \multimap B_m]$. Therefore, $n + 1 = m$ and $n = 1$, hence $m = 2$. Thus, the identity above becomes $[[A_1] \multimap A_0, A_1] = [[B_1] \multimap B_1, [B_2] \multimap B_2]$. As a consequence, $A_1 = A_0 = [A] \multimap A$ and $B_1 = B_2 = A$, for any linear type A. So, every derivation in NI of $(\lambda x.xx)\lambda y.y$ is necessarily of the form below, for any linear type A.

$$\dfrac{\dfrac{\vdots\, \mathcal{D}^{\delta,1}_{[A]\multimap A,[A]\multimap A}}{\vdash \lambda x.xx : \big[[[A]\multimap A] \multimap [A]\multimap A,\ [A]\multimap A\big] \multimap [A]\multimap A} \qquad \dfrac{\dfrac{\vdots\, \mathcal{D}^I_{[A]\multimap A}}{\vdash \lambda y.y : [[A]\multimap A] \multimap [A]\multimap A} \quad \dfrac{\vdots\, \mathcal{D}^I_A}{\vdash \lambda y.y : [A] \multimap A}}{\vdash \lambda y.y : \big[[[A]\multimap A] \multimap [A]\multimap A,\ [A]\multimap A\big]}\,!}{\vdash (\lambda x.xx)\lambda y.y : [A] \multimap A}\,@$$

Exercise 9

Prove rigorously the two lemmas on p. 13 and the two lemmas on p. 16 of Day 4.

Lemma (Typing $h\beta$-normal forms, p. 13 of Day 4)**.** *Let t be $h\beta$-normal. If $\mathcal{D} \rhd_{\mathsf{NI}} \Gamma \vdash t : A$ then $|t|_{h\beta} \le |\mathcal{D}|$.*

Proof. Since t is $h\beta$-normal, $t = \lambda x_n \dots \lambda x_1.yt_1 \dots t_m$ for some $m, n \in \mathbb{N}$. We prove the statement by induction on $|t|_{h\beta} \in \mathbb{N}$. Cases (as A is a linear type, the last rule in $\mathcal{D}$ cannot be !):

- $n = 0 = m$: Then, $t = y$ and hence $\mathcal{D}$ is necessarily as below, with $\Gamma = y : [A]$ and $|\mathcal{D}| = 1 = |t|_{h\beta}$.

$$\mathcal{D} = \dfrac{}{y : [A] \vdash y : A}\,\mathsf{var}$$

- $n = 0$, $m > 0$: Then, $t = yt_1 \dots t_m$. Let $t' = yt_1 \dots t_{m-1}$, so $t = t't_m$ (this makes sense because $m > 0$). By necessity, $\mathcal{D}$ is as below, with $\Gamma = \Gamma' \uplus \Gamma_m$.

$$\mathcal{D} = \dfrac{\dfrac{\vdots\, \mathcal{D}'}{\Gamma' \vdash t' : M \multimap A} \qquad \dfrac{\vdots\, \mathcal{D}_m}{\Gamma_m \vdash t_m : M}}{\Gamma' \uplus \Gamma_m \vdash t't_m : A}\,@$$

  As t' is $h\beta$-normal with $|t'|_{h\beta} < 1 + |t'|_{h\beta} = |t|_{h\beta}$, we have $|\mathcal{D}'| \ge |t'|_{h\beta}$ by induction hypothesis. Therefore, $|\mathcal{D}| = 1 + |\mathcal{D}'| + |\mathcal{D}_m| \ge 1 + |\mathcal{D}'| \ge 1 + |t'|_{h\beta} = |t|_{h\beta}$.

- $n > 0$: Then, $t = \lambda x_n \dots \lambda x_1.yt_1 \dots t_m$. Let $t' = \lambda x_{n-1} \dots \lambda x_1.yt_1 \dots t_m$, so $t = \lambda x_n.t'$ (this makes sense because $n > 0$). By necessity, $\mathcal{D}$ is as below, with $A = M \multimap B$.

$$\mathcal{D} = \cfrac{\begin{array}{c}\vdots\,\mathcal{D}' \\ \Gamma, x_n : M \vdash t' : B\end{array}}{\Gamma \vdash \lambda x_n.t' : M \multimap B}\,\lambda$$

Since t' is $h\beta$-normal with $|t'|_{h\beta} < 1 + |t'|_{h\beta} = |t|_{h\beta}$, we have $|\mathcal{D}'| \geq |t'|_{h\beta}$ by induction hypothesis. Therefore, $|\mathcal{D}| = 1 + |\mathcal{D}'| \geq 1 + |t'|_{h\beta} = |t|_{h\beta}$. □

Lemma (Typability of $h\beta$-normal forms, p. 16 of Day 4)**.** *If t be $h\beta$-normal, then there is $\mathcal{D} \triangleright_{\mathsf{NI}} \Gamma \vdash t : A$ with $|t|_{h\beta} = |\mathcal{D}|$, for some environment Γ and linear type A.*

Proof. To have the right induction hypothesis, we prove the following stronger statement:

> If t be $h\beta$-normal, then there is a derivation $\mathcal{D} \triangleright_{\mathsf{NI}} \Gamma \vdash t : A$ with $|t|_{h\beta} = |\mathcal{D}|$, for some environment Γ and linear type A. If, moreover, $t = yt_1 \dots t_m$ for some $m \in \mathbb{N}$ and terms $t_1, \dots, t_m$, then for every linear type A and $k \in \mathbb{N}$, there is an environment Γ and a derivation $\mathcal{D} \triangleright_{\mathsf{NI}} \Gamma \vdash t : \underbrace{[\,] \multimap \cdots \multimap [\,]}_{k \text{ times } [\,]} \multimap A$, with $|\mathcal{D}|_\lambda = 0$, $|\mathcal{D}|_{\mathsf{var}} = 1$ and $|\mathcal{D}|_@ = m$.

Since t is $h\beta$-normal, $t = \lambda x_n \dots \lambda x_1.yt_1 \dots t_m$ for some $m, n \in \mathbb{N}$. We prove the stronger statement by induction on $|t|_{h\beta} \in \mathbb{N}$. Cases:

- $n = 0 = m$: Then $t = y$, which is not an abstraction. Let A be a linear type and $k \in \mathbb{N}$. Let $\mathcal{D}$ be as below, hence $|\mathcal{D}| = 1 = |t|_{h\beta}$ and $|\mathcal{D}|_\lambda = 0$, $|\mathcal{D}|_{\mathsf{var}} = 1$ and $|\mathcal{D}|_@ = 0 = m$.

$$\mathcal{D} = \cfrac{}{y : [\underbrace{[\,] \multimap \cdots \multimap [\,]}_{k \text{ times } [\,]} \multimap A] \vdash y : \underbrace{[\,] \multimap \cdots \multimap [\,]}_{k \text{ times } [\,]} \multimap A}\,\mathsf{var}$$

- $n = 0$, $m > 0$: Then $t = yt_1 \dots t_m$, which is not an abstraction. Let A be a linear type and $k \in \mathbb{N}$. Let $t' = yt_1 \dots t_{m-1}$, so $t = t't_m$ (this makes sense because $m > 0$). As t' is $h\beta$-normal and not an abstraction, with $|t'|_{h\beta} < 1 + |t'|_{h\beta} = |t|_{h\beta}$, then by induction hypothesis there is a derivation $\mathcal{D}' \triangleright_{\mathsf{NI}} \Gamma \vdash t' : \underbrace{[\,] \multimap \cdots \multimap [\,]}_{k+1 \text{ times } [\,]} \multimap A$ with $|\mathcal{D}'| = |t'|_{h\beta}$ and $|\mathcal{D}'|_\lambda = 0$, $|\mathcal{D}'|_{\mathsf{var}} = 1$ and $|\mathcal{D}'|_@ = m - 1$. Let $\mathcal{D}$ be as below.

$$\mathcal{D} = \cfrac{\begin{array}{c}\vdots\,\mathcal{D}' \\ \Gamma \vdash t' : \overbrace{[\,] \multimap \cdots \multimap [\,]}^{k+1 \text{ times } [\,]} \multimap A\end{array} \qquad \cfrac{}{\vdash t_m : [\,]}\,!}{\Gamma \vdash t't_m : \underbrace{[\,] \multimap \cdots \multimap [\,]}_{k \text{ times } [\,]} \multimap A}\,@$$

Hence, $|\mathcal{D}| = 1 + |\mathcal{D}'| = 1 + |t'|_{h\beta} = |t|_{h\beta}$ with $|\mathcal{D}|_\lambda = |\mathcal{D}'|_\lambda = 0$, $|\mathcal{D}|_{\mathsf{var}} = |\mathcal{D}'|_{\mathsf{var}} = 1$ and $|\mathcal{D}|_@ = 1 + |\mathcal{D}'|_@ = 1 + m - 1 = m$.

- $n > 0$: Then $t = \lambda x_n \dots \lambda x_1.yt_1 \dots t_m$, which is an abstraction because $n > 0$. Let $t' = \lambda x_{n-1} \dots \lambda x_1.yt_1 \dots t_m$, so $t = \lambda x_n.t'$ (this makes sense because $n > 0$). As t' is $h\beta$-normal with $|t'|_{h\beta} < 1 + |t'|_{h\beta} = |t|_{h\beta}$, by induction hypothesis there is $\mathcal{D}' \triangleright_{\mathsf{NI}} \Gamma, x_n : M \vdash t' : B$ for some environment $\Gamma, x_n : M$ and linear type B, with $|\mathcal{D}| = |t'|_{h\beta}$. Let $\mathcal{D}$ be as below, hence $|\mathcal{D}| = 1 + |\mathcal{D}'| = 1 + |t'|_{h\beta} = |t|_{h\beta}$.

$$\mathcal{D} = \cfrac{\begin{array}{c}\vdots\,\mathcal{D}' \\ \Gamma, x_n : M \vdash t' : B\end{array}}{\Gamma \vdash \lambda x_n.t' : M \multimap B}\,\lambda$$

□

Exercises from Day 5 (https://pageperso.lis-lab.fr/~giulio.guerrieri/ECI2024/day5.pdf)

Exercise 6

Prove rigorously the two lemmas on p. 7 and the the lemma on p. 9 of Day 5.

Lemma (Spreading of shrinkingness, p. 7 of Day 5)**.** *Let t be β-normal and not an abstraction. Let $\mathcal{D} \triangleright_{\mathsf{NI}} \Gamma \vdash t : A$. If Γ is co-shrinking then A is co-shrinking.*

Proof. Since t is β-normal and not an abstraction, $t = yt_1 \dots t_m$ for some $m \in \mathbb{N}$ with β-normal $t_1, \dots, t_m$. We proceed by induction on $m \in \mathbb{N}$ (as A is a linear type, the last rule of $\mathcal{D}$ cannot be !). Cases:

- $m = 0$: Then, $t = y$ and thus $\mathcal{D}$ is as below, with $\Gamma = y : [A]$. Since Γ is co-shrinking, so are $[A]$ and hence A.

$$\mathcal{D} = \dfrac{}{y : [A] \vdash y : A}\,\mathsf{var}$$

- $m > 0$: Then, $t = yt_1 \dots t_m$. Let $t' = yt_1 \dots t_{m-1}$, so $t = t't_m$ (this makes sense because $m > 0$). Thus, $\mathcal{D}$ is as below, with $\Gamma = \Gamma' \uplus \Gamma_m$.

$$\mathcal{D} = \dfrac{\begin{array}{c}\vdots\,\mathcal{D}'\\ \Gamma' \vdash t' : M \multimap A\end{array} \qquad \begin{array}{c}\vdots\,\mathcal{D}_m\\ \Gamma_m \vdash t_m : M\end{array}}{\Gamma' \uplus \Gamma_m \vdash t't_m : A}\,@$$

  Since Γ is co-shrinking, so is Γ'. We can then apply the induction hypothesis to $\mathcal{D}' \triangleright_{\mathsf{NI}} \Gamma' \vdash t' : M \multimap A$, because t' is β-normal and not an abstraction: thus, $M \multimap A$ is co-shrinking. Hence, A is co-shrinking too. □

Lemma (Typing β-normal forms in a co-shrinking environment, p. 7 of Day 5)**.** *Let t be β-normal and let $\mathcal{D} \triangleright_{\mathsf{NI}} \Gamma \vdash t : A$. If Γ is co-shrinking and (A is shrinking or t is not an abstraction), then $|t| \leq |\mathcal{D}|$.*

Proof. Since t is β-normal, $t = \lambda x_n \dots \lambda x_1.yt_1 \dots t_m$ for some $m, n \in \mathbb{N}$, with $t_1, \dots, t_m$ β-normal. We proceed by induction on the size $|t| \in \mathbb{N}$ of t. Cases (as A is a linear type, the last rule in $\mathcal{D}$ cannot be !):

- $n = 0 = m$: Then, $t = y$ and hence $\mathcal{D}$ is necessarily as below, with $\Gamma = y : [A]$ and $|\mathcal{D}| = 1 = |t|$.

$$\mathcal{D} = \dfrac{}{y : [A] \vdash y : A}\,\mathsf{var}$$

- $n = 0$, $m > 0$: Then, $t = yt_1 \dots t_m$. Let $t' = yt_1 \dots t_{m-1}$, so $t = t't_m$ (this makes sense because $m > 0$). By necessity, $\mathcal{D}$ is as below, with $\Gamma = \Gamma' \uplus \Gamma_m$ and $\Gamma_m = \biguplus_{i=1}^{k} \Gamma_m^i$ and $M = [A_1, \dots, A_k]$ for some $k \in \mathbb{N}$.

$$\mathcal{D} = \dfrac{\begin{array}{c}\vdots\,\mathcal{D}'\\ \Gamma' \vdash t' : M \multimap A\end{array} \qquad \dfrac{\left(\begin{array}{c}\vdots\,\mathcal{D}_m^i\\ \Gamma_m^i \vdash t_m : A_i\end{array}\right)_{1 \leq i \leq k}}{\Gamma_m \vdash t_m : M}\,!}{\Gamma' \uplus \Gamma_m \vdash t't_m : A}\,@$$

  Since Γ is co-shrinking, so is Γ'. We can then apply the induction hypothesis to $\mathcal{D}' \triangleright_{\mathsf{NI}} \Gamma' \vdash t' : M \multimap A$, because t' is β-normal and not an abstraction with $|t'| < 1 + |t'| + |t_m| = |t|$: thus, $|\mathcal{D}'| \geq |t'|$. By the lemma above (spreading of shrinkingness), $M \multimap A$ is co-shrinking, which entails that: A is co-shrinking, M is shrinking and hence $k > 0$ (that is, $M \neq [\,]$), and A_i is shrinking for all $1 \leq i \leq k$. Since Γ is co-shrinking, so is Γ_m^i for all $1 \leq i \leq k$. We can then apply the induction hypothesis to $\mathcal{D}_m^i \triangleright_{\mathsf{NI}} \Gamma_m^i \vdash t' : A_i$ for all $1 \leq i \leq k$, because t_m is β-normal with $|t_m| < 1 + |t'| + |t_m| = |t|$: thus, $|\mathcal{D}_m^i| \geq |t|$ for all $1 \leq i \leq k$. So, $|\mathcal{D}| = 1 + |\mathcal{D}'| + \sum_{i=1}^{k} |\mathcal{D}_m^i| \geq 1 + |\mathcal{D}'| + |\mathcal{D}_m^1| \geq 1 + |t'| + |t_m| = |t|$ (the first inequality hold because $k > 0$).

- $n > 0$: Then, $t = \lambda x_n \dots \lambda x_1.yt_1 \dots t_m$ which is an abstraction. Let $t' = \lambda x_{n-1} \dots \lambda x_1.yt_1 \dots t_m$, so $t = \lambda x_n.t'$ (this makes sense because $n > 0$). Thus, $\mathcal{D}$ is as below, with $A = M \multimap B$ shrinking, as t is an abstraction.

$$\mathcal{D} = \dfrac{\begin{array}{c}\vdots\,\mathcal{D}'\\ \Gamma, x_n : M \vdash t' : B\end{array}}{\Gamma \vdash \lambda x_n.t' : M \multimap B}\,\lambda$$

  Since $M \multimap B$ is shrinking, so is B and M is co-shrinking. Therefore, $\Gamma, x_n : M$ is co-shrinking. We can then apply the induction hypothesis to $\mathcal{D}' \triangleright_{\mathsf{NI}} \Gamma, x_n : M \vdash t' : B$, because t' is β-normal with $|t'| < 1 + |t'| = |t|$: thus, $|\mathcal{D}'| \geq |t'|$. Hence, $|\mathcal{D}| = 1 + |\mathcal{D}'| \geq 1 + |t'| = |t|$. □

Lemma (Shrinking typability of β-normal forms, p. 9 of Day 5)**.** *If t be β-normal, then there is a shrinking derivation $\mathcal{D} \triangleright_{\mathsf{NI}} \Gamma \vdash t : A$ with $|t| = |\mathcal{D}|$, for some environment Γ and linear type A.*

Proof. To have the right induction hypothesis, we prove the following stronger statement:

If t be β-normal, then there is a shrinking derivation $\mathcal{D} \rhd_{\mathsf{NI}} \Gamma \vdash t : A$ with $|t| = |\mathcal{D}|$, for some environment Γ and linear type A. If, moreover, $t = yt_1 \dots t_m$ for some $m \in \mathbb{N}$ and β-normal $t_1, \dots, t_m$, then for every $k \in \mathbb{N}$ and co-shrinking linear type A and shrinking linear types $A_1, \dots, A_k$, there is a derivation $\mathcal{D} \rhd_{\mathsf{NI}} \Gamma \vdash t : [A_1] \multimap \dots \multimap [A_k] \multimap A$ for some co-shrinking environment Γ.

Since t is β-normal, $t = \lambda x_n \dots \lambda x_1.yt_1 \dots t_m$ for some $m, n \in \mathbb{N}$ and β-normal $t_1, \dots, t_m$. We prove the stronger statement by induction on $|t| \in \mathbb{N}$. Cases:

- $n = 0 = m$: Then $t = y$, which is not an abstraction. Let $k \in \mathbb{N}$ and A be a co-shrinking linear type and $A_1 \dots, A_k$ be shrinking linear types, thus $[A_1] \multimap \dots \multimap [A_k] \multimap A$ and $[[A_1] \multimap \dots \multimap [A_k] \multimap A]$ are co-shrinking. Let $\mathcal{D}$ be as below, so $|\mathcal{D}| = 1 = |t|$ and $y : [[A_1] \multimap \dots \multimap [A_k] \multimap A]$ is a co-shrinking environment.

$$\mathcal{D} = \frac{}{y : [[A_1] \multimap \dots \multimap [A_k] \multimap A] \vdash y : [A_1] \multimap \dots \multimap [A_k] \multimap A}\,\mathsf{var}$$

  In the particular case where $k = 0$ and $A = X$ (note that X is shrinking and co-shrinking), $\mathcal{D} \rhd_{\mathsf{NI}} y : [X] \vdash y : X$ is a shrinking derivation, since $y : [X]$ is a co-shrinking environment and X is a shrinking linear type.

- $n = 0$, $m > 0$: Then $t = yt_1 \dots t_m$, which is not an abstraction, with $t_1, \dots, t_m$ β-normal. Let $k \in \mathbb{N}$ and A be a co-shrinking linear type and $A_1, \dots A_k$ be shrinking linear types. Let $t' = yt_1 \dots t_{m-1}$, so $t = t't_m$ (this makes sense because $m > 0$). As t_m is β-normal, then by induction hypothesis there is a shrinking derivation $\mathcal{D}_m \rhd_{\mathsf{NI}} \Gamma_m \vdash t_m : B$ with $|\mathcal{D}_m| = |t_m|$, hence Γ_m is co-shrinking and B is shrinking. As t' is β-normal and not an abstraction, then by induction hypothesis there is a derivation $\mathcal{D}' \rhd_{\mathsf{NI}} \Gamma \vdash t' : [B] \multimap [A_1] \multimap \dots \multimap [A_k] \multimap A$ for some co-shrinking Γ', with $|\mathcal{D}'| = |t'|$. Let $\mathcal{D}$ be as below, hence $\Gamma \uplus \Gamma_m$ is a co-shrinking environment (because so are Γ' and Γ_m) and $|\mathcal{D}| = 1 + |\mathcal{D}'| + |\mathcal{D}_m| = 1 + |t'| + |t_m| = |t|$.

$$\mathcal{D} = \frac{\begin{array}{c}\vdots\,\mathcal{D}' \\ \Gamma' \vdash t' : [B] \multimap [A_1] \multimap \dots \multimap [A_k] \multimap A\end{array} \qquad \begin{array}{c}\vdots\,\mathcal{D}_m \\ \Gamma_m \vdash t_m : B\end{array}}{\Gamma' \uplus \Gamma_m \vdash t't_m : [A_1] \multimap \dots \multimap [A_k] \multimap A}\,@$$

  In the particular case where $k = 0$ and $A = X$ (note that X is shrinking and co-shrinking), $\mathcal{D} \rhd_{\mathsf{NI}} \Gamma' \uplus \Gamma_m \vdash t : X$ is a shrinking derivation, since $\Gamma' \uplus \Gamma_m$ is a co-shrinking environment and X is a shrinking linear type.

- $n > 0$: Then $t = \lambda x_n \dots \lambda x_1.yt_1 \dots t_m$, which is an abstraction because $n > 0$. Let $t' = \lambda x_{n-1} \dots \lambda x_1.yt_1 \dots t_m$, so $t = \lambda x_n.t'$ (this makes sense because $n > 0$). As t' is β-normal, by induction hypothesis there is a shrinking derivation $\mathcal{D}' \rhd_{\mathsf{NI}} \Gamma, x_n : M \vdash t' : B$ for some environment $\Gamma, x_n : M$ and linear type B, with $|\mathcal{D}'| = |t'|$. Let $\mathcal{D}$ be as below, hence $|\mathcal{D}| = 1 + |\mathcal{D}'| = 1 + |t'| = |t|$ and Γ is a co-shrinking environment (since so is $\Gamma, x_n : M$) and $M \multimap B$ is a shrinking linear type (because M is co-shrinking and B is shrinking).

$$\mathcal{D} = \frac{\begin{array}{c}\vdots\,\mathcal{D}' \\ \Gamma, x_n : M \vdash t' : B\end{array}}{\Gamma \vdash \lambda x_n.t' : M \multimap B}\,\lambda$$

□