# The $\lambda$-calculus: from simple types to non-idempotent intersection types

https://pageperso.lis-lab.fr/~giulio.guerrieri/ECI2024.html

Final exam — ECI 2024

Due on Wednesday 14 August 2024 anywhere on Earth. Send to giulio.guerrieri@gmail.com

## Exercise 1 (4 points)

1. (2 points) Prove that $X \Rightarrow Y, X \Rightarrow (Y \Rightarrow Z) \vdash X \Rightarrow Z$ by using $\mathsf{ND}$ or $\mathsf{ND}_{\text{seq}}$.
2. (2 points) Find the simply typed $\lambda$-term associated with the derivation found in the previous point.

## Exercise 2 (2 points)

Find the type and derivation associated with the simply typed $\lambda$-term (in Church-style) $\lambda x^{Z \Rightarrow Y \Rightarrow X}.\lambda y^{Z \Rightarrow Y}.\lambda z^{Z}.xz(yz)$.

## Exercise 3 (3 points)

Let $=_\beta$ be the reflexive, transitive and symmetric closure of $\beta$-reduction in the untyped $\lambda$-calculus, that is, for every untyped $\lambda$-terms, $t =_\beta u$ if and only if there is a finite sequence $(t_i)_{0 \leq i \leq n}$ of terms for some $n \in \mathbb{N}$ such that $t_0 = t$ and $t_n = u$, and $t_i \to_\beta t_{i+1}$ or $t_{i+1} \to_\beta t_i$ for every $0 \leq i < n$ (note that if $n = 0$ then $t = u$).

Prove that if $t =_\beta u$ then there is a term $s$ such that $t \to_\beta^* s$ and $u \to_\beta^* s$.

*Hint:* Proceed by induction on $n \in \mathbb{N}$ (for the $n$ in the definition of $=_\beta$ above) and use the confluence of $\to_\beta$.

## Exercise 4 (4 points)

Construct an untyped $\lambda$-term $F$ such that $Fxy \to_\beta^* FyxF$.

*Hint:* Use the fixpoint combinator $\Theta$ and get inspired by the method used to prove that the factorial is representable in the untyped $\lambda$-calculus, as explained in the lecture of Day 3.

## Exercise 5 (4 points)

Find a derivation in $\mathsf{NI}$ with conclusion $f : M \vdash \lambda a.f(aa) : C$, for some multi type $M$ and linear type $C$.

## Exercise 6 (3 points)

Prove that a linear type $M \multimap A$ is shrinking if and only if the multi type $M$ is co-shrinking and the linear type $A$ is shrinking.