The λ-calculus: from simple types to non-idempotent intersection types

https://pageperso.lis-lab.fr/~giulio.guerrieri/ECI2024.html/

Solutions to the final exam

Due on Wednesday 14 August 2024 anywhere on Earth. Send to giulio.guerrieri@gmail.com

Exercise 1 (4 points)

1. (2 points) Prove that $X \Rightarrow Y, X \Rightarrow (Y \Rightarrow Z) \vdash X \Rightarrow Z$ by using ND or ND_{seq}.
2. (2 points) Find the simply typed λ-term (in Curry-style or Church-style) associated with the derivation found in the previous point.

Solution to Exercise 1

1. In NI:

$$\dfrac{\dfrac{\dfrac{X \Rightarrow (Y \Rightarrow Z) \qquad [X]^*}{Y \Rightarrow Z}{\scriptstyle \Rightarrow_e} \qquad \dfrac{X \Rightarrow Y \qquad [X]^*}{Y}{\scriptstyle \Rightarrow_e}}{Z}{\scriptstyle \Rightarrow_e}}{X \Rightarrow Z}{\scriptstyle \Rightarrow_i^*}$$

In NI_{seq}:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{X \Rightarrow Y,\ X \Rightarrow (Y \Rightarrow Z),\ X \vdash X \Rightarrow (Y \Rightarrow Z)}{\scriptstyle \mathsf{ax}} \quad \dfrac{}{X \Rightarrow Y,\ X \Rightarrow (Y \Rightarrow Z),\ X \vdash X}{\scriptstyle \mathsf{ax}}}{X \Rightarrow Y,\ X \Rightarrow (Y \Rightarrow Z),\ X \vdash Y \Rightarrow Z}{\scriptstyle \Rightarrow_e} \quad \dfrac{\dfrac{}{X \Rightarrow Y,\ X \Rightarrow (Y \Rightarrow Z),\ X \vdash X \Rightarrow Y}{\scriptstyle \mathsf{ax}} \quad \dfrac{}{X \Rightarrow Y,\ X \Rightarrow (Y \Rightarrow Z),\ X \vdash X}{\scriptstyle \mathsf{ax}}}{X \Rightarrow Y,\ X \Rightarrow (Y \Rightarrow Z),\ X \vdash Y}{\scriptstyle \Rightarrow_e}}{X \Rightarrow Y,\ X \Rightarrow (Y \Rightarrow Z),\ X \vdash Z}{\scriptstyle \Rightarrow_e}}{X \Rightarrow Y,\ X \Rightarrow (Y \Rightarrow Z) \vdash X \Rightarrow Z}{\scriptstyle \Rightarrow_i}$$

2. In Curry-style:

$$\dfrac{\dfrac{\dfrac{z : X \Rightarrow (Y \Rightarrow Z) \qquad [x : X]^*}{zx : Y \Rightarrow Z}{\scriptstyle \Rightarrow_e} \qquad \dfrac{y : X \Rightarrow Y \qquad [x : X]^*}{yx : Y}{\scriptstyle \Rightarrow_e}}{zx(yx) : Z}{\scriptstyle \Rightarrow_e}}{\lambda x.zx(yx) : X \Rightarrow Z}{\scriptstyle \Rightarrow_i^*}$$

In Church-style:

$$\dfrac{\dfrac{\dfrac{z : X \Rightarrow (Y \Rightarrow Z) \qquad [x : X]^*}{zx : Y \Rightarrow Z}{\scriptstyle \Rightarrow_e} \qquad \dfrac{y : X \Rightarrow Y \qquad [x : X]^*}{yx : Y}{\scriptstyle \Rightarrow_e}}{zx(yx) : Z}{\scriptstyle \Rightarrow_e}}{\lambda x^X.zx(yx) : X \Rightarrow Z}{\scriptstyle \Rightarrow_i^*}$$

Exercise 2 (2 points)

Find the type and derivation associated with the simply typed λ-term (in Church-style) $\lambda x^{Z \Rightarrow Y \Rightarrow X}.\lambda y^{Z \Rightarrow Y}.\lambda z^Z.xz(yz)$.

Solution to Exercise 2

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[x : Z \Rightarrow Y \Rightarrow X]^{\bullet} \qquad [z : Z]^*}{xz : Y \Rightarrow X}{\scriptstyle \Rightarrow_e} \qquad \dfrac{[y : Z \Rightarrow Y]^{\circ} \qquad [z : Z]^*}{yz : Y}{\scriptstyle \Rightarrow_e}}{xz(yz) : X}{\scriptstyle \Rightarrow_e}}{\lambda z^Z.xz(yz) : Z \Rightarrow X}{\scriptstyle \Rightarrow_i^*}}{\lambda y^{Z \Rightarrow Y}.\lambda z^Z.xz(yz) : (Z \Rightarrow Y) \Rightarrow Z \Rightarrow X}{\scriptstyle \Rightarrow_i^{\circ}}}{\lambda x^{Z \Rightarrow Y \Rightarrow X}.\lambda y^{Z \Rightarrow Y}.\lambda z^Z.xz(yz) : (Z \Rightarrow Y \Rightarrow X) \Rightarrow (Z \Rightarrow Y) \Rightarrow Z \Rightarrow X}{\scriptstyle \Rightarrow_i^{\bullet}}$$

Exercise 3 (3 points)

Let $=_\beta$ be the reflexive, transitive and symmetric closure of β-reduction in the untyped λ-calculus, that is, for every untyped λ-terms, $t =_\beta u$ if and only if there is a finite sequence $(t_i)_{0\leq i\leq n}$ of terms for some $n \in \mathbb{N}$ such that $t_0 = t$ and $t_n = u$, and $t_i \to_\beta t_{i+1}$ or $t_{i+1} \to_\beta t_i$ for every $0 \leq i < n$ (note that if $n = 0$ then $t = u$).

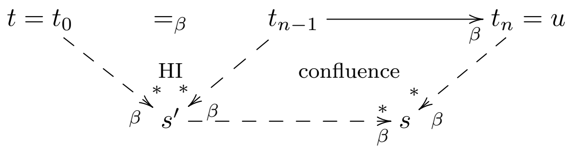
Prove that if $t =_\beta u$ then there is a term s such that $t \to_\beta^* s$ and $u \to_\beta^* s$.

Hint: Proceed by induction on $n \in \mathbb{N}$ (for the n in the definition of $=_\beta$ above) and use the confluence of $\to_\beta$.

Solution to Exercise 3

By definition of $t =_\beta u$, there is a finite sequence $(t_i)_{0\leq i\leq n}$ of terms for some $n \in \mathbb{N}$ such that $t_0 = t$ and $t_n = u$, and $t_i \to_\beta t_{i+1}$ or $t_{i+1} \to_\beta t_i$ for every $0 \leq i < n$. We proceed by induction on $n \in \mathbb{N}$. Cases:

- $n = 0$: Then, $t = u$ and the statement to prove holds by taking $s = t$.
- $n > 0$: Thus, $t =_\beta t_{n-1}$ and $(t_{n-1} \to_\beta u$ or $u \to_\beta t_{n-1})$. By induction hypothesis applied to $t =_\beta t_{n-1}$ (since $n - 1 < n$), there is a term s' such that $t \to_\beta^* s'$ and $t_{n-1} \to_\beta^* s'$. There are two cases:
  1. if $u = t_n \to_\beta t_{n-1}$, then $u \to_\beta^* s'$ and the statement to prove holds by taking $s = s'$;
  2. if $t_{n-1} \to_\beta t_n = u$, then $t_{n-1} \to_\beta^* s'$ and $t_{n-1} \to_\beta u$ and hence, by confluence of $\to_\beta$, there is a term s such that $s' \to_\beta^* s$ and $u \to_\beta^* s$. As $t \to_\beta^* s'$, then $t \to_\beta^* s$ and we are done. Graphically:



Exercise 4 (4 points)

Construct an untyped λ-term F such that $Fxy \to_\beta^* FyxF$.

Hint: Use the fixpoint combinator Θ and get inspired by the method used to prove that the factorial is representable in the untyped λ-calculus, as explained in the lecture of Day 3.

Solution to Exercise 4

$Fxy \to_\beta^* FyxF$ follows from $F \to_\beta^* \lambda x.\lambda y.FyxF$, which in turn follows from $F \to_\beta^* (\lambda f.\lambda x.\lambda y.fyxf)F$. Note that F is a fixed point of $\lambda f.\lambda x.\lambda y.fyxf$. Let $F = \Theta(\lambda f.\lambda x.\lambda y.fyxf)$, where Θ is the fixpoint combinator defined in the lecture of Day 3. Now, $F = \Theta(\lambda f.\lambda x.\lambda y.fyxf) \to_\beta^* (\lambda f.\lambda x.\lambda y.fyxf)(\Theta(\lambda f.\lambda x.\lambda y.fyxf)) = (\lambda f.\lambda x.\lambda y.fyxf)F \to_\beta \lambda x.\lambda y.FyxF$. Therefore, $Fxy \to_\beta *(\lambda x.\lambda y.FyxF)xy \to_\beta^* FyxF$.

Exercise 5 (4 points)

Find a derivation in NI with conclusion $f : M \vdash \lambda a.f(aa) : C$, for some multi type M and linear type C.

Solution to Exercise 5

All the derivations in NI with conclusion $f : M \vdash \lambda a.f(aa) : C$, for any multi type M and linear type C, have the form below, where $N = [[A_1^1, \ldots, A_{n_1}^1] \multimap A_0^1, A_1^1, \ldots, A_{n_1}^1, \ldots\ldots, [[A_1^m, \ldots, A_{n_m}^m] \multimap A_0^m, A_1^m, \ldots, A_{n_m}^m]]$ and $M' = [A_0^1, \ldots, A_0^m]$.

$$
\dfrac{
  \dfrac{
    \dfrac{}{f:[M' \multimap A] \vdash f : M' \multimap A}\,\mathsf{var}
    \qquad
    \dfrac{
      \left(
        \dfrac{
          \dfrac{}{a:[[A_1^j,\ldots,A_{n_j}^j] \multimap A_0^j] \vdash a : [A_1^j,\ldots,A_{n_j}^j] \multimap A_0^j}\,\mathsf{var}
          \quad
          \dfrac{\left(\dfrac{}{a:[A_i^j] \vdash a : A_i^j}\,\mathsf{var}\right)_{1\leq i\leq n_j}}{a:[A_1^j,\ldots,A_{n_j}^j] \vdash aa : [A_1^j,\ldots,A_{n_j}^j]}\,!
        }{
          a:[[A_1^j,\ldots,A_{n_j}^j] \multimap A_0^j, A_1^j,\ldots,A_{n_j}^j] \vdash aa : A_0^j
        }\,@
      \right)_{1\leq j\leq m}
    }{
      a:N \vdash aa : [A_0^1,\ldots,A_0^m]
    }\,!
  }{
    f:[M' \multimap A], a:N \vdash f(aa) : A
  }\,@
}{
  f:[M' \multimap A] \vdash \lambda a.f(aa) : N \multimap A
}\,\lambda
$$

Therefore, a derivation in NI with conclusion $f : M \vdash \lambda a.f(aa) : C$, for some multi type M and linear type C, is the following, obtained from the one above taking $m = 1$ and $n_1 = 0$ and $A = X$ and $A^1_0 = Y$:

$$
\dfrac{\dfrac{}{f : [[Y] \multimap X] \vdash f : [Y] \multimap X}\,\mathsf{var} \qquad \dfrac{\dfrac{\dfrac{}{a : [] \multimap Y] \vdash a : [] \multimap Y}\,\mathsf{var} \quad \dfrac{}{\vdash aa : []}\,!}{a : [[] \multimap Y] \vdash aa : Y}\,@}{a : [[] \multimap Y] \vdash aa : [Y]}\,!}{\dfrac{f : [[Y] \multimap X],\, a : [[] \multimap Y] \vdash f(aa) : X}{f : [[Y] \multimap X] \vdash \lambda a.f(aa) : [[] \multimap Y] \multimap X}\,\lambda}\,@
$$

Exercise 6 (3 points)

Prove that a linear type $M \multimap A$ is shrinking if and only if the multi type M is co-shrinking and the linear type A is shrinking.

Solution to Exercise 6

⇒: Let $N \in \mathsf{oc}_+(A)$. Then, $N \in \mathsf{oc}_+(M \multimap A)$ and hence $|N| \geq 1$ since $M \multimap A$ is shrinking. Thus, A is shrinking.

To prove that M is co-shrinking, we show that every $B \in M$ is co-shrinking. Let $B \in M$. Let $N \in \mathsf{oc}_-(B)$. Then, $N \in \mathsf{oc}_-(M)$ and hence $N \in \mathsf{oc}_+(M \multimap A)$. So, $|N| \geq 1$ since $M \multimap A$ is shrinking. Therefore, B is co-shrinking.

⇐: Let $N \in \mathsf{oc}_+(M \multimap A)$. Then, $N \in \mathsf{oc}_-(M)$ or $N \in \mathsf{oc}_+(A)$. In the first case, $|N| \geq 1$ since M is co-shrinking. In the second case, $|N| \geq 1$ since A is shrinking. In either case, $|N| \geq 1$. Therefore, $M \multimap A$ is shrinking.