The λ-calculus: from simple types to non-idempotent intersection types

Day 5: More about non-idempotent intersection types for the λ-calculus

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Outline

1. Characterizing leftmost-outermost normalization in NI
2. Some final remarks about non-idempotent intersection types
3. Conclusion, exercises and bibliography

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What can we do with non-idempotent intersection types?

Goal. We want to characterize all and only the $\ell\beta$-normalizing terms via NI.

Motivation 1. $\to_{\ell\beta}$ is a normalizing strategy for $\to_\beta$: reaches a β-normal form if it exists.

Motivation 2. The number of $\to_{\ell\beta}$ steps is a reasonable cost model.

Bonus. We use the same type system NI (same rules), we just consider specific types.
$\rightsquigarrow$ NI is versatile. Also, some results are already proven and can be used immediately.

To achieve this qualitative characterization, we need to prove two properties.

1. Correctness: if a term is typable in NI with specific types then it is $\ell\beta$-normalizing.
2. Completeness: if a term is $\ell\beta$-normalizing then it is typable in NI with specific types.

Bonus. We can extract some quantitative information from NI about:

1. the length of evaluation (the number of $\ell\beta$-steps to reach the β-normal form);
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3. A multi type M is shrinking (resp. co-shrinking) if so is every $A \in M$.
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Ingredients to prove correctness

Def. The size $|t|$ of a term t is defined by induction on t as follows:

$$|x| = 1 \qquad |\lambda x.t| = 1 + |t| \qquad |st| = 1 + |s| + |t|$$

Lemma (Spreading of shrinkingness)

Let $t \neq \lambda x.s$ be β-normal and $\mathcal{D} \triangleright_{\mathsf{NI}} \Gamma \vdash t : A$. If Γ is co-shrinking then A is co-shrinking.

Proof. By induction on $m \in \mathbb{N}$, as $t = xt_1 \dots t_m$ for some $m \in \mathbb{N}$, β-normal $t_1, \dots, t_m$. □

Lemma (Typing β-normal forms in a co-shrinking environment)

Let t be β-normal and $\mathcal{D} \triangleright_{\mathsf{NI}} \Gamma \vdash t : A$. If Γ is co-shrinking and (A is shrinking or t is not an abstraction), then $|t| \leq |\mathcal{D}|$.

Proof. Every β-normal term is of the form $t = \lambda x_n. \dots \lambda x_1.yt_1 \dots t_m$ for some $m, n \in \mathbb{N}$, with $t_1, \dots, t_m$ β-normal. The lemma is proved by induction on $|t| \in \mathbb{N}$. We use the lemma above if $n = 0$ and $m > 0$. □

Ingredients to prove correctness

Def. The size $|t|$ of a term t is defined by induction on t as follows:

$$|x| = 1 \qquad |\lambda x.t| = 1 + |t| \qquad |st| = 1 + |s| + |t|$$

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Correctness of NI: shrinking typability implies $\ell\beta$-normalization

Proposition (Quantitative subject reduction for shrinking derivations)

If $\mathcal{D} \triangleright_{\mathsf{NI}} \Gamma \vdash t : A$ is shrinking and $t \to_{\ell\beta} t'$, then there is $\mathcal{D}' \triangleright_{\mathsf{NI}} \Gamma \vdash t' : A$ with $|\mathcal{D}| > |\mathcal{D}'|$.

Proof. By induction on the definition $t \to_{\ell\beta} t'$ (p. 6, Day 3). The only non-trivial case is when $t = (\lambda x.u)s \to_{\ell\beta} u\{s/x\} = t'$: so, $\mathcal{D}$ must have the form below, with $\Gamma = \Gamma' \uplus \Gamma''$.

$$\mathcal{D} = \dfrac{\dfrac{\begin{matrix}\vdots\, \mathcal{D}_u \\ \Gamma' x : M \vdash u : A\end{matrix}}{\Gamma' \vdash \lambda x.u : M \multimap A}\,\lambda \qquad \begin{matrix}\vdots\, \mathcal{D}_s \\ \Gamma'' \vdash s : M\end{matrix}}{\Gamma' \uplus \Gamma'' \vdash (\lambda x.u)s : A}\,@$$

By substitution lemma, there is $\mathcal{D}' \triangleright_{\mathsf{NI}} \Gamma \vdash u\{s/x\} : A$ with $|\mathcal{D}'| = |\mathcal{D}_u| + |\mathcal{D}_s| - |M| < |\mathcal{D}_u| + |\mathcal{D}_s| + 2 = |\mathcal{D}|$. □

Rmk. The quantitative aspect of subject reduction (i.e. $|\mathcal{D}| > |\mathcal{D}'|$) is false:

- if $\mathcal{D}$ is not shrinking, e.g. $\lambda x.x(\delta\delta) \to_{\beta} \lambda x.x(\delta\delta)$ with $\delta = \lambda z.zz$, see Day 4, p. 10;
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Proof. By induction on $|\mathcal{D}|$. If t is β-normal, then the claim follows from the lemma about typing β-normal forms, taking $s = t$ and $k = 0$.
Otherwise, $t \to_{\ell\beta} t'$ and by quantitative subject reduction there is $\mathcal{D}' \rhd_{\mathsf{NI}} \Gamma \vdash t' : A$ with $|\mathcal{D}| > |\mathcal{D}'|$. By induction hypothesis, $t' \to_{\ell\beta}^* s$ in k $\ell\beta$-steps for some β-normal s with $|\mathcal{D}'| \geq k + |s|$. Hence, $t \to_{\ell\beta}^* s$ in $k+1$ $\ell\beta$-steps and $|\mathcal{D}| \geq |\mathcal{D}'| + 1 \geq k + 1 + |s|$. □

Ingredients to prove completeness

Rmk. Completeness is the converse of correctness, so their needed ingredients are "dual".

Lemma (Shrinking typability of β-normal forms)

If t is β-normal, then there is a shrinking $\mathcal{D} \triangleright_{\mathsf{NI}} \Gamma \vdash t : A$ for some Γ, A, with $|\mathcal{D}| = |t|$.

Proof. Every β-normal term is of the form $t = \lambda x_n. \dots \lambda x_1 . y t_1 \dots t_m$ for some $m, n \in \mathbb{N}$ with $t_1, \dots, t_m$ β-normal. To have the right induction hypothesis, for $n = 0$ we also have to prove that, for all $k \in \mathbb{N}$ and co-shrinking A and shrinking $A_1, \dots, A_n$, there is $\mathcal{D} \triangleright_{\mathsf{NI}} \Gamma \vdash y t_1 \dots t_m : [A_1] \multimap \cdots \multimap [A_k] \multimap A$ for some co-shrinking environment Γ.
The stronger statement is proved by induction on $|t| \in \mathbb{N}$. □

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Rmk. We have seen (in day 2) that subject expansion fails with simple types.

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If $t \to_{\ell\beta}^{k} s$ with s β-normal, then there is shrinking $\mathcal{D} \rhd_{\mathsf{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| \geq k + |s|$.

Proof. By induction on $k \in \mathbb{N}$. If $k = 0$, then $t = s$ and typability of β-normal concludes. Otherwise $k > 0$ and $t \to_{\ell\beta} t' \to_{\ell\beta}^{k-1} s$. By induction hypothesis, there is $\mathcal{D}' \rhd_{\mathsf{NI}} \Gamma \vdash t' : A$ with $|\mathcal{D}'| \geq k - 1 + |s|$. By quantitative subject expansion, there is $\mathcal{D} \rhd_{\mathsf{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| > |\mathcal{D}'|$, therefore $|\mathcal{D}| \geq |\mathcal{D}'| + 1 \geq k + |s|$. □

Summing up: characterization of leftmost-outermost normalization

Putting together correctness and completeness of NI, we obtain:

Corollary (Characterization of leftmost-outermost normalization)

A term t is $\ell\beta$-normalizing if and only if there is $\mathcal{D} \triangleright_{\mathsf{NI}} \Gamma \vdash t : A$. Moreover, $|\mathcal{D}| \geq k + |s|$ if $t \to_{\ell\beta}^{k} s$ with s β-normal.

Rmk. The quantitative information about

- the length k of evaluation (left reduction) from t to its β-normal form s, and
- the head size $|s|$ of the β-normal term s

are in the size $|\mathcal{D}|$ of $\mathcal{D}$ without performing head reduction $\to_{\ell\beta}$ or knowing s.

Rmk. $|\mathcal{D}|$ is an upper bound to k plus $|s|$ together. NI can be refined so that one can:

1. disentangle the information about k and $|s|$ by means of two different sizes of $\mathcal{D}$,
2. obtain the exact values of k and $|s|$ from these two sizes of $\mathcal{D}$.

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Outline

1 Characterizing leftmost-outermost normalization in NI

2 Some final remarks about non-idempotent intersection types

3 Conclusion, exercises and bibliography

The leftmost-outermost reduction is a normalizing strategy

Ex. Let $\delta = \lambda y.yy$, $t = (\lambda z.x)(\delta\delta)$: $t \to_\beta t$ but all derivations of t have the same size.

Proposition (Qualitative subject reduction and expansion)

1. If $\mathcal{D} \triangleright_{\mathsf{NI}} \Gamma \vdash t : A$ and $t \to_\beta t'$, then there is $\mathcal{D}' \triangleright_{\mathsf{NI}} \Gamma \vdash t' : A$ with $|\mathcal{D}| \geq |\mathcal{D}'|$.
2. If $\mathcal{D}' \triangleright_{\mathsf{NI}} \Gamma \vdash t' : A$ and $t \to_\beta t'$, then there is $\mathcal{D} \triangleright_{\mathsf{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| \geq |\mathcal{D}'|$.

Proof. By induction on the definition of $t \to_\beta t'$ (p. 6 on Day 3). The proof is the same as for $\to_{h\beta}$ or $\to_{\ell\beta}$, except that there are more cases, for which $|\mathcal{D}|$ may not decrease. □

Corollary (leftmost-outermost reduction is a normalizing strategy for $\to_\beta$)

If $t \to_\beta^* s$ with s β-normal, then $t \to_{\ell\beta}^* s$.

Proof. Use shrinking typability of β-normal forms, qualitative subject expansion, correctness for shrinking NI, and confluence of $\to_\beta$. □

Rmk. The corollary is a non-trivial property that has apparently nothing to do with NI.
$\rightsquigarrow$ NI is useful to prove general properties of the untyped λ-calculus.

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The philosophy behind the proofs characterizing normalizations via NI

Our proofs to characterize normalization for $\to_r$ ($r \in \{h\beta, \ell\beta\}$) follow the same pattern:

normalizability for $\to_r$	completeness $\Longrightarrow$	typability in NI (for some types)	correctness $\Longrightarrow$	strong normalizability for $\to_r$

Indeed, correctness is proved via quantitative subject reduction (p. 15 Day 4/p. 8 Day 5): after every single r-step from a typable term, the size of the derivation decreases.

$\rightsquigarrow$ This is a way to prove that normalization and strong normalization coincide for $\to_r$ (t is r-normalizing iff t is strongly r-normalizing, see definitions on p. 11 of Day 3).

$\to_{h\beta}/\to_{\ell\beta}$ is deterministic $\rightsquigarrow$ trivially normaliz. and strong normaliz. coincide for it.

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What we have learned today?

1. How to use the non-idempotent intersection type system NI with specific types.
2. Characterization of leftmost-outermost normalization via NI.
3. A combinatorial proof for that characterization.
4. How to extract quantitative information from shrinking derivations in NI.

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Exercises

1. Find all shrinking derivations of $x\!:\!M \vdash xx\!:\!C$, for any linear C and any multi M.
2. Find all shrinking derivations of $x\!:\!M, y\!:\!N \vdash xy\!:\!C$, for any linear C and multi M, N.
3. Find all derivations of $\vdash (\lambda x.xx)\lambda y.y : A$ for any shrinking linear type A.
4. Prove that there is no derivation of $\vdash \lambda x.x(\delta\delta) : C$ for any shrinking linear type C.
5. Prove that all derivations of $\vdash (\lambda z.x)(\delta\delta) : C$ for any shrinking linear type C have the same size. Deduce that quantitative subject reduction on shrinking derivations (proposition on p. 7) does not hold in general when $t \to_\beta t'$ instead of $t \to_{\ell\beta} t'$.
6. Prove rigorously the two lemmas on p. 7 and the lemma on p. 9.
7. Prove rigorously the quantitative subject reduction (p. 8) and expansion (p. 10), by induction on the definition of $t \to_{\ell\beta} t'$ (see Day 3, p. 9).
8. Prove rigorously the qualitative versions of subject reduction and expansion (Proposition on p. 12).
9. Prove rigorously that $\to_{\ell\beta}$ is a normalizing strategy for $\to_\beta$ (Corollary on p. 12).
10. Do we really need quantitative subject expansion for shrinking derivations (proposition on p. 10) to prove completeness of shrinking NI (theorem on p. 10)? *Hint*: Use the qualitative version of subject expansion (Proposition on p. 12).

Bibliography

- For an (almost gentle) introduction to non-idempotent intersection types:

Antonio Bucciarelli, Delia Kesner, Daniel Ventura. *Non-Idempotent Intersection types for the Lambda-Calculus*. Logic Journal of the IGPL, vol. 25, issue 4, pp. 431–464, 2017. https://doi.org/10.1093/jigpal/jzx018

- For a very advanced study about non-idempotent intersection types:

Beniamino Accattoli, Stéphan Graham-Lengrand, Delia Kesner. *Tight typings and split bounds, fully developed*. Journal of Functional Programming, vol. 30, 14 pages, 2020. https://doi.org/10.1017/S095679682000012X