The λ-calculus: from simple types to non-idempotent intersection types

Day 4: Non-idempotent intersection types for the λ-calculus

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Outline

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The λ-calculus between simple types and the untyped one

The simply typed λ-calculus:

1. has very nice operational properties (e.g. normalization, confluence);
2. has a clear logical meaning (Curry-Howard correspondence);
3. is not very expressive (recursion cannot be represented, Turing-completeness fails).

The untyped λ-calculus:

1. has some very nice properties (e.g. confluence, Turing-completeness);
2. misses some nice properties (e.g. normalization);
3. has no logical meaning;
4. contains diverging terms without any meaning (e.g. $\delta\delta$).

Questions.

1. Is there a more liberal type system which only takes the pros of the two worlds?
2. Can it characterize all and only the "meaningful" terms of the untyped λ-calculus?

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The syntax for non-idempotent intersection types

We fix a countably infinite set of atoms, denoted by $X, Y, Z, \ldots$.

$$
\begin{array}{rll}
\text{Linear types:} & A, B ::= X \mid M \multimap A & \\
\text{Multi types:} & M, N ::= [A_1, \ldots, A_n] & (\text{with } n \in \mathbb{N}) \\
\text{(Non-idempotent intersection) types:} & S, T ::= A \mid M &
\end{array}
$$

where $[A_1, \ldots, A_n]$ with $n \in \mathbb{N}$ is a finite multiset ($[\,]$ is the empty multiset for $n = 0$).

Idea. $[A_1, \ldots, A_n]$ stands for a conjunction $A_1 \wedge \cdots \wedge A_n$ where $\wedge$ is:

- commutative $A \wedge B \equiv B \wedge A$ (multisets do not take order into account);
- associative $A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$ (multisets are associative);
- non-idempotent $A \wedge A \not\equiv A$ (multisets take multiplicites into account).

Def. A judgment is a sequent of the form $\Gamma \vdash t : T$ where

1. t is a term, T is a type, Γ is an environment, that is,
2. Γ is a function from variables to multi types such that $\{x \mid \Gamma(x) \neq [\,]\}$ is a finite set.

Notation. $\uplus$ is the multiset union (e.g. $[A, B] \uplus [A] = [A, A, B] \neq [A, B]$) whose unit is $[\,]$. Extended to type environments pointwise: $(\Gamma \uplus \Delta)(x) = \Gamma(x) \uplus \Delta(x)$.

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The typing rules for non-idempotent intersection type system NI

Notation. An environment Γ is denoted by $x_1 : M_1, \dots, x_n : M_n$ if:
variables $x_1, \dots, x_n$ are pariwise distinct and $\Gamma(x) = \begin{cases} M_i & \text{if } x = x_i \text{ for some } 1 \leq i \leq n, \\ [\,] & \text{otherwise.} \end{cases}$

Typing rules for NI: $\dfrac{}{x : [A] \vdash x : A}\,\text{var}$

$\dfrac{\Gamma, x : M \vdash t : A}{\Gamma \vdash \lambda x.t : M \multimap A}\,\lambda$ $\qquad$ $\dfrac{\Gamma \vdash s : M \multimap A \quad \Delta \vdash t : M}{\Gamma \uplus \Delta \vdash st : A}\,@$ $\qquad$ $\dfrac{(\Gamma_i \vdash t : A_i)_{1 \leq i \leq n} \quad n \in \mathbb{N}}{\biguplus_{i=1}^{n} \Gamma_i \vdash t : [A_1, \dots, A_n]}\,!$

Idea. A term typed $t : [A, A, B]$ means that, during evaluation, t can be used:

- once as a data of type B, and
- twice as a data of type A.

Notation. $\mathcal{D} \triangleright_{\mathrm{NI}} \Gamma \vdash t : T$ means that $\mathcal{D}$ is a derivation in NI with conclusion $\Gamma \vdash t : T$.
$\Gamma \vdash_{\mathrm{NI}} t : T$ means that there is a derivation $\mathcal{D} \triangleright_{\mathrm{NI}} \Gamma \vdash t : T$.

Rmk. $\vdash_{\mathrm{NI}} t : [\,]$ for every term t (take ! with no premises).

Def. The size $|\mathcal{D}|$ of a derivation $\mathcal{D}$ is the number of its rules, not counting the rules !.
$|\mathcal{D}|_{\text{var}}$ (resp. $|\mathcal{D}|_\lambda$; $|\mathcal{D}|_@$) is the number of rules var (resp. λ; @) in $\mathcal{D}$.

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Some examples of derivations in NI

Ex. Find all the derivations with conclusion $\vdash \lambda x.x : C$, for any linear type C.

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Some examples of derivations in NI

Ex. Find all the derivations with conclusion $\vdash \lambda x.x : C$, for any linear type C.

$$\mathcal{D}^I_A = \cfrac{\cfrac{}{x : [A] \vdash x : A}\,\text{var}}{\vdash \lambda x.x : [A] \multimap A}\,\lambda \qquad \text{for any linear type } A.$$

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Ex. Find all the derivations with conclusion $\vdash \lambda x.xx : C$, for any linear type C.

$$\mathcal{D}^{\delta,n}_{A_0,\dots,A_n} = \dfrac{\dfrac{\dfrac{}{x : [[A_1,\dots,A_n] \multimap A_0] \vdash x : [A_1,\dots,A_n] \multimap A_0}\text{var} \quad \dfrac{\left(\dfrac{}{x : [A_i] \vdash x : A_i}\text{var}\right)_{1 \leq i \leq n}}{x : [A_1,\dots,A_n] \vdash x : [A_1,\dots,A_n]}!}{x : [[A_1,\dots,A_n] \multimap A_0, A_1,\dots,A_n] \vdash xx : A_0}@}{\vdash \lambda x.xx : [[A_1,\dots,A_n] \multimap A_0, A_1,\dots,A_n] \multimap A_0}\lambda$$

for any $n \in \mathbb{N}$ and any linear types $A_0,\dots,A_n$ (in particular, for $n = 0$, $\vdash \lambda x.x : [[\,] \multimap A_0] \multimap A_0$).

More examples of derivations in NI

Ex. Find all the derivations with conclusion $\vdash (\lambda x.x)\lambda y.y : C$, for any linear type C.

Ex. Find a derivation with conclusion $\vdash (\lambda x.xx)\lambda y.y : C$, for some linear type C.

Rmk. In the derivation $\mathcal{D}^{II}_A$ (resp. $\mathcal{D}^{\delta,I}_A$) the rule ! has 1 premise (resp. 2 premises) because 1 copy (resp. 2 copies) of $\lambda y.y$ is (resp. are) needed in the evaluation $(\lambda x.x)\lambda y.y \to_{h\beta} \lambda y.y$ (resp. $(\lambda x.xx)\lambda y.y \to_{h\beta} (\lambda y.y)\lambda y.y$).

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for any linear type A (actually, all derivations for $(\lambda x.xx)\lambda y.y$ have the form above).

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Oh no! More examples of derivations in NI!

Ex. Find a derivation with conclusion $\vdash (\lambda x.xx)((\lambda y.y)\lambda z.z) : C$, for some linear type C.

Rmk. In the derivation $\mathcal{D}_A^{\delta,II}$, the rule ! has 2 premises because 2 copies of $(\lambda y.y)\lambda z.z$ are needed in the evaluation $(\lambda x.xx)((\lambda y.y)\lambda z.z) \to_{h\beta} ((\lambda y.y)\lambda z.z)((\lambda y.y)\lambda z.z)$.

In turn, in each of the derivations $\mathcal{D}^{II}_{[[A]\multimap A]\multimap[A]\multimap A}$ and $\mathcal{D}^{II}_{[A]\multimap A}$ the rule ! has 2 premises, hence the derivation $\mathcal{D}_A^{\delta,II}$ has 4 subderivations with conclusion $\lambda x.x$, because 4 copies of $\lambda x.x$ are needed in the evaluation $(\lambda x.xx)((\lambda y.y)\lambda z.z) \to_{h\beta} ((\lambda y.y)\lambda z.z)((\lambda y.y)\lambda z.z)$.

Ex. Find a derivation with conclusion $\vdash (\lambda x.xx)\lambda y.yy : C$, for some linear type C.

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for any linear type A (actually, all derivations for $(\lambda x.xx)((\lambda y.y)\lambda z.z)$ have that form).

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Rmk. In the derivation $\mathcal{D}^{\delta,II}_A$, the rule ! has 2 premises because 2 copies of $(\lambda y.y)\lambda z.z$ are needed in the evaluation $(\lambda x.xx)((\lambda y.y)\lambda z.z) \to_{h\beta} ((\lambda y.y)\lambda z.z)((\lambda y.y)\lambda z.z)$.
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Good luck!

Enough with the examples of derivations, old man!

Ex. Find all derivations with conclusion $\vdash \lambda x.x((\lambda y.yy)\lambda z.zz) : C$, for any linear type C.

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This is a good exercise, old man!

Outline

What can we do with non-idempotent intersection types?

Goal. We want to characterize all and only the $h\beta$-normalizing terms via NI.

Motivation. There are many theoretical reasons to say "meaningful" = $h\beta$-normalizing.

To achieve this qualitative characterization, we need to prove two properties.

1. Correctness: if a term is typable in NI then it is $h\beta$-normalizing.
2. Completeness: if a term is $h\beta$-normalizing then it is typable in NI.

Bonus. We can extract some quantitative information from NI about:

1. the length of evaluation (the number of $h\beta$-steps to reach the $h\beta$-normal form);
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Ingredients to prove correctness

Def. The head size $|t|_{h\beta}$ of a term t is defined by induction on t as follows:

$$|x|_{h\beta} = 1 \qquad |\lambda x.t|_{h\beta} = 1 + |t|_{h\beta} \qquad |st|_{h\beta} = 1 + |s|_{h\beta}$$

Lemma (Typing $h\beta$-normal forms)

Let t be $h\beta$-normal. If $\mathcal{D} \triangleright_{\mathsf{NI}} \Gamma \vdash t : A$ then $|t|_{h\beta} \leq |\mathcal{D}|$.

Proof. Every $h\beta$-normal term is of the form $t = \lambda x_n. \dots \lambda x_1.yt_1 \dots t_m$ for some $m, n \in \mathbb{N}$. The lemma is proved by induction on $|t|_{h\beta} \in \mathbb{N}$. □

Notation. For a finite multiset M over a set X, its cardinality is $|M| = \sum_{x \in X} M(x) \in \mathbb{N}$.

Lemma (Substitution)

If $\mathcal{D} \triangleright_{\mathsf{NI}} \Gamma, x : M \vdash t : A$ and $\mathcal{D}' \triangleright_{\mathsf{NI}} \Delta \vdash s : M$, then there is $\mathcal{D}'' \triangleright_{\mathsf{NI}} \Gamma \uplus \Delta \vdash t\{s/x\} : A$ with $|\mathcal{D}''| = |\mathcal{D}| + |\mathcal{D}'| - |M|$.

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A graphical view to the substitution lemma

Like natural deduction, derivations in NI can be depicted by a tree-like structure where:

- edges are labeled by typed terms, nodes are the typing rules,
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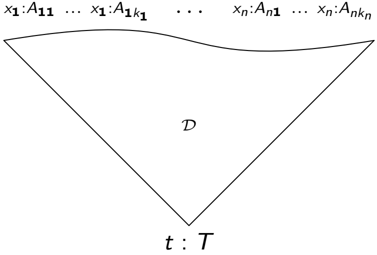

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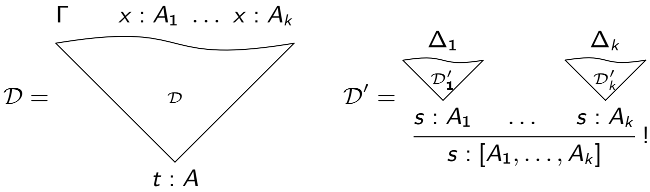

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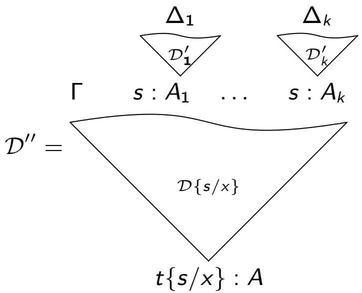

Correctness of NI: typability implies $h\beta$-normalization

Proposition (Quantitative subject reduction)

If $\mathcal{D} \triangleright_{\mathsf{NI}} \Gamma \vdash t : A$ and $t \to_{h\beta} t'$, then there is $\mathcal{D}' \triangleright_{\mathsf{NI}} \Gamma \vdash t' : A$ with $|\mathcal{D}| > |\mathcal{D}'|$.

Proof. By induction on the definition $t \to_{h\beta} t'$ (p. 6, Day 3). The only non-trivial case is when $t = (\lambda x.u)s \to_{h\beta} u\{s/x\} = t'$: so, $\mathcal{D}$ must have the form below, with $\Gamma = \Gamma' \uplus \Gamma''$.

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- if $t \to_\beta t'$ instead of $t \to_{h\beta} t'$, e.g. $\lambda x.x(\delta\delta) \to_\beta \lambda x.x(\delta\delta)$ with $\delta = \lambda z.zz$, see p. 10;
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Theorem (Correctness of NI)

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By substitution lemma, there is $\mathcal{D}' \rhd_{\mathsf{NI}} \Gamma \vdash u\{s/x\} : A$ with $|\mathcal{D}'| = |\mathcal{D}_u| + |\mathcal{D}_s| - |M| < |\mathcal{D}_u| + |\mathcal{D}_s| + 2 = |\mathcal{D}|$. □

Rmk. The quantitative aspect of subject reduction (i.e. $|\mathcal{D}| > |\mathcal{D}'|$) is false:

- if $t \to_\beta t'$ instead of $t \to_{h\beta} t'$, e.g. $\lambda x.x(\delta\delta) \to_\beta \lambda x.x(\delta\delta)$ with $\delta = \lambda z.zz$, see p. 10;
- if $\mathcal{D}$ and $\mathcal{D}'$ are derivations in the simply typed λ-calculus, instead of NI.

Theorem (Correctness of NI)

If $\mathcal{D} \rhd_{\mathsf{NI}} \Gamma \vdash t : A$ then there is s $h\beta$-normal such that $t \overbrace{\to_{h\beta} \cdots \to_{h\beta}}^{k\ h\beta\text{-steps}} s$ and $|\mathcal{D}| \geq k + |s|_{h\beta}$.

Proof. By induction on $|\mathcal{D}|$. If t is $h\beta$-normal, then the claim follows from the lemma about typing $h\beta$-normal forms, taking $s = t$ and $k = 0$.

Correctness of NI: typability implies $h\beta$-normalization

Proposition (Quantitative subject reduction)

If $\mathcal{D} \triangleright_{\mathsf{NI}} \Gamma \vdash t : A$ and $t \to_{h\beta} t'$, then there is $\mathcal{D}' \triangleright_{\mathsf{NI}} \Gamma \vdash t' : A$ with $|\mathcal{D}| > |\mathcal{D}'|$.

Proof. By induction on the definition $t \to_{h\beta} t'$ (p. 6, Day 3). The only non-trivial case is when $t = (\lambda x.u)s \to_{h\beta} u\{s/x\} = t'$: so, $\mathcal{D}$ must have the form below, with $\Gamma = \Gamma' \uplus \Gamma''$.

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Proof. By induction on $|\mathcal{D}|$. If t is $h\beta$-normal, then the claim follows from the lemma about typing $h\beta$-normal forms, taking $s = t$ and $k = 0$.
Otherwise, $t \to_{h\beta} t'$ and by quantitative subject reduction there is $\mathcal{D}' \triangleright_{\mathsf{NI}} \Gamma \vdash t' : A$ with $|\mathcal{D}| > |\mathcal{D}'|$. By induction hypothesis, $t' \to_{h\beta}^* s$ in k $h\beta$-steps for some $h\beta$-normal s with $|\mathcal{D}'| \geq k + |s|_{h\beta}$. Hence, $t \to_{h\beta}^* s$ in $k{+}1$ $h\beta$-steps and $|\mathcal{D}| \geq |\mathcal{D}'| + 1 \geq k + 1 + |s|_{h\beta}$. □

Ingredients to prove completeness

Rmk. Completeness is the converse of correctness, so their needed ingredients are "dual".

Lemma (Typability of $h\beta$-normal forms)

If t is $h\beta$-normal, then there is $\mathcal{D} \rhd_{\mathsf{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| = |t|_{h\beta}$.

Proof. Every $h\beta$-normal term is of the form $t = \lambda x_n. \dots \lambda x_1.yt_1 \dots t_m$ for some $m, n \in \mathbb{N}$. For $n = 0$, we prove by induction on $m \in \mathbb{N}$ the stronger property that, for all $k \in \mathbb{N}$ and linear A, there is $\mathcal{D} \rhd_{\mathsf{NI}} y : [A_k] \vdash yt_1 \dots t_m : A_k$ with $|\mathcal{D}| = m + 1 = |\mathcal{D}|_@ + |\mathcal{D}|_{\mathsf{var}}$ and

$$A_k = \overbrace{[\,] \multimap \cdots \multimap [\,]}^{k \text{ times } [\,]} \multimap A \qquad (\text{note that } |yt_1 \dots t_m|_{h\beta} = m + 1 \text{ and } |\mathcal{D}|_@ = m).$$

The lemma including the stronger statement is proved by induction on $|t|_{h\beta} \in \mathbb{N}$. □

Lemma (Anti-substitution)

If $\mathcal{D} \rhd_{\mathsf{NI}} \Gamma \vdash t\{s/u\} : A$, then there are environments Γ' and Γ'', a multi type M and derivations $\mathcal{D}' \rhd_{\mathsf{NI}} \Gamma', x : M \vdash t : A$ and $\mathcal{D}'' \rhd_{\mathsf{NI}} \Gamma'' \vdash s : M$ such that $\Gamma = \Gamma' \uplus \Gamma''$ and $|\mathcal{D}| = |\mathcal{D}'| + |\mathcal{D}''| - |M|$.

Proof. By structural induction on t. The base case is when t is a variable (either x or other than x). The other cases follow easily from the inductive hypothesis. □

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Completeness of NI: $h\beta$-normalization implies typability

Proposition (Quantitative subject expansion)

If $\mathcal{D}' \rhd_{\mathsf{NI}} \Gamma \vdash t' : A$ and $t \to_{h\beta} t'$, then there is $\mathcal{D} \rhd_{\mathsf{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| > |\mathcal{D}'|$.

Proof. By induction on the definition $t \to_{h\beta} t'$ (p. 6, Day 3). The only non-trivial case is when $t = (\lambda x.u)s \to_{h\beta} u\{s/x\} = t'$: as $\mathcal{D}' \rhd_{\mathsf{NI}} \Gamma \vdash t' : A$, by the anti-substitution lemma

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Rmk. We have seen (in day 2) that subject expansion fails with simple types.

Notation. Given $k \in \mathbb{N}$, we write $t \to_{h\beta}^{k} s$ if $t \overbrace{\to_{h\beta} \cdots \to_{h\beta}}^{k\ h\beta\text{-steps}} s$ (thus $t \to_{h\beta}^{0} s$ means $t = s$).

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Completeness of NI: $h\beta$-normalization implies typability

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Proof. By induction on the definition $t \to_{h\beta} t'$ (p. 6, Day 3). The only non-trivial case is when $t = (\lambda x.u)s \to_{h\beta} u\{s/x\} = t'$: as $\mathcal{D}' \triangleright_{\mathsf{NI}} \Gamma \vdash t' : A$, by the anti-substitution lemma

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Proof. By induction on $k \in \mathbb{N}$.

Completeness of NI: $h\beta$-normalization implies typability

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Proof. By induction on the definition $t \to_{h\beta} t'$ (p. 6, Day 3). The only non-trivial case is when $t = (\lambda x.u)s \to_{h\beta} u\{s/x\} = t'$: as $\mathcal{D}' \triangleright_{\mathsf{NI}} \Gamma \vdash t' : A$, by the anti-substitution lemma

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Theorem (Completeness of NI)

If $t \to_{h\beta}^{k} s$ with s $h\beta$-normal, then there is $\mathcal{D} \triangleright_{\mathsf{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| \geq k + |s|_{h\beta}$.

Proof. By induction on $k \in \mathbb{N}$. If $k = 0$, then $t = s$ and typability of $h\beta$-normal concludes.

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Proposition (Quantitative subject expansion)

If $\mathcal{D}' \rhd_{\mathsf{NI}} \Gamma \vdash t' : A$ and $t \to_{h\beta} t'$, then there is $\mathcal{D} \rhd_{\mathsf{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| > |\mathcal{D}'|$.

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Proof. By induction on $k \in \mathbb{N}$. If $k = 0$, then $t = s$ and typability of $h\beta$-normal concludes. Otherwise $k > 0$ and $t \to_{h\beta} t' \to_{h\beta}^{k-1} s$. By induction hypothesis, there is $\mathcal{D}' \rhd_{\mathsf{NI}} \Gamma \vdash t' : A$ with $|\mathcal{D}'| \geq k - 1 + |s|_{h\beta}$. By quantitative subject expansion, there is $\mathcal{D} \rhd_{\mathsf{NI}} \Gamma \vdash t : A$ with $|\mathcal{D}| > |\mathcal{D}'|$, therefore $|\mathcal{D}| \geq |\mathcal{D}'| + 1 \geq k + |s|_{h\beta}$. □

Summing up: characterization of head normalization

Putting together correctness and completeness of NI, we obtain:

Corollary (Characterization of head normalization)

A term t is $h\beta$-normalizing if and only if there is $\mathcal{D} \triangleright_{\mathsf{NI}} \Gamma \vdash t : A$. Moreover, $|\mathcal{D}| \geq k + |s|_{h\beta}$ if $t \to_{h\beta}^{k} s$ with s $h\beta$-normal.

Rmk. The quantitative information about

- the length k of evaluation (head reduction) from t to its $h\beta$-normal form s, and
- the head size $|s|_{h\beta}$ of the $h\beta$-normal term s

are in the size $|\mathcal{D}|$ of $\mathcal{D}$ without performing head reduction $\to_{h\beta}$ or knowing s.

Rmk. $|\mathcal{D}|$ is an upper bound to k plus $|s|_{h\beta}$ together. NI can be refined so that one can:

1. disentangle the information about k and $|s|_{h\beta}$ by means of two different sizes of $\mathcal{D}$,
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Outline

What we have learned today?

1. The non-idempotent intersection type system NI.
2. Characterization of head normalization via NI.
3. A combinatorial proof for that characterization.
4. How to extract quantitative information from derivations in NI.

Questions?

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Questions?

Exercises

1. Find all the derivations of $x : M \vdash xx : C$, for any linear type C and any multi type M.
2. Find all the derivations of $x : M, y : N \vdash xy : C$, for any linear C and any multi M, N.
3. Prove that all derivations in NI for $(\lambda x.xx)\lambda y.y$ have the form $\mathcal{D}_A^{\delta,I}$ shown on p.8, for any linear type A.
4. Prove that there is no derivation of $\vdash (\lambda x.xx)\lambda y.yy : C$, for any linear type C.
5. Find a derivation of $\vdash \lambda a.\lambda f.f(aaf) : C$, for some linear type C.
6. Find a derivation of $\vdash (\lambda a.\lambda f.f(aaf))\lambda a.\lambda f.f(aaf) : C$, for some linear type C.
7. Find all the derivations of $\vdash \lambda a.\lambda f.f(aaf) : C$, for any linear type C.
8. Find all the derivations of $\vdash (\lambda a.\lambda f.f(aaf))\lambda a.\lambda f.f(aaf) : C$, for any linear type C.
9. Prove rigorously the two lemmas on p. 13 and the two lemmas on p. 16.
10. Prove rigorously the quantitative subject reduction (p. 15) and expansion (p. 17), by induction on the definition of $t \to_{h\beta} t'$ (see Day 3, p. 9).

Bibliography

- For an (almost gentle) introduction to non-idempotent intersection types:

Antonio Bucciarelli, Delia Kesner, Daniel Ventura. *Non-Idempotent Intersection types for the Lambda-Calculus*. Logic Journal of the IGPL, vol. 25, issue 4, pp. 431–464, 2017. https://doi.org/10.1093/jigpal/jzx018

- For a very advanced study about non-idempotent intersection types:

Beniamino Accattoli, Stéphan Graham-Lengrand, Delia Kesner. *Tight typings and split bounds, fully developed*. Journal of Functional Programming, vol. 30, 14 pages, 2020. https://doi.org/10.1017/S095679682000012X