The λ-calculus: from simple types to non-idempotent intersection types

Day 1: Natural deduction for minimal logic and cut-elimination.

Giulio Guerrieri

Department of Informatics, University of Sussex (Brighton, UK)
✉ g.guerrieri@sussex.ac.uk https://pageperso.lis-lab.fr/~giulio.guerrieri/

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Outline

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Why this course?

We will talk about logic, proofs, abstract models of computation.

This a pen-and-paper course. You won't use a computer. Why is this computer science?

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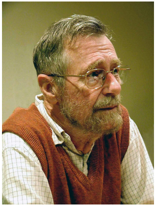

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Is this course really useless?

We will talk about the λ-calculus, a model of computation that can be seen as:

- a minimal prototype of functional programming languages such as Haskell, OCaml;
- a minimal prototype of many proof assistants such as Agda, Coq, Lean.

Also, many mainstream languages (e.g. Java, Python, Scala) implement some λ-features.

$\leadsto$ Learning λ-features (higher-order computation) is needed to be a good programmer.

Ex. After this course, you understand what happens in OCaml interpreter when you write:

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# (fun x -> x * x) 3 ;;
- :  int = 9
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(or equivalently, `let x = 3 in x * x ;;`)

not from a engineering point of view, but from a conceptual point of view.

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Objectives of the course

This is an introductory course to the λ-calculus and to its links to proof-theory.

- I present natural deduction as a formalism to write formal proofs in minimal logic, and I define cut-elimination.
- I present the simply typed λ-calculus as a shorthand for natural deduction in minimal logic (Curry-Howard correspondence).
- I forget the logical content of the simply typed λ-calculus and we get the untyped λ-calculus, a Turing-complete model of computation.
- I introduce a more liberal typing system, non-idempotent intersection types, to characterize termination of head and full evaluation in the untyped λ-calculus.
- I extract some quantitative information from non-idempotent intersection type system, such the length of the evaluation or the size of the result.

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1. Day 1: Natural deduction for minimal logic.
2. Day 2: The simply typed λ-calculus and the Curry-Howard correspondence.
3. Day 3: The untyped λ-calculus.
4. Day 4: Non-idempotent intersection types for the λ-calculus.
5. Day 5: More about non-idempotent intersection types for the λ-calculus.

Rmk.

- If you already know natural deduction and its normalization, skip day 1.
- If you already know the λ-calculus, skip days 2–3.
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Logic is the study of correct reasoning, that is, of deductively valid inferences.

- How conclusions follow from premises due to the structure of arguments alone.
- Independent of the topic and content of sentences.

Ex. Consider the following sentences:

1. "If it rains, then it rains"
2. "If Cordoba is the capital of Argentina, then Cordoba is the capital of Argentina".

Both sentences have the same structure ("pattern")

If A then A

which is always true independently of the content of A.
Logic studies the "patterns" that are always true, and how to prove them.

There are many logics: classical, intuitionistic, modal, first-order, higher-order, ...

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Minimal logic: the implicational fragment of propositional intuitionistic logic

Language of minimal logic: implicational fragment of propositional intuitionistic logic.

Def. Given a countably infinite set of propositional variables, denoted by $X, Y, Z, \dots$, formulas are defined by the BNF grammar below:

$$A, B, C ::= X \mid (A \Rightarrow B)$$

This is a shorthand for an inductive definition of the set of formulas. That is:

- Every propositional variable is a formula.
- If A and B are formulas, then $(A \Rightarrow B)$ is a formula (called implication).
- Nothing else is a formula.

Notation.

- The outermost parentheses are often omitted: $A \Rightarrow B := (A \Rightarrow B)$.
- $\Rightarrow$ is right-associative: $A \Rightarrow B \Rightarrow C := (A \Rightarrow (B \Rightarrow C))$.

Rmk. In minimal logic there is no conjunction, disjunction, negation, falsehood.

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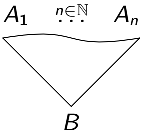
Natural deduction for minimal logic, informally

Natural deduction (ND) is a formalism to represent proofs in minimal logic (and others).

A proof in ND is a finite, vaguely tree-like structure (this is more a graphical illusion):

- edges are labeled by formulas, nodes are inference rules $\dfrac{A_1 \quad \overset{n\in\mathbb{N}}{\dots} \quad A_n}{B}$;
- leaves are hypotheses (they are finitely many, possibly none) or dead leaves;
- the root is the (unique) conclusion.



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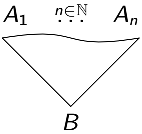

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Natural deduction for minimal logic, slightly more formally

Notation. $\begin{array}{c}\vdots\,\mathcal{D}\\ B\end{array}$ means that $\mathcal{D}$ is a derivation with conclusion B and some hypotheses.

Def. A derivation $\mathcal{D}$ in ND is

- either A (for any formula A), which is both the conclusion and the hypothesis of $\mathcal{D}$;
- or it is obtained from derivations $\mathcal{D}'$, $\mathcal{D}_1$, $\mathcal{D}_2$ by applying one of the inference rules

$$\frac{\begin{array}{c}\vdots\,\mathcal{D}_1\\ A \Rightarrow B\end{array} \qquad \begin{array}{c}\vdots\,\mathcal{D}_2\\ A\end{array}}{B}\Rightarrow_e$$

$\Rightarrow$ elimination $\qquad\qquad$ $\Rightarrow$ introduction

where the hypotheses of $\mathcal{D}$ are

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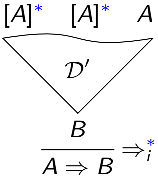

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Examples of derivations in ND

Notation. $A_1, \ldots, A_n \vdash B$ means that there is a derivation in ND with conclusion B and hypothesis *among* $A_1, \ldots, A_n$.

1. Prove that $A \vdash A$, and $\vdash A \Rightarrow A$, and $B \vdash A \Rightarrow A$, and $\vdash A \Rightarrow B \Rightarrow A$.

2. Give two distinct derivations of $A \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.

3. Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

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Soundness and completeness of ND with respect to minimal logic

Def. A (finite) multiset over a set X is a (finite) set of occurrences of elements of X.
Idea. A multiset takes into account the number of copies (not the order) of its elements.

Notation. Given a finite multiset $\Gamma = A_1, \dots, A_n$ of formulas, with $n \in \mathbb{N}$ ($\Gamma = \emptyset$ if n = 0)

- $\mathcal{D} \triangleright_{\mathrm{ND}} \Gamma \vdash A$ means that $\mathcal{D}$ is a derivation with conclusion A and hypotheses among the formulas in Γ;
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Theorem (Soundness and completeness)

$A_1, \dots, A_n \vdash B$ if and only if $A_1 \Rightarrow \cdots \Rightarrow A_n \Rightarrow B$ is valid in minimal logic.

Proof. Omitted. □

Moral. The syntactic approach (ND) is equivalent to the semantic one (Kripke models).

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An alternative presentation of ND via sequents

Def. A sequent is a pair $\Gamma \vdash A$ of a finite multiset Γ of formulas and a formula A.

Def. A derivation in $\mathsf{ND}_{\mathsf{seq}}$ is a tree built up from the inference rules below.

$$\dfrac{}{\Gamma, A \vdash A}\,\mathsf{ax} \qquad \dfrac{\Gamma \vdash A \Rightarrow B \qquad \Gamma \vdash A}{\Gamma \vdash B}\Rightarrow_e \qquad \dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\Rightarrow_i$$

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Proposition

$\Gamma \vdash A$ in ND if and only if $\Gamma \vdash A$ is derivable in $\mathsf{ND}_{\mathsf{seq}}$.

Proof. Every $\mathcal{D} \rhd_{\mathsf{ND}} \Gamma \vdash A$ can be translated into a $\mathcal{D}' \rhd_{\mathsf{ND}_{\mathsf{seq}}} \Gamma \vdash A$ and vice versa.

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Examples of derivations in $\mathrm{ND_{seq}}$

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$$\dfrac{}{A \vdash A}\,\mathsf{ax} \qquad \dfrac{\dfrac{}{A \vdash A}\,\mathsf{ax}}{\vdash A \Rightarrow A}\,\Rightarrow_i \qquad \dfrac{\dfrac{}{A, B \vdash A}\,\mathsf{ax}}{B \vdash A \Rightarrow A}\,\Rightarrow_i \qquad \dfrac{\dfrac{\dfrac{}{A, B \vdash A}\,\mathsf{ax}}{A \vdash B \Rightarrow A}\,\Rightarrow_i}{\vdash A \Rightarrow B \Rightarrow A}\,\Rightarrow_i$$

2. Give two distinct derivations of $A \vdash A \Rightarrow A$, and two distinct ones of $\vdash A \Rightarrow A \Rightarrow A$.

$$\dfrac{\dfrac{}{\overline{A}, A \vdash A}\,\mathsf{ax}}{A \vdash A \Rightarrow A}\,\Rightarrow_i \qquad \dfrac{\dfrac{}{A, \overline{A} \vdash A}\,\mathsf{ax}}{A \vdash A \Rightarrow A}\,\Rightarrow_i \qquad \dfrac{\dfrac{\dfrac{}{\overline{A}, A \vdash A}\,\mathsf{ax}}{A \vdash A \Rightarrow A}\,\Rightarrow_i}{\vdash A \Rightarrow A \Rightarrow A}\,\Rightarrow_i \qquad \dfrac{\dfrac{\dfrac{}{A, \overline{A} \vdash A}\,\mathsf{ax}}{A \vdash A \Rightarrow A}\,\Rightarrow_i}{\vdash A \Rightarrow A \Rightarrow A}\,\Rightarrow_i$$

3. Prove $\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow C) \Rightarrow A \Rightarrow C$ and $\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A, A \Rightarrow B, A \Rightarrow B \Rightarrow C \vdash A}\,\mathbf{ax} \quad \dfrac{}{A, A \Rightarrow B, A \Rightarrow B \Rightarrow C \vdash A \Rightarrow B \Rightarrow C}\,\mathbf{ax}}{A, A \Rightarrow B, A \Rightarrow B \Rightarrow C \vdash B \Rightarrow C}\,\Rightarrow_e \quad \dfrac{\dfrac{}{A, A \Rightarrow B, A \Rightarrow B \Rightarrow C \vdash A \Rightarrow B}\,\mathbf{ax} \quad \dfrac{}{A, A \Rightarrow B, A \Rightarrow B \Rightarrow C \vdash A}\,\mathbf{ax}}{A, A \Rightarrow B, A \Rightarrow B \Rightarrow C \vdash B}\,\Rightarrow_e}{A, A \Rightarrow B, A \Rightarrow (B \Rightarrow C) \vdash C}\,\Rightarrow_e}{A \Rightarrow B, A \Rightarrow (B \Rightarrow C) \vdash A \Rightarrow C}\,\Rightarrow_i}{A \Rightarrow (B \Rightarrow C) \vdash (A \Rightarrow B) \Rightarrow (A \Rightarrow C)}\,\Rightarrow_i}{\vdash (A \Rightarrow (B \Rightarrow C)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow C)}\,\Rightarrow_i$$

Outline

The inversion principle in natural deduction

Def. Let $\mathcal{D}$ a derivation in ND.

- A cut-formula is a formula in $\mathcal{D}$ that is conclusion of a $\Rightarrow_i$ and left premise of a $\Rightarrow_e$.
- A redex is a pair $\Rightarrow_i/\Rightarrow_e$ containing a cut-formula.

Inversion principle. A redex proving B by means of $\Rightarrow_e$, having proved its premises $A \Rightarrow B$ and A, the former by means of $\Rightarrow_i$ with a proof of B from A, amounts to concatenate a proof of A with a proof of B from A (substitution of hypotheses A for a derivation of A).

Mathematically, "amounts to" can be seen as a rewrite relation $\rightarrow_{\text{cut}}$ (cut-elimination).

Rmk. All the discharged hypotheses are replaced by (copies of) the derivation of A.

Rmk. $\rightarrow_{\text{cut}}$ preserves the conclusion, but may discard or copy some hypotheses.

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$$\dfrac{\dfrac{\begin{matrix}[A]^* \\ \vdots\,\mathcal{D} \\ B\end{matrix}}{A \Rightarrow B}\Rightarrow_i^* \qquad \begin{matrix}\vdots\,\mathcal{D}' \\ A\end{matrix}}{B}\Rightarrow_e \qquad \text{amounts to derive} \qquad \begin{matrix}\vdots\,\mathcal{D}' \\ A \\ \vdots\,\mathcal{D} \\ B\end{matrix}$$

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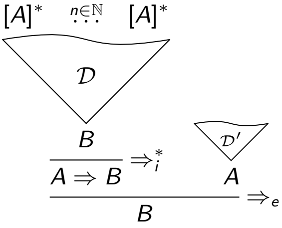

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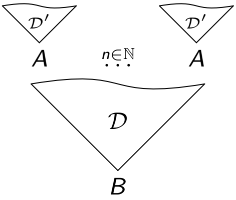

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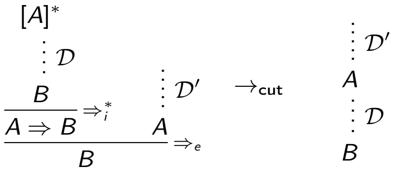

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Examples of cut-elimination steps in ND

The cut-formula is in blue.

$$\dfrac{\dfrac{[X \Rightarrow X]^*}{(X \Rightarrow X) \Rightarrow X \Rightarrow X}\Rightarrow_i^* \qquad \dfrac{[X]^\dagger}{X \Rightarrow X}\Rightarrow_i^\dagger}{X \Rightarrow X}\Rightarrow_e \quad \to_{\text{cut}} \quad \dfrac{[X]^\dagger}{X \Rightarrow X}\Rightarrow_i^\dagger$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A \Rightarrow (B \Rightarrow A)]^\dagger \quad [A]^\circ}{B \Rightarrow A}\Rightarrow_e \quad \dfrac{[A \Rightarrow B]^* \quad [A]^\circ}{B}\Rightarrow_e}{A}\Rightarrow_e}{A \Rightarrow A}\Rightarrow_i^\circ}{(A \Rightarrow B) \Rightarrow (A \Rightarrow A)}\Rightarrow_i^*}{(A \Rightarrow (B \Rightarrow A)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow A)}\Rightarrow_i^\dagger \quad \dfrac{\dfrac{[A]^\dagger}{B \Rightarrow A}\Rightarrow_i}{A \Rightarrow (B \Rightarrow A)}\Rightarrow_i^\dagger}{(A \Rightarrow B) \Rightarrow (A \Rightarrow A)}\Rightarrow_e \quad \to_{\text{cut}} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A]^\dagger}{B \Rightarrow A}\Rightarrow_i}{A \Rightarrow (B \Rightarrow A)}\Rightarrow_i^\dagger \quad [A]^\circ}{B \Rightarrow A}\Rightarrow_e \quad \dfrac{[A \Rightarrow B]^* \quad [A]^\circ}{B}\Rightarrow_e}{A}\Rightarrow_e}{A \Rightarrow A}\Rightarrow_i^\circ}{(A \Rightarrow B) \Rightarrow (A \Rightarrow A)}\Rightarrow_i^*$$

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$$\dfrac{\dfrac{[X \Rightarrow X]^*}{(X \Rightarrow X) \Rightarrow X \Rightarrow X}\Rightarrow_i^* \qquad \dfrac{[X]^\dagger}{X \Rightarrow X}\Rightarrow_i^\dagger}{X \Rightarrow X}\Rightarrow_e \quad \rightarrow_{\mathbf{cut}} \quad \dfrac{[X]^\dagger}{X \Rightarrow X}\Rightarrow_i^\dagger$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A \Rightarrow (B \Rightarrow A)]^\dagger \quad [A]^\circ}{B \Rightarrow A}\Rightarrow_e \quad \dfrac{[A \Rightarrow B]^* \quad [A]^\circ}{B}\Rightarrow_e}{A}\Rightarrow_e}{A \Rightarrow A}\Rightarrow_i^\circ}{(A \Rightarrow B) \Rightarrow (A \Rightarrow A)}\Rightarrow_i^*}{(A \Rightarrow (B \Rightarrow A)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow A)}\Rightarrow_i^\dagger \quad \dfrac{\dfrac{[A]^\dagger}{B \Rightarrow A}\Rightarrow_i}{A \Rightarrow (B \Rightarrow A)}\Rightarrow_i^\dagger}{(A \Rightarrow B) \Rightarrow (A \Rightarrow A)}\Rightarrow_e \quad \rightarrow_{\mathbf{cut}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A]^\dagger}{B \Rightarrow A}\Rightarrow_i}{A \Rightarrow (B \Rightarrow A)}\Rightarrow_i^\dagger \quad [A]^\circ}{B \Rightarrow A}\Rightarrow_e \quad \dfrac{[A \Rightarrow B]^* \quad [A]^\circ}{B}\Rightarrow_e}{A}\Rightarrow_e}{A \Rightarrow A}\Rightarrow_i^\circ}{(A \Rightarrow B) \Rightarrow (A \Rightarrow A)}\Rightarrow_i^*$$

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$$\dfrac{\dfrac{[X \Rightarrow X]^{*}}{(X \Rightarrow X) \Rightarrow X \Rightarrow X}\Rightarrow_i^{*} \qquad \dfrac{[X]^{\dagger}}{X \Rightarrow X}\Rightarrow_i^{\dagger}}{X \Rightarrow X}\Rightarrow_e \quad \to_{\mathsf{cut}} \quad \dfrac{[X]^{\dagger}}{X \Rightarrow X}\Rightarrow_i^{\dagger}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A \Rightarrow (B \Rightarrow A)]^{\dagger} \quad [A]^{\circ}}{B \Rightarrow A}\Rightarrow_e \quad \dfrac{[A \Rightarrow B]^{*} \quad [A]^{\circ}}{B}\Rightarrow_e}{A}\Rightarrow_e}{A \Rightarrow A}\Rightarrow_i^{\circ}}{(A \Rightarrow B) \Rightarrow (A \Rightarrow A)}\Rightarrow_i^{*}}{(A \Rightarrow (B \Rightarrow A)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow A)}\Rightarrow_i^{\dagger} \quad \dfrac{\dfrac{[A]^{\dagger}}{B \Rightarrow A}\Rightarrow_i}{A \Rightarrow (B \Rightarrow A)}\Rightarrow_i^{\dagger}}{(A \Rightarrow B) \Rightarrow (A \Rightarrow A)}\Rightarrow_e$$

$$\to_{\mathsf{cut}} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A]^{\dagger}}{B \Rightarrow A}\Rightarrow_i}{A \Rightarrow (B \Rightarrow A)}\Rightarrow_i^{\dagger} \quad [A]^{\circ}}{B \Rightarrow A}\Rightarrow_e \quad \dfrac{[A \Rightarrow B]^{*} \quad [A]^{\circ}}{B}\Rightarrow_e}{A}\Rightarrow_e}{A \Rightarrow A}\Rightarrow_i^{\circ}}{(A \Rightarrow B) \Rightarrow (A \Rightarrow A)}\Rightarrow_i^{*}$$

Examples of cut-elimination steps in ND

The cut-formula is in blue.

$$\dfrac{\dfrac{[X \Rightarrow X]^*}{\color{blue}{(X \Rightarrow X) \Rightarrow X \Rightarrow X}}\Rightarrow_i^* \qquad \dfrac{[X]^\dagger}{X \Rightarrow X}\Rightarrow_i^\dagger}{X \Rightarrow X}\Rightarrow_e \quad \to_{\mathbf{cut}} \quad \color{red}{\dfrac{[X]^\dagger}{X \Rightarrow X}\Rightarrow_i^\dagger}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A \Rightarrow (B \Rightarrow A)]^\dagger \quad [A]^\circ}{B \Rightarrow A}\Rightarrow_e \quad \dfrac{[A \Rightarrow B]^* \quad [A]^\circ}{B}\Rightarrow_e}{A}\Rightarrow_e}{A \Rightarrow A}\Rightarrow_i^\circ}{(A \Rightarrow B) \Rightarrow (A \Rightarrow A)}\Rightarrow_i^*}{\color{blue}{(A \Rightarrow (B \Rightarrow A)) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow A)}}\Rightarrow_i^\dagger \quad \dfrac{\dfrac{[A]^\dagger}{B \Rightarrow A}\Rightarrow_i}{A \Rightarrow (B \Rightarrow A)}\Rightarrow_i^\dagger}{(A \Rightarrow B) \Rightarrow (A \Rightarrow A)}\Rightarrow_e \quad \to_{\mathbf{cut}} \quad \dfrac{\dfrac{\dfrac{\dfrac{\color{red}{\dfrac{\dfrac{[A]^\dagger}{B \Rightarrow A}\Rightarrow_i}{A \Rightarrow (B \Rightarrow A)}\Rightarrow_i^\dagger} \quad [A]^\circ}{B \Rightarrow A}\Rightarrow_e \quad \dfrac{[A \Rightarrow B]^* \quad [A]^\circ}{B}\Rightarrow_e}{A}\Rightarrow_e}{A \Rightarrow A}\Rightarrow_i^\circ}{(A \Rightarrow B) \Rightarrow (A \Rightarrow A)}\Rightarrow_i^*$$

Other examples of cut-elimination steps in ND

The cut-formula is in blue.

$$
\dfrac{\dfrac{\dfrac{\dfrac{(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X) \quad [X \Rightarrow X]^\circ}{B \Rightarrow X \Rightarrow X}\Rightarrow_e \quad \dfrac{(X \Rightarrow X) \Rightarrow B \quad [X \Rightarrow X]^\circ}{B}\Rightarrow_e}{X \Rightarrow X}\Rightarrow_e}{\color{blue}{(X \Rightarrow X) \Rightarrow X \Rightarrow X}}\Rightarrow_i^\circ \quad \dfrac{[X]^*}{X \Rightarrow X}\Rightarrow_i^*}{X \Rightarrow X}\Rightarrow_e
$$

$\downarrow$cut

$$
\dfrac{\dfrac{(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X) \quad \dfrac{[X]^*}{X \Rightarrow X}\Rightarrow_i^*}{B \Rightarrow X \Rightarrow X}\Rightarrow_e \quad \dfrac{(X \Rightarrow X) \Rightarrow B \quad \dfrac{[X]^*}{X \Rightarrow X}\Rightarrow_i^*}{B}\Rightarrow_e}{X \Rightarrow X}\Rightarrow_e
$$

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$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X) \quad [X \Rightarrow X]^\circ}{B \Rightarrow X \Rightarrow X}\Rightarrow_e
\quad
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}{X \Rightarrow X}\Rightarrow_e
}{\color{blue}{(X \Rightarrow X) \Rightarrow X \Rightarrow X}}\Rightarrow_i^\circ
\quad
\dfrac{[X]^*}{X \Rightarrow X}\Rightarrow_i^*
}{X \Rightarrow X}\Rightarrow_e
$$

$$\downarrow_{\text{cut}}$$

$$
\dfrac{
\dfrac{(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X) \quad \color{red}{\dfrac{[X]^*}{X \Rightarrow X}\Rightarrow_i^*}}{B \Rightarrow X \Rightarrow X}\Rightarrow_e
\quad
\dfrac{(X \Rightarrow X) \Rightarrow B \quad \color{red}{\dfrac{[X]^*}{X \Rightarrow X}\Rightarrow_i^*}}{B}\Rightarrow_e
}{X \Rightarrow X}\Rightarrow_e
$$

Cut-elimination (aka normalization) theorem in natural deduction

Def. The size of a formula A is the number of occurrences of $\Rightarrow$ in A.
The weight of a redex is the size of its cut-formula.
The weight $w(\mathcal{D})$ of a derivation $\mathcal{D}$ is the finite multiset of the weights of its redexes.

Rmk. A multiset over a set S can be seen as a function $m \colon S \to \mathbb{N}$.

Idea. $m(x) \in \mathbb{N}$ is the multiplicity of x, the number of copies of x in the multiset m.

Def. Let $(S, \prec)$ be an ordered set and m, n be multisets over S: $m \prec_{mul} n$ if $m \neq n$ and for all $x \in S$ such that $m(x) > n(x)$ there is $y \in S$ such that $x \prec y$ and $m(y) < n(y)$.

Ex. $[1,2,2] \prec_{mul} [1,2,2,3,3,3] \prec_{mul} [2,3,3,3,3]$. $\quad <_{mul}$ from $(\mathbb{N}, <)$ is a well-ordering.

Theorem (Cut-elimination, aka normalization [Gentzen 1936, Prawitz 1965])

If $\mathcal{D} \rhd_{\mathsf{ND}} \Gamma \vdash A$, then there is $\mathcal{D}' \rhd_{\mathsf{ND}} \Gamma \vdash A$ without redexes such that $\mathcal{D} \to_{\mathsf{cut}}^{*} \mathcal{D}'$.

Proof. If $\mathcal{D}$ is without redexes, we are done. Otherwise, take a redex r in $\mathcal{D}$ such that there are no redexes above the $\Rightarrow_e$ in r (such a r exists because $\mathcal{D}$ is finite!). Apply $\to_{\mathsf{cut}}$ to r to get $\mathcal{D}_1 \rhd_{\mathsf{ND}} \Gamma \vdash A$ where redexes are not duplicated (as r is an uppermost redex), new redexes can be created but have a lower weight (smaller cut-formula). Therefore, $w(\mathcal{D}) \succ_{mul} w(\mathcal{D}_1)$. By induction hypothesis on the weight of derivations, we conclude. □

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The cut-elimination theorem in an example

The weight of the derivation $\mathcal{D}$ below is $w(\mathcal{D}) = [3, 9]$.

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{[(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)]^{\dagger} \quad [X \Rightarrow X]^{\circ}}{B \Rightarrow (X \Rightarrow X)}\Rightarrow_e
\quad
\dfrac{[(X \Rightarrow X) \Rightarrow B]^{*} \quad [X \Rightarrow X]^{\circ}}{B}\Rightarrow_e
}{X \Rightarrow X}\Rightarrow_e
}{(X \Rightarrow X) \Rightarrow (X \Rightarrow X)}\Rightarrow_i^{\circ}
\quad
\dfrac{[X]^{\bullet}}{X \Rightarrow X}\Rightarrow_i^{\bullet}
}{X \Rightarrow X}\Rightarrow_e
}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}\Rightarrow_i^{*}
}{((X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)) \Rightarrow ((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}\Rightarrow_i^{\dagger}
\quad
\dfrac{\dfrac{[X \Rightarrow X]^{\dagger}}{B \Rightarrow X \Rightarrow X}\Rightarrow_i}{(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)}\Rightarrow_i^{\dagger}
}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}\Rightarrow_e
$$

$\downarrow$cut We fire the red redex (uppermost in $\mathcal{D}$).

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{[(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)]^{\dagger} \quad \dfrac{[X]^{\bullet}}{X \Rightarrow X}\Rightarrow_i^{\bullet}}{B \Rightarrow (X \Rightarrow X)}\Rightarrow_e
\quad
\dfrac{[(X \Rightarrow X) \Rightarrow B]^{*} \quad \dfrac{[X]^{\bullet}}{X \Rightarrow X}\Rightarrow_i^{\bullet}}{B}\Rightarrow_e
}{X \Rightarrow X}\Rightarrow_e
}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}\Rightarrow_i^{*}
}{((X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)) \Rightarrow ((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}\Rightarrow_i^{\dagger}
\quad
\dfrac{\dfrac{[X \Rightarrow X]^{\dagger}}{B \Rightarrow X \Rightarrow X}\Rightarrow_i}{(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)}\Rightarrow_i^{\dagger}
}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}\Rightarrow_e
$$

The weight of the derivation $\mathcal{D}'$ above is $w(\mathcal{D}') = [9] \prec_{mul} [3, 9] = w(\mathcal{D})$.

The cut-elimination theorem in an example

The weight of the derivation $\mathcal{D}$ below is $w(\mathcal{D}) = [3, 9]$.

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{[(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)]^{\dagger} \quad [X \Rightarrow X]^{\circ}}{B \Rightarrow (X \Rightarrow X)}\Rightarrow_e
\quad
\dfrac{[(X \Rightarrow X) \Rightarrow B]^{*} \quad [X \Rightarrow X]^{\circ}}{B}\Rightarrow_e
}{X \Rightarrow X}\Rightarrow_e
}{(X \Rightarrow X) \Rightarrow (X \Rightarrow X)}\Rightarrow_i^{\circ}
\quad
\dfrac{[X]^{\bullet}}{X \Rightarrow X}\Rightarrow_i^{\bullet}
}{X \Rightarrow X}\Rightarrow_e
}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}\Rightarrow_i^{*}
}{((X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)) \Rightarrow ((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}\Rightarrow_i^{\dagger}
\quad
\dfrac{\dfrac{[X \Rightarrow X]^{\dagger}}{B \Rightarrow X \Rightarrow X}\Rightarrow_i}{(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)}\Rightarrow_i^{\dagger}
}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}\Rightarrow_e
$$

$\downarrow$cut We fire the red redex (uppermost in $\mathcal{D}$).

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{[(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)]^{\dagger} \quad \dfrac{[X]^{\bullet}}{X \Rightarrow X}\Rightarrow_i^{\bullet}}{B \Rightarrow (X \Rightarrow X)}\Rightarrow_e
\quad
\dfrac{[(X \Rightarrow X) \Rightarrow B]^{*} \quad \dfrac{[X]^{\bullet}}{X \Rightarrow X}\Rightarrow_i^{\bullet}}{B}\Rightarrow_e
}{X \Rightarrow X}\Rightarrow_e
}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}\Rightarrow_i^{*}
}{((X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)) \Rightarrow ((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}\Rightarrow_i^{\dagger}
\quad
\dfrac{\dfrac{[X \Rightarrow X]^{\dagger}}{B \Rightarrow X \Rightarrow X}\Rightarrow_i}{(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)}\Rightarrow_i^{\dagger}
}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}\Rightarrow_e
$$

The weight of the derivation $\mathcal{D}'$ above is $w(\mathcal{D}') = [9] \prec_{mul} [3, 9] = w(\mathcal{D})$.

Some consequences of cut-elimination

Prop. If $\mathcal{D} \rhd_{\mathsf{ND}} \Gamma \vdash A$ is without redexes, then in $\mathcal{D}$ there are only subformulas of Γ or A.

Corollary (Subformula property)

If $\Gamma \vdash A$ in ND then there is $\mathcal{D} \rhd_{\mathsf{ND}} \Gamma \vdash A$ only containing subformulas of Γ and A.

Proof. By cut-elimination, there is $\mathcal{D}$ with no redexes. By Prop. above, we conclude. □

Moral. If you are searching for a $\mathcal{D} \rhd_{\mathsf{ND}} \Gamma \vdash A$, just look at the subformulas of Γ and A.

Corollary (Consistency of ND)

Some formulas are not provable in ND.

Proof. $\nvdash X$ in ND, otherwise there would be $\mathcal{D} \rhd_{\mathsf{ND}} \vdash X$ with the subformula property by Cor. above, but the last rule of $\mathcal{D}$ could neither be $\Rightarrow_i$ (because X is not an implication) nor $\Rightarrow_e$ (by the subformula property) nor an hypothesis (since $\mathcal{D}$ has no hypotheses). □

Rmk. Consistency of ND already follows from soundness of ND. Who cares aboout $\to_{\mathsf{cut}}$?

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Rmk. Consistency of ND already follows from soundness of ND. Who cares aboout $\to_{\mathsf{cut}}$?

Cut-elimination is not deterministic

In the derivation below, there are two redexes. We can fire either of them.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)]^\dagger \quad [X \Rightarrow X]^\circ}{B \Rightarrow (X \Rightarrow X)}\Rightarrow_e \quad \dfrac{[(X \Rightarrow X) \Rightarrow B]^* \quad [X \Rightarrow X]^\circ}{B}\Rightarrow_e}{\dfrac{X \Rightarrow X}{\color{red}{(X \Rightarrow X) \Rightarrow (X \Rightarrow X)}}\color{red}{\Rightarrow_i^\circ}}\Rightarrow_e \quad \dfrac{[X]^\bullet}{X \Rightarrow X}\Rightarrow_i^\bullet}{X \Rightarrow X}\color{red}{\Rightarrow_e}}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}\Rightarrow_i^*}{\color{blue}{((X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)) \Rightarrow (((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X))}}\color{blue}{\Rightarrow_i^\dagger} \quad \dfrac{\dfrac{[X \Rightarrow X]^\dagger}{B \Rightarrow X \Rightarrow X}\Rightarrow_i}{(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)}\Rightarrow_i^\dagger}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}\color{blue}{\Rightarrow_e}$$

Questions: Keep on performing cut-elimination steps starting from a same derivation:

1. Do we eventually obtain the same derivation?
2. Is there a cut-elimination sequence leading to a derivation without redexes?
3. Does every cut-elimination sequence eventually lead to a derivation without redexes?

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In the derivation below, there are two redexes. We can fire either of them.

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{[(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)]^{\dagger} \quad [X \Rightarrow X]^{\circ}}{B \Rightarrow (X \Rightarrow X)}\Rightarrow_e
        \quad
        \dfrac{[(X \Rightarrow X) \Rightarrow B]^{*} \quad [X \Rightarrow X]^{\circ}}{B}\Rightarrow_e
      }{
        \dfrac{X \Rightarrow X}{{\color{red}(X \Rightarrow X) \Rightarrow (X \Rightarrow X)}}{\color{red}\Rightarrow_i}^{\circ}
      }\Rightarrow_e
      \quad
      \dfrac{[X]^{\bullet}}{X \Rightarrow X}\Rightarrow_i^{\bullet}
    }{
      \dfrac{X \Rightarrow X}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}\Rightarrow_i^{*}
    }{\color{red}\Rightarrow_e}
  }{
    {\color{blue}((X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)) \Rightarrow ((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X))}
  }{\color{blue}\Rightarrow_i}^{\dagger}
  \quad
  \dfrac{\dfrac{[X \Rightarrow X]^{\dagger}}{B \Rightarrow X \Rightarrow X}\Rightarrow_i}{(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)}\Rightarrow_i^{\dagger}
}{
  ((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)
}{\color{blue}\Rightarrow_e}
$$

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    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{[(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)]^{\dagger} \quad [X \Rightarrow X]^{\circ}}{B \Rightarrow (X \Rightarrow X)}{\scriptstyle \Rightarrow_e}
          \quad
          \dfrac{[(X \Rightarrow X) \Rightarrow B]^{*} \quad [X \Rightarrow X]^{\circ}}{B}{\scriptstyle \Rightarrow_e}
        }{
          \dfrac{X \Rightarrow X}{\color{red}{(X \Rightarrow X) \Rightarrow (X \Rightarrow X)}}{\color{red}{\scriptstyle \Rightarrow_i^{\circ}}}
        }{\scriptstyle \Rightarrow_e}
        \quad
        \dfrac{[X]^{\bullet}}{X \Rightarrow X}{\scriptstyle \Rightarrow_i^{\bullet}}
      }{X \Rightarrow X}{\color{red}{\scriptstyle \Rightarrow_e}}
    }{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}{\scriptstyle \Rightarrow_i^{*}}
  }{\color{blue}{((X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)) \Rightarrow (((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X))}}{\color{blue}{\scriptstyle \Rightarrow_i^{\dagger}}}
  \quad
  \dfrac{\dfrac{[X \Rightarrow X]^{\dagger}}{B \Rightarrow X \Rightarrow X}{\scriptstyle \Rightarrow_i}}{(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)}{\scriptstyle \Rightarrow_i^{\dagger}}
}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}{\color{blue}{\scriptstyle \Rightarrow_e}}
$$

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\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{[(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)]^{\dagger} \quad [X \Rightarrow X]^{\circ}}{B \Rightarrow (X \Rightarrow X)}{\scriptstyle \Rightarrow_e}
          \quad
          \dfrac{[(X \Rightarrow X) \Rightarrow B]^{*} \quad [X \Rightarrow X]^{\circ}}{B}{\scriptstyle \Rightarrow_e}
        }{X \Rightarrow X}{\scriptstyle \Rightarrow_e}
      }{\color{red}(X \Rightarrow X) \Rightarrow (X \Rightarrow X)}{\color{red}\scriptstyle \Rightarrow_i^{\circ}}
      \quad
      \dfrac{[X]^{\bullet}}{X \Rightarrow X}{\scriptstyle \Rightarrow_i^{\bullet}}
    }{X \Rightarrow X}{\color{red}\scriptstyle \Rightarrow_e}
  }{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}{\scriptstyle \Rightarrow_i^{*}}
  }{\color{blue}((X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)) \Rightarrow (((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X))}{\color{blue}\scriptstyle \Rightarrow_i^{\dagger}}
  \quad
  \dfrac{\dfrac{[X \Rightarrow X]^{\dagger}}{B \Rightarrow X \Rightarrow X}{\scriptstyle \Rightarrow_i}}{(X \Rightarrow X) \Rightarrow (B \Rightarrow X \Rightarrow X)}{\scriptstyle \Rightarrow_i^{\dagger}}
}{((X \Rightarrow X) \Rightarrow B) \Rightarrow (X \Rightarrow X)}{\color{blue}\scriptstyle \Rightarrow_e}
$$

Questions: Keep on performing cut-elimination steps starting from a same derivation:

1. Do we eventually obtain the same derivation?
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3. Does every cut-elimination sequence eventually lead to a derivation without redexes?

Outline

What have we learned today?

1. How to write formal proofs in minimal logic using natural deduction.
2. The procedure of cut elimination for natural deduction in minimal logic.
3. The normalization theorem (and its proof) for natural deduction in minimal logic.

Questions?

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Questions?

Exercises

1. Prove the following facts, using ND and ND_{seq}.
   1. $\vdash X \Rightarrow ((X \Rightarrow Y) \Rightarrow Y)$.
   2. $(X \Rightarrow Y) \Rightarrow (X \Rightarrow Z) \vdash Y \Rightarrow X \Rightarrow Z$.
   3. $(X \Rightarrow Y) \Rightarrow X \vdash Y \Rightarrow X$.
   4. $X \Rightarrow (Y \Rightarrow Z) \vdash Y \Rightarrow X \Rightarrow Z$.
   5. $X \Rightarrow Y \Rightarrow Z, X \Rightarrow Y \vdash X \Rightarrow Z$.
   6. $(X \Rightarrow X) \Rightarrow Y \vdash (Y \Rightarrow Z) \Rightarrow Z$.
2. Show that $\nvdash (X \Rightarrow Y) \Rightarrow X$, i.e. $(X \Rightarrow Y) \Rightarrow X$ is not derivable with no hypothesis.
   Hint: Use the subformula property (do you really need it?).
3. Perform all possible cut-elimination steps from the derivation on p. 24, until you get a derivation without redexes. Is it always the same?
4. Order the following multisets over $\mathbb{N}$ according to the multiset order $\prec_{mul}$.

   $$[1, 1] \qquad [0, 2] \qquad [1] \qquad [0, 0, 2] \qquad [\,] \qquad [0, 3] \qquad [0, 2, 2]$$

5. Prove in a rigorous way the proposition on p. 15.
   Hint: Proceed by structural induction on a derivation in ND for the left-to-right part, and by structural induction on the a derivation in ND_{seq} for the right-to-left part.
6. For any formula B, prove that if $\Gamma \vdash A$ is derivable in ND_{seq}, then so is $\Gamma, B \vdash A$.
7. For any formula B, prove that if $\Gamma, B, B \vdash A$ is derivable in ND_{seq} then so is $\Gamma, B \vdash A$.

Bibliography

- For more about natural deduction:

Dag Prawitz. *Natural Deduction: a Proof-Theoretical Study*. Mineola, N.Y.: Dover Publications, 1965 (reprint 2006). [Chapters 1–2, 4]

Jean-Yves Girard, Yves Lafont, Paul Taylor. *Proofs and Types*. Cambridge Tracts in Theoretical Computer Science, series number 7, Cambridge University Press, 1989. `https://www.paultaylor.eu/stable/prot.pdf`. [Chapter 2]

- For more about proof theory:

Anne S. Troelstra, Helmut Schwichtenberg. *Basic Proof Theory*. Cambridge Tracts in Theoretical Computer Science, series number 43, Cambridge University Press, 2nd edition, 2000. [Chapters 2, 6]