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Distance Functions on Digital Pictures

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Abstract—This paper describes algorithms for computing various functions on a digital picture which depend on the distance to a given subset of the picture. The algorithms involve local operations which are performed repeatedly, "in parallel", on every picture element and its immediate neighbors. Applications to the detection of "clusters" and "regularities" in a picture, and to the dissection of a region into "pieces", are also described.

1. DISTANCE FUNCTIONS

LET A be the set of all pairs of integers (i, j). The function f from $A \times A$ into the nonnegative integers is called

(a) *Positive definite*, if $f(x, y) = 0$ if and only if $x = y$.
(b) *Symmetric*, if $f(x, y) = f(y, x)$ for all x, y in A.
(c) *Triangular*, if $f(x, z) \leq f(x, y)+f(y, z)$ for all x, y, z in A.

If f satisfies (a–c), it is called a *distance function*.

Proposition 1. If f and g satisfy (a), so do $f+g$ and $\max(f, g)$; similarly for (b) and (c).

Proof: Let $f(x, y)+g(x, y) = 0$; then since $f \geq 0$ and $g \geq 0$, we must have $f(x, y) = g(x, y) = 0$, so that $x = y$. Clearly $f(y, x)+g(y, x) = f(x, y)+g(x, y)$, while $f(x, z)+g(x, z) \leq f(x, y)+f(y, z)+g(x, y)+g(y, z) = f(x, y)+g(x, y)+f(y, z)+g(y, z)$. The proofs for $\max(f, g)$ are analogous.

COROLLARY. If $f_1, \ldots, f_n$ satisfy (a), (b) or (c), so do $f_1+\cdots+f_n$ and $\max(f_1, \ldots, f_n)$.

Proposition 2. $f[(i, j), (h, k)] = |i-h|$ and $g[(i, j), (h, k)] = |j-k|$ are distance functions.

Proof: (a)–(b) are clear; as for (c), note that $f[(m, n), (p, q)] = f[(m+r, n), (p+r, q)]$ for any m, so that it suffices to prove $f[(i, j), (h, k)] \leq f[(i, j), (0, 0)]+f[(h, k), (0, 0)]$, i.e. $|i-h| \leq |i|+|h|$, which is a well known property of absolute value, and similarly for g.

COROLLARY. $d_1[(i, j), (h, k)] = |i-h|+|j-k|$ and $d_2[(i, j), (h, k)] = \max(|i-h|, |j-k|)$ are distance functions. (d_1 is sometimes called *city block distance*.)

Proposition 3. The set of (i, j) at distance $\leq r$ from $(0, 0)$ is:

(a) For d_1: A *diamond* (i.e. a square with sides inclined at $\pm 45°$) with vertices at $(0, \pm r)$ and $(\pm r, 0)$
(b) For d_2: A *square* (with sides horizontal and vertical) with vertices at $(r, \pm r)$ and $(-r, \pm r)$.

Proof: (a) This is the set of (i, j) such that $|i|+|j| \leq r$; thus $|j| \leq r$, and for any such j we have $|j|-r \leq i \leq r-|j|$.

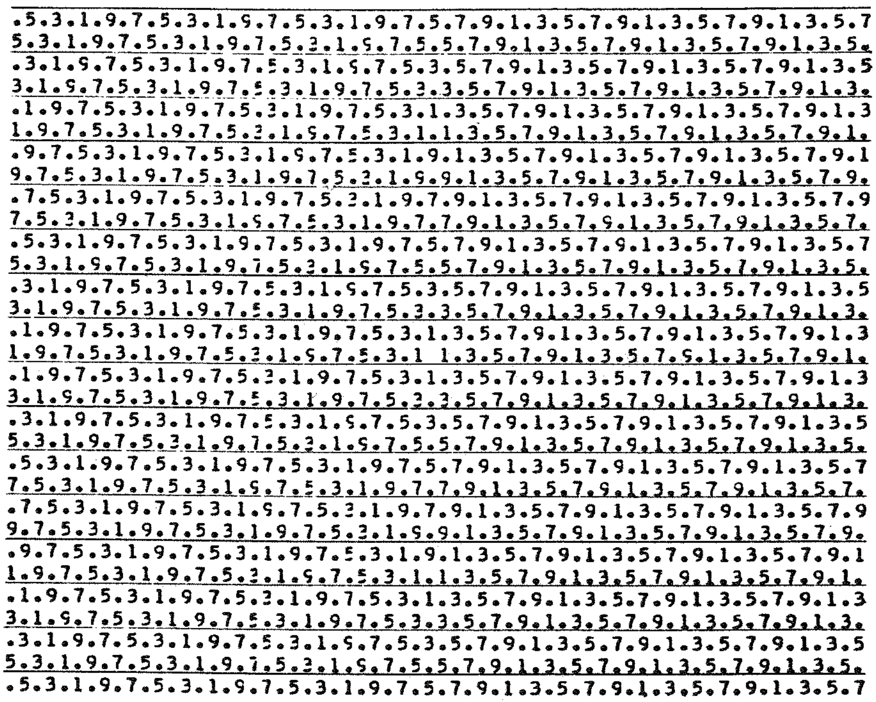

FIG. 1. City block distances (d_1) from a single point.

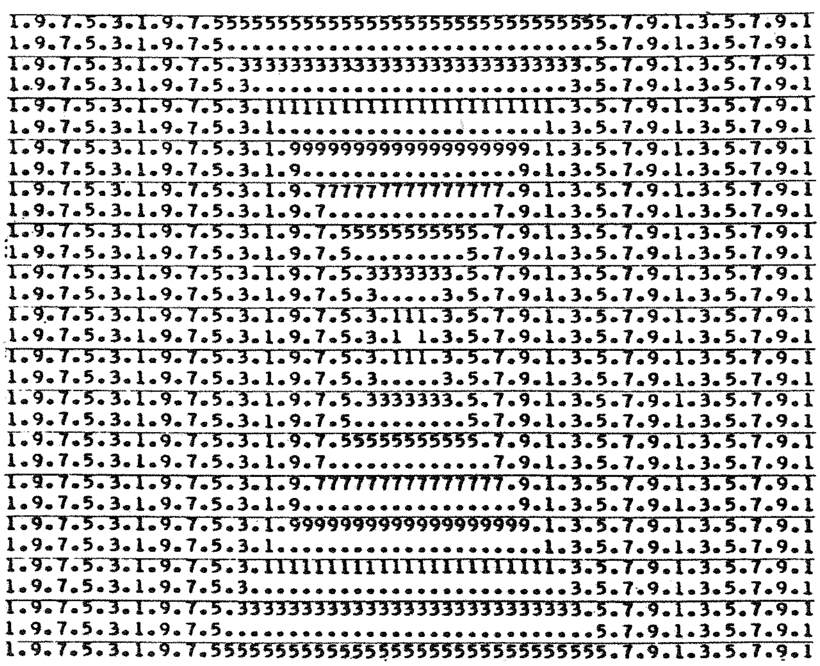

FIG. 2. "Square" distances (d_2) from a single point.

(b) This is the set of (i, j) such that $\max(|i|, |j|) \leq r$, so that $|j| \leq r$, and for any such j we have $-r \leq i \leq r$.

Figures 1–2 show the distances d_1 and d_2 from a single centrally located point. For clarity, only the odd distances are printed out modulo 10, while the points at even distances are represented by dots; the original point is blank.

Proposition 4.

$$d_3[(i, j), (h, k)] = \max\left(|i-h|, \tfrac{1}{2}(|i-h|+(i-h)) - \left(\left[\frac{i}{2}\right] - \left[\frac{h}{2}\right]\right) + k - j,\right.$$
$$\left.\tfrac{1}{2}(|i-h|-(i-h)) + \left(\left[\frac{i}{2}\right] - \left[\frac{h}{2}\right]\right) + j - k\right)$$

is a distance function, where $[x]$ means the greatest integer not exceeding x.

Proof: If $d_3 = 0$, since $|i-h| \geq 0$ we must have $|i-h| = 0$, so that $i = h$; hence $\max(k-j, j-k) \leq 0$, i.e. $|j-k| = 0$, so that $j = k$. Symmetry is clear, since interchanging (i, j) and (h, k) just interchanges the second and third arguments of the max. As for triangularity, note that for all (i, j), (h, k), (m, n) we have

$$\frac{i-h}{2} - \left(\left[\frac{i}{2}\right] - \left[\frac{h}{2}\right]\right) + k - j = \frac{i-m}{2} - \left(\left[\frac{i}{2}\right] - \left[\frac{m}{2}\right]\right) + n - j + \frac{m-h}{2} - \left(\left[\frac{m}{2}\right] - \left[\frac{h}{2}\right]\right) + k - n,$$

while

$$\frac{|i-h|}{2} \leq \frac{|i-m|}{2} + \frac{|m-h|}{2}$$

by Proposition 2. Adding, we have the triangularity of the second argument of the max, and similarly for the third argument, while the first argument is triangular by Proposition 2; now use the Corollary to Proposition 1.

The set of (i, j) such that $d_3[(i, j), (0, 0)] \leq r$ is approximately a *hexagon* centered at $(0, 0)$. [To see this, use the fact that $d_3[(i, j), (h, k)] =$

$$|i-h| + \max\left(0, k-j-\left[\frac{|i-h|}{2}\right], j-k-\left[\frac{|i-h|+1}{2}\right]\right), \quad \text{if } i \text{ is odd;}$$

$$|i-h| + \max\left(0, k-j-\left[\frac{|i-h|+1}{2}\right], j-k-\left[\frac{|i-h|}{2}\right]\right), \quad \text{if } i \text{ is even.]}$$

Figure 3 shows the distances d_3 from a single point, analogously to Figs. 1–2. Figure 4 similarly shows the distances d_3 from a single point to the points of a "staggered" array in which, on alternating rows, only odd numbered or only even numbered elements are used. Note that this array can be regarded as hexagonal, since each element has exactly six "neighbors," two on its own row and two each on the rows above and below it.

Proposition 5. $g[(i, j), (h, k)] = [2(|i-h|+|j-k|+1)/3]$ is a distance function.

Proof: If $g = 0$ we have $2(|i-h|+|j-k|+1)/3 \leq \frac{2}{3}$, so that $|i-h|+|j-k| \leq 0$, proving positive definiteness, while symmetry is clear. For triangularity, as in Proposition 2 it

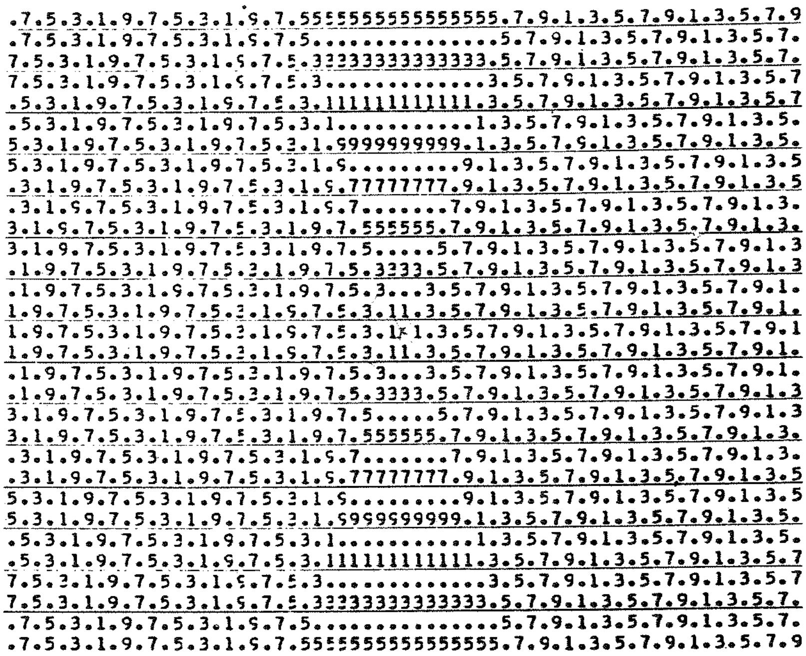

FIG. 3. "Hexagonal" distances (d_3) from a single point (Cartesian array).

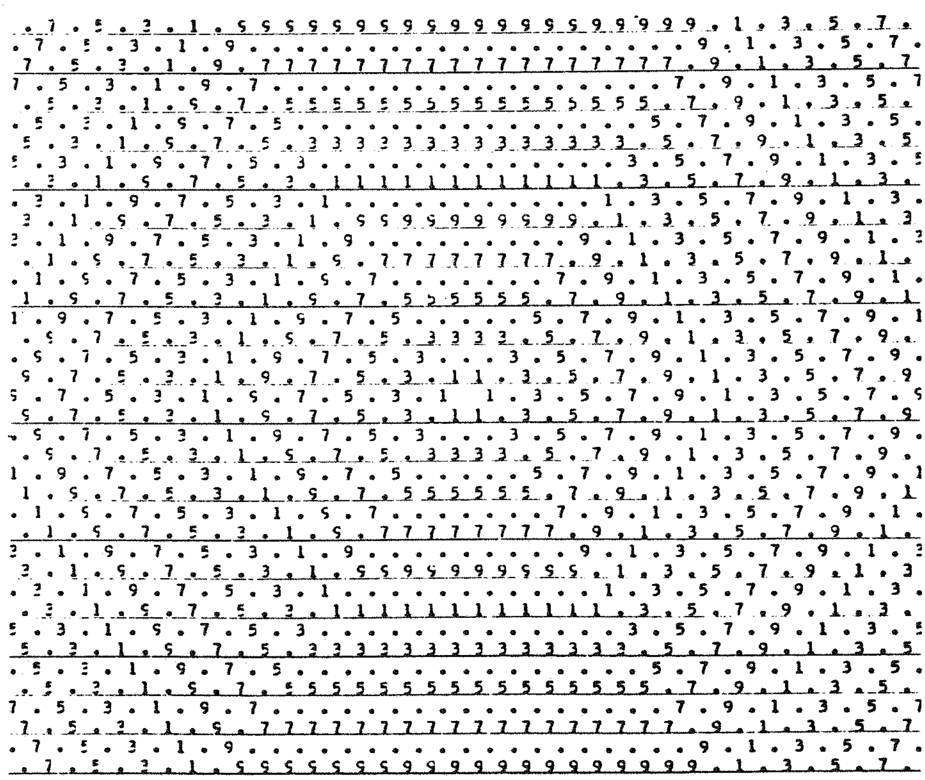

FIG. 4. Hexagonal distances from a single point (staggered array).

suffices to prove that

$$\left[\frac{2(|i-h|+|j-k|+1)}{3}\right] \le \left[\frac{2(|i|+|j|+1)}{3}\right] + \left[\frac{2(|h|+|k|+1)}{3}\right].$$

Now $|i-h|+|j-k| \le |i|+|h|+|j|+|k|$. If this inequality is strict, we have $|i-h|+|j-k| \le |i|+|h|+|j|+|k|-1$, so that

$$\begin{aligned}\left[\frac{2(|i-h|+|j-k|+1)}{3}\right] &\le \frac{2(|i-h|+|j-k|+1)}{3} \\ &\le \frac{2(|i|+|j|+|h|+|k|)}{3} \\ &= \frac{2(|i|+|j|+1)}{3} - \frac{2}{3} + \frac{2(|h|+|k|+1)}{3} - \frac{2}{3} \\ &\le \left[\frac{2(|i|+|j|+1)}{3}\right] + \left[\frac{2(|h|+|k|+1)}{3}\right],\end{aligned}$$

as required. Thus it remains only to consider the case where $|i-h|+|j-k| = |i|+|h|+|j|+|k|$, so that

$$\begin{aligned}\frac{2(|i-h|+|j-k|+1)}{3} &= \frac{2(|i|+|j|+|h|+|k|+1)}{3} \\ &= \frac{2(|i|+|j|+1)}{3} + \frac{2(|h|+|k|+1)}{3} - \frac{2}{3}.\end{aligned}$$

If the fractional part of the left member is $\frac{2}{3}$, or if the sum of the fractional parts of the first two terms in the right member is not more than $\frac{2}{3}$, the desired inequality follows. The same is true if the fractional part of the left member is $\frac{1}{3}$, and those of the first two terms in the right member are not both $\frac{2}{3}$. Hence we need only consider cases in which the fractional part of the left member is 0 or $\frac{1}{3}$, and those of the first two terms in the right member are $\frac{1}{3}$ and $\frac{2}{3}$ or are both $\frac{2}{3}$, i.e.

(1) $2(|i|+|j|+|h|+|k|+1) \equiv 0$ or $1 \pmod 3$;
$2(|i|+|j|+1) \equiv 2(|h|+|k|+1) \equiv 2 \pmod 3$

(2) $2(|i|+|j|+|h|+|k|+1) \equiv 0 \pmod 3$;
$2(|i|+|j|+1) \equiv 1 \pmod 3$ and $2(|h|+|k|+1) \equiv 2 \pmod 3$ (or vice versa).

In case (1) we have $|i|+|j|+1 \equiv |h|+|k|+1 \equiv 1 \pmod 3$, i.e. $|i|+|j| \equiv |h|+|k| \equiv 0 \pmod 3$, so that $2(|i|+|j|+|h|+|k|+1) \equiv 2 \pmod 3$, contradiction. Similarly, in case (2) we have $|i|+|j| \equiv 1 \pmod 3$ and $|h|+|k| \equiv 0 \pmod 3$, so that $2(|i|+|j|+|h|+|k|+1) \equiv 1 \pmod 3$, contradiction.

COROLLARY. $d_4 = \max(g, d_2)$ is a distance function.
Readily, the set of (i, j) such that $g[(i, j), (0, 0)] \le r$ is a diamond with vertices at

$$\left(0, \pm\left[\frac{3r}{2}\right]\right) \quad \text{and} \quad \left(\pm\left[\frac{3r}{2}\right], 0\right).$$

Indeed, this is the set of (i, j) such that

$$\left\lceil \frac{2(|i|+|j|+1)}{3} \right\rceil \leq r,$$

so that

$$|j| \leq \left[\frac{3r}{2}\right] \text{—for, if } |j| > \left[\frac{3r}{2}\right]$$

we would have

$$\left\lceil \frac{2(|i|+|j|+1)}{3} \right\rceil \geq \left\lceil \left(2\left[\frac{3r}{2}\right]+4\right)/3 \right\rceil \geq \left\lceil \left(2\left(\frac{3r}{2}-\frac{1}{2}\right)+4\right)/3 \right\rceil = r+1.$$

Moreover, for any such j we have

$$\frac{2(|i|+|j|+1)}{3} \leq r+\frac{2}{3},$$

i.e. $|i| \leq 3r/2-|j|$. It follows that the set of (i, j) such that $d_4[(i, j), (0, 0)] \leq r$ is an *octagon* with vertices at

$$\left(r, \pm\left[\frac{r}{2}\right]\right), \quad \left(-r, \pm\left[\frac{r}{2}\right]\right), \quad \left(\left[\frac{r}{2}\right], \pm r\right), \quad \text{and} \quad \left(-\left[\frac{r}{2}\right], \pm r\right).$$

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1..9..7..5..3..1..9..7..555555555555555..7..9..1..3..5..7..9..1
1.9..7..5..3..1..9..7..5...............5..7..9..1..3..5..7..9.1
1.9.7..5..3..1..9..7..5..3333333333333..5..7..9..1..3..5..7.9.1
1.9.7.5..3..1..9..7..5..3.............3..5..7..9..1..3..5.7.9.1
1.9.7.5.3..1..9..7..5..3..11111111111..3..5..7..9..1..3.5.7.9.1
1.9.7.5.3.1..9..7..5..3..1...........1..3..5..7..9..1.3.5.7.9.1
1.9.7.5.3.1.9..7..5..3..1..999999999..1..3..5..7..9.1.3.5.7.9.1
1.9.7.5.3.1.9.7..5..3..1..9.........9..1..3..5..7.9.1.3.5.7.9.1
1.9.7.5.3.1.9.7.5..3..1..9..7777777..9..1..3..5.7.9.1.3.5.7.9.1
1.9.7.5.3.1.9.7.5.3..1..9..7.......7..9..1..3.5.7.9.1.3.5.7.9.1
1.9.7.5.3.1.9.7.5.3.1..9..7..55555..7..9..1.3.5.7.9.1.3.5.7.9.1
1.9.7.5.3.1.9.7.5.3.1.9..7..5.....5..7..9.1.3.5.7.9.1.3.5.7.9.1
1.9.7.5.3.1.9.7.5.3.1.9.7..5..333..5..7.9.1.3.5.7.9.1.3.5.7.9.1
1.9.7.5.3.1.9.7.5.3.1.9.7.5..3...3..5.7.9.1.3.5.7.9.1.3.5.7.9.1
1.9.7.5.3.1.9.7.5.3.1.9.7.5.3..1..3.5.7.9.1.3.5.7.9.1.3.5.7.9.1
1.9.7.5.3.1.9.7.5.3.1.9.7.5.3.1.1.3.5.7.9.1.3.5.7.9.1.3.5.7.9.1
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1.9.7.5.3.1.9.7.5.3.1.9.7.5..3...3..5.7.9.1.3.5.7.9.1.3.5.7.9.1
1.9.7.5.3.1.9.7.5.3.1.9.7..5..333..5..7.9.1.3.5.7.9.1.3.5.7.9.1
1.9.7.5.3.1.9.7.5.3.1.9..7..5.....5..7..9.1.3.5.7.9.1.3.5.7.9.1
1.9.7.5.3.1.9.7.5.3.1..9..7..55555..7..9..1.3.5.7.9.1.3.5.7.9.1
1.9.7.5.3.1.9.7.5.3..1..9..7.......7..9..1..3.5.7.9.1.3.5.7.9.1
1.9.7.5.3.1.9.7.5..3..1..9..7777777..9..1..3..5.7.9.1.3.5.7.9.1
1.9.7.5.3.1.9.7..5..3..1..9.........9..1..3..5..7.9.1.3.5.7.9.1
1.9.7.5.3.1.9..7..5..3..1..999999999..1..3..5..7..9.1.3.5.7.9.1
1.9.7.5.3.1..9..7..5..3..1...........1..3..5..7..9..1.3.5.7.9.1
1.9.7.5.3..1..9..7..5..3..11111111111..3..5..7..9..1..3.5.7.9.1
1.9.7.5..3..1..9..7..5..3.............3..5..7..9..1..3..5.7.9.1
1.9.7..5..3..1..9..7..5..3333333333333..5..7..9..1..3..5..7.9.1
1.9..7..5..3..1..9..7..5...............5..7..9..1..3..5..7..9.1
1..9..7..5..3..1..9..7..555555555555555..7..9..1..3..5..7..9..1
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FIG. 5. "Octagonal" distances (d_4) from a single point.

[This is not a regular octagon; the sides have the right orientations, but the wrong lengths. Note that for $r = 1$ the octagon degenerates into a diamond.] To see this, note first that from d_2 we have $|j| \le r$, and for any such j, also $|i| \le r$. If $|j| \le [r/2]$ and $|i| \le r$, we have

$$\left[\frac{2(|i|+|j|+1)}{3}\right] \le \left[\frac{2(r+[r/2]+1)}{3}\right] \le \left[2\left(\frac{3r}{2}+1\right)/3\right] = \left[r+\frac{2}{3}\right] = r.$$

so that $d_4 \le r$ is also satisfied. On the other hand, if $[r/2] < |j|$ and

$$\left[\frac{2(|i|+|j|+1)}{3}\right] \le r,$$

we have

$$|i| \le \frac{3r}{2} - |j| < \frac{3r}{2} - \left[\frac{r}{2}\right] \le r,$$

so that $d_2 \le r$ is also satisfied. Figure 5 shows the distances d_4 from a single point.

2. ALGORITHMS FOR THE DISTANCE FUNCTIONS

In this section we describe algorithms for computing distance functions d_1–d_4 by performing repeated local operations. To this end, we first show that these functions have certain special properties.

Proposition 6. Distance functions d_1–d_3 all have the following property: For any distinct x, y in $A \times A$, there exists $z \in A \times A$ such that $d(x, z) = 1$ and $d(z, y) = d(x, y) - 1$.

Proof: The set of $(u, v) \in A \times A$ such that $d_r[(i, j), (u, v)] = 1$ is readily

(a) For d_1: $\{(i-1, j), (i+1, j), (i, j-1), (i, j+1)\}$

(b) For d_2: $\{(i-1, j-1),\ (i-1, j),\ (i-1, j+1),\ (i, j-1),\ (i, j+1),\ (i+1, j-1),\ (i+1, j),\ (i+1, j+1)\}$

(c) For d_3: $\{(i-1, j-1), (i-1, j), (i, j-1), (i, j+1), (i+1, j-1), (i+1, j)\}$, if i is odd; $\{(i-1, j), (i-1, j+1), (i, j-1), (i, j+1), (i+1, j), (i+1, j+1)\}$, if i is even.

Let $d_r[(i, j), (h, k)] = d$, $r = 1, 2, 3$, where $d > 0$. In case (a), if $h < i$, then $d_1[(i-1, j), (h, k)] = d-1$; if $h > i$, then $d_1[(i+1, j), (h, k)] = d-1$; if $k < j$, then $d_1[(i, j-1), (h, k)] = d-1$; and if $k > j$, then $d_1[(i, j+1), (h, k)] = d-1$—where at least one of these must be true since $(i, j) \ne (h, k)$.

In case (b), similarly, if i, j, h, k are related as indicated, then $d_2[(u, v), (h, k)] = d-1$ for the indicated (u, v):

Relations	(u, v)	Relations	(u, v)
$i < h, j < k$	$(i-1, j-1)$	$i > h, j > k$	$(i+1, j+1)$
$i < h, j = k$	$(i-1, j)$	$i > h, j = k$	$(i+1, j)$
$i < h, j > k$	$(i-1, j+1)$	$i > h, j < k$	$(i+1, j-1)$
$i = h, j < k$	$(i, j-1)$	$i = h, j > k$	$(i, j+1)$

In case (c), analogously, we have

Relations	(u, v), i odd	(u, v), i even
$i < h, j \le k$	$(i+1, j)$	$(i+1, j+1)$
$i < h, j \ge k$	$(i+1, j-1)$	$(i+1, j)$
$i = h, j < k$	$(i, j+1)$	$(i, j+1)$
$i = h, j > k$	$(i, j-1)$	$(i, j-1)$
$i > h, j \le k$	$(i-1, j)$	$(i-1, j+1)$
$i > h, j \ge k$	$(i-1, j-1)$	$(i-1, j)$

Note that d_4 does not have the property of Proposition 6. In fact, readily the set of (u, v) such that $d_4[(i, j), (u, v)] = 1$ is the same as for d_1. Let $|i-h| = |j-k| = 3$; then $d_4[(i, j), (h, k)] = 4$, but none of the (u, v) are at distance 3 from (h, k). However, d_4 has a weaker property which we can use instead:

Proposition 7. Let $d_4(x, y) = d > 0$; then there exists z such that $d_4(z, y) = d-1$, where $d_1(z, x) = 1$ if d is odd, and $d_2(z, x) = 1$ if d is even.

Proof: As usual, it suffices to prove the case where $y = (0, 0)$. Let $x = (i, j)$, so that

$$d = \max\left\{|i|, |j|, \left[\frac{2(|i|+|j|+1)}{3}\right]\right\}.$$

If $d = |i|$ is a strict maximum, we can decrease d by 1 by taking $z = (u, v)$ where $|u| = |i|-1$, $v = j$; similarly, if $d = |j|$ is a strict maximum, we can take $u = i$, $|v| = |j|-1$. If $|i| = [2(|i|+|j|+1)/3] > 0$, then readily $|i| > |j|$, so that $|u| = |i|-1$, $v = j$ decreases d by 1 unless $|i|+|j|$ is divisible by 3; similarly if $|j| = [2(|i|+|j|+1)/3] > 0$, using $u = i$, $|v| = |j|-1$; while if $|i| = |j|$, then

$$\left[\frac{2(|i|+|j|+1)}{3}\right]$$

is a strict maximum. Thus we are done unless $|i|+|j| = 3r$ (say); but then

$$\left[\frac{2(|i|+|j|+1)}{3}\right] = 2r,$$

so that in the remaining cases d is even, and we can decrease it by 1 by taking $|u| = |i|-1$, $|v| = |j|-1$.

We can now define the desired algorithms. Let B be a binary picture, i.e. an m by n matrix $(b_{ij}) = (b_{ij}^{(0)})$ in which each b_{ij} is 0 or 1. Let d be a distance function which has the property of Proposition 6, and let f be the local operation on B defined by

$$f(b_{ij}) = \min\{b_{uv} | d[(u, v), (i, j)] \le 1\}.$$

Let B_0 be the set of (h, k) such that $b_{hk} = 0$, and let

$$d[(i, j), B_0] = \min\{d[(i, j), (h, k)] \,|\, (h, k) \in B_0\}$$

be the distance from (i, j) to the nearest element of B_0. (We assume from now on that B_0 is nonempty.) Let $b_{ij}^{(r+1)} = f(b_{ij}^{(r)})$, $r = 0, 1, \ldots$

Proposition 8. $b_{ij}^{(r)} = 1$ for all $r < d[(i,j), B_0]$;
0 for all $r \geq d[(i,j), B_0]$.

Proof: Since B is a binary picture, clearly $b_{ij}^{(r)} \leq 1$ for any r; and once it becomes 0, it remains 0. The Proposition is true if $d[(i,j), B_0] = 0$, i.e. if $(i,j) \in B_0$, since then $b_{ij}^{(0)} = 0$. Suppose it proved for all (h, k) such that $d[(h,k), B_0] = s-1$, and let $d[(i,j), B_0] = s$. By induction hypothesis and Proposition 6, we have $b_{uv}^{(s-1)} = 0$ for some (u, v) such that $d[(u,v),(i,j)] = 1$; hence $b_{ij}^{(s)} = 0$ by definition of f. It remains only to show that we cannot have $b_{hk}^{(r)} = 0$ for any $(h,k) \in B$ and any $r < d[(h,k), B_0]$. But indeed, if not, let q be the smallest such r, and let $b_{hk}^{(q)} = 0$, where $q < d[(h,k), B_0]$; clearly $q \neq 0$. By triangularity, we have $d[(u,v), B_0] \geq d-1$ for any (u,v) such that $d[(h,k),(u,v)] \leq 1$; hence $q-1 < d-1 \leq d[(u,v), B_0]$, so that by the minimality of q we have $b_{uv}^{(q-1)} = 1$, whence $b_{hk}^{(q)} = 1$ by definition of f, contradiction.

Corollary. If $d \leq d_1$, then $\sum_{r=0}^{m+n} b_{ij}^{(r)} = d[(i,j), B_0]$ for all $(i,j) \in B$.

Proof: This distance must be less than $m+n$.

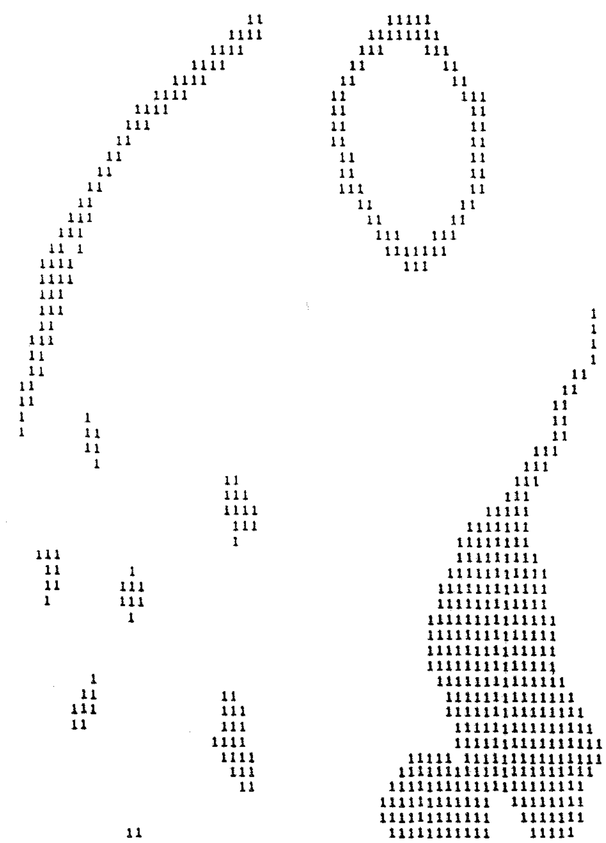


Fig. 6. Test picture.

In particular, the hypotheses of Proposition 7 and its Corollary hold for the distance functions d_1, d_2 and d_3, so that the foregoing defines local operations (call them f_1, f_2, f_3) whose repeated application gives rise to a picture $(\sum_{r=0}^{m+n} b_{ij}^{(r)})$ in which the (i, j) element is the distance from (i, j) to the nearest zero in B.

We can obtain analogous results for the distance function d_4 by using Proposition 7; in fact, we can readily prove

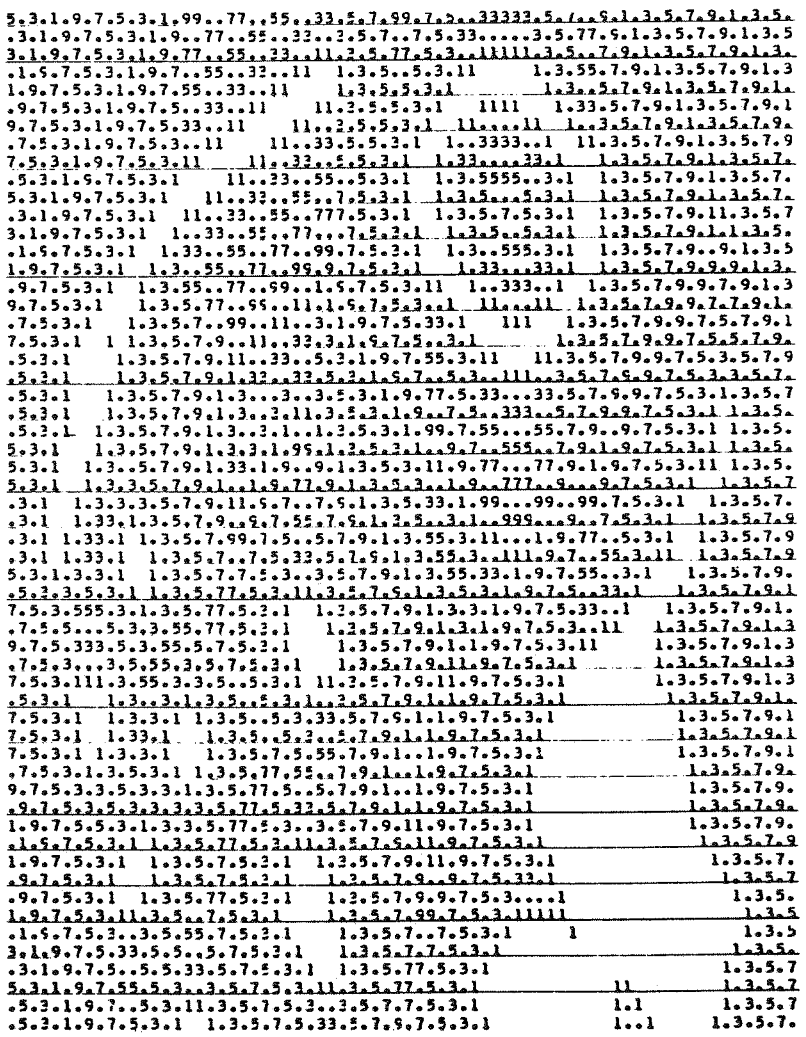


FIG. 7. City block distances (d_1) from the blanks.

Proposition 9. Let $b_{ij}^{(2k+1)} = f_1(b_{ij}^{(2k)})$, $k = 0, 1, \ldots$; $b_{ij}^{(2k+2)} = f_2(b_{ij}^{(2k+1)})$, $k = 0, 1, \ldots$ Then $b_{ij}^{(r)} = 1$ for all $r < d_4[(i, j), B_0]$, and $=0$ for all $r \geq d_4[(i, j), B_0]$.

COROLLARY. $\Sigma_{r=0}^{m+n} b_{ij}^{(r)} = d_4[(i, j), B_0]$ for all $(i, j) \in B$.

The results of applying these algorithms to the binary picture in Fig. 6 are shown as Figs. 7–10, respectively.

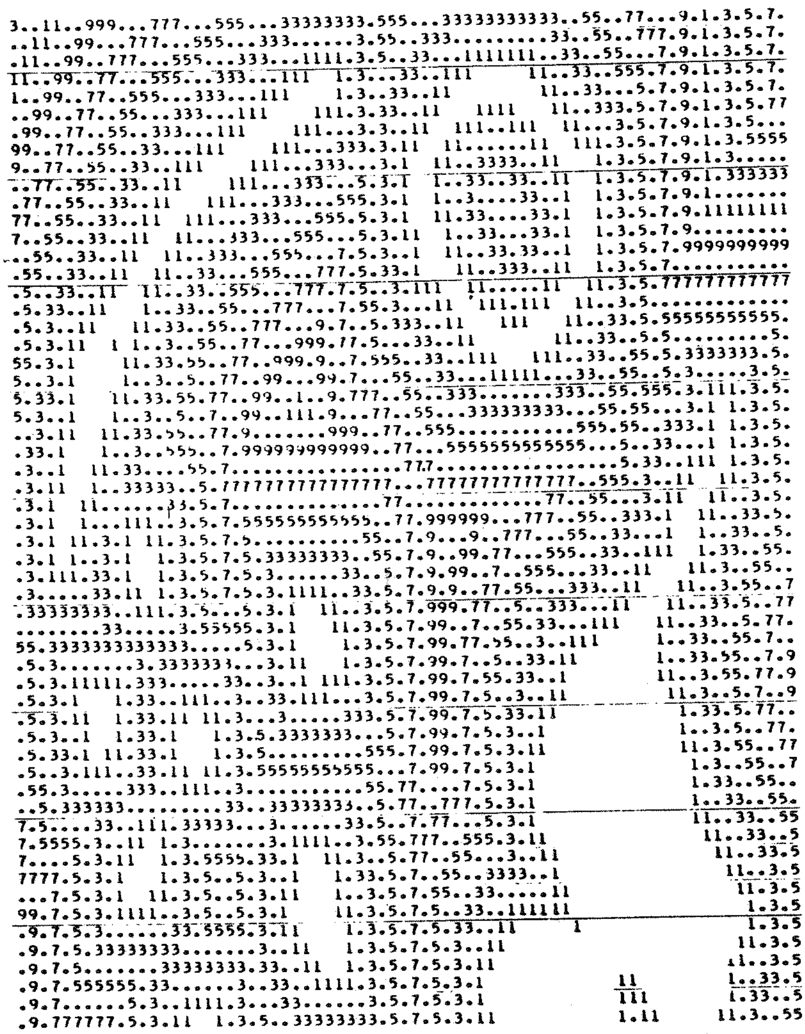


FIG. 8. Square distances (d_2) from the blanks.

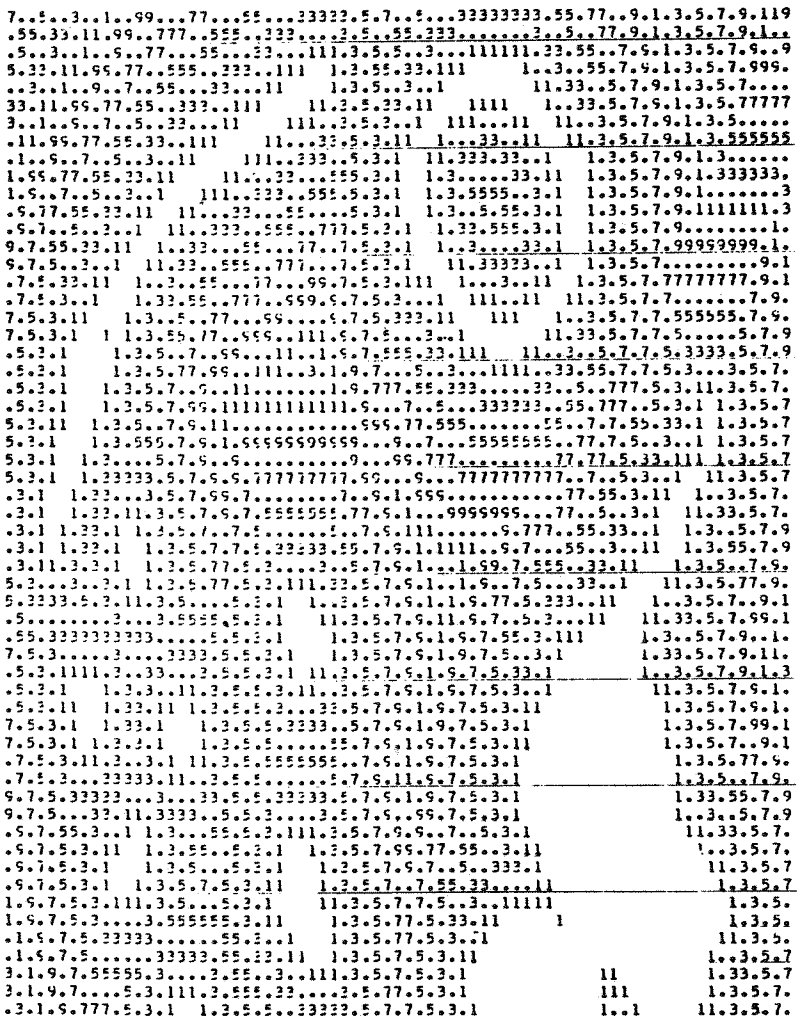


FIG. 9. Hexagonal distances (d_3) from the blanks.

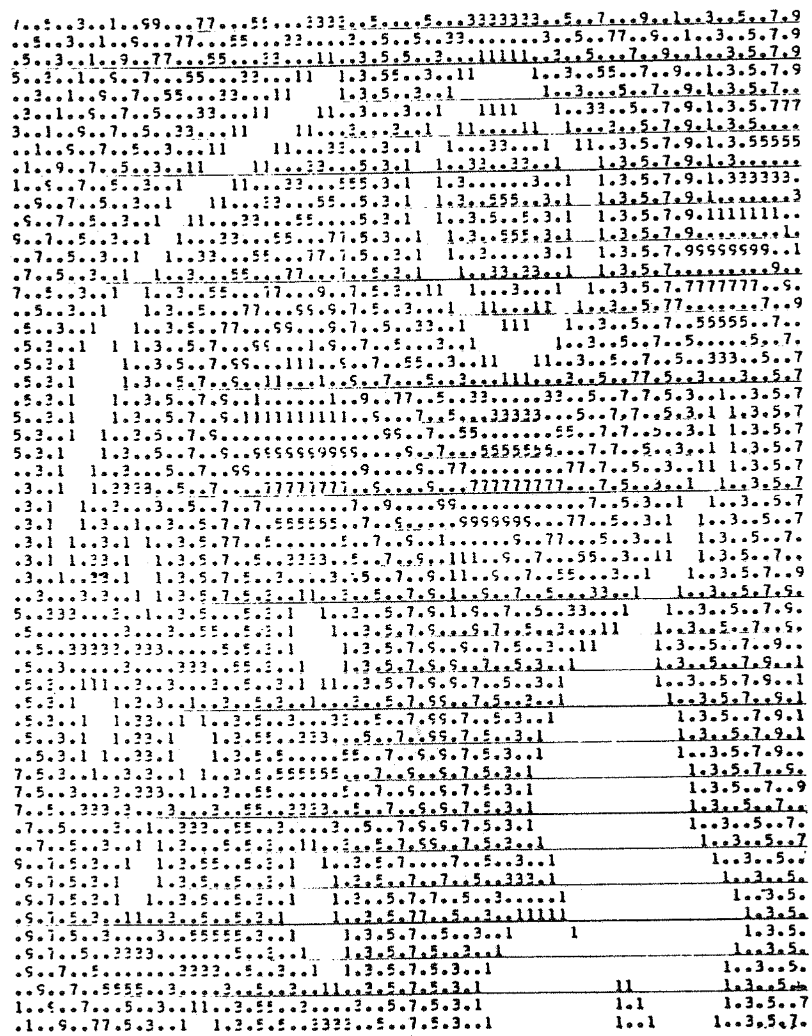
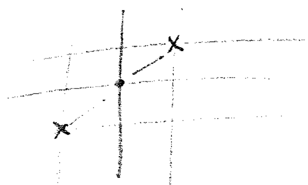


FIG. 10. Octagonal distances (d_4) from the blanks.

3. APPROXIMATIONS TO EUCLIDEAN DISTANCE

As shown by Figs. 1–5, the "octagonal" distance function d_4 is a better approximation to Euclidean distance on $A \times A$ than are the "city block," "square," and "hexagonal" distance functions $d_1 - d_3$. In fact, it is an adequate approximation for many purposes, particularly if (as in Fig. 6) the distances to B_0 never get very great. Figure 11 shows nearest-integer Euclidean distances from a single point (0, 0)—in other words, $d[(i, j), (0, 0)] = d$ means that $d - \frac{1}{2} < \sqrt{i^2 + j^2} < d + \frac{1}{2}$. [It is interesting to note that "nearest integer to Euclidean distance" itself is *not* a distance function. For example, let $(i, j) = (1, 1)$, $(h, k) = (-1, -1)$; then $\sqrt{(i-h)^2 + (j-k)^2} = \sqrt{8}$, the nearest integer to which is 3, but $\sqrt{i^2 + j^2} = \sqrt{h^2 + k^2} = \sqrt{2}$, the nearest integer to which is 1, so that triangularity is



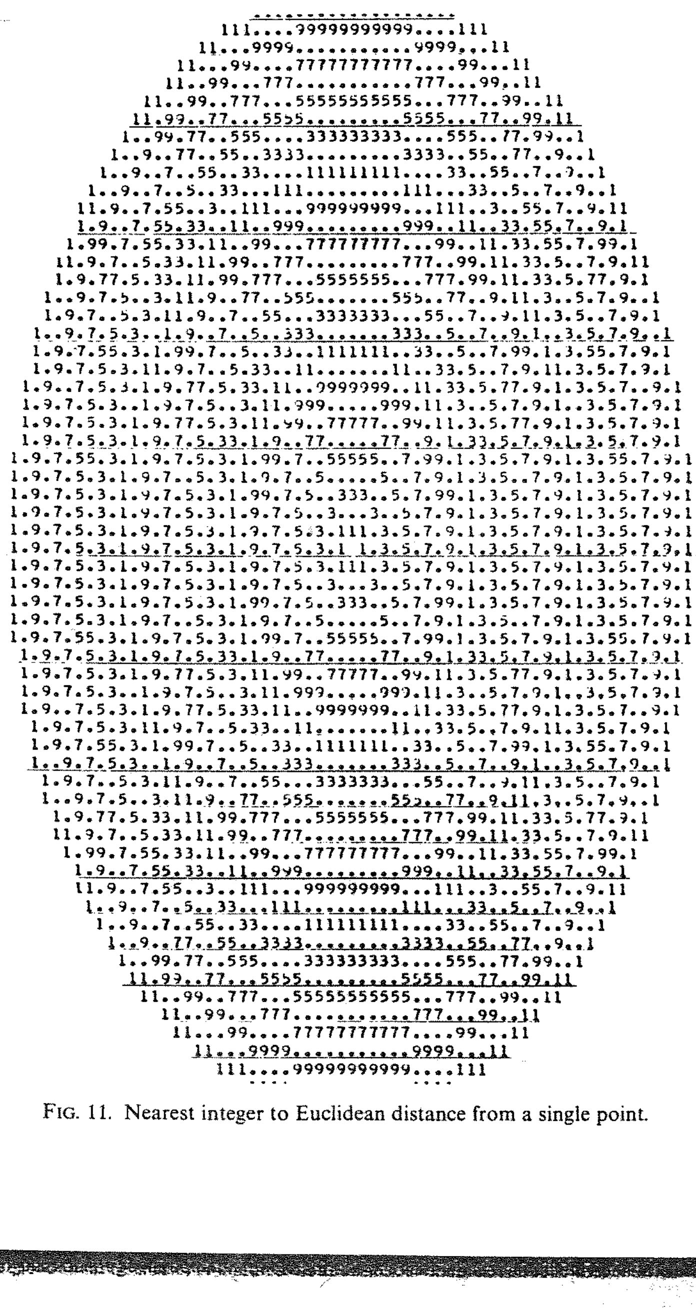


FIG. 11. Nearest integer to Euclidean distance from a single point.

violated. Similarly, "greatest integer not exceeding Euclidean distance" is not a distance function, as shown by the example $(i, j) = (2, 3)$, $(h, k) = (-1, -1)$.]

The algorithm for computing d_4 was obtained by alternating the algorithms for d_1 and d_2, and is a better approximation to Euclidean distance than either of them. This suggests that other distance functions, and perhaps still better approximations, might be obtained by "mixing" these (or other) algorithms in other ways. In this section we show that octagons are the best "circles" which can be obtained using only isotropic local operations of the type used in Propositions 8–9. We also show that arbitrarily close approximations to regular octagons can be obtained.

Let $O(h, k)$ be the octagon with vertices at $(k, \pm h)$, $(-k, \pm h)$, $(\pm h, k)$, and $(\pm h, -k)$, where $0 \le h \le k$. Note that if $h = 0$, this octagon degenerates into a diamond, while if $h = k$, it degenerates into a square. It is easily verified that

Proposition 10. Let $B = (b_{ij})$ be a binary picture in which $B_0 \cong O(h, k)$, where $h > 0$, and let $C = (f_1(b_{ij}))$, $D = (f_2(b_{ij}))$, and $E = (f^*(b_{ij}))$, where $f^*(b_{ij}) = \min(b_{ij}, b_{i-1,j-1}, b_{i-1,j+1}, b_{i+1,j-1}, b_{i+1,j+1})$. Then $C_0 \cong O(h, k+1)$, while $D_0 = E_0 \cong O(h+1, k+1)$.

This proposition implies that if d is a distance function which is computed by iterating the isotropic local operations f_1, f_2 and f^* in any sequence, and if the locus of points at distance $\le r$ from a given point is an octagon (with $h > 0$), then so is the locus of points at distance $\le s$, for any $s \ge r$. We assume here and from now on that the border elements of all pictures considered are nonzero, so that the octagons and other figures described are all contained inside the picture.

Proposition 11. Let P be a picture resulting from the application of f_1's, f_2's and f^*'s, in any sequence, to a binary picture B in which B_0 is a single element. Then unless no f_2's were used, and no f_1 followed any f^*, we have $P_0 \cong O(h, k)$ for some $0 < h \le k$.

Proof: By Proposition 10, as soon as such an octagon is obtained, the following stages are all such octagons. It thus suffices to show that when f_2 is applied after any sequence of f_1's and/or f^*'s, or f_1 is applied after a sequence of (f_1's followed by) f^*'s, the result is such an octagon. But readily, if h f_1's are applied to B followed by k f^*'s, the result is congruent to the octagon $O(k, h+k)$ with some of its elements deleted, no two of them horizontally or vertically adjacent (see Fig. 12; all of the missing elements are on the boundary of the octagon if $h > 0$). Application of f_1 then gives the octagon $O(k, h+k+1)$ with no elements missing, while application of f_2 similarly gives $O(k+1, h+k+1)$.

a) Five f*'s

```
5 5 5 5 5 5
 4 4 4 4 4
5 3 3 3 3 5
 4 2 2 2 4
5 3 1 1 3 5
 4 2 0 2 4
5 3 1 1 3 5
 4 2 2 2 4
5 3 3 3 3 5
 4 4 4 4 4
5 5 5 5 5 5
```

b) One f_1 followed by four f*'s

```
 5 5 5 5 5
55454545455
 443434344
55332323355
 442212244
55331013355
 442212244
55332323355
 443434344
55454545455
 5 5 5 5 5
```

c) Two f_1's followed by three f*'s

```
  5 5 5 5
 554545455
55443434455
 443323344
55332123355
 442101244
55332123355
 443323344
55443434455
 554545455
  5 5 5 5
```

FIG. 12. Results of applying f_1's followed by f^*'s to a picture containing a single zero (numbers indicate the steps at which elements became zero).

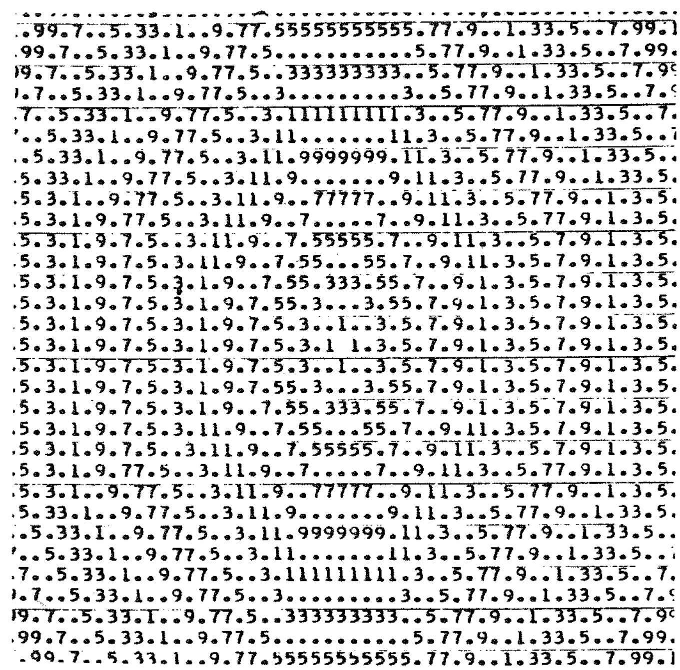


FIG. 13. Octagonal distances (f_1, f_2, f_1 repeated) from a single point.

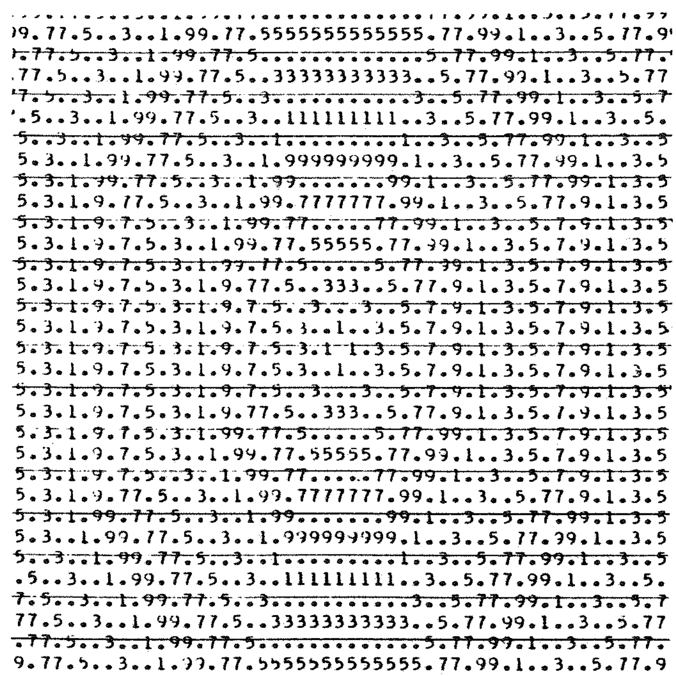


FIG. 14. Octagonal distances (f_1, f_2, f_1, f_2, f_1 repeated) from a single point.

By Proposition 11, an octagon is the best approximation to a circle which can be obtained by iterating f_1, f_2 and f^* in any sequence, since an octagon is certainly "better" than an octagon with deletions. Clearly any octagon $O(h, k)$ can be obtained using a suitable sequence of f_1's and f_2's—e.g. $k-h$ f_1's followed by h f_2's. The (Euclidean) length of the horizontal and vertical sides of $O(h, k)$ is $2h$, and the length of its diagonal sides is $(k-h)\sqrt{2}$. Thus $O(h, k)$ can never be a perfectly regular octagon, but it can be made to approximate such an octagon arbitrarily closely by taking $2h/(k-h)$ to be a good approximation to $\sqrt{2}$. The octagons obtained using d_+ (i.e. by alternating f_1 and f_2) are not very regular, since for r even their horizontal and vertical sides have length r and their diagonal sides have length $(r/2)\sqrt{2}$, while for r odd these lengths are $r-1$ and $(r+1/2)\sqrt{2}$. A more regular octagon can be obtained by repeating a sequence of f_1's and f_2's in which the ratio of the number of f_1's to the number of f_2's is closer to $\sqrt{2}$. The octagons obtained by repeating the sequence (f_1, f_2, f_1) are shown in Fig. 13; those obtained using the sequence $(f_1, f_2, f_1, f_2, f_1)$ are shown in Fig. 14.

4. APPLICATIONS OF DISTANCE FUNCTIONS

The following applications of distance functions illustrate the importance of using functions which approximate Euclidean distance reasonably closely.

(a) *Cluster detection*

Let P be a binary picture containing a scattering of 0's, some of which are grouped into clusters, as shown by the W's in Fig. 15. Let $P^{(r)}$ be the binary picture in which $p_{ij}^{(r)} = 0$ if and only if $d[(i, j), P_0] \leq r$, where d is a distance function. Let $P^{(r,s)}$ be the binary picture in which $p_{ij}^{(r,s)} = 0$ if and only if $d[(i, j), P_0^{(r)}] \geq s$. We can think of $P^{(r)}$ as "expanding" each zero in P into a "disk" of radius r, and of $P^{(r,s)}$ as "re-contracting" the set of zeros in $P^{(r)}$ by s. In particular, taking $s = r$, it is clear that the disk in $P^{(r)}$ which arises from an *isolated* zero in P re-contracts into an isolated zero in $P^{(r,r)}$. (In fact, readily $P_0^{(r,r)} \supseteq P_0$ for any r.) On the other hand, if there is a *cluster* of zeros in P, the expansion process (for sufficiently large r) will "fuse" it into a solid mass of zeros, and the re-contraction will leave a central core of this mass intact. Figures 16–17 show the $P^{(3)}$ and $P^{(3,3)}$ which arise from the P of Fig. 15, using the distance function f_3. (One clearly wants to use a distance function which is as Euclidean as possible, so that the "closeness" required to define a cluster will not depend strongly on direction.)

The clusters which are detected by the foregoing procedure become sparser as r increases. If there is no prior information about expected degrees of sparseness, one can construct $P^{(r,r)}$ for a range of r's. Any connected component of zeros in $P^{(r,r)}$ whose area is large compared to r^2 must arise from a cluster of zeros in P.

(b) *Elongated part detection*

A procedure exactly "dual" to the foregoing can be used to detect elongated parts of a connected set of zeros in a binary picture P. Let $Q^{(r)}$ be the binary picture in which $q_{ij}^{(r)} = 0$ if and only if $d[(i, j), P-P_0] \geq r$, and let $Q^{(r,s)}$ be the binary picture in which $q_{ij}^{(r,s)} = 0$ if and only if $d[(i, j), Q_0^{(r)}] \leq s$. In effect, $Q^{(r)}$ contracts the set of zeros in P by r steps, and $Q^{(r,s)}$ then re-expands the zeros by s steps. Readily, $Q_0^{(r,r)} \subseteq P_0$ for any r; let $\bar{Q}^{(r,r)}$ be the binary picture in which $\bar{Q}_0^{(r,r)} = P_0 - Q_0^{(r,r)}$. Then any connected component of

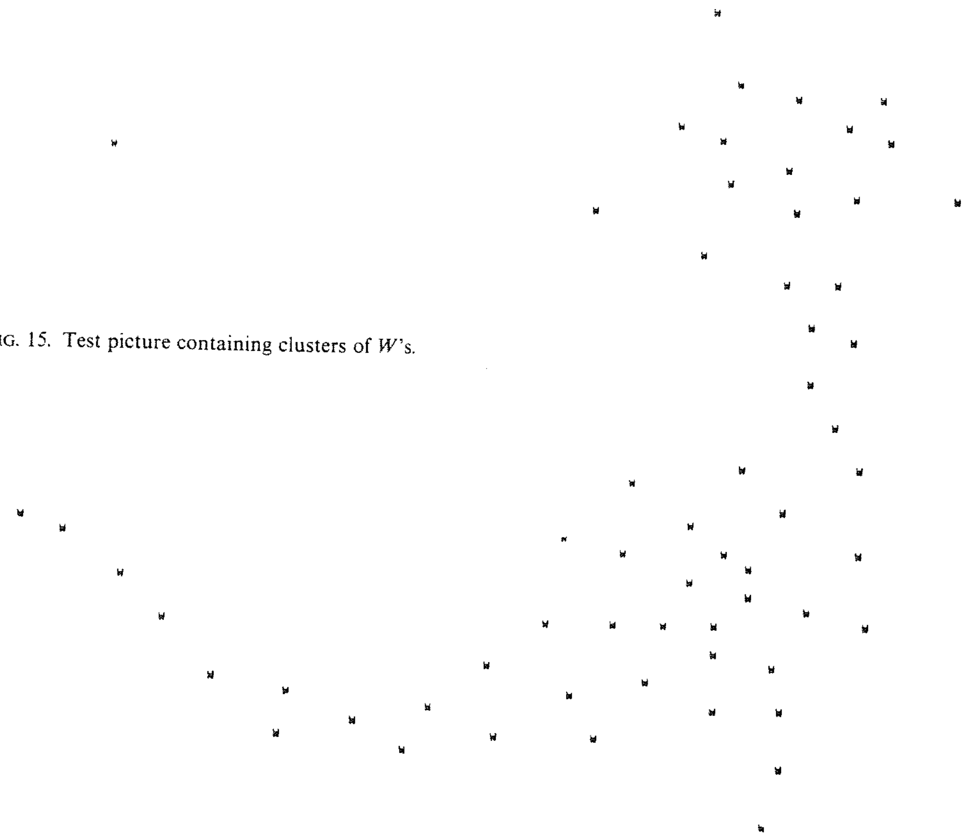


FIG. 15. Test picture containing clusters of W's.

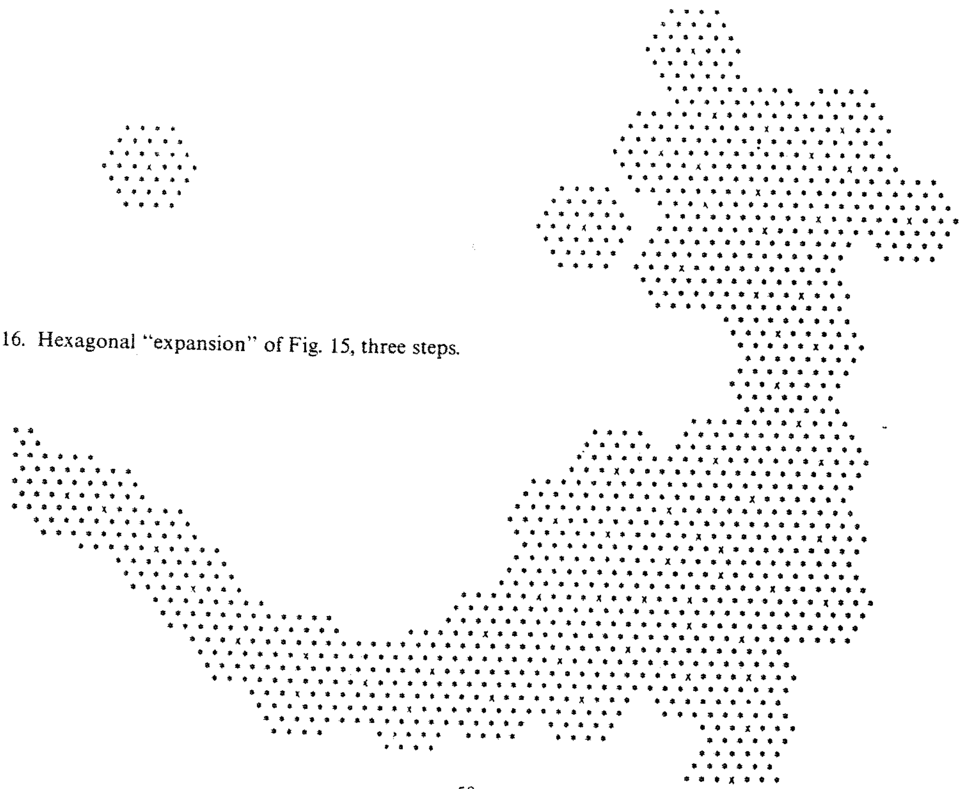


FIG. 16. Hexagonal "expansion" of Fig. 15, three steps.

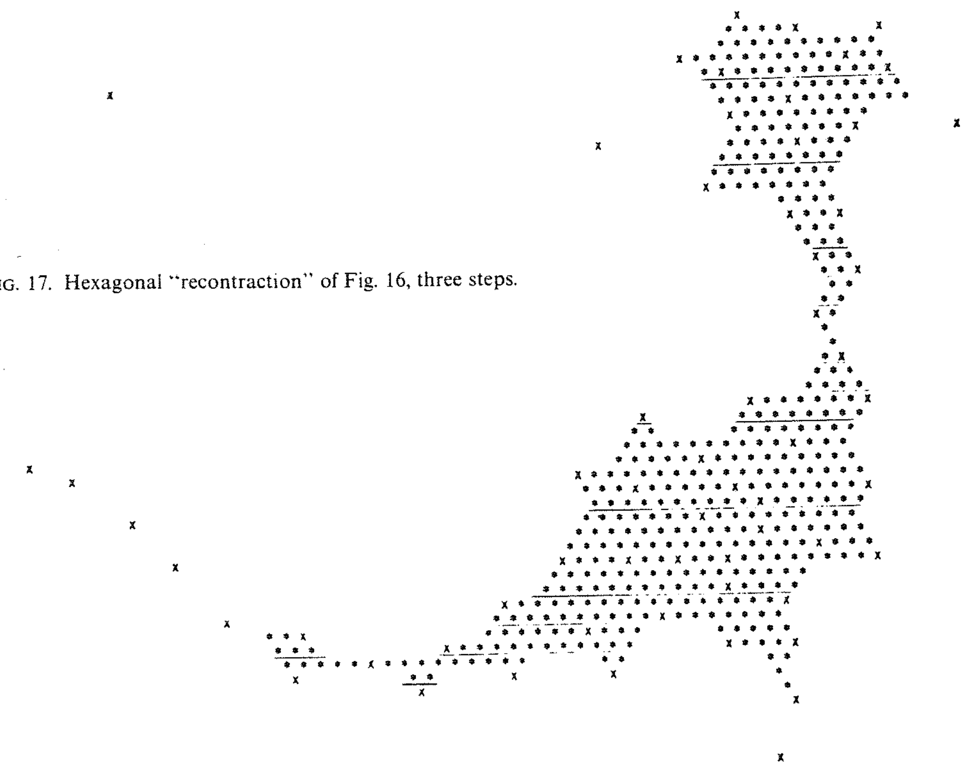


FIG. 17. Hexagonal "recontraction" of Fig. 16, three steps.

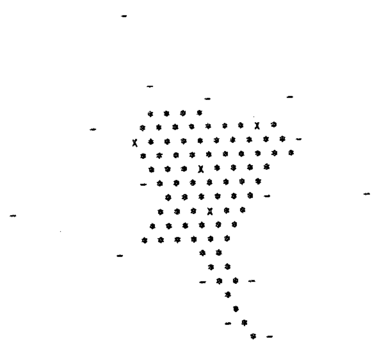
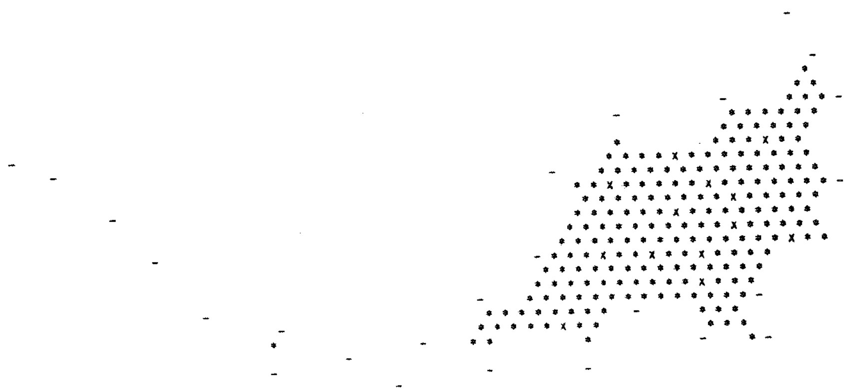


FIG. 18. Hexagonal contraction of Fig. 17, one step.



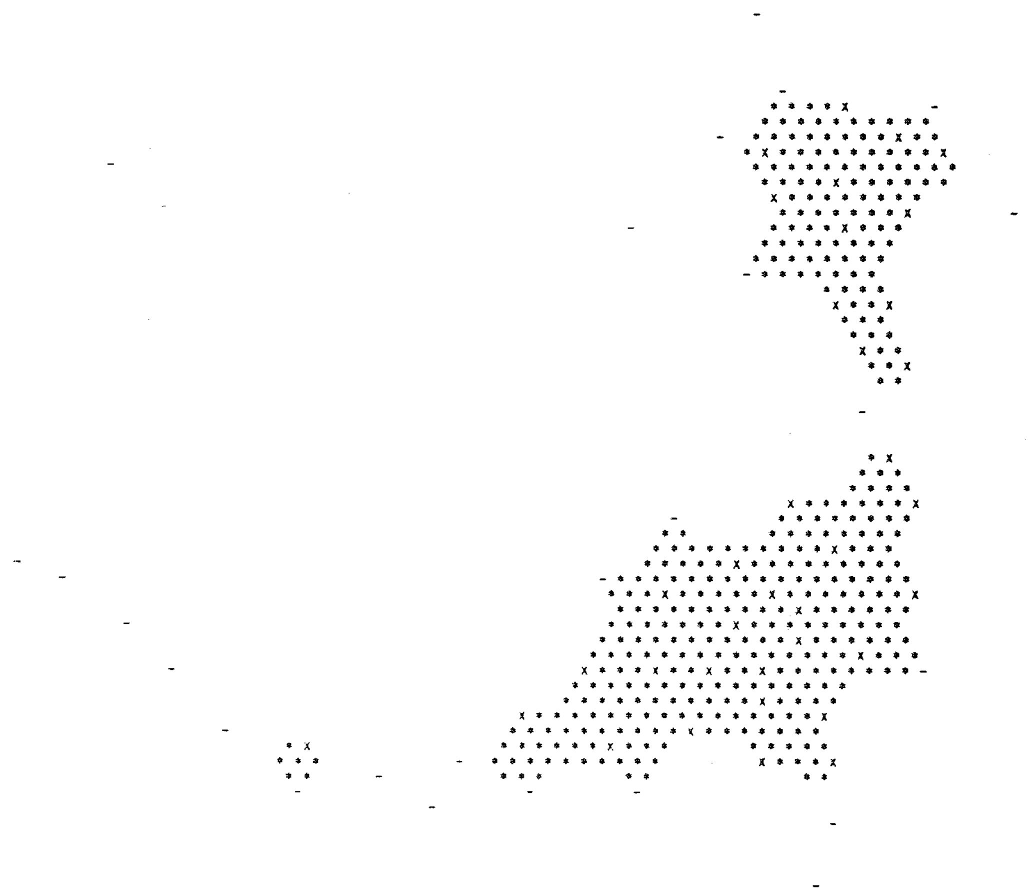


FIG. 19. Hexagonal re-expansion of Fig. 18, one step.

zeros in $\bar{Q}^{(r,r)}$ whose area is large compared to r^2 must be elongated, since every point of it is at distance at most r from its complement. [The "duality" of the cluster and elongated part detection procedures can also be expressed as follows: Inside a cluster of 0's, the set of 1's is everywhere elongated.]

Figures 18–19 show the $Q^{(1)}$ and $Q^{(1,1)}$ resulting from the P of Fig. 17, again using f_3. (Here too the importance of using a near-Euclidean distance is clear.) Note that this procedure will eventually (as r increases) detect any elongated part of P_0, no matter how wide; it does not depend on the elongated parts having a narrow range of widths ("strokes," "tracks" and the like). The contraction/re-expansion process is somewhat analogous to, but simpler and probably more effective than, the "skeleton" process for detecting elongated parts of objects proposed by ROSENFELD and PFALTZ.[1]

(c) *Regularity detection*

A process of expansion and re-contraction can also be used to detect "regularity" in a scattering of zeros—i.e. near-constancy of the distances between near neighbors. For example, let r_{ij} be the smallest r for which $p_{ij}^{(r,r+1)} \neq 0$. (Readily, $p_{ij}^{(r,r+1)} \neq 0$ means that (i, j) is inside a cluster of zeros of "sparseness" $\leq r$.) We call r_{ij} the *radius of fusion* at (i, j). Intuitively, a region in which the zeros are regularly distributed will have a relatively constant radius of fusion. Such regions can thus be detected by appropriately level slicing this radius. The radii of fusion for the picture in Fig. 20 are shown in Fig. 21.

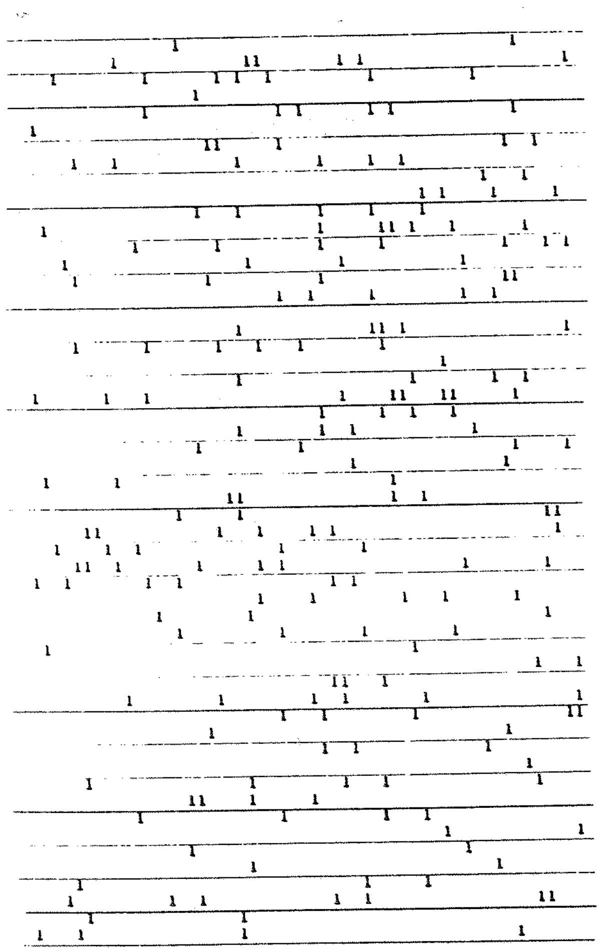


FIG. 20. Random point pattern.

5. PATH COUNTING

In this and the next section we describe other types of functions on a digital picture which depend on distance to a subset.

Let S be any subset of $A \times A$. By a *d-path* of length r from (i, j) to S, where S is a subset of $A \times A$ and d is a distance function, is meant an $r+1$-tuple $(i_0, j_0), \ldots, (i_r, j_r)$ such that $(i_0, j_0) = (i, j)$; (i_r, j_r) is in S; and $d[(i_{s-1}, j_{s-1}), (i_s, j_s)] = 1$, $1 \leq s \leq r$. For example, if $d = d_1$, each element of a path is a horizontal or vertical neighbor of the preceding

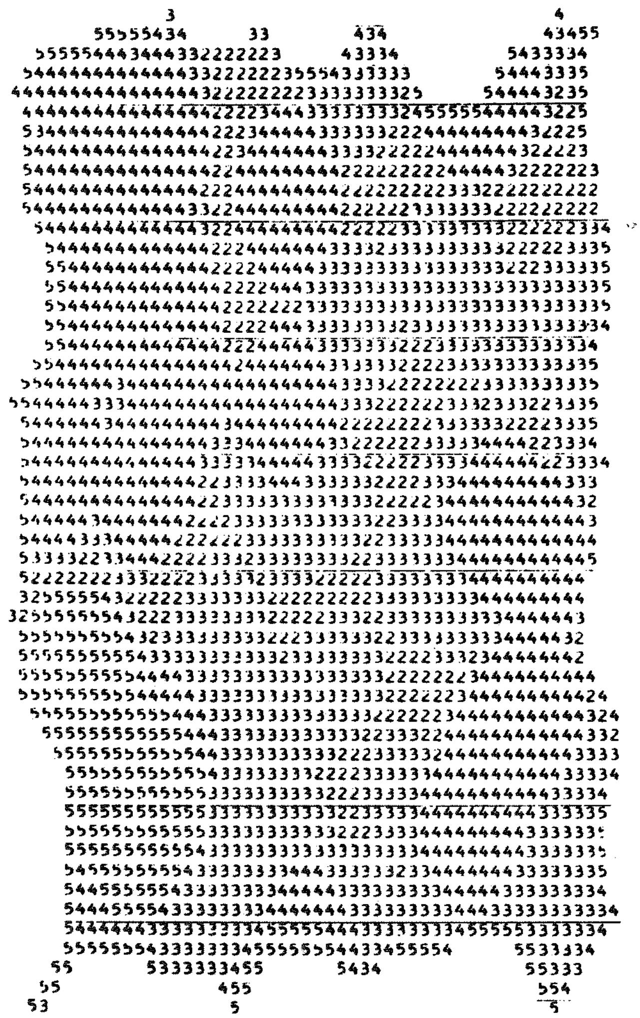


FIG. 21. Radii of fusion for Fig. 20.

element, so that the path can be regarded as a series of horizontal and vertical "steps." If $d = d_2$, diagonal steps are also allowed; if $d = d_4$, they are allowed only at every other step. For $d = d_3$, the path can be regarded as a series of steps on a hexagonal grid. Note that a d-path can cross or even retrace itself repeatedly, and that it may pass through the set S many times.

Let C_1 be the set of elements (h, k) in the binary picture C such that $c_{hk} = 1$, and let $N_d^{(r)}(i, j)$ be the number of d-paths of length r from (i, j) to C_1. Let g_d be the local operation defined by

$$g_d(c_{ij}) = \sum_{d[(u,v),(i,j)]=1} c_{uv},$$

and let $c_{ij}^{(r)}$ be the (i, j) element of the picture obtained by applying g_d to C r times, i.e.

$$c_{ij}^{(r)} = \sum_{d[(u,v),(i,j)]=1} c_{uv}^{(r-1)}, \qquad r = 1, 2, \ldots,$$

where $c_{hk}^{(0)}$ means c_{hk}.

Proposition 12. $c_{ij}^{(r)} = N_d^{(r)}(i, j)$.

Proof: This is clear for $r = 0$, since the number of d-paths of length 0 from (i, j) to C_1 is 1 if $(i, j) \in C_1$, and 0 otherwise. (This is why the set was taken to be C_1, rather than C_0 as in Section 2.) If it is true for $r-1$, its truth for r follows from the fact that any path of length r from C_1 to (i, j) consists of a path of length $r-1$ from C_1 to exactly one of the (u, v)'s, together with the step from (u, v) to (i, j). (This argument is valid even if not all of the (u, v)'s are inside the picture.)

In short: Repetition of the local operation g_d generates a stack of integer-valued pictures $C^{(r)} = (c_{ij}^{(r)})$, $r = 0, 1, \ldots$, in which $c_{ij}^{(r)}$ is the number of d-paths of length r from C_1 to (i, j).

Knowledge of the number of paths of each length from any point to a given set S is of interest in the following type of situation: Suppose that every point of S has a certain effect on each of its neighbors, and that this effect is transmitted from neighbor to neighbor, with attenuation at each step. Then the total effect at the point (i, j) is given by an expression of the form $\Sigma a_r N^{(r)}(i, j)$, where a_r is the magnitude of the effect due to a single point after attenuation along an r-step path. (The transmission of the effect is assumed here to be instantaneous.) Processes of this type have been considered in connection with the study of lateral inhibition in neural networks (e.g. Beddoes *et al.*[2]).

To get an idea of the convergence properties of $\sum N^{(r)}(i, j)$, suppose that S consists of a single point, which we may assume without loss of generality to be $(0, 0)$.

Proposition 13. The number of d_1-paths of length r from $(0, 0)$ to (i, j) is

$$N_1^{(r)}(i, j) = \binom{r}{\frac{r-|i+j|}{2}} \cdot \binom{r}{\frac{r-|i-j|}{2}},$$

where the binomial coefficient $\binom{r}{x}$ is defined to be 0 if x is not an integer or is not in the interval $[0, r]$.

Proof: This is evident for $r = 0$; suppose it true for $r-1$. Then by definition of g_{d_1} we have

$$\begin{aligned} N_1^{(r)}(i, j) &= \binom{r-1}{\frac{r-1-|i+j-1|}{2}}\binom{r-1}{\frac{r-1-|i-j-1|}{2}} + \binom{r-1}{\frac{r-1-|i+j+1|}{2}}\binom{r-1}{\frac{r-1-|i-j+1|}{2}} \\ &\quad + \binom{r-1}{\frac{r-1-|i+j-1|}{2}}\binom{r-1}{\frac{r-1-|i-j+1|}{2}} + \binom{r-1}{\frac{r-1-|i+j+1|}{2}}\binom{r-1}{\frac{r-1-|i-j-1|}{2}} \\ &= \left[\binom{r-1}{\frac{r-1-|i+j-1|}{2}} + \binom{r-1}{\frac{r-1-|i+j+1|}{2}}\right] \\ &\quad \times \left[\binom{r-1}{\frac{r-1-|i-j-1|}{2}} + \binom{r-1}{\frac{r-1-|i-j+1|}{2}}\right]. \end{aligned}$$

The Proposition then follows from the following

LEMMA.

$$\binom{a-1}{\frac{a-1-|b+1|}{2}}+\binom{a-1}{\frac{a-1-|b-1|}{2}}=\binom{a}{\frac{a-|b|}{2}}$$

for all positive integers a and all integers b.

Proof: If $|b+1| \not\equiv a-1 \pmod 2$, each of these binomial coefficients is zero; we may thus suppose that $(a-1)-|b+1| = 2k$ (say). We consider three cases:

(1) If $b = 0$, then $a = 2k+2$, and we must show that

$$\binom{2k+1}{k}+\binom{2k+1}{k}=\binom{2k+2}{k+1};$$

but since

$$\binom{2k+1}{k}=\binom{2k+1}{k+1},$$

this is just the familiar recurrence relation for the binomial coefficients.

(2) If $b > 0$, we must show that

$$\binom{a-1}{k}+\binom{a-1}{k+1}=\binom{a}{k+1},$$

which is the familiar recurrence relation if $k+1 \geq 0$, and every term of which is zero if $k+1 < 0$.

(3) If $b < 0$, we must show that

$$\binom{a-1}{k}+\binom{a-1}{k-1}=\binom{a}{k},$$

which follows analogously.

COROLLARY. Let $a_r \leq (4+\varepsilon)^{-r}$, where $\varepsilon > 0$; then $\Sigma a_r N_1^{(r)}(i,j)$ converges for all (i,j).

Proof: $N_1^{(r)}(i,j) < 2^r 2^r = 4^r$ if $r > 0$; hence

$$\Sigma a_r N_1^{(r)}(i,j) < \Sigma 4^r(4+\varepsilon)^{-r} = 1\Big/\left(1-\frac{4}{4+\varepsilon}\right) = \frac{4+\varepsilon}{\varepsilon}.$$

In short: An effect which falls off by a factor of more than 4 at each step has a finite magnitude at any point, even though it is summed over all paths of all possible lengths to the point. Similar results hold for d_2-paths and d_3-paths.

Under some circumstances it may be of interest to consider only d-paths which do not pass through the set C_1. Readily, the number of such paths of length r is just $d_{ij}^{(r)}$, where

$$d_{ij}^{(1)} = c_{ij}^{(1)}, \quad \text{and} \quad d_{ij}^{(r)} = \sum_{\substack{d[(u,v),(i,j)]=1 \\ c_{uv}\neq 1}} d_{uv}^{(r-1)}, \qquad r = 2, 3, \ldots$$

In other words, this number is obtained by omitting from g_d, except at its first application, any terms corresponding to elements of C_1.

It may also be of interest to count the *shortest* d-paths from (i, j) to C_1, which may be done as follows: Let

$$e_{ij}^{(r)} = e_{ij}^{(r-1)} \qquad \text{if } e_{ij}^{(r-1)} \neq 0;$$
$$= \sum_{d[(u,v),(i,j)]=1} e_{uv}^{(r-1)} \quad \text{if } e_{ij}^{(r-1)} = 0,$$

where $e_{ij}^{(0)} = c_{ij}$. Thus if $c_{ij} = 1$, i.e. $(i, j) \in C_1$, we have $e_{ij}^{(r)} = 1$ for all r; while if $c_{ij} = 0$, we readily have $e_{ij}^{(r)} = 0$ as long as $r < d[(i, j), C_1]$, and thereafter it has a constant, nonzero value. Since a shortest path from C_1 to (i, j) is a shortest path from C_1 to any point on the path, it follows readily that this constant value is just the desired number of shortest paths.

6. POINT COUNTING

We next consider the problem of counting the number $M_d^{(r)}(i, j)$ of points (h, k) of the given set S such that $d[(i, j), (h, k)] = r$. Knowledge of this number is of interest in a situation where each point of S has an effect on other points which depends on their distances from it. The total effect of S on the point (i, j) is then given by an expression of the form $\Sigma a_r M_d^{(r)}(i, j)$, where a_r is the magnitude of the effect which a single point of S has on a point at distance r from it.

It is interesting to note that $M_d^{(r)}(i, j)$ cannot in general be computed, even for $r = 2$, by twice iterating any isotropic local operation. To see this, consider the four pictures shown in Fig. 22, where the points of S are indicated by asterisks. Since each of the points

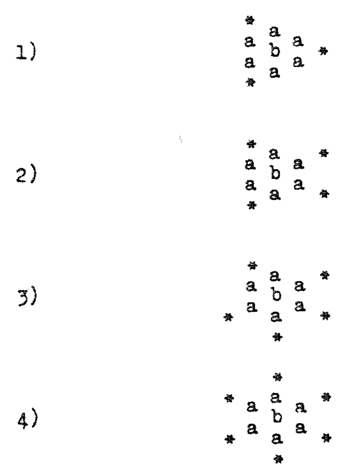


FIG. 22. Examples showing that $M_d^{(2)}(i, j)$ cannot in general be computed using isotropic local operations (hexagonal grid).

marked "a" has exactly one neighbor in S, any isotropic local operation f must give each of them the same value a. Similarly, the points marked "b" have no neighbors in S, so that f must give each of them the same value b. Thus when f is applied a second time, the points marked "b" and their six neighbors have the same values in each of the four cases, so that the "b" points must again all be given the same value by the second application of f. Thus this value cannot be equal to the number of points of S at distance 2 from the "b" (since this number is different in each case), no matter how f is defined.

We now show that $M_d^{(r)}(i,j)$ *can* be computed using simple nonisotropic local operations. Let $H^{(r)}(i,j) = \{(i,j-r),(i,j+r)\}$ be the set of points at distance r from (i,j) in the horizontal direction. Then readily we have (see Fig. 23)

$$H^{(1)}(i,j-1) \cup H^{(1)}(i,j+1) = H^{(2)}(i,j) \cup H^{(0)}(i,j)$$
$$H^{(1)}(i,j-1) \cap H^{(1)}(i,j+1) = H^{(0)}(i,j)$$

and

$$H^{(r)}(i,j-1) \cup H^{(r)}(i,j+1) = H^{(r+1)}(i,j) \cup H^{(r-1)}(i,j)$$
$$H^{(r)}(i,j-1) \cap H^{(r)}(i,j+1) = \phi \qquad r = 2,3,\ldots$$

$$H^{(1)}(i,j-1) = \{(i,j-2),(i,j)\}$$
$$H^{(1)}(i,j+1) = \{(i,j),(i,j+2)\}$$
$$H^{(2)}(i,j) = \{(i,j-2),(i,j+2)\}$$
$$H^{(0)}(i,j) = \{(i,j)\}$$

$$H^{(r)}(i,j-1) = \{(i,j-r-1),(i,j+r-1)\}$$
$$H^{(r)}(i,j+1) = \{(i,j-r+1),(i,j+r+1)\}$$
$$H^{(r+1)}(i,j) = \{(i,j-r-1),(i,j+r+1)\}$$
$$H^{(r-1)}(i,j) = \{(i,j-r+1),(i,j+r-1)\}$$

FIG. 23. Verification of the horizontal direction set recurrence.

Let F be any additive set function, so that $F(\phi) = 0$, and $F(A \cup B) + F(A \cap B) = F(A) + F(B)$ for any sets A, B. Since clearly $H^{(r+1)}(i,j) \cap H^{(r-1)}(i,j) = \phi$, $r = 1, 2, \ldots$, we have

$$F[H^{(2)}(i,j)] = F[H^{(1)}(i,j-1)] + F[H^{(1)}(i,j+1)] - 2F[H^{(0)}(i,j)]$$

and

$$F[H^{(r+1)}(i,j)] = F[H^{(r)}(i,j-1)] + F[H^{(r)}(i,j+1)] - F[H^{(r-1)}(i,j)], \qquad r = 2,3,\ldots$$

We thus have an expression for $F[H^{(r+1)}(i,j)]$ in terms of previously computed F's of (i,j) and its two horizontal neighbors. Since the number of elements of a given set S which are in $H^{(r)}(i,j)$ is certainly an additive set function, we have thus proved

Proposition 14. Let C be a binary picture, and let $h_{ij}^{(r)}$ be the number of elements of C_1 in $\{(i,j-r),(i,j+r)\}$; thus $h_{ij}^{(0)} = c_{ij}$, and $h_{ij}^{(1)} = c_{i,j-1} + c_{i,j+1}$. Moreover, we have

$$h_{ij}^{(2)} = h_{i,j-1}^{(1)} + h_{i,j+1}^{(1)} - 2h_{ij}^{(0)},$$

while

$$h_{ij}^{(r+1)} = h_{i,j-1}^{(r)} + h_{i,j+1}^{(r)} - h_{ij}^{(r-1)}, \qquad r = 2,3,\ldots$$

Using this result, we can now find an expression for the number $x_{ij}^{(r)}$ of elements (h,k) of C_1 such that $d_1[(h,k),(i,j)] = r$. Let $X^{(r)}(i,j)$ be the set of (u,v) such that $d_1[(u,v),(i,j)] = r$.

Then readily we have

$$X^{(r)}(i-1,j) \cup X^{(r)}(i+1,j) \cup H^{(r+1)}(i,j) = X^{(r+1)}(i,j) \cup X^{(r-1)}(i,j);$$

$$X^{(r)}(i-1,j) \cap X^{(r)}(i+1,j) = H^{(r-1)}(i,j), \qquad r = 1, 2, \ldots$$

(See Fig. 24, where the elements of $X^{(r)}(i-1,j)$ are indicated by $-$'s, and the elements of $X^{(r)}(i+1,j)$ by $+$'s.) We can now apply any additive set function F to these sets to obtain an expression for $F[X^{(r+1)}(i,j)]$ in terms of $F[H^{(r+1)}(i,j)]$ and previously computed F's of (i,j) and its two vertical neighbors. We thus have

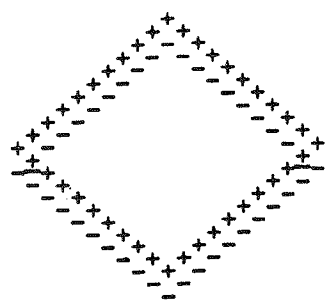


FIG. 24. Case $r = 10$ of the d_1 set recurrence.

Proposition 15. Let $x_{ij}^{(r)}$ be the number of elements (h,k) of C_1 such that $d_1[(h,k),(i,j)] = r$. Then

$$x_{ij}^{(0)} = c_{ij},$$

$$x_{ij}^{(1)} = c_{i-1,j}+c_{i+1,j}+c_{i,j-1}+c_{i,j+1},$$

and

$$x_{ij}^{(r+1)} = x_{i-1,j}^{(r)}+x_{i+1,j}^{(r)}-x_{ij}^{(r-1)}+h_{ij}^{(r+1)}-h_{ij}^{(r-1)}, \qquad r = 1, 2, \ldots$$

(Clearly an alternative method of computing $x_{ij}^{(r)}$ would be to first compute the number $v_{ij}^{(r)}$ of elements of C_1 at distance r from (i,j) in the vertical direction, using the recurrence

$$v_{ij}^{(r+1)} = v_{i-1,j}^{(r)}+v_{i+1,j}^{(r)}-v_{ij}^{(r-1)},$$

and then to compute

$$x_{ij}^{(r+1)} = x_{i,j-1}^{(r)}+x_{i,j+1}^{(r)}-x_{ij}^{(r-1)}+v_{ij}^{(r+1)}-v_{ij}^{(r-1)}.)$$

For the distance function d_2, the situation is slightly more complicated (see Fig. 25), but we can analogously prove

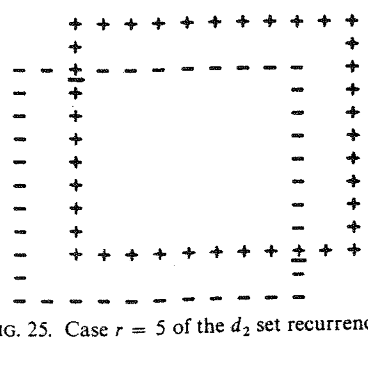


FIG. 25. Case $r = 5$ of the d_2 set recurrence.

Proposition 16. Let $d_{ij}^{(r)}$ be the number of elements of C_1 in $\{(i+r, j+r), (i-r, j-r)\}$; then

$$d_{ij}^{(0)} = c_{ij},$$

$$d_{ij}^{(1)} = c_{i+1,j+1} + c_{i-1,j-1},$$

$$d_{ij}^{(2)} = d_{i+1,j+1}^{(1)} + d_{i-1,j-1}^{(1)} - 2d_{ij}^{(0)},$$

and

$$d_{ij}^{(r+1)} = d_{i+1,j+1}^{(r)} + d_{i-1,j-1}^{(r)} - d_{ij}^{(r-1)}, \qquad r = 2, 3, \ldots$$

Proposition 17. Let $\delta_{ij}^{(r)}$ be the number of elements of C_1 in $\{(i+r, j+r-1), (i+r-1, j+r), (i-r, j-r+1), (i-r+1, j-r)\}$; then

$$\delta_{ij}^{(0)} = \delta_{ij}^{(1)} = x_{ij}^{(1)}, \quad \text{and} \quad \delta_{ij}^{(r+1)} = \delta_{i+1,j+1}^{(r)} + \delta_{i-1,j-1}^{(r)} - \delta_{ij}^{(r-1)}, \qquad r = 1, 2, \ldots$$

Proposition 18. Let $y_{ij}^{(r)}$ be the number of elements (h, k) of C_1 such that $d_2[(h, k), (i, j)] = r$. Then

$$y_{ij}^{(0)} = c_{ij};$$

$$y_{ij}^{(1)} = c_{i-1,j-1} + c_{i-1,j} + c_{i-1,j+1} + c_{i,j-1} + c_{i,j+1} + c_{i+1,j-1} + c_{i+1,j} + c_{i+1,j+1};$$

and

$$y_{ij}^{(r+1)} = y_{i-1,j+1}^{(r)} + y_{i+1,j-1}^{(r)} - y_{ij}^{(r-1)} + \delta_{ij}^{(r+1)} - \delta_{ij}^{(r)} + d_{ij}^{(r)} - d_{ij}^{(r-2)}, \qquad r = 1, 2, \ldots$$

(Here too there is an obvious alternative approach in which the roles of the two diagonal directions are interchanged.)

For convenience in describing an analogous algorithm for d_3, we regard it as applying to a hexagonal array, on which we define coordinates using the horizontal and 60° directions as axes. We then have (see Fig. 26)

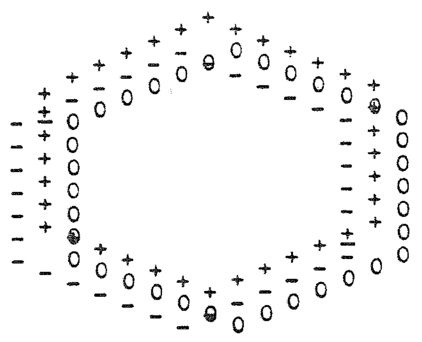


FIG. 26. Case $r = 6$ of the d_3 set recurrence.

Proposition 19. Let $z_{ij}^{(r)}$ be the number of elements (h, k) of C_1 such that $d_3[(h, k), (i, j)] = r$. Then

$$z_{ij}^{(r+1)} = z_{i,j+1}^{(r)} + z_{i-1,j-1}^{(r)} + z_{i+1,j}^{(r)} - z_{ij}^{(r)} - z_{ij}^{(r-1)}$$
$$+ c_{i-r,j} + c_{i,j-r} + c_{i+r,j+r} + c_{i,j+r-1} + c_{i+r-1,j} + c_{i-r+1,j-r-1}$$
$$- c_{i-r+1,j} - c_{i,j-r+1} - c_{i+r-1,j-r+1} - c_{i,j+r-2} - c_{i+r-2,j}$$
$$- c_{i-r+2,j-r+2}, \qquad r = 2, 3, \ldots$$

Here the c's can be "computed" by repeated translation of the picture; $z_{ij}^{(2)}$ can be computed directly in the same way. (One could also define an analogous algorithm using $z_{i,j-1}^{(r)}$, $z_{i-1,j}^{(r)}$, and $z_{i+1,j+1}^{(r)}$.) Evidently, these algorithms have little advantage over direct computation of $z_{ij}^{(r)}$ by repeated ("spiraling") translation of the picture, unless r is very large. The situation is even worse in the case of d_4, for which an analogous algorithm would involve at least ten recurrence relations, as well as many single-point "correction terms."

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